Power Flow Theory

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1 Nomenclature

$$
\begin{aligned}
y_{ik} &= g_{ik} + jb_{ik} = \text{complex admittance along branch } ik. \\
Y_{ik} &= G_{ik} + jB_{ik} = \text{nodal admittance matrix element } i, k. \\
&= \begin{cases} -y_{ik} & \text{if } i \neq k \\ y_i + \sum_l y_{il} & \text{if } i = k \end{cases} \\
V_i &= \text{complex voltage at bus } i. \\
M_i &= \text{voltage magnitude at bus } i,\ M_i := |V_i|. \\
I_{\mathrm{br}i} &= \text{complex current injection from branches at bus } i. \\
I_{\mathrm{ld}i} &= \text{total complex current injection from load at bus } i. \\
I_{\mathrm{bus}i} &= \text{complex current injection due to the bus at bus } i. \\
S_{\mathrm{bus}i} &= \text{complex power injection due to bus } i. \\
S_{ci} &= \text{constant power component of load at bus } i. \\
I_{ci} &= \text{constant current injection component of load.} \\
y_{ci} &= \text{constant impedance component of load.} \\
\delta_{ik} &= \text{the Kronecker delta, } \delta_{ik} = 1 \text{ if } i = k, \text{ 0 otherwise.}
\end{aligned}
$$

2 Power flow equations in current form

For simplicity, we write all bus quantities as injections *into* the bus. Thus a normal load will use quantities expressed as negative injections, while a generator will have positive injections.

Each bus has an associated load, containing a constant power component, a constant current component, and a constant shunt impedance current. As well as receiving injections from branches and loads, busses provide their own injections according to their bus type. PQ busses provide a specified complex power injection. PV busses instead keep the voltage magnitude of the bus constant while providing a specified real power injection. Slack busses keep the complex voltage of the bus constant.

The total current injection into bus i is:

$$I_i = I_{\mathrm{br},i} + I_{\mathrm{ld}i} + I_{\mathrm{bus},i} = 0 \tag{1}$$

$$I_{\mathrm{bri}} = -\sum_{k=0}^{N} Y_{ik} V_k \tag{2}$$

$$I_{\mathrm{ld}i} = \frac{S_{ci}^*}{V_i^*} + I_{ci} - y_{ci} V_i \tag{3}$$

$$I_{\mathrm{bus}i} = \frac{S_{\mathrm{bus}i}^*}{V_i^*} \tag{4}$$

which is zero, due to Kirchoff's current conservation law. Thus,

$$I_i = \frac{S_{ci}^* + S_{\mathrm{bus}i}^*}{V_i^*} + I_{ci} - y_{ci} V_i - \sum_{k=0}^{N} Y_{ik} V_k = 0 \tag{5}$$

or, absorbing y_c into Y and S_{bus} into S_c we have

$$I_i = \frac{S_{ci}'^*}{V_i^*} + I_{ci} - \sum_{k=0}^{N} Y_{ik}' V_k = 0 \tag{6}$$

where

$$Y_{ik}' = Y_{ik} + y_{ci} \delta_{ki} \tag{7}$$

and $S_c' = S_c + S_{\mathrm{bus}}$.

Real and imaginary components are:

$$I_{Ri} = \frac{P_{ci}' V_{Ri} + Q_{ci}' V_{Ii}}{M_i^2} + I_{cRi} + \sum_{k=0}^{N} \left(-G_{ik}' V_{Rk} + B_{ik}' V_{Ik} \right) \tag{8}$$

$$I_{Ii} = \frac{P_{ci}' V_{Ii} - Q_{ci}' V_{Ri}}{M_i^2} + I_{cIi} + \sum_{k=0}^{N} \left(-G_{ik}' V_{Ik} - B_{ik}' V_{Rk} \right) = 0 \tag{9}$$

2.1 Newton-Raphson equations

For PQ busses, the unknowns are the real and imaginary parts of V, so this equation can be solved using the Newton-Raphson method. Letting the function to which we want to find the zero be $f = \{I_R, I_I\}$, the unknows be $x = \{V_R, V_I\}$, we wish to solve $f(x) = 0$. Using the Jacobian

$$J_{ik}(x) = \frac{\partial f_i(x)}{\partial x_k} \tag{10}$$

the NR method calculates the update to x at each iteration as the solution to the linear equations

$$-f_{(n)} = J(x_{(n)})(x_{(n+1)} - x_{(n)}) = J(x_{(n)})\Delta x_{(n,n+1)} \tag{11}$$

