Assignment 4

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Problem 1

Let (X, ρ) be a metric space, and $f : [0, \infty) \to [0, \infty)$ be an increasing concave function such that $f(r) = 0$ if and only if $r = 0$. Prove that $f \circ \rho$ is also a metric on X. Hint: f being concave means that $\forall p, q \in [0, \infty)$, we have $f(tp+(1-t)q) \geq tf(p)+(1-t)f(q)$ $\forall t \in [0, 1]$. We can show here that f is subadditive in the following way: (i) take $q = 0$, we can see that $f(tp) \geq tf(p)$; (ii) $f(p) + f(q) = f(\frac{p}{p+1}(p + q)) + f(\frac{q}{p+q}(p + q)) \geq \frac{p}{p+q}f(p + q) + \frac{q}{p+q}f(p + q) = f(p + q)$

To be a metric space, $f \circ \rho$ must satisfy the following axioms: non-negativity, symmetry, and the triangle inequality.

Non-negativity: Since, by definition, f is an increasing function that maps zero to zero, we know that $\forall\ p \in [0, \infty)\ .\ f(p) \in [0, \infty)\ \geq 0$. Similarly, since ρ is a metric it is a function $\rho : X \times X \to [0, \infty)$. Thus, $\forall x, y \in X\ .\ f(\rho(x, y)) \in [0, \infty)$ so it satisfies non-negativity.

Symmetry: Consider arbitrary $x, y \in X$. Then, $\exists z \in [0, \infty)$ such that $\rho(x, y) = \rho(y, x) = z$. Because f is a well-defined function, we know that $f(z) = f(z)$. Thus $\forall x, y \in X\ .\ f(\rho(x, y)) = f(\rho(y, x))$. Thus, $f \circ \rho$ satisfies symmetry.

Triangle-inequality: We first note that $f(\rho(x, y))+f(\rho(y, z)) \geq f(\rho(x, y)+\rho(y, z))$ because f is subadditive. We also note that $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ because of the triangle inequality in the metric ρ. This implies that $f(\rho(x, y) + \rho(y, z)) \geq f(\rho(x, z))$ because f is an increasing function. Thus we know that $f(\rho(x, z)) \geq f(\rho(x, z))$, which means that $f \circ \rho$ satisfies the triangle inequality.

Problem 2

Let $X = \mathbb{R}^2$. Define $\rho_1(x, y) \equiv |x_1 - y_1| + |x_2 - y_2|, \rho_2(x, y) \equiv \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ and $\rho_{max}(x, y) \equiv max(|x_1 - y_1|, |x_2 - y_2|)$. Prove that $\rho_1, \rho_2, \rho_{max}$ are uniformly equivalent.

Let $a = |x_1 - y_1|, b = |x_2 - y_2|$. Then we note that $\rho_1^2 = (a+b)^2 = a^2+2ab+b^2$, and similarly $\rho_2^2 = a^2+b^2$. But since $(a - b)^2 \geq 0 \to a^2 + b^2 \geq 2ab$, we know that $2 \cdot \rho_2^2 \geq \rho_1^2$. Thus taking the square-root of both sides gives us the constant $c_1 = \sqrt{2}$ to show that $c_1\rho_2 \geq \rho_1$. It is also clear that $a^2 + 2ab + b^2 \geq a^2 + b^2 \to \rho_1^2 \geq \rho_2^2$ which gives us the constant $c_1 = \sqrt{1} = 1$. Thus ρ_1, ρ_2 are uniformly equivalent.

We will now show that ρ_1 is uniformly equivalent to ρ_{max}. WLOG we can assume that $a \geq b$. Thus, $2a = 2\rho_{max} \geq \rho_1$. Thus we know that the constant $c_1 = 2$ satisfies $c_1\rho_{max} \geq \rho_1$. It is also clear that since $b \geq 0$, we know that the constant $c_2 = 1$ satisfies $c_2 \cdot \rho_1 \geq \rho_{max}$. Thus, ρ_{max} and ρ_1 are uniformly equivalent.

Since ρ_1 is uniformly equivalent to both ρ_2 and ρ_{max}, it is obvious that ρ_2 must be uniformly equivalent to ρ_{max}.

Problem 3

Consider $a, b \in \mathbb{R}$. Prove the following statements

a)

The set $X = (a, b)$, with the metric $\rho(x, y) = |x - y|$, is open

Take arbitrary $c \in (a, b)$, and define $\epsilon = \frac{min(\rho(a,c),\rho(b,c))}{2}$ so $B_\epsilon(c) \subseteq (a, b)$ and thus (a, b) is open.

b)

The set $X = [a, b]$, with the metric $\rho(x, y) = |x - y|$, is closed.

We will show that $X^C = (-\infty, a) \cup (b, \infty)$ is open. Take arbitrary $c \in X^C$, then either $c < a$ or $c > b$. If $c < a$ then we can take $\epsilon = \frac{\rho(a,c)}{2}$ to get the ball $B_\epsilon(c) \subseteq (-\infty, a) \subseteq X^C$. Similarly if $c > b$ then we can take $\epsilon = \frac{\rho(b,c)}{2}$ to get the ball $B_\epsilon(c) \subseteq (b, \infty) \subseteq X^C$. Thus X^C is open, which implies that X is closed.

c)

The set $X = (a, b]$, with the metric $\rho(x, y) = |x - y|$, is neither open nor closed.

Since b is the greatest element of the set, there is no positive ε such that $b + \varepsilon \in (a, b]$. Thus $(a, b]$ cannot be open.

However, $X^C = (-\infty, a] \cup (b, \infty)$ is also not open because no element of (a, b) can be included in an open ball centered on a. Since $a \neq b$, we know that such an element exists, and therefore no satisfying ball can exist. Thus, since X^C is not open, we know that X cannot be closed.

Problem 4

Let X be any non-empty set. We define $\rho : X \times X \to [0, \infty)$ as:

$$\rho(x, y) = \begin{cases} 0, if \ x = y \\ 1, if \ x \neq y \end{cases}$$

Then it can be shown that ρ is a metric on X. Therefore, (X, ρ) is a metric space. Such a metric space is often called a discrete metric space. Let (X, ρ) be a discrete metric space. Prove the following statements.

i)

An open ball in X is either a set with only one element (that is, a singleton) or all of X.

Consider some open ball in X $B_r(x)$ with some center x. Because the radius of $B_r(x)$ must be positive, we know that it contains at least one element x. If it contains some $y \neq x$ that implies that the radius must be at least 1, since $\forall y \in x \ . \ x \neq y \to \rho(x, y) = 1$. But if the radius is at least one the fact that $\forall y \in x \ . \ x \neq y \to \rho(x, y) = 1$ implies that the open ball must also contain all of X.

ii)

All subsets of X are both open and closed.

Lemma: every subset of X is open. Consider some subset S of X. Then we can take a ball with radius of $0 < \varepsilon < 1$, which will mean that this ball contains every element of X, but does not contain any element not in X. Thus, S is open.

Second, consider the complement of a subset in X referred to as S^C. Note that $S^C = X - S \subseteq X$. Thus, by the previous lemma, we know that S^C must be similarly open.

Thus, any subset of X is clopen.

Problem 5

Let (X, ρ) be a metric space and $S \subseteq X$ a subset. Denote by $\overline{S}$ the set of points of closure of S. Prove that $\overline{S}$ is a closed set.

To prove that $\overline{S}$ is closed, we must prove that $\overline{S}^C$ is open. The elements of $\overline{S}^C$ are the elements such that $\neg \forall \varepsilon > 0 \ . \ \exists y \in B_\varepsilon(x)$ such that $y \in S$.