Assignment 3

Rushi Shah

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Problem 1

Prove that the set $S \equiv \{5, 10, 15, 20, \cdots\}$ is countable by constructing a one-to-one function from S onto $\mathbb{N}$

First note that $S = \{5, 10, 15, 20, \cdots\} = \{5k \mid k \in \mathbb{N}\}$. Thus, we can construct a one-to-one function $f : S \to \mathbb{N}$ by defining $f(5k) \mapsto k$.

Problem 2

Part a

The union of two finite sets is finite.

Consider two finite sets: X with cardinality n and Y with cardinality m such that WLOG $X = \{1, \ldots, n\}$ and $Y = \{1, \ldots, m\}$. Then there exists a function with $Dom(\{1, \ldots, n\} \cup \{1, \ldots, m\})$ that is bijective with $\{1, \ldots, n + m\}$

$$f(z) = \begin{cases} z, z \in \{1, \ldots, n\} \\ n + z, z \in \{1, \ldots, m\} \end{cases}$$

Thus, $X \cup Y$ is finite with cardinality $n + m$.

Part b

The union of a finite set and a countable set is countable.

Consider a finite set that has cardinality n which (WLOG) can be stated as $X = \{1, \ldots, n\}$ and a countable set which (WLOG) can be stated as $Y = \mathbb{N}$. Then we can define a bijection $f : X \cup Y \to \mathbb{N}$

$$f(z) = \begin{cases} z, z \in X \\ n + z, z \in Y \end{cases}$$

Thus $X \cup Y$ is countable.

Part c

The union of two countable sets is countable.

Consider two countable sets $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$. Then we can define a bijection $f : \mathbb{N} \to X \cup Y$

$$f(z) = \begin{cases} x_z, z \text{ is even} \\ y_z, z \text{ is odd} \end{cases}$$

Thus $X \cup Y$ is countable.

Problem 3

Rational numbers are defined as real numbers that can be written in the form $\frac{m}{n}, n \neq 0$ with m and n integers without common factors. The set of rational numbers, $\mathbb{Q}$, can be split into three parts, the positive ones $\mathbb{Q}_+$, the negative ones $\mathbb{Q}_-$, and the set that contains only zero $\{0\}$, $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$. Prove that $\mathbb{Q}$ is countable.

Hint: We can first show that $\mathbb{Q}_+$ is countable by constructing the function $f : \frac{m}{n} \mapsto (m, n), f : \mathbb{Q} \mapsto U \subset \mathbb{N} \times \mathbb{N}$ that is one-to-one with $Dom(f) = \mathbb{Q}_+$. $\mathbb{N} \times \mathbb{N}$ is countable, so U is countable, so $\mathbb{Q}_+$ is countable. We then use the results in the previous problem 2.

We can define $f_1 : \mathbb{Q}_+ \to U_1 \subset \mathbb{N} \times \mathbb{N}$ by $f_1(\frac{m}{n}) \mapsto (m,n)$. Note that negative rationals can all be represented with positive denominators by multiplying any negative rational with a negative denominator by $\frac{-1}{-1} = 1$. Thus, we can define $f_2 : \mathbb{Q}_- \to U_2 \subset \mathbb{N} \times \mathbb{N}$ by $f_2(\frac{m}{n}) \mapsto (|m|, n)$. With these two functions, we see that $\mathbb{Q}_-, \mathbb{Q}_+$ are both countable. Note that it is clear that $\{0\}$ is finite. Since the union of countable sets and the union of finite sets are countable by the previous problem, we can see that $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ is countable.

Problem 4

Let ρ be a metric on X. Prove that the following are also metrics

Part a

$\rho_1 \equiv 5\rho$

We must show that ρ_1 satisfies non-negativity, symmetry, and the triangle inequality.

Non-negativity: Take arbitrary $x, y \in X$. We know that $\rho(x,y) = 0$ iff $x = y$. But $5\rho(x,y) = 0$ iff $\rho(x,y) = 0$ so $\rho_1(x,y) = 0$ iff $x = y$. Similarly, $5\rho(x,y) \geq 0$ when $\rho(x,y) \geq 0$. Since $\rho(x,y) \geq 0 \;\forall\; x, y \in X$, we know that ρ_1 satisfies non-negativity.

