M365C: Real Analysis I
Homework # 04
Handout: 02/08/2018, Thursday
Due: 02/15/2018, Thursday

Submission. Please make your homework neat and stapled. Note that *no late homework will be accepted without compelling reasons.*

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1 To be Graded

Problem 1. Let (X, ρ) be a metric space, and $f\colon [0, \infty) \to [0, \infty)$ be an increasing concave function such that $f(r) = 0$ if and only if $r = 0$. Prove that $f \circ \rho$ is also a metric on X. *Hint: f being concave means that $\forall p, q \in [0, \infty)$ we have $f(tp + (1-t)q) \geq tf(p) + (1-t)f(q)$ $\forall t \in [0, 1]$. We can show here that f is subadditive in the following way: (i) take $q = 0$, we can see that $f(tp) \geq tf(p)$; (ii) $f(p) + f(q) = f(\frac{p}{p+q}(p+q)) + f(\frac{q}{p+q}(p+q)) \geq \frac{p}{p+q} f(p+q) + \frac{q}{p+q} f(p+q) = f(p+q)$.*

Problem 2. Let $X = \mathbb{R}^2$. Define $\rho_1(x, y) \equiv |x_1 - y_1| + |x_2 - y_2|$, $\rho_2(x, y) \equiv \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ and $\rho_{max}(x, y) \equiv \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Prove that ρ_1, ρ_2 and ρ_{max} are uniformly equivalent.

Problem 3. Let $a, b \in \mathbb{R}$ be two real numbers. Prove the following statements.
(i) The set $X = (a, b)$, with the metric $\rho(x, y) = |x - y|$, is open;
(ii) The set $X = [a, b]$, with the metric $\rho(x, y) = |x - y|$, is closed;
(iii) The set $X = (a, b]$, with the metric $\rho(x, y) = |x - y|$, is neither open nor closed.

Problem 4. Let X be any non-empty set. We define $\rho : X \times X \to [0, \infty)$ as:

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Then it can be shown that ρ is a metric on X. Therefore, (X, ρ) is a metric space. Such a metric space is often called a *discrete metric space*. Let (X, ρ) be a discrete metric space. Prove the following statements.
(i) An open ball in X is either a set with only one element (that is, a *singleton*) or all of X.
(ii) All subsets of X are both open and closed.

Problem 5. Let (X, ρ) be a metric space and $S \subseteq X$ a subset. Denote by $\overline{S}$ the set of points of closure of S. Prove that $\overline{S}$ is a closed set.

2 Reading Assignments

- Review Lecture Notes # 6 and # 7;
- Review Sections 2.2 and 3.1 of the textbook;