MONASH University

MONASH BUSINESS SCHOOL

ETC3550 Applied forecasting for business and economics

Ch3. The forecasters’ toolbox

OTexts.org/fpp3/

Outline

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1. Some simple forecasting methods
2. Box-Cox transformations
3. Residual diagnostics
4. Evaluating forecast accuracy
5. Prediction intervals

Some simple forecasting methods

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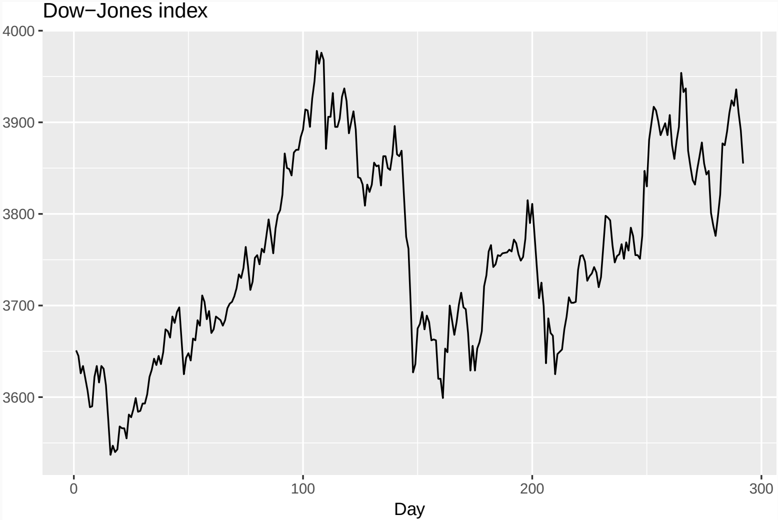

How would you forecast these data?

Some simple forecasting methods

Average method

- Forecast of all future values is equal to mean of historical data $\{y_1, \dots, y_T\}$.
- Forecasts: $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \cdots + y_T)/T$

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Naïve method

- Forecasts equal to last observed value.
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- Consequence of efficient market hypothesis.

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- Forecasts: $\hat{y}_{T+h|T} = y_T$.
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Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts: $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$, where m = seasonal period and k is the integer part of $(h-1)/m$.

Some simple forecasting methods

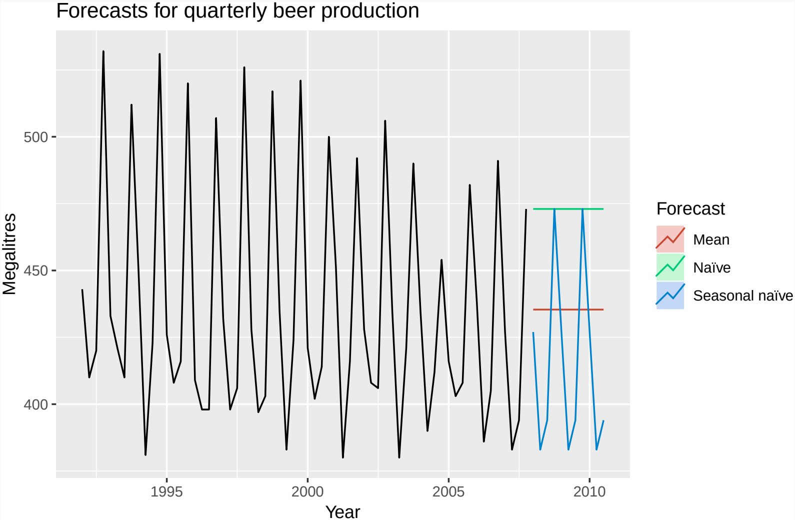
Drift method

- Forecasts equal to last value plus average change.
- Forecasts:

$$\begin{aligned}\hat{y}_{T+h|T} &= y_T + \frac{h}{T-1}\sum_{t=2}^{T}(y_t - y_{t-1}) \\ &= y_T + \frac{h}{T-1}(y_T - y_1).\end{aligned}$$

- Equivalent to extrapolating a line drawn between first and last observations.

