Assignment 2

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Problem 1

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions whose analytical expressions are given below. In each case, determine the domain and compute a formula for $f \circ g$ and $g \circ f$

a)

$f(x) = 2x + 1$ *and* $g(x) = e^x$

$$f \circ g = f(g(x)) = 2(e^x) + 1$$

$$Dom(f \circ g) = \{x | x \in Dom(g), g(x) \in Dom(f)\} = \mathbb{R}$$

$$g \circ f = g(f(x)) = e^{2x+1}$$

$$Dom(g \circ f) = \{x | x \in Dom(f), f(x) \in Dom(g)\} = \mathbb{R}$$

b)

$f(x) = 4 - x^2$ *and* $g(x) = ln\ x$

$$f \circ g = f(g(x)) = 4 - (ln\ x)^2$$

$$Dom(f \circ g) = \{x | x \in Dom(g), g(x) \in Dom(f)\} = (0, \infty)$$

$$g \circ f = g(f(x)) = ln\ (4 - x^2)$$

$$Dom(g \circ f) = \{x | x \in Dom(f), f(x) \in Dom(g)\}$$

$$\begin{aligned} f(x) &\in Dom(g) \\ (4 - x^2) &\in (0, \infty) \\ 4 - x^2 &> 0 \\ 4 &> x^2 \\ x &\in (-2, 2) \end{aligned}$$

so $Dom(g \circ f) = (-2, 2)$

Problem 2

Each of the following functions has domain $Dom(f) = \{x \in \mathbb{R} | x \neq 0\}$. For each, determine the range of the function and whether it is injective and surjective as a function $f : Dom(f) \mapsto \mathbb{R}$:

a)

$f(x) = \frac{1}{x}$

Range is $(-\infty, 0) \cup (0, \infty)$. Function is injective, but not surjective.

b)

$f(x) = ln\ |x|$

Range is $\mathbb{R}$. Function is not injective, but it is surjective.

c)

$f(x) = \frac{1}{x^2}$

Range is positive real numbers. Not injective, not surjective.

Problem 3

Let X be the set of all students on the campus of UT Austin. Define the binary relation $\sim$ as follows. For any two students x and y, we say $x \sim y$ if x and y have the same birthday. Show that $\sim$ is an equivalence relation on X.

An equivalence relation must satisfy three axioms: reflexivity, symmetry, and transitivity.

Reflexivity: $x \sim x\ .\ \forall x \in X$: Obviously any student has the same birthday as themselves.

Symmetry: $x \sim y \to y \sim x\ .\ \forall x, y \in X$: If student x has the same birthday as student y, then obviously student y has the same birthday as student x.

Transitivity: $x \sim y, y \sim z \to x \sim z\ .\ \forall x, y, z \in X$: If student x has the same birthday as y, and y has the same birthday as z, then obviously x has the same birthday as z.

Problem 4

Let $X = \mathbb{R}$ be the set of real numbers. Define the "square" relation $R = \{(x,y)|x^2 = y^2\}$. Show that R is an equivalence relation.

Define $\sim$ as the binary relation R. An equivalence relation must satisfy three axioms: reflexivity, symmetry, and transitivity.

Reflexivity: $x \sim x\ .\ \forall x \in X$: Given some $(x, x) \in R$, we can see that $x \sim x$ because $x^2 = x^2$ by the reflexivity of equality over real numbers.

Symmetry: $x \sim y \to y \sim x\ .\ \forall x, y \in X$: Given some $(x, y) \in R$, we can see that $y^2 = x^2 \to y \sim x$ when $x \sim y$ because of the symmetry of equality over real numbers.

Transitivity: $x \sim y, y \sim z \to x \sim z\ .\ \forall x, y, z \in X$: $x \sim y \to x^2 = y^2$ and $y \sim z \to y^2 = z^2$. Thus by the transitivity of equality over the real numbers we can see that $x^2 = y^2 = z^2 \to x \sim z$.

Problem 5

Let X be the set of fractions: $X = \{\frac{p}{q}|p, q \in \mathbb{Z}, q \neq 0\}$. Define a binary relation R on X by: $\frac{a}{b} R \frac{c}{d}$ iff $ad = bc$. Show that R is an equivalence relation.

Define $\sim$ as a relation R. An equivalence relation must satisfy: reflexivity, symmetry, and transitivity.

Reflexivity: $x \sim x\ .\ \forall x \in X$: Given some $x = \frac{p}{q} \in X$ we can see that $pq = pq \to \frac{p}{q} R \frac{p}{q}$, so $x \sim x$.

Symmetry: $x \sim y \to y \sim x\ .\ \forall x, y \in X$: Given some $\frac{a}{b} \sim \frac{c}{d}$ we know that $ab = bc$ and $ab, bc \in \mathbb{R}$. Due to the symmetry of equality of real numbers we can see that $bc = ab \to \frac{c}{d} \sim \frac{a}{b}$.

Transitivity: $x \sim y, y \sim z \to x \sim z\ .\ \forall x, y, z \in X$: Given some $\frac{a}{b} \sim \frac{c}{d}, \frac{c}{d} \sim \frac{e}{f}$ we can see that $ad = bc, cf = de$. Since $b \neq 0, d \neq 0, f \neq 0$ we know that $ad = bc \to a = \frac{bc}{d}$ and $cf = de \to e = \frac{cf}{d}$. Thus:

$$af = \frac{bc}{d} f = b \frac{cf}{d} = be \quad \square$$