Assignment 8

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Problem 1

Let f be a real-valued function defined as $f(x) = x^2$ on $Dom(f) = \mathbb{R}$, and let $\varepsilon > 0$ be given. Find a δ so that $|x - 1| \leq \delta$ implies $|f(x) - 1| \leq \varepsilon$.

We want to show

$$\begin{aligned}
|f(x) - 1| &\leq \varepsilon \\
|x^2 - 1| &\leq \varepsilon \\
|(x + 1)(x - 1)| &\leq \varepsilon \\
|(x - 1 + 2))(x - 1)| &\leq \varepsilon \\
|x - 1 + 2||x - 1| &\leq \varepsilon \\
(|x - 1| + |2|)|x - 1| &\leq \varepsilon \\
(\delta + 2)\delta &\leq \varepsilon \\
\delta^2 + 2\delta &\leq \varepsilon
\end{aligned}$$

We can solve $\delta^2 + 2\delta \leq \varepsilon$ to get an equation for δ in terms of ε. Take $\delta = \sqrt{\varepsilon + 1} - 1$.

Problem 2

Prove that the real-valued function $f(x) = 1/x$ defined on $Dom(f) = [1, \infty)$ is uniformly continuous under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$.

To show uniform continuity we would like to show that given any $\varepsilon > 0$, we can select a δ independent of x such that $\rho(x, y) \leq \delta \to \rho(f(x), f(y)) \leq \varepsilon$. So consider some $\varepsilon > 0$, and x, y such that $|x - y| \leq \delta$.

$$\begin{aligned}
\rho(f(x), f(y)) &\leq \varepsilon \\
|f(x) - f(y)| &\leq \varepsilon \\
|1/x - 1/y| &\leq \varepsilon \\
\left|\frac{y - x}{xy}\right| &\leq \varepsilon \\
\frac{|y - x|}{|xy|} &\leq \varepsilon \\
|y - x| &\leq \varepsilon \cdot |x| \cdot |y| \\
\delta &\leq \varepsilon \cdot x \cdot y \\
\delta &\leq \varepsilon \leq \varepsilon \cdot x \cdot y
\end{aligned}$$

Where we know that $\varepsilon \leq \varepsilon \cdot x \cdot y$ since $x, y \in [1, \infty)$. Thus we can select $\delta \leq \varepsilon$ independent of x to satisfy uniform continuity. For example, we can always select $\delta = \varepsilon/2$.

Problem 3

Let (X, σ) be a metric space and x_0 a point in X. Prove that the function $f : X \to \mathbb{R}$ defined as $f(x) = \sigma(x, x_0)$ is continuous under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$.

To show continuity we would like to show that given any $\varepsilon > 0$, we can select a δ such that $\forall\, a, b \,.\, \sigma(a,b) \leq \delta \to |f(a) - f(b)| \leq \varepsilon$

We can use the triangle inequality of σ to note that

$$\begin{aligned} |f(a) - f(b)| &\leq \varepsilon \\ |\sigma(a, x_0) - \sigma(b, x_0)| &\leq \varepsilon \\ |\sigma(a,b)| \leq |\sigma(a, x_0) - \sigma(b, x_0)| &\leq \varepsilon \\ \delta &\leq \varepsilon \end{aligned}$$

Thus by taking $\delta \leq \varepsilon$, for example $\delta = \varepsilon/2$, we can satisfy the criteria for continuity.

Problem 4

Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces, and $f : X \to Y$ a function with domain $Dom(f) = X$ and range $Ran(f) = Y$. Prove that f is continuous if and only if $f^{-1}[\tilde{Y}]$ (the preimage of $\tilde{Y}$) is open for every open set $\tilde{Y}$ in Y.

If f is continuous, then for every open set $\tilde{Y} \in Y$ we know $f^{-1}[\tilde{Y}]$ is open.

Since f is continuous, we know that for any $\varepsilon > 0$, if $f(y) \in B_\varepsilon(f(x))$ then there exists some δ such that $y \in B_\delta(x)$. Since $\tilde{Y}$ is an open set in Y, we know that for any $f(x) \in \tilde{Y}$, there is some $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq \tilde{Y}$. Thus for any $f(x) \in \tilde{Y}$, we can take the corresponding ε and plug it into the def'n of continuous to see that there exists some δ such that $B_\delta(x) \subseteq X$. This implies that every $x \in X$ has an open ball of some positive radius that is a subset of X, which means that $f^{-1}[\tilde{Y}]$ is open.

If for every open set $\tilde{Y} \in Y$ we know that $f^{-1}[\tilde{Y}]$ is open, then f must be continuous.

We would like to show that for any $\varepsilon > 0$, there exists some δ such that $\rho_X(x, y) < \delta$ implies $\rho_Y(f(x), f(y)) < \varepsilon$. Since $\tilde{Y}$ is open, we know that for any $f(x) \in \tilde{Y}$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq (\tilde{Y})$. Since $f^{-1}[\tilde{Y}]$ is also open, we know that for any x there exists some $\delta > 0$ such that $B_\delta(x) \subseteq X$. So for any $\varepsilon > 0$, we can select $\tilde{Y} = B_\varepsilon(f(x))$, which gives us the corresponding $f^{-1}[\tilde{Y}]$ with the ball $B_\delta(x)$. The connection between this x, δ, and ε satisfies the criteria for f to be continuous.

Problem 5

Let f be the function on $[0, 1]$ given by

$$f(x) = \begin{cases} 1 \; if \; x \neq \frac{1}{2} \\ 2 \; if \; x = \frac{1}{2} \end{cases}$$

Prove that f is Riemann integrable and compute $\int_0^1 f(x)dx$. Hint: for each $\varepsilon > 0$, find a partition P so that $U_P(f) - L_P(f) \leq \varepsilon$.

Given any ε, we can select the partition $P = \left\{0, \frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}, 1\right\}$. Thus

$$\begin{aligned} U_P(f) &= 1\left(\frac{1}{2} - \frac{\varepsilon}{2} - 0\right) + 2\left(\frac{1}{2} + \frac{\varepsilon}{2} - \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\right) + 1\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\right) \\ &= 1 + \varepsilon \end{aligned}$$

$$\begin{aligned} L_P(f) &= 1\left(\frac{1}{2} - \frac{\varepsilon}{2} - 0\right) + 1\left(\frac{1}{2} + \frac{\varepsilon}{2} - \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\right) + 1\left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\right) \\ &= 1 \end{aligned}$$

So, $U_P(f) - L_P(f) = 1 + \varepsilon - 1 = \varepsilon$, which by the Riemann integrability test, shows that f is integrable. The value of $\int_0^1 f(x)dx = sup_P[L_P(f)] = 1$.