Some simple forecasting methods



Some simple forecasting methods

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- Mean: `MEAN(y)`
- Naïve: `NAIVE(y)`
- Seasonal naïve: `SNAIVE(y)`
- Drift: `RW(y ~ drift())`

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Your turn

- Use these four functions to produce forecasts for `goog` and `auscafe`.
- Plot the results using `autoplot()`.

Outline

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Mathematical transformations for stabilizing variation

Square root	$w_t = \sqrt{y_t}$	$\downarrow$
Cube root	$w_t = \sqrt[3]{y_t}$	Increasing
Logarithm	$w_t = \log(y_t)$	strength

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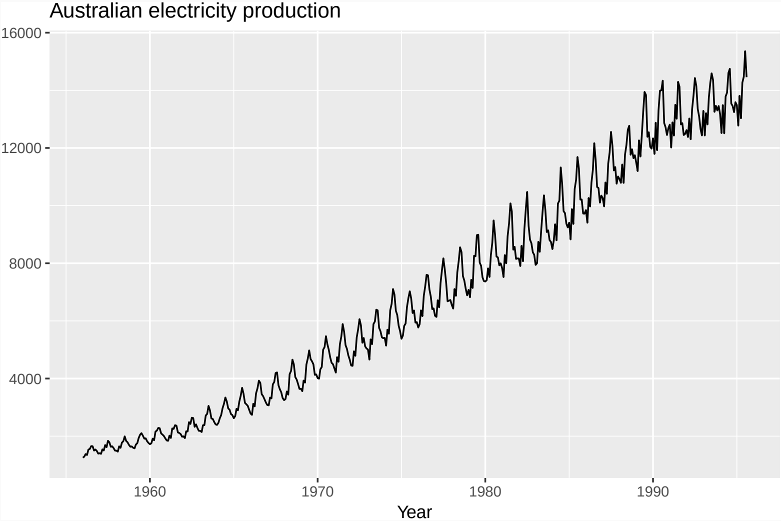
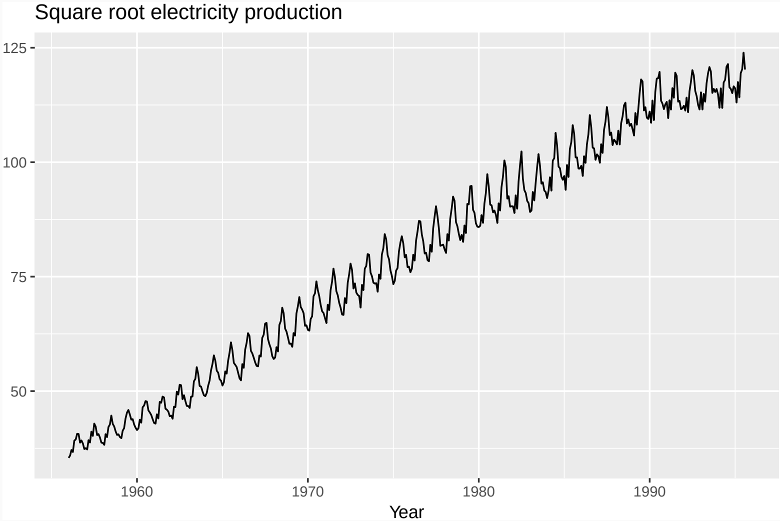
Logarithms, in particular, are useful because they are more interpretable: changes in a log value are **relative (percent) changes on the original scale**.