The Jacobian is given by:

$$\frac{\partial I_{Ri}}{\partial V_{Rk}} = \left[-\frac{2V_{Rk}(P'_{ck}V_{Rk} + Q'_{ck}V_{Ik})}{M_k^4} + \frac{P'_{ck}}{M_k^2} \right] \delta_{ik} - G'_{ik} \tag{12}$$

$$\frac{\partial I_{Ri}}{\partial V_{Ik}} = \left[-\frac{2V_{Ik}(P'_{ck}V_{Rk} + Q'_{ck}V_{Ik})}{M_k^4} + \frac{Q'_{ck}}{M_k^2} \right] \delta_{ik} + B'_{ik} \tag{13}$$

$$\frac{\partial I_{Ii}}{\partial V_{Rk}} = \left[-\frac{2V_{Rk}(P'_{ck}V_{Ik} - Q'_{ck}V_{Rk})}{M_k^4} - \frac{Q'_{ck}}{M_k^2} \right] \delta_{ik} - B'_{ik} \tag{14}$$

$$\frac{\partial I_{Ii}}{\partial V_{Ik}} = \left[-\frac{2V_{Ik}(P'_{ck}V_{Ik} - Q'_{ck}V_{Rk})}{M_k^4} + \frac{P'_{ck}}{M_k^2} \right] \delta_{ik} - G'_{ik} \tag{15}$$

$$\tag{16}$$

2.2 PV busses

For a PV bus k, the power flow equations also hold, but with Q_{bus} being considered as a variable rather than a constant, and an extra constraint:

$$\Delta M_k^2 = V_{Rk}^2 + V_{Ik}^2 - M_{\text{PV}k}^2 = 0 \tag{17}$$

The corresponding rows in the NR equation may be solved by hand, with the following update:

$$\Delta V_{Rk} = \frac{M_{\text{PV}k}^2 - V_{Rk}^2 - V_{Ik}^2 - 2V_{Ik}\Delta V_{Ik}}{2V_{Rk}} \tag{18}$$

Thus, ΔV_R may be eliminated from the NR equations. First write the Jacobian as if all busses were PQ. Let k be a PV bus. Take the column corresponding to ΔV_{Rk}, and add its product with $-V_{Ik}/V_{Rk}$ to the matching column for ΔV_{Ik}. Add its product with $(M_{\text{PV}k}^2 - V_{Rk}^2 - V_{Ik}^2)/(2V_{Rk})$ to f. The column and the corresponding element of x will now be replaced to correspond to Q_k. Set the column to zero, and set the block diagonal elements, using:

$$\frac{\partial I_{Rk}}{\partial Q_{\text{bus}k}} = \frac{V_{Ik}}{M_k^2} \tag{19}$$

$$\frac{\partial I_{Ik}}{\partial Q_{\text{bus}k}} = -\frac{V_{Rk}}{M_k^2} \tag{20}$$

3 Multi phase lines

Lines are typically handled in terms of a, b, c and d matrices. Assume for discussion that there are three phases, so these are 3×3 matrices, and V and I are assumed to be 3-vectors.

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \tag{21}$$

Be careful of signs! Conventionally, the equations are written in terms of currents entering and leaving the line. But we'll move to a nodal admittance model, and hence all currents are treated as injections. This is why we have a negative sign on b and d.

These equations can be transformed to the following:

$$\begin{bmatrix} I_0 \\ I_1 \end{bmatrix} = \begin{bmatrix} db^{-1} & c - db^{-1}a \\ -b^{-1} & b^{-1}a \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} = Y \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} \tag{22}$$

which defines the nodal admittance matrix Y.

What is the interpretation of a, b, c, d? From Kersting [**?**], using a pi-model of a transmission line, Z being the matrix of self and cross impedances of the line, U being the identity matrix and $Y_s/2$ being the shunt admittance matrix for each of the two legs of the pi, we have

$$a = d = U + \frac{1}{2} Z Y_s \tag{23}$$

$$b = Z \tag{24}$$

$$c = Y_s + \frac{1}{4} Y_s Z Y_s \tag{25}$$

and thus

$$Y = \begin{bmatrix} Z^{-1} + \frac{1}{2} Z Y_s Z^{-1} & \frac{1}{2} Y_s + \frac{1}{2} Y_s Z Y_s - Z^{-1} - \frac{1}{2} Z Y_s Z^{-1} - \frac{1}{4} Z Y^2 \\ -Z^{-1} & Z^{-1} + \frac{1}{2} Y \end{bmatrix} \tag{26}$$

3.1 Short three wire lines

For three phase delta lines of up to around 80 km in length, the shunt admittance Y_s is small enough to neglect, and thus the nodal admittance matrix is:

$$Y = \begin{bmatrix} Z^{-1} & -Z^{-1} \\ -Z^{-1} & Z^{-1} \end{bmatrix} \tag{27}$$

which is very reminiscent of the expression used in the single wire definition of nodal admittance.

4 Transformers

4.1 Single phase transformers

For an ideal transformer with a single turns ratio $r = n_0/n_1$, we have

$$\begin{bmatrix} V_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & -1/n \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \tag{28}$$

This can't be modelled correctly in the formalism of nodal admittance.

