Assignment 7

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Problem 1

Let $x, y \in \mathbb{R}$ be such that $x \neq y$. Prove that $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$

Because $x \neq y$ we know that there exists some $\varepsilon' > 0$ such that $\rho(x, y) > \varepsilon'$. Then by taking any $\varepsilon = \varepsilon'/2$ we can see that $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$. We can prove this claim by contradiction: assume there was some element $z \in B_\varepsilon(x) \cap B_\varepsilon(y)$. This implies that $\rho(x, z) < \varepsilon$ and $\rho(z, y) < \varepsilon$, so $\rho(x, y) + \rho(y, z) < 2\varepsilon$. But since we know $\rho(x, y) = 2\varepsilon$, the strict inequality for $\rho(x, y) + \rho(y, z) < 2\varepsilon$ violates the triangle inequality between x, y, and z.

Problem 2

Let $X = [0, \infty)$ and take $\mathcal{O}_k = (-10, k), k \geq 1$. Prove that:

Part i

$O = \{\mathcal{O}_k\}_{k=1}^\infty$ is a open cover of X;

Note that we say that $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ is an open cover of X if $X \subseteq \bigcup_{\alpha \in A} \mathcal{O}_\alpha$.

Consider an arbitrary element $x \in X$. Because we know that x is a real number greater than or equal to zero, we know that we can take $k = x + 1$ to get $\mathcal{O}_k = (-10, x + 1)$. For this arbitrary value of x and this value of k, it is clear that $x \in \mathcal{O}_k$, so we know that

$$X \subseteq \bigcup_{k \in [1, \infty)} \mathcal{O}_k$$

Thus O is an open cover of X.

Part ii

O has no finite subcover, i.e. X is not compact.

Proof by contradiction: assume there exists some finite subcover of O:

$$\exists \{\alpha_k\}_{k=1}^K \subseteq [1, \infty) \ s.t. \ X \subseteq \bigcup_{k=1}^K \mathcal{O}_{\alpha_k}$$

In other words, there is a finite subsequence of $[1, \infty)$ such that the union of each of the $\mathcal{O}$s is also a cover of X. However, it is evident that by the definition of $\mathcal{O}_k = (-10, k)$ that $K + 1 \notin \bigcup_{k=1}^K \mathcal{O}_{\alpha_k}$. Thus $X \not\subseteq \bigcup_{k=1}^K \mathcal{O}_{\alpha_k}$, so O has no finite subcover and X is not compact.

Problem 3

Let (X, ρ) be a metric space with ρ the discrete metric. Prove that (X, ρ) is compact if and only if X is a finite set.

<u>If X is a finite set then (X, ρ) is compact</u>

Consider any infinite sequence in X. Because there are an infinite number of terms in the sequence, but only a finite number of elements in X, there is some element $x \in X$ for each sequence that appears infinitely many times in that sequence. Thus there obviously exists a subsequence that converges to x for any sequence. Therefore, the (X, ρ) is sequentially compact, which implies it is compact.

<u>If (X, ρ) is compact then X is a finite set</u>

Proof by contradiction: assume (X, ρ) is a compact, X is an infinite set, and ρ is the discrete metric. But because X is infinite, there exists a sequence $\{x_n\}$ such that $x_n \neq x_{n'}$ for all $n' < n$. This implies that $\rho(x_n, x_{n'}) = 1$ for all $n > n'$ for all n'. No subsequence of this sequence can converge (because we can always choose $\varepsilon < 1$), and therefore (X, ρ) cannot be sequentially compact, and since we are in a topology this implies (X, ρ) cannot be compact. This contradicts our assumption, and thus we know that X cannot be infinite.

Problem 4

Let (X, σ) be a metric space, and $f(x) : Dom(f) = X \mapsto \mathbb{R}$ be a continuous function (under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$). Prove that $|f(x)|$ is a continuous function on $Dom(f)$

Since f is continuous, we know that for any $\varepsilon > 0$ and $x \in Dom(f)$ we can select $\delta(\varepsilon, x)$ such that $\sigma(x, y) \leq \delta$ implies that $\rho(f(x), f(y)) \leq \varepsilon$.

We want to show that for any $\varepsilon > 0$ and $x \in Dom(f)$ we can select $\delta(\varepsilon, x)$ such that $\sigma(x, y) \leq \delta$ implies that $\rho(|f(x)|, |f(y)|) \leq \varepsilon$. If we select $\delta(\varepsilon, x)$ in the same way for $|f(x)|$ as we did for $f(x)$ we can see that

$$\begin{aligned}
\rho(|f(x)|, |f(y)|) &= ||f(x)| - |f(y)|| \\
&\leq |f(x) - f(y)| \\
&\leq \varepsilon
\end{aligned}$$

and thus $|f(x)|$ is clearly a continuous function on $Dom(f)$.

Problem 5

Let (X, σ) be a metric space, $f : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$ be both Lipschitz on X (under the usual euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$). Prove that $f + g$ is also Lipschitz on X.

Since f is Lipschitz, we know that there exists some M_f such that $\rho(f(x), f(y)) \leq M\sigma(x, y)$ for all $x, y \in Dom(f)$. Since g is also Lipschitz, we know that there exists some M_g such that $\rho(g(x), g(y)) \leq M\sigma(x, y)$ for all $x, y \in Dom(g)$. Now consider $M = 2 * max(M_f, M_g)$.

$$\begin{aligned}
\rho(f(x) + g(x), f(y) + g(y)) &= |f(x) + g(x) - (f(y) + g(y))| \\
&= |f(x) - f(y) + g(x) - g(y)| \\
&\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&\leq \rho(f(x), f(y)) + \rho(g(x), g(y)) \\
&\leq \frac{M}{2}\sigma(x, y) + \frac{M}{2}\sigma(x, y) \\
&\leq M\sigma(x, y)
\end{aligned}$$

Thus we have found some M such that $\rho(f(x) + g(x), f(y) + g(y)) \leq M\sigma(x, y)$ for all $x, y \in Dom(f + g)$, so $f + g$ is Lipschitz on X.

Problem 6

Let (X, σ) be a metric space, $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be both uniformly continous on X (under the usual Euclidean metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$). Prove that $f + g$ is also uniformly continuous on X.

Since f is uniformly continuous, we know that $\forall \varepsilon > 0, \exists \delta_f(\varepsilon) > 0$ such that $\sigma(x, y) \leq \delta_f$ implies $\rho(f(x), f(y)) \leq \varepsilon$ for all $x, y \in Dom(f)$.

Since g is also uniformly continuous, we know that $\forall \varepsilon > 0, \exists \delta_g(\varepsilon) > 0$ such that $\sigma(x, y) \leq \delta_g$ implies $\rho(g(x), g(y)) \leq \varepsilon$ for all $x, y \in Dom(g)$.

So consider an arbitrary ε and take $\delta = min(\delta_f(\frac{\varepsilon}{2}), \delta_g(\frac{\varepsilon}{2}))$. Then

$$\begin{aligned}
\rho(f(x)+g(x), f(y)+g(y)) &= |f(x)+g(x)-(f(y)+g(y))| \\
&= |f(x)-f(y)+g(x)-g(y)| \\
&\leq |f(x)-f(y)|+|g(x)-g(y)| \\
&\leq \rho(f(x), f(y)) + \rho(g(x), g(y)) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq \varepsilon
\end{aligned}$$

Thus given any ε we are able to select a δ such that $\sigma(x, y) \leq \delta$ implies that $\rho(f(x)+g(x), f(y)+g(y)) \leq \varepsilon$. This means that $f+g$ is also uniformly continuous on X.