Assignment 5

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February 22, 2018

Problem 1

Prove directly, using the definition of convergence, that each of the following sequences converges in the metric space (X, ρ) with $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$:

Part a

The sequence $\{x_n\}$ with $x_n = 1 + \frac{10}{\sqrt{n}}$

Proof that $x_n \to 1$. Consider some $\varepsilon > 0$, then we want to show that there exists some N such that any $n > N$ satisfies $|1 + \frac{10}{\sqrt{n}} - 1| < \varepsilon$. We can select this $N = (\frac{10}{\varepsilon})^2$ to satisfy the inequality.

Part b

The sequence $\{x_n\}$ with $x_n = 3 + 2^{-n}$

Proof that $x_n \to 3$. Consider some $\varepsilon > 0$, then we want to show that there exists some N such that any $n > N$ satisfies $|3 + 2^{-n} - 3| < \varepsilon$. We can select this $N = log_2(\frac{1}{\varepsilon})$ to satisfy the inequality.

Part c

The sequence $\{x_n\}$ with $x_n = \frac{2n+3}{n+1}$

Proof that $x_n \to 2$. Consider some $\varepsilon > 0$, then we want to show that there exists some N such that any $n > N$ satisfies $|2 - \frac{2n+3}{n+1}| < \varepsilon$. Note that $\frac{2n+3}{n+1} = \frac{n}{n} \cdot \frac{2+\frac{3}{n}}{1+\frac{1}{n}} = \frac{2+\frac{3}{n}}{1+\frac{1}{n}} = \frac{2}{1+\frac{1}{n}} + \frac{\frac{3}{n}}{1+\frac{1}{n}} = \frac{2}{1+\frac{1}{n}} + \frac{3}{n+1}$

It is clear that $x_n = \frac{3}{n+1} \to 0$ with $N_1 = \frac{3}{\varepsilon} - 1$, and $x_n = \frac{1}{n} \to 0$ with some $N_2 = \frac{1}{\varepsilon}$. Thus we can take $N = max(N_1, N_2)$ to see that $\frac{2}{1+\frac{1}{n}} + \frac{3}{n+1} \to 2$.

Problem 2

Let (X, ρ) be a discrete metric space, and $\{x_n\}$ a sequence in X. Prove that $x_n \to x$ if and only if there exists a $N \in \mathbb{N}$ such that $x_n = x \forall n \geq N$.

<u>If $x_n \to x$ then there exists a $N \in \mathbb{N}$ such that $x_n = x \forall n \geq N$:</u>

$x_n \to x$ implies that $\forall \varepsilon > 0. \exists N(\varepsilon). \rho(x, x_n) \leq \varepsilon \forall n \geq N$. We will show that this N satisfies the property that $x_n = x \forall n \geq N$ by contradiction. Assume it did not, so there would exist some $n \geq N$ such that $x_n \neq x$, which means that $\rho(x_n, x) = 1$ because ρ is a discrete metric space. But, then we could select any $\varepsilon > 1$ such that $\varepsilon > \rho(x, x_n)$, which contradicts our hypothesis.

<u>If there exists a $N \in \mathbb{N}$ such that $x_n = x \forall n \geq N$ then $x_n \to x$:</u>

Since $x_n = x \forall n \geq N$, we know that $\rho(x_n, x) = 0 \forall n \geq N$ because ρ is the discrete metric. Thus, for any positive ε, it is clear that $\rho(x, x_n) < \varepsilon$, which implies that $x_n \to x$.

Problem 3

Let ρ, σ be two uniformly equivalent metrics defined on X and $\{x_n\}$ be a sequence in X. Show that $x_n \to x$ in metric ρ iff $x_n \to x$ in metric σ.

Assume $x_n \to x$ in ρ, which means that $\forall \varepsilon > 0. \exists N(\varepsilon). \rho(x, x_n) \leq \varepsilon \forall n \geq N$. Also, since ρ, σ are uniformly equivalent, we know that $\exists c > 0. c\sigma \leq \rho$.

From #2, we know that there exists N such that $\rho(x, x_n) = 0$ for all $n \geq N$. But since ρ, σ are uniformly equivalent, there exists some constant $c > 0$ such that $\sigma \leq \rho \cdot c$. But for $n > N$, $c \cdot \rho(x, x_n) = 0$ because $\rho(x, x_n) = 0$. Thus since $\sigma(x, x_n) \geq 0$ (since its a metric) and $\sigma(x, x_n) \leq 0$ (as shown above) we know that $\sigma(x, x_n) = 0 \forall n \geq N$ (by anti-symmetry). Thus, again by the lemma proved in question 2, $x_n \to x$ in σ.