M365C: Real Analysis I
Homework # 06
Handout: 02/22/2018, Thursday
Due: 03/1/2018, Thursday

Submission. Please make your homework neat and stapled. Note that *no late homework will be accepted without compelling reasons.*

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1 To be Graded

Problem 1. Prove directly, by verifying the definition, that each of the following sequences is a Cauchy sequence in the metric space (X, ρ) with $X = \mathbb{R}$ and $\rho(x, y) = |x - y|$:
(a) The sequence $\{x_n\}$ with $x_n = \dfrac{1}{\sqrt{n}}$;
(b) The sequence $\{x_n\}$ with $x_n = \dfrac{\cos n}{2n}$.

Problem 2. Let (X, σ) be a metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X. Prove that the sequence of real numbers $\{s_n\}$, defined as $s_n = \sigma(x_n, y_n)$, converges in the usual Euclidean metric $\rho(x, y) = |x - y|$.

Problem 3. Let (X, σ) be a metric space and $\{x_n\}$ a Cauchy sequence in X. Let $\{y_n\}$ be another sequence in X such that $\sigma(x_n, y_n) \to 0$ (in the usual Euclidean metric $\rho(x, y) = |x - y|$). Prove that
(a) $\{y_n\}$ is a Cauchy sequence;
(b) $y_n \to y \in X$ iff $x_n \to y \in X$ for the same y.

Problem 4. Let X be a non-empty set and ρ the discrete metric on X, meaning that: $\forall x, y \in X$, we have

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Show that (X, ρ) is a complete metric space.

2 Reading Assignments

- Review Lecture Notes # 8 and # 9;
- Review Sections 2.2, 2.3 and 3.3 of the textbook;