Variance stabilization



Variance stabilization



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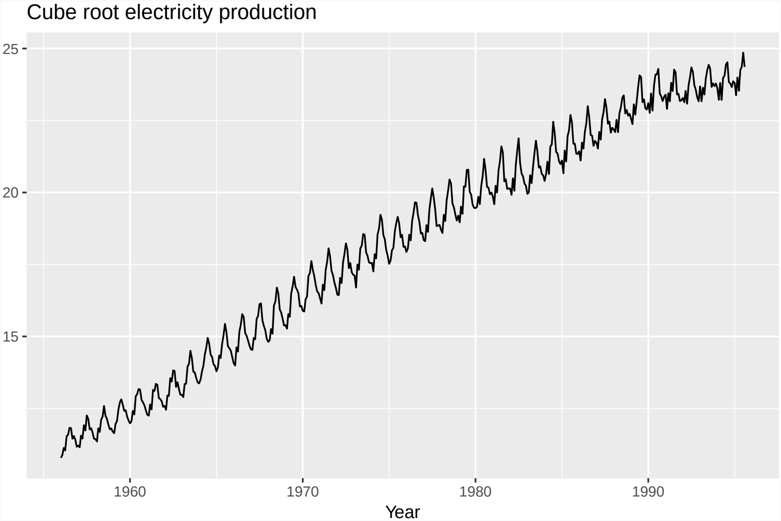

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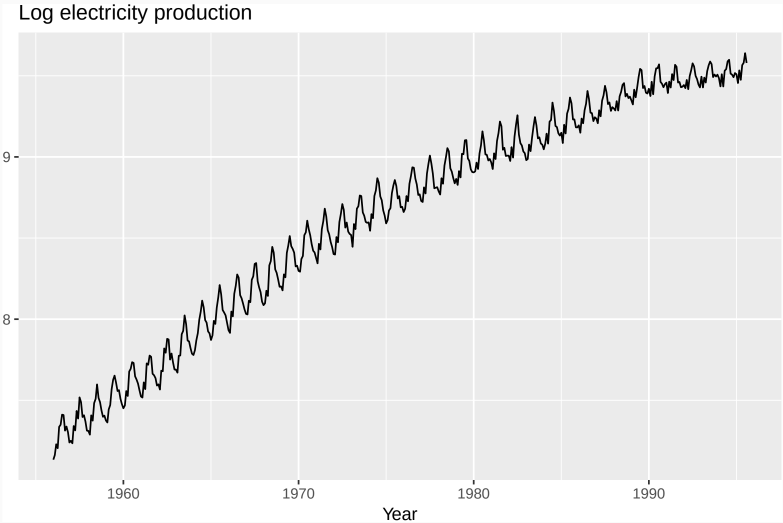

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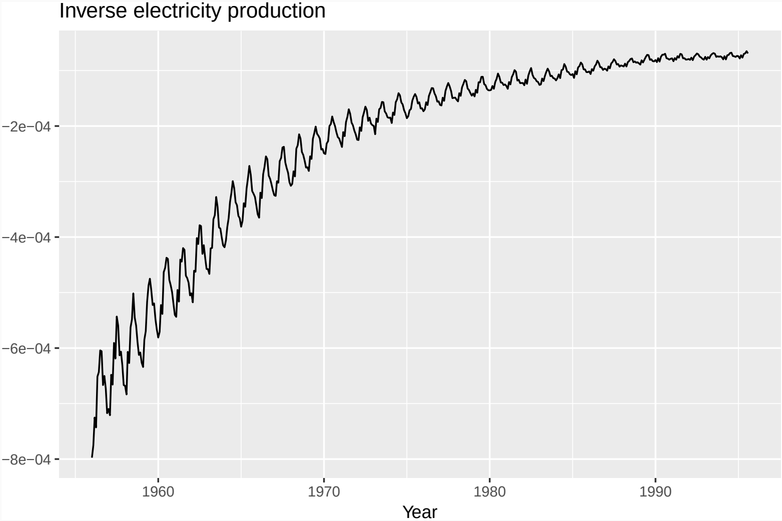

Box-Cox transformations

Each of these transformations is close to a member of the family of **Box-Cox transformations**:

$$w_t = \begin{cases} \log(y_t), & \lambda = 0; \\ (y_t^\lambda - 1)/\lambda, & \lambda \neq 0. \end{cases}$$

