Assignment 1

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January 23, 2018

Problem 1

Let $X = \{1, 2, a\}$. Find the power set of X, $P(X)$.

$P(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{a\}, \{1, a\}, \{2, a\}, \{1, 2, a\}\}$

Problem 2

For each $n \in \mathbb{N}$, let $A_n = \{(n + 1)k : k \in \mathbb{N}\}$

a)

What is $A_1 \cap A_2$?

$A_1 = \{2k : k \in \mathbb{N}\}$, $A_2 = \{3k : k \in \mathbb{N}\}$, so $A_1 \cap A_2$ is the set of all natural numbers divisible by both 2 and 3. This is the set of all numbers that are divisible by 6, which can be represented by $\{6k = (5 + 1)k : k \in \mathbb{N}\} = A_5$.

b)

Determine the sets $\cup\{A_n : n \in \mathbb{N}\}$ and $\cap\{A_n : n \in \mathbb{N}\}$.

$\cup\{A_n : n \in \mathbb{N}\} = \mathbb{N} - \{1\}$ because every natural number greater than one can be written as mk where $m > 1 \in \mathbb{N}$ and $k \in \mathbb{N}$. Similarly, $\cap\{A_n : n \in \mathbb{N}\} = \emptyset$ because no natural number is divisible by every natural number.

Problem 3

Let A and B be two sets. Prove that $A \subseteq B$ iff $A \cap B = A$

First we will show that if $A \subseteq B$ then $A \cap B = A$. To do so, we must show that $A \subseteq A \cap B$ and $A \cap B \subseteq A$. Given any element $a \in A$, we know that $a \in B$ since $A \subseteq B$. Thus, $a \in A \cap B$, which implies that $A \subseteq A \cap B$. Similarly, given an element $a \in A \cap B$, it must also be in A by the definition of intersection, so $A \subseteq A \cap B$.

Second we will show that if $A \cap B = A$ then $A \subseteq B$. Given any element $a \in A$, we know that $a \in A \cap B$, which implies $a \in B$. Thus, $A \subseteq B$.

Problem 4

Let A, B, and C be arbitrary sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

First we will show that if an arbitrary $x \in A \cap (B \cup C)$ then $x \in (A \cap B) \cup (A \cap C)$. If $x \in A \cap (B \cup C)$, we know $x \in A$ and $x \in B \cup C$. $x \in B \cup C$ implies $x \in B$ or $x \in C$. Since in either case we know $x \in A$, we can conclude that $x \in (A \cap B) \cup (A \cap C)$.

Second we will show that if an arbitrary $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap (B \cup C)$. $x \in (A \cap B) \cup (A \cap C)$ means that $x \in A \cap B$ or $x \in A \cap C$. In either case $x \in A$, and depending on the case $x \in B$ or $x \in C$. Thus we can conclude that $x \in (A \cap B) \cup (A \cap C)$.

Problem 5

Let $\mathbb{N}$ be the set of natural numbers, and $|$ be the relation of divisibility (i.e. we say $y \in \mathbb{N}$ divides $x \in \mathbb{N}$, denoted by $y|x$, if there exists an integer n such that $x = ny$). Prove that $|$ is an ordering relation on $\mathbb{N}$

Reflexivity: Since 1 is the multiplicative identity in the natural numbers, $x = 1 \cdot x \forall x \in \mathbb{N}$. Therefore, 1 is the integer that shows that $x = nx$ for all x. In other words $x|x \forall x \in \mathbb{N}$, so $|$ is reflexive over the natural numbers.

Anti-symmetry: consider $x, y \in \mathbb{N}$ where $x|y$ and $y|x$. $x = my$ and $y = nx$ for some $m, n \in \mathbb{Z}$. But by substituting we can see that $x = my = mnx$ which implies $mn = 1$. Thus we know that $x = y$ and $|$ is anti-symmetric over the natural numbers.

Transitivity: Consider x, y, z such that $x|y, y|z$. Thus $y = nx$, and $z = my$ for integers m, n. But by substituting we can see $z = my = mnx$, and $mn \in \mathbb{Z}$, so $x|z$. Thus $|$ is transitive over the natural numbers.

Therefore, $|$ is an ordering relation because it satisfies reflexivity, anti-symmetry, and transitivity.

Problem 6

Let $\leq$ be an ordering relation on the set X. We define the inverse of $\leq$, denoted by $\geq$, as follows: $\forall x, y \in X, x \geq y$ iff $y \leq x$. Prove that $\geq$ is an ordering relation on X.

Reflexivity: Consider any $x \in X$. Since $x \leq x$ by the reflexivity of $\leq$, we know that $x \geq x$ by the def'n of $\geq$. Thus, $\geq$ is reflexive over X.

Anti-symmetry: consider any $x, y \in X$ where $x \geq y$ and $y \geq x$. Note that $x \geq y \to y \leq x$ and $y \geq x \to x \leq y$. Since we know $x \leq y, y \leq x$ we can say $x = y$ by the anti-symmetry of $\leq$. Thus we know that $x = y$ and $\geq$ is anti-symmetric over X.

Transitivity: Consider x, y, z such that $x \geq y, y \geq z$. Note that $x \geq y \to y \leq x$, and $y \geq z \to z \leq y$. By the transitivity of $\leq$ we know that $z \leq x$, and by the def'n of $\geq$ this shows that $x \geq z$. Since $x \geq z$ we know $\geq$ is transitive over X.

Therefore, $\geq$ is an ordering relation because it satisfies reflexivity, anti-symmetry, and transitivity.

Problem 7

Let $(X, \leq)$ be a totally-ordered space and $Y \subseteq X$ an nonempty subset of X. Let α be a lower bound of Y and β an upper bound of Y. Prove that $\alpha \leq \beta$.

α is a lower bound of Y if $\alpha \leq y \ \forall y \in Y$. Similarly β is an upper bound of Y if $y \leq \beta \ \forall y \in Y$. Since Y is nonempty, we know there exists some $y \in Y$ such that $\alpha \leq y$ and such that $y \leq \beta$. By the transitivity of $\leq$ we can see that $\alpha \leq \beta \quad \square$.