Assignment 10

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Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^3$. Prove that is differentiable everywhere on $\mathbb{R}$ by showing that the limit of the difference quotient exists.

To show that the function f is differentiable everywhere, we must show that for any $x \in \mathbb{R}$ the following limit exists:

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^2+2hx+h^2)(x+h) - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3+2hx^2+h^2x+x^2h+2h^2x+h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3hx^2+h^2x+2h^2x+h^3}{h} \\
&= \lim_{h\to 0} 3x^2+hx+2hx+h^2 \\
&= 3x^2
\end{aligned}
$$

Problem 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function on a closed finite interval $[a, b] \subseteq \mathbb{R}$. Prove that f is Lipschitz continuous on $[a, b]$. Hint: an easy way to go is to use the Mean Value Theorem.

We would like to show that $\forall\ x, y \in [a, b]\ .\ \exists\ m \geq 0\ .\ |f(x) - f(y)| \leq m \cdot |x - y|$. Thus if we fix $x, y \in [a, b]$ (WLOG such that $x \leq y$), by Lagrange's MVT we know that there exists some $c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$ which implies $f'(c) \cdot (y - x) = f(y) - f(x)$. We can take the absolute values of both sides to see that we have found such an $m = f'(c)$ to satisfy the property of Lipschitz continuity.

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\forall x \in \mathbb{R}\ .\ f(x) \geq 0$. Assume that $f(x)^2$ is differentiable. Is $f(x)$ necessarily differentiable? If it is, prove so. Otherwise, explain why not.

The claim is false, we will provide a counterexample. Consider the function $f(x) = |x|$. This function is non-negative everywhere, but not differentiable at $x = 0$. However, $|x|^2 = x^2$ for all real x, and we know that x^2 is differentiable. Thus $f(x)^2$ is differentiable does not imply that $f(x)$ is necessarily differentiable.

Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be both continuously differentiable. Assume that $f(0) = g(0)$ and $f'(x) \leq g'(x), \forall x \geq 0$. Prove that $f(x) \leq g(x), \forall x \geq 0$.

Consider functions $f(x), g(x) : \mathbb{R} \to \mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) \leq g'(x), \forall x \geq 0$. Then, we can define a function $h(x) = f(x) - g(x)$, and we know that $h'(x) = f'(x) - g'(x)$ by the linearity of derivatives. since $f'(x) \leq g'(x)$, we know that $h'(x) \leq 0$ for all x. By the def'n of h, we know that $h(0) = f(0) - g(0) = 0$. Thus, h is a function that starts at zero and is monotonically decreasing. Thus $h(x) \leq 0 \to f(x) \leq g(x), \forall x \geq 0$ as desired.