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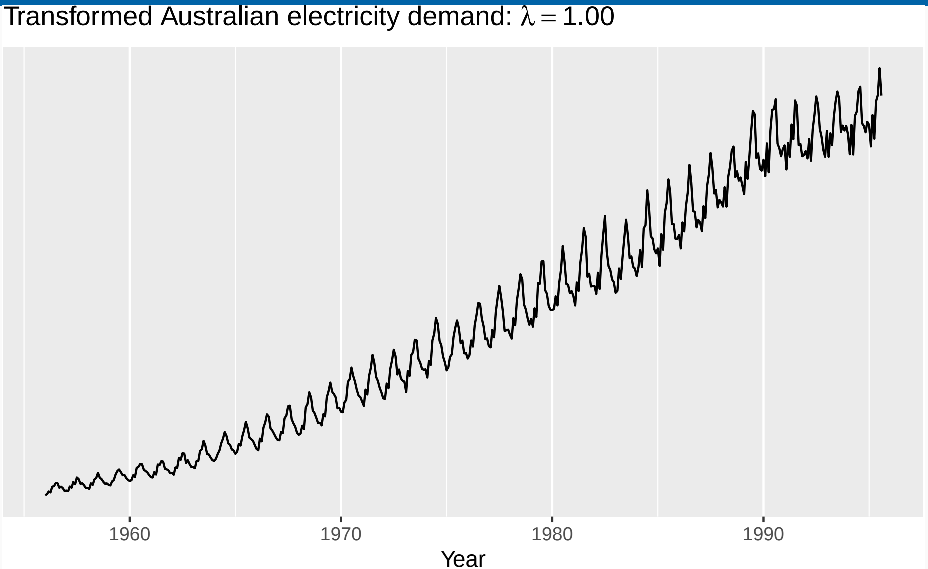
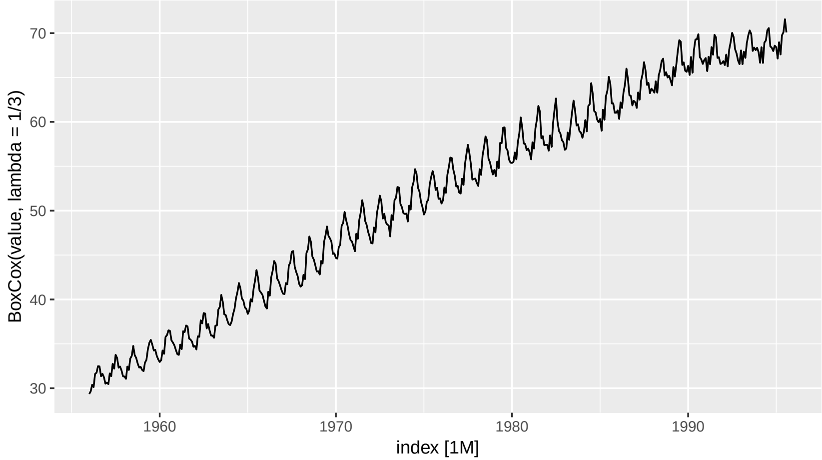
- $\lambda = 1$: (No substantive transformation)
- $\lambda = \frac{1}{2}$: (Square root plus linear transformation)
- $\lambda = 0$: (Natural logarithm)
- $\lambda = -1$: (Inverse plus 1)

Box-Cox transformations



Box-Cox transformations

```r
elec %>% autoplot(BoxCox(value,lambda=1/3))
```



Box-Cox transformations

- y_t^λ for λ close to zero behaves like logs.
- If some $y_t = 0$, then must have $\lambda > 0$
- if some $y_t < 0$, no power transformation is possible unless all y_t adjusted by **adding a constant to all values**.
- Simple values of λ are easier to explain.
- Results are relatively insensitive to λ.
- Often no transformation ($\lambda = 1$) needed.
- Transformation can have very large effect on PI.
- Choosing $\lambda = 0$ is a simple way to force forecasts to be positive

Automated Box-Cox transformations

```
forecast::BoxCox.lambda(elec$value)
```

```
## [1] -0.1143683
```

Automated Box-Cox transformations

```r
forecast::BoxCox.lambda(elec$value)
```

```
## [1] -0.1143683
```

- This attempts to balance the seasonal fluctuations and random variation across the series.
- Always check the results.
- A low value of λ can give extremely large prediction intervals.