A real transformer includes a leakage impedance (due to finite resistance of copper windings and core losses) and a shunt magnetising impedance. The latter is often large and may often be ignored. The nodal admittance matrix may be then derived:

$$\begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} y_l/|a|^2 & -y_l/a^* \\ -y_l/a & y_l \end{bmatrix} \begin{bmatrix} V_P \\ V_S \end{bmatrix} \tag{29}$$

where $a = V_P/V_S = N_P/N_S$ for an ideal transformer.

4.2 Three-phase transformers

The nodal admittance matrices of three-phase transformers may be derived from the single-phase expression, above, combined with information about the connections between phases.

4.2.1 Delta-GWye

Considering Fig. 1, we have,

$$I_{AB} = \frac{y_l}{|a|^2}(V_A - V_B) - \frac{y_l}{a^*}V_a \tag{30}$$

$$I_{BC} = \frac{y_l}{|a|^2}(V_B - V_C) - \frac{y_l}{a^*}V_b \tag{31}$$

$$I_{CA} = \frac{y_l}{|a|^2}(V_C - V_A) - \frac{y_l}{a^*}V_c \tag{32}$$

$$I_a = -\frac{y_l}{a}(V_A - V_B) + y_l V_a \tag{33}$$

$$I_b = -\frac{y_l}{a}(V_B - V_C) + y_l V_b \tag{34}$$

$$I_c = -\frac{y_l}{a}(V_C - V_A) + y_l V_c \tag{35}$$

$$\tag{36}$$

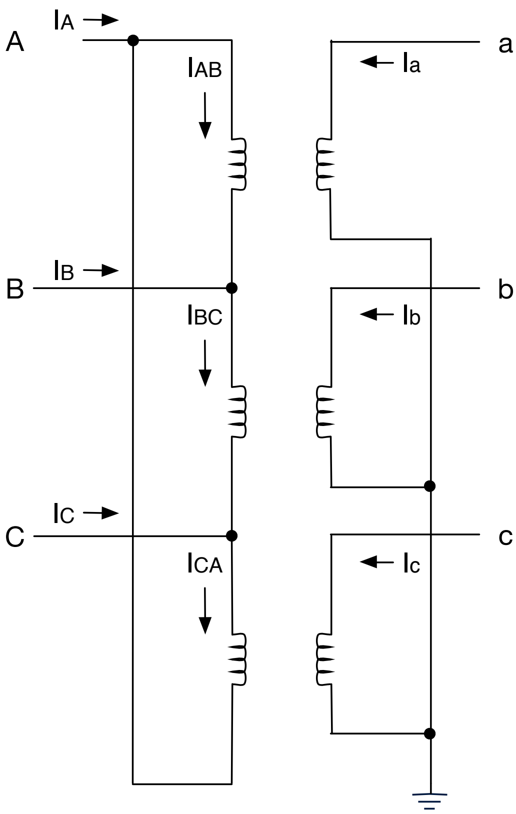

Figure 1: Schematic of a Delta-GWye transformer

Also, by the KCL, we have

$$\begin{aligned}
I_A &= I_{AB} - I_{CA} && (37)\\
&= \frac{y_l}{|a|^2}(2V_A - V_B - V_C) + \frac{y_l}{a^*}(V_c - V_a) && (38)\\
I_B &= I_{BC} - I_{AB} && (39)\\
&= \frac{y_l}{|a|^2}(2V_B - V_C - V_A) + \frac{y_l}{a^*}(V_a - V_b) && (40)\\
I_C &= I_{CA} - I_{BC} && (41)\\
&= \frac{y_l}{|a|^2}(2V_C - V_A - V_B) + \frac{y_l}{a^*}(V_b - V_c) && (42)
\end{aligned}$$

So we can immediately write down the nodal admittance relationship:

$$\begin{bmatrix} I_A \\ I_B \\ I_C \\ I_a \\ I_b \\ I_c \end{bmatrix} = y_l \begin{bmatrix} 2/|a|^2 & -1/|a|^2 & -1/|a|^2 & -1/a^* & 0 & 1/a^* \\ -1/|a|^2 & 2/|a|^2 & -1/|a|^2 & 1/a^* & -1/a^* & 0 \\ -1/|a|^2 & -1/|a|^2 & 2/|a|^2 & 0 & 1/a^* & -1/a^* \\ -1/a & 1/a & 0 & 1 & 0 & 0 \\ 0 & -1/a & 1/a & 0 & 1 & 0 \\ 1/a & 0 & -1/a & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \\ V_a \\ V_b \\ V_c \end{bmatrix} \quad (43)$$

with the nodal admittance matrix being specified by the matrix on the right, including the factor of y_l.