# Assignment 9

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April 4, 2018

## Problem 1

*The function $f$ defined as follows is called the Dirichlet function:*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

*where $\mathbb{Q}$ is the set of rational numbers. Prove that $f$ is NOT Riemann integrable on $[0,1]$. Hint: using the fact that any interval of real numbers contains both rationals and irrationals.*

To show that the function $f$ is not Riemann integrable, we must show that

$$\underset{P}{sup}\{L_P(f)\} \neq \underset{P}{inf}\{U_P(f)\}$$

Since any interval of real numbers contains both rational and irrational numbers, we know that for any partition $P$ with $N$ intervals, $m_i(f) = 0$ and $M_i(f) = 1$ for all $1 \leq i \leq N$.

By def'n, for any $P$ with $N$ intervals, and since $M_i(f) = 1, m_i(f) = 0$ are constants, we know that $L_P(f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1}) = m_i \sum_{i=1}^{N} x_i - x_{i-1} = m_i \cdot (x_n - x_0) = m_i = 0$ and $U_P(f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1}) = M_i \sum_{i=1}^{N} x_i - x_{i-1} = M_i \cdot (x_n - x_0) = M_i = 1$.

Thus $\underset{P}{sup}\{L_P(f)\} = 0$ and $\underset{P}{inf}\{U_P(f)\} = 1$, so $\underset{P}{sup}\{L_P(f)\} \neq \underset{P}{inf}\{U_P(f)\}$ and the function is not Riemann integrable.

## Problem 2

*Let $f$ and $g$ be both Riemann integrable on $[a,b]$. Prove that $f\ g$ is Riemann integrable on $[a,b]$. Hint: One strategy is to first show that $f^2$ is integrable if $f$ is integrable and then use the fact that $f\ g = ((f+g)^2 - f^2 - g^2)/2$.*

Lemma: if $f$ is integrable then $f^2$ is integrable. Since $f$ is integrable, we know that $\forall\, \varepsilon > 0 \,.\, \exists\, P \,.\, U_P(f) - L_P(f) \leq \varepsilon$. We would like to show that $\forall\, \varepsilon > 0 \,.\, \exists\, P \,.\, U_P(f^2) - L_P(f^2) \leq \varepsilon$.

Consider

$$\begin{aligned} |f^2(x) - f^2(y)| = |(f(x) + f(y))(f(x) - f(y))| &\leq |f(x) + f(y)||f(x) - f(y)| \\ &\leq 2B|f(x) - f(y)| \\ &\leq 2B|M_i(f) - m_i(f)| \end{aligned}$$

But also

$$|f^2(x) - f^2(y)| \leq |M_i(f^2) - m_i(f^2)|$$

by the least upper bound property of $f^2$. Because $|M_i(f^2) - m_i(f^2)|$ is the least upper bound, we can see that $|M_i(f^2) - m_i(f^2)| \leq 2B|M_i(f) - m_i(f)|$.

Now we can consider

$$\begin{aligned} U_P(f)^2 - L_P(f^2) &= \sum_{i=1}^{N} (M_i(f^2) - m_i(f^2))(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{N} (2B(M_i(f) - m_i(f))(x_i - x_{i-1})) \\ &\leq 2B(U_P(f) - L_P(f)) \\ &\leq 2B\varepsilon \end{aligned}$$

Which, by the Riemann integrability test, implies that $f^2$ is integrable if $f$ is integrable.

Because $f\ g = ((f+g)^2 - f^2 - g^2)/2$ and integrability is closed under addition, we know that $f\ g$ is integrable as desired.

# Problem 3

*Let $f$ be a continuous function on $[a, b]$ and suppose that $\delta > 0$.*

## a)

*Prove that $\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^b f(x)dx$.*

Consider a partition $P$ with $N$ intervals of size $\delta$ so $x_1 = x_0 + \delta = a + \delta$ and $b = x_N = a + N\delta$. Then $b = a + N\delta \to \delta = \frac{b-a}{N}$ so $\lim_{\delta \to 0} \int_{a+\delta}^b f(x)dx = \lim_{N \to \infty} \sum_{i=1}^N f(x_i^*)(x_i - x_{i-1})$ by the def'n of a Riemann sum. But $\lim_{N \to \infty} \sum_{i=1}^N f(x_i^*)(x_i - x_{i-1}) = \int_a^b f(x)dx$ which gives the desired result. Thus $\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^b f(x)dx$. This result was also proved in theorem 14.2.

## b)

*Prove that $\int_a^b f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^{b-\delta} f(x)dx$.*

We can consider the open interval $(a, b)$ to apply theorem 14.3 which gives us the points $a_1 = a, a_K = b$ and gives us the following equality

$$\int_a^b f(x)dx = \sum_{j=1}^{K+1} \lim_{\delta \to 0} \int_{a_{j-1}+\delta}^{a_j - \delta} f(x)dx = \lim_{\delta \to 0} \int_{a+\delta}^{b-\delta} f(x)dx$$

as desired.