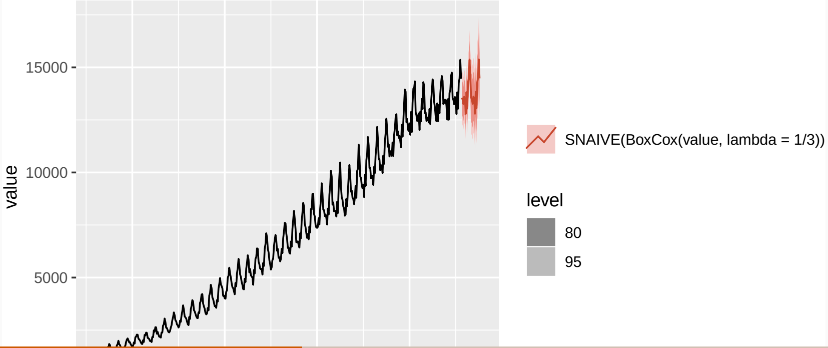
Back-transformation

We must reverse the transformation (or *back-transform*) to obtain forecasts on the original scale. The reverse Box-Cox transformations are given by

$$y_t = \begin{cases} \exp(w_t), & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda}, & \lambda \neq 0. \end{cases}$$

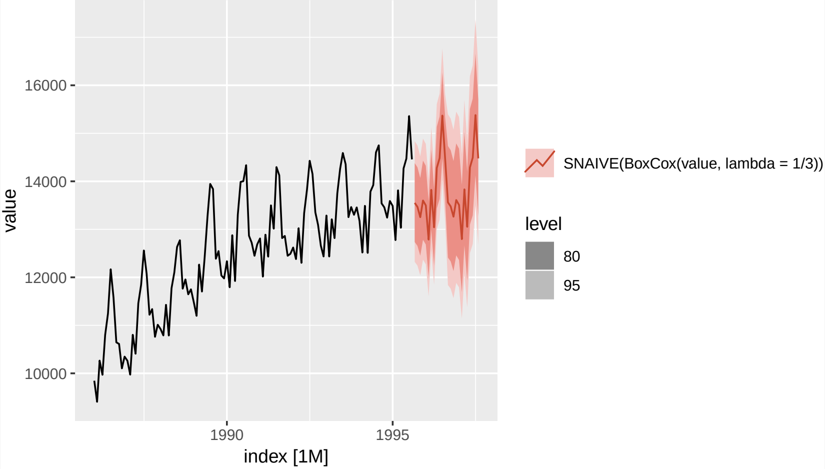
Back-transformation

```
fc <- elec %>%
  model(SNAIVE(BoxCox(value, lambda=1/3))) %>%
  forecast()
fc %>% autoplot(elec)
```



Back-transformation

```
fc %>% autoplot(filter(elec, year(index) > 198
```



Your turn

Find a Box-Cox transformation that works for the `gas` data.

Bias adjustment

- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

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Back-transformed means

Let X be have mean μ and variance σ^2.

Let $f(x)$ be back-transformation function, and $Y = f(X)$.

Taylor series expansion about μ:

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2 f''(\mu).$$

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$$\text{E}[Y] = \text{E}[f(X)] = f(\mu) + \tfrac{1}{2}\sigma^2 f''(\mu)$$

Bias adjustment

Box-Cox back-transformation:

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f(x) = \begin{cases} e^x & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} e^x & \lambda = 0; \\ (1 - \lambda)(\lambda x + 1)^{1/\lambda - 2} & \lambda \neq 0. \end{cases}$$

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$$\mathrm{E}[Y] = \begin{cases} e^{\mu}\left[1 + \frac{\sigma^2}{2}\right] & \lambda = 0; \\ (\lambda\mu + 1)^{1/\lambda}\left[1 + \frac{\sigma^2(1-\lambda)}{2(\lambda\mu+1)^2}\right] & \lambda \neq 0. \end{cases}$$