Symmetry: Take arbitrary $x, y \in X$:

$$\rho_1(x,y) = 5\rho(x,y) = 5\rho(y,x) = \rho_1(x,y)$$

Triangle inequality: Take arbitrary $x, y, z \in X$:

$$\begin{aligned} \rho(x,y) &\leq \rho(x,z) + \rho(z,y) \\ 5 \cdot \rho(x,y) &\leq 5 \cdot (\rho(x,z) + \rho(z,y)) \\ \rho_1(x,y) &\leq 5\rho(x,z) + 5\rho(z,y) \\ &\leq \rho_1(x,z) + \rho_1(z,y) \end{aligned}$$

Part b

$\rho_2 \equiv min\{1, \rho\}$

Non-negativity:

Since $0 < 1$, $\rho_2(x,y) = 0$ iff $\rho(x,y) = 0$ iff $x = y$. Similarly, since ρ is never less than zero, and 1 is positive, $\rho_2(x,y) \geq 0 \;\forall\; x, y \in X$.

Symmetry:

$$\rho_2(x,y) = min(1, \rho(x,y)) = min(1, \rho(y,x)) = \rho_2(x,y)$$

Triangle inequality:

We would like to show that $\rho_2(x,y) \leq \rho_2(x,z) + \rho_2(z,y)$

1. $min(1, \rho(x,y)) = 1, min(1, \rho(x,z)) = 1, min(1, \rho(z,y)) = 1, 1 \leq 1 + 1$.
2. $min(1, \rho(x,y)) = \rho(x,y), min(1, \rho(x,z)) = 1, min(1, \rho(z,y)) = 1, \rho(x,y) \leq \rho(x,z) + \rho(z,y)$ and $\rho(x,z) > 1, \rho(z,y) > 1$, so $\rho(x,y) \leq 1 + 1$.
3. $min(1, \rho(x,y)) = 1, min(1, \rho(x,z)) = \rho(x,z), min(1, \rho(z,y)) = 1$. $\rho(x,z) > 1$ gives us $1 \leq 1 + \rho(x,z)$.
4. $min(1, \rho(x,y)) = 1, min(1, \rho(x,z)) = 1, min(1, \rho(z,y)) = \rho(z,y)$. WLOG from #3.
5. $min(1, \rho(x,y)) = \rho(x,y), min(1, \rho(x,z)) = \rho(x,z), min(1, \rho(z,y)) = 1$, $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ and $\rho(z,y) > 1$, so $\rho(x,y) \leq \rho(x,z) + 1$
6. $min(1, \rho(x,y)) = \rho(x,y), min(1, \rho(x,z)) = 1, min(1, \rho(z,y)) = \rho(z,y)$ WLOG from # 5

7. $min(1, \rho(x,y)) = 1, min(1, \rho(x,z)) = \rho(x,z), min(1, \rho(z,y)) = \rho(z,y)$, $1 < 1 + 1$ and $\rho(x,z) \geq 1, \rho(z,y) \geq 1$, so $1 < \rho(x,z) + \rho(z,y)$

8. $min(1, \rho(x,y)) = \rho(x,y), min(1, \rho(x,z)) = \rho(x,z), min(1, \rho(z,y)) = \rho(z,y)$, triangle inequality from ρ.

Problem 5

Let X = (0, 1]. Define $\rho(x,y) \equiv \left|\frac{1}{x} - \frac{1}{y}\right|$. Prove that ρ is a metric on X.

Non-negativity:
If $x = y$ then $\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{x}\right| = |0| = 0$.
If $\rho(x,y) = 0$ then $x = y$:

$$\begin{aligned} \rho(x,y) &= 0 \\ \left|\frac{1}{x} - \frac{1}{y}\right| &= 0 \\ \frac{1}{x} &= \frac{1}{y} \\ x &= y \end{aligned}$$

Symmetry:

$$\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|-\left(\frac{1}{y} - \frac{1}{x}\right)\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = \rho(y,x)$$

Triangle inequality:
For all $x, y, z \in X$, we must show that $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

$$\begin{aligned} \rho(x,y) &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \left|\frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y}\right| \\ &= \left|\left(\frac{1}{x} - \frac{1}{z}\right) + \left(\frac{1}{z} - \frac{1}{y}\right)\right| \\ &\leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = \rho(x,z) + \rho(z,y) \end{aligned}$$