Assignment 6

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Problem 1

Prove directly, by verifying the definition, that each of the following sequences is a Cauchy sequence in the metric space (X, ρ) with $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$:

Part a

The sequence $\{x_n\}$ with $x_n = \frac{1}{\sqrt{n}}$

Note that $|a_n - a_m| = |\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}| \leq |\frac{1}{\sqrt{n}}| + |\frac{1}{\sqrt{m}}| \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}$ because $n, m > 0$. Thus, we need to find N such that $n, m > N$ solves the inequality $\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}} \leq \varepsilon\ \forall \varepsilon > 0$. The smallest such n, m can be taken to be N, so we need to solve the inequality $\frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} \leq \varepsilon$ which gives us $\frac{2}{\varepsilon} \leq \sqrt{N}$. Thus, we can select $N > \frac{4}{\varepsilon^2}$ which demonstrates that the sequeunce is Cauchy.

Part b

The sequence $\{x_n\}$ with $x_n = \frac{cos\ n}{2n}$

Note that $|a_n - a_m| = |\frac{cos\ n}{2n} - \frac{cos\ m}{2m}| \leq |\frac{cos\ n}{2n}| + |\frac{cos\ m}{2m}| \leq \frac{1}{2n} + \frac{1}{2m}$, where the last inequality holds because of the possible values of cos and the fact that $n, m > 0$. We want to select N such that $n, m \geq N$ satisfy $\frac{1}{2n} + \frac{1}{2m} < \varepsilon\ \forall \varepsilon > 0$. We can take the smallest such $n, m = N$ to get $\frac{1}{2N} + \frac{1}{2N} = \frac{2}{2N} = \frac{1}{N} < \varepsilon\ \rightarrow \frac{1}{\varepsilon} < N$. Thus, by selecting $N > \frac{1}{\varepsilon}$ we demonstrate the sequence is Cauchy.

Problem 2

Let (X, σ) be a metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X. Prove that the sequence of real numbers $\{s_n\}$, defined as $s_n = \sigma(x_n, y_n)$, converges in the usual Euclidean metric $\rho(x, y) = |x - y|$

We want to show that there exists some N such that for any $\varepsilon > 0$ and any $n, m \geq N$ we know that $\rho(s_n, s_m) \leq \varepsilon$. This can be transformed as follows:

$$\rho(s_n, s_m) = \rho(\sigma(x_n, y_n), \sigma(x_m, y_m)) = |\sigma(x_n, y_n) - \sigma(x_m, y_m)|$$

But note that x_n can be arbitrarily close to x_m with sufficiently large $n, m > N_1$and y_n can be arbitrarily close to y_m with sufficiently large $n, m > N_2$. Thus, for any $\varepsilon > 0$, we can take the corresponding $N = max(N_1, N_2)$ and $n, m > N$ to get $|\sigma(x_n, y_n) - \sigma(x_m, y_m)| < \varepsilon$ as desired. Thus, s_n is a Cauchy sequence in $\mathbb{R}$. Since the reals are complete, we know that $\{s_n\}$ converges.

Problem 3

Let (X, σ) be a metric space and $\{x_n\}$ a Cauchy sequence in X. Let $\{y_n\}$ be another sequence in X such that $\sigma(x_n, y_n) \to 0$ in the standard euclidean metric. Prove that:

Part a

$\{y_n\}$ is a Cauchy sequence. Since x_n is Cauchy, $\forall \varepsilon > 0. \exists N_1. \rho(x_n, x_m) \leq \varepsilon\ \forall n, m \geq N_1$. Similarly, since $\sigma(x_n, y_n)$ converges to 0, we know there exists some N_2 such that $\forall \varepsilon\ .\ \sigma(x_n, y_n) \leq \varepsilon \forall n \geq N_2$. Take $N = max(N_1, N_2)$, then we can see that for any $\varepsilon > 0$ and $n, m \geq N$, $\sigma(y_n, y_m) \leq \sigma(x_n, x_m) \leq \varepsilon$ so $\{y_n\}$ is Cauchy.

Part b

$y_n \to y \in X$ iff $x_n \to y \in X$ for the same y.

First we will show that if $x_n \to y \in X$ then $y_n \to y \in X$. Since $x_n \to y \in X$ we know that there is some N_1 such that $\forall \varepsilon \;.\; \forall n > N_1 \;.\; \sigma(x_n, y) < \varepsilon$. But since we also know that $\sigma(x_n, y_n) \to 0$, we know that there is some N_2 such that $\forall \varepsilon \;.\; \forall n > N_2 \;.\; \sigma(x_n, y_n) < \varepsilon$. Thus, we can see that by taking $N = max(N_1, N_2)$, x_n is arbitrarily close to y and y_n is arbitrarily close to x_n so y_n is arbitrarily close to y for any $n > N$. Thus $y_n \to y$.

WLOG we can see that if $y_n \to y \in X$ then $x_n \to y \in X$.

Problem 4

Let X be a non-empty set and ρ the discrete metric on X, meaning that $\forall x, y \in X$, we have

$$\rho(x,y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$$

Show that (X, ρ) is a complete metric space.

Consider an arbitrary Cauchy sequence $\{x_n\}$ in X. Then $\forall \varepsilon > 0 \;.\; \exists N \;.\; \forall n, m > N \;.\; \rho(x_n, x_m) < \varepsilon$. But since $\rho(x_n, x_m)$ is either 0 or 1, and we can select $\varepsilon < 1$ we know that there exists some N such that $\rho(x_n, x_m) = 0$ for all $n, m > N$. Thus, we have proven that for every Cauchy sequence there exists some N such that the sequence must converge to $x_N \in X$.