Bias adjustment

```
eggs <- as_tsibble(fma::eggs)
fit <- eggs %>% model(RW(log(value) ~ drift()))
fc <- fit %>% forecast(h=50)
fc_biased <- fit %>% forecast(h=50, bias_adjust = FALSE)
eggs %>% autoplot(value) +
  autolayer(fc_biased, series="Simple back transformation", level
  autolayer(fc, series="Bias adjusted", level = NULL) +
  guides(colour=guide_legend(title="Forecast"))
```

Outline

Fitted values

- $\hat{y}_{t|t-1}$ is the forecast of y_t based on observations $y_1, \dots, y_t$.
- We call these "fitted values".
- Sometimes drop the subscript: $\hat{y}_t \equiv \hat{y}_{t|t-1}$.
- Often not true forecasts since parameters are estimated on all data.

For example:

- $\hat{y}_t = \bar{y}$ for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T-1)$ for drift method.

Forecasting residuals

Residuals in forecasting: difference between observed value and its fitted value: $e_t = y_t - \hat{y}_{t|t-1}$.

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Assumptions

1. $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
2. $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

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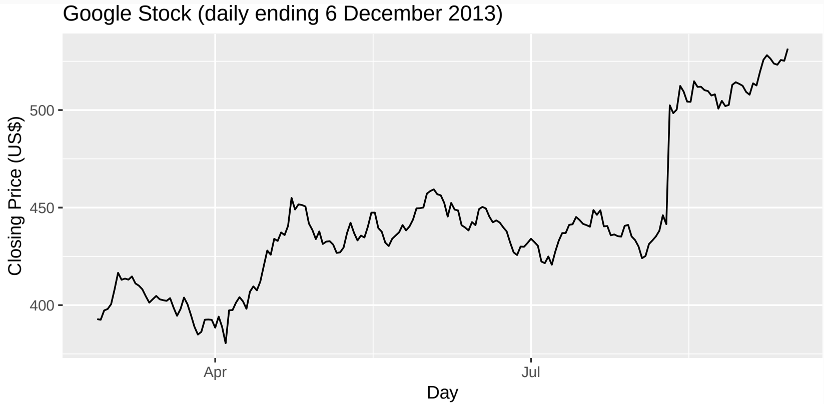
Useful properties (for prediction intervals)

3. $\{e_t\}$ have constant variance.
4. $\{e_t\}$ are normally distributed.

Example: Google stock price

```r
goog200 %>% autoplot(Price) +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



Example: Google stock price

Naïve forecast:

$$\hat{y}_{t|t-1} = y_{t-1}$$

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$$e_t = y_t - y_{t-1}$$

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Naïve forecast:

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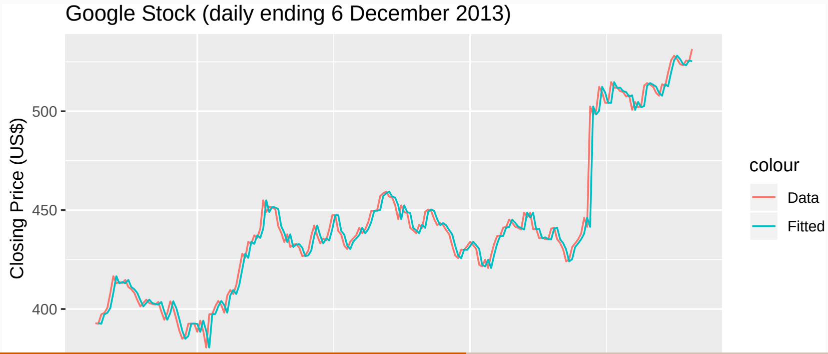
$$e_t = y_t - y_{t-1}$$

Note: e_t are one-step-forecast residuals

Example: Google stock price

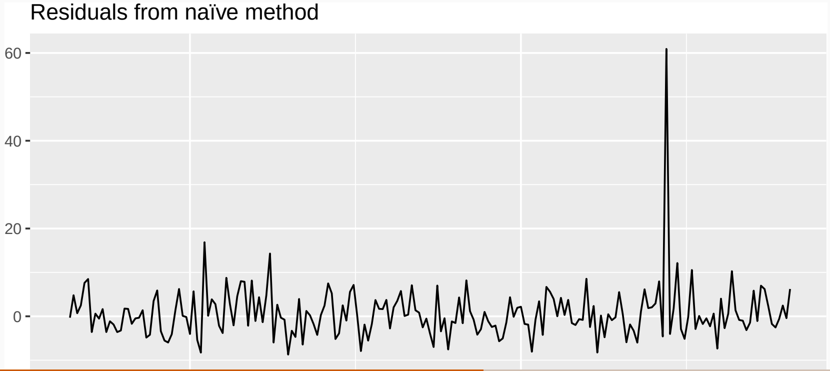
```
fit <- goog200 %>% model(NAIVE(Price))
augment(fit) %>%
  ggplot(aes(x = Time)) +
  geom_line(aes(y = Price, colour = "Data")) +
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  xlab("Day") + ylab("Closing Price (US$)") +
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



