Assignment 7

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March 28, 2018

Problem 1

Let f be a real-valued function defined as $f(x) = x^2$ on $Dom(f) = \mathbb{R}$, and let $\varepsilon > 0$ be given. Find a δ so that $|x-1| \leq \delta$ implies $|f(x) - 1| \leq \varepsilon$.

We want to show

$$\begin{aligned}
|f(x) - 1| &\leq \varepsilon \\
|x^2 - 1| &\leq \varepsilon \\
|(x+1)(x-1)| &\leq \varepsilon \\
|(x-1+2))(x-1)| &\leq \varepsilon \\
|x-1+2||x-1| &\leq \varepsilon \\
(|x-1| + |2|)|x-1| &\leq \varepsilon \\
(\delta + 2)\delta &\leq \varepsilon \\
\delta^2 + 2\delta &\leq \varepsilon
\end{aligned}$$

We can solve $\delta^2 + 2\delta \leq \varepsilon$ to get an equation for δ in terms of ε. Take $\delta = \sqrt{\varepsilon + 1} - 1$.

Problem 2

Prove that the real-valued function $f(x) = 1/x$ defined on $Dom(f) = [1, \infty)$ is uniformly continuous under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$.

To show uniform continuity we would like to show that given any $\varepsilon > 0$, we can select a δ independent of x such that $\rho(x, y) \leq \delta \to \rho(f(x), f(y)) \leq \varepsilon$. So consider some $\varepsilon > 0$, and x, y such that $|x - y| \leq \delta$.

$$\begin{aligned}
\rho(f(x), f(y)) &\leq \varepsilon \\
|f(x) - f(y)| &\leq \varepsilon \\
|1/x - 1/y| &\leq \varepsilon \\
|\frac{y-x}{xy}| &\leq \varepsilon \\
\frac{|y-x|}{|xy|} &\leq \varepsilon \\
|y - x| &\leq \varepsilon \cdot |x| \cdot |y| \\
\delta &\leq \varepsilon \cdot x \cdot y \\
\delta &\leq \varepsilon \leq \varepsilon \cdot x \cdot y
\end{aligned}$$

Where we know that $\varepsilon \leq \varepsilon \cdot x \cdot y$ since $x, y \in [1, \infty)$. Thus we can select $\delta \leq \varepsilon$ independent of x to satisfy uniform continuity. For example, we can always select $\delta = \varepsilon/2$.

Problem 3

Let (X, σ) be a metric space and x_0 a point in X. Prove that the function $f : X \to \mathbb{R}$ defined as $f(x) = \sigma(x, x_0)$ is continuous under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$.

To show continuity we would like to show that given any $\varepsilon > 0$, we can select a δ such that $\forall\, a, b\,.\, \sigma(a,b) \leq \delta \to |f(a) - f(b)| \leq \varepsilon$

We can use the triangle inequality of σ to note that

$$\begin{aligned} |f(a) - f(b)| &\leq \varepsilon \\ |\sigma(a, x_0) - \sigma(b, x_0)| &\leq \varepsilon \\ |\sigma(a,b)| \leq |\sigma(a, x_0) - \sigma(b, x_0)| &\leq \varepsilon \\ \delta &\leq \varepsilon \end{aligned}$$

Thus by taking $\delta \leq \varepsilon$, for example $\delta = \varepsilon/2$, we can satisfy the criteria for continuity.

Problem 4

Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, and $f : X \to Y$ a function with domain $Dom(f) = X$ and range $Ran(f) = Y$. Prove that f is continuous if and only if $f^{-1}[\tilde{Y}]$ (the preimage of $\tilde{Y}$) is open for every open set $\tilde{Y}$ in Y.

<u>If f is continuous, then $f^{-1}[\tilde{Y}]$ is open for every open set $\tilde{Y} \in Y$.</u>

Recall that if $\exists$ a limit point a not in set A, then A is open. Also recall that a limit point a is a point such that $\exists$ a non-constant sequence of $\{a_n\} \subseteq A$ such that $\{a_n\} \to a$. Thus we must show that there exists $\{x_n\} \subseteq f^{-1}[\tilde{Y}]$ such that $\{x_n\} \to x$ where $x \notin f^{-1}[\tilde{Y}]$. Consider an open set $\tilde{Y} \in Y$. Since this set is open, there exists some non-constant sequence $\{y_n\} \to y$ where $y \notin \tilde{Y}$ but $\{y_n\} \subseteq \tilde{Y}$. Thus, we can take the preimage of all $\{y_n\}$ to be a non-constant sequence $\{x_n\} \subseteq f^{-1}[\tilde{Y}]$. But, it is also clear that $x = f^{-1}(y) \notin f^{-1}[\tilde{Y}]$ because $y \notin \tilde{Y}$. Because f is continuous, we know that$\{x_n\} \to x$ where $x \notin f^{-1}[\tilde{Y}]$. Thus, we know that $f^{-1}[\tilde{Y}]$ is open.

<u>If for every open set $\tilde{Y} \in Y$ we know that $f^{-1}[\tilde{Y}]$ is open, then f must be continuous.</u>

As shown in class, an equivalent condition for continuity is $\forall\, \{x_n\} \subseteq X, x_n \to x_0$ implies $f(x_n) \to f(x_0)$. Consider some arbitrary $\{x_n\} \subseteq X$.

Problem 5

Let f be the function on $[0,1]$ given by

$$f(x) = \begin{cases} 1 \; if \;\; x \neq \frac{1}{2} \\ 2 \; if \;\; x = \frac{1}{2} \end{cases}$$

Prove that f is Riemann integrable and compute $\int_0^1 f(x)dx$. Hint: for each $\varepsilon > 0$, find a partition P so that $U_P(f) - L_P(f) \leq \varepsilon$.

Given any ε, we can select the partition $P = \left\{0, \frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, 1\right\}$. Thus

$$\begin{aligned} U_P(f) &= 1\left(\frac{1}{2} - \frac{\varepsilon}{2} - 0\right) + 2\left(\frac{1}{2} + \frac{\varepsilon}{2} - \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\right) + 1\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\right) \\ &= 1 + \varepsilon \end{aligned}$$

$$\begin{aligned} L_P(f) &= 1\left(\frac{1}{2} - \frac{\varepsilon}{2} - 0\right) + 1\left(\frac{1}{2} + \frac{\varepsilon}{2} - \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\right) + 1\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\right) \\ &= 1 \end{aligned}$$

So, $U_P(f) - L_P(f) = 1 + \varepsilon - 1 = \varepsilon$, which by the Riemann integrability test, shows that f is integrable. The value of $\int_0^1 f(x)dx = sup_P[L_P(f)] = 1$.