Assignment 11

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Problem 1

Let $\{f_n\}$ be a sequence of functions defined on $[a,b]$ and g be a continuous function on $[a,b]$.

a

Prove that if $f_n \to f$ pointwise, then $g \cdot f_n \to g \cdot f$ pointwise.

Since g is continuous, we know it is bounded, so the sup exists; let $c = sup_{x\in[a,b]}|g|$. Then since $f_n \to f$ pointwise, we know that $|f_n - f| \leq \varepsilon$ for some N. Thus we also know for any $n \geq N$, $c \cdot |f_n - f| \leq c \cdot \varepsilon$. This lets us conclude $|c \cdot f_n - c \cdot f| \leq c \cdot \varepsilon$ for any $\varepsilon > 0$. This implies that $c \cdot f_n \to c \cdot f$ pointwise. Since c is the sup of $|g|$, $|g \cdot f_n - g \cdot f| = |g||f_n - f| \leq (sup_{x\in[a,b]}|g|)|f_n - f| = c \cdot |f_n - f|$. This implies that $g \cdot f_n \to g \cdot f$ pointwise as well.

See also Corollary 18.5 in the lecture notes.

b

Prove that if $f_n \to f$ uniformly, then $g \cdot f_n \to g \cdot f$ uniformly.

Since g is continuous, we know it is bounded, so the sup exists; let $c = sup_{x\in[a,b]}|g|$. Then since $f_n \to f$ uniformly, we know that $|f_n - f| \leq \varepsilon$ for some N not dependent on x. Thus we also know for any $n \geq N$, $c \cdot |f_n - f| \leq c \cdot \varepsilon$. This lets us conclude $|c \cdot f_n - c \cdot f| \leq c \cdot \varepsilon$ for any $\varepsilon > 0$. This implies that $c \cdot f_n \to c \cdot f$ uniformly. Since c is the sup of $|g|$, $|g \cdot f_n - g \cdot f| = |g||f_n - f| \leq (sup_{x\in[a,b]}|g|)|f_n - f| = c \cdot |f_n - f|$. This implies that $g \cdot f_n \to g \cdot f$ uniformly as well.

See also Corollary 18.5 in the lecture notes.

c

If we assume further that $f_n(x)(n \geq 1)$ are bounded functions and $f_n \to f$ in the sup norm, does $g \cdot f_n \to g \cdot f$ in the sup norm? If so, prove it. Otherwise, give a counter example.

It does, because $\|g \cdot f_n - g\ f\|_\infty \leq (sup_{x\in[a,b]}|g|)\|f_n - f\|_\infty$. Since g is continuous, we know it is bounded, so the sup exists; let $c = sup_{x\in[a,b]}|g|$. Then since $f_n \to f$ in the sup norm, we know that $|f_n - f| \leq \varepsilon$ for some N. Thus we also know for any $n \geq N$, $c \cdot |f_n - f| \leq c \cdot \varepsilon$. This lets us conclude $|c \cdot f_n - c \cdot f| \leq c \cdot \varepsilon$ for any $\varepsilon > 0$. This implies that $c \cdot f_n \to c \cdot f$ in the sup norm. Since c is the sup of $|g|$, $|g \cdot f_n - g \cdot f| = |g||f_n - f| \leq (sup_{x\in[a,b]}|g|)|f_n - f| = c \cdot |f_n - f|$. This implies that $g \cdot f_n \to g \cdot f$ in the sup norm as well.

See also Corollary 18.5 in the lecture notes.

Problem 2

Prove that $f_n(x) = \left(x - \frac{1}{n}\right)^2$ converges uniformly on any finite interval. Hint: $f_n \to x^2$.

Consider some finite interval that is (WLOG) $[a,b]$. Then when we take $f(x) = x^2$ we can see that

$$|f_n(x) - f(x)| = \left|\left(x - \frac{1}{n}\right)^2 - x^2\right| = \left|x^2 - \frac{2x}{n} + \frac{1}{n^2} - x^2\right| = \left|\frac{2x}{n} + \frac{1}{n^2}\right| \leq \frac{2}{n}|x| + \frac{1}{n^2}$$

We will first prove that for any $\varepsilon_1 \geq 0$ we can select a sufficiently large N_1 such that $\frac{2}{n}|x| \leq \varepsilon_1, \forall n \geq N_1$ and for any $\varepsilon_2 \geq 0$ we can select a sufficiently large N_2 such that $\frac{1}{n^2} \leq \varepsilon_2, \forall n \geq N_2$.

It is clear that for any $\varepsilon_1 \geq 0$ we can select a sufficiently large N_1 such that $\frac{2}{n}|x| \leq \varepsilon_1, \forall n \geq N_1$. On the interval $[a,b]$, we can take the constant $c = max(|a|,|b|)$ to see that we must select a value for N such that $\forall n \geq N$. $\frac{2}{n}|x| \leq \frac{2 \cdot c}{n} \leq \varepsilon_1$. Clearly any $N \geq \frac{2 \cdot c}{\varepsilon_1}$ suffices.

It is similarly clear that for any $\varepsilon_2 \geq 0$ we can select a sufficiently large N_2 such that $\frac{1}{n^2} \leq \varepsilon_2, \forall n \geq N_2$. In this case we must simply select $N \geq \sqrt{\frac{1}{\varepsilon_2}}$.

Now, we know that for any ε we can select $\varepsilon_1, \varepsilon_2$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$ and take $N = max(N_1, N_2)$ which is sufficient to show that $|f_n(x) - f(x)| \leq \varepsilon, \forall n \geq N$. This lets us conclude that $f_n(x) = \left(x - \frac{1}{n}\right)^2$ converges uniformly on the interval $[a, b]$.

Problem 3

Let f and g be continuous functions on $[a, b]$.

a

Use the triangle inequality to prove that

$$|\|f\|_\infty - \|g\|_\infty| \leq \|f - g\|_\infty$$

Note that

$$\begin{aligned}\|f\|_\infty &= \|f - g + g\|_\infty \\ &\leq \|f - g\|_\infty + \|g\|_\infty\end{aligned}$$

which implies that $\|f\|_\infty - \|g\|_\infty \leq \|f - g\|_\infty$. And by symmetry we can take the absolute value to see that $|\|f\|_\infty - \|g\|_\infty| \leq \|f - g\|_\infty$.

See also Lemma 17.4 in the lecture notes.

b

Suppose $f_n \to f$ in the sup norm. Prove that $\|f_n\|_\infty \to \|f\|_\infty$

Since $f_n \to f$ in the sup norm we know that $\|f_n(x) - f(x)\|_\infty \to 0$. This means that for any $\varepsilon > 0$ we know that for any $n \geq N$ we have $\|f_n(x) - f(x)\| \leq \varepsilon$ for some N. But by part a we know that $\|f_n(x) - f(x)\| \geq |\|f_n\|_\infty - \|f\|_\infty|$, so we can use the same N such such that for any $n \geq N$ we know that $|\|f_n\|_\infty - \|f\|_\infty| \leq \varepsilon$ for every $n \geq N$.

See also Corollary 18.6 in the lecture notes.