

Assignment 2

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Problem 1

Prove that the set $S \equiv \{5, 10, 15, 20, \dots\}$ is countable by constructing a one-to-one function from S onto \mathbb{N}

First note that $S = \{5, 10, 15, 20, \dots\} = \{5k \mid k \in \mathbb{N}\}$. Thus, we can construct a one-to-one function $f : S \rightarrow \mathbb{N}$ by defining $f(5k) \mapsto k$.

Problem 2

Part a

The union of two finite sets is finite.

Consider two finite sets: X with cardinality n and Y with cardinality m such that WLOG $X = \{1, \dots, n\}$ and $Y = \{1, \dots, m\}$. Then there exists a function with $\text{Dom}(\{1, \dots, n\} \cup \{1, \dots, m\})$ that is bijective with $\{1, \dots, n+m\}$

$$f(z) = \begin{cases} z, & z \in \{1, \dots, n\} \\ n+z, & z \in \{1, \dots, m\} \end{cases}$$

Thus, $X \cup Y$ is finite with cardinality $n+m$.

Part b

The union of a finite set and a countable set is countable.

Consider a finite set that has cardinality n which (WLOG) can be stated as $X = \{1, \dots, n\}$ and a countable set which (WLOG) can be stated as $Y = \mathbb{N}$. Then we can define a bijection $f : X \cup Y \rightarrow \mathbb{N}$

$$f(z) = \begin{cases} z, & z \in X \\ n+z, & z \in Y \end{cases}$$

Thus $X \cup Y$ is countable.

Part c

The union of two countable sets is countable.

Consider two countable sets $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$. Then we can define a bijection $f : \mathbb{N} \rightarrow X \cup Y$

$$f(z) = \begin{cases} x_z, & z \text{ is even} \\ y_z, & z \text{ is odd} \end{cases}$$

Thus $X \cup Y$ is countable.

Problem 3

Rational numbers are defined as real numbers that can be written in the form $\frac{m}{n}$, $n \neq 0$ with m and n integers without common factors. The set of rational numbers, \mathbb{Q} , can be split into three parts, the positive ones \mathbb{Q}_+ , the negative ones \mathbb{Q}_- , and the set that contains only zero $\{0\}$, $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$. Prove that \mathbb{Q} is countable.

Hint: We can first show that \mathbb{Q}_+ is countable by constructing the function $f : \frac{m}{n} \mapsto (m, n)$, $f : \mathbb{Q} \mapsto U \subset \mathbb{N} \times \mathbb{N}$ that is one-to-one with $\text{Dom}(f) = \mathbb{Q}_+$. $\mathbb{N} \times \mathbb{N}$ is countable, so U is countable, so \mathbb{Q}_+ is countable. We then use the results in the previous problem 2.

We can define $f_1 : \mathbb{Q}_+ \rightarrow U_1 \subset \mathbb{N} \times \mathbb{N}$ by $f_1(\frac{m}{n}) \mapsto (m, n)$. Note that negative rationals can all be represented with positive denominators by multiplying any negative rational with a negative denominator by $\frac{-1}{-1} = 1$. Thus, we can define $f_2 : \mathbb{Q}_- \rightarrow U_2 \subset \mathbb{N} \times \mathbb{N}$ by $f_2(\frac{m}{n}) \mapsto (|m|, n)$. With these two functions, we see that $\mathbb{Q}_-, \mathbb{Q}_+$ are both countable. Note that it is clear that $\{0\}$ is finite. Since the union of countable sets and the union of finite sets are countable by the previous problem, we can see that $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ is countable.

Problem 4

Let ρ be a metric on X . Prove that the following are also metrics

Part a

$$\rho_1 \equiv 5\rho$$

Part b

$$\rho_2 \equiv \min\{1, \rho\}$$

Problem 5

Let $X = (0, 1]$. Define $\rho(x, y) \equiv \left| \frac{1}{x} - \frac{1}{y} \right|$. Prove that ρ is a metric on X .