Example: Google stock price

```r
residuals(fit) %>%
  autoplot(.resid) + xlab("Day") + ylab("") +
  ggtitle("Residuals from naïve method")
```

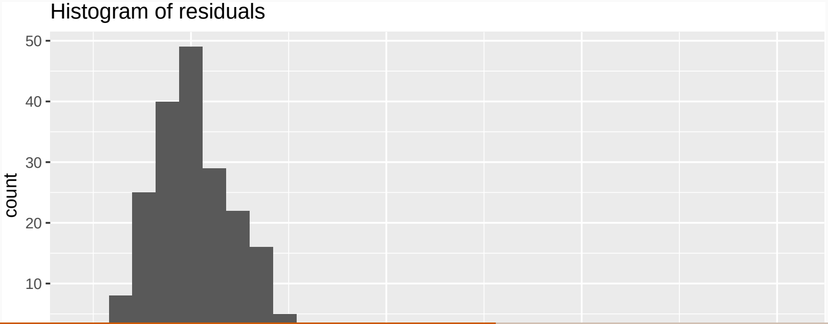
```
## Warning: Removed 1 rows containing missing
## values (geom_path).
```



Example: Google stock price

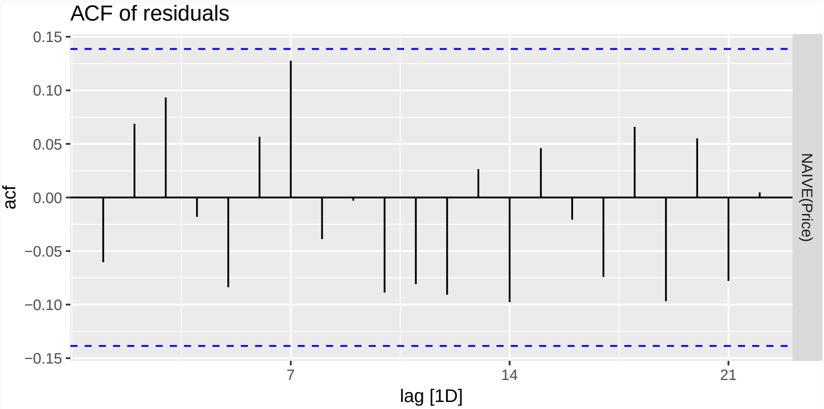
```r
residuals(fit) %>%
  ggplot(aes(x = .resid)) +
  geom_histogram() +
  ggtitle("Histogram of residuals")
```

```
## stat_bin() using bins = 30. Pick
## better value with binwidth.
```



Example: Google stock price

```
residuals(fit) %>% ACF(.resid) %>% autoplot() + ggtitle("ACF
```



ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.
- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.
- We *expect* these to look like white noise.

Portmanteau tests

Consider a *whole set* of r_k values, and develop a test to see whether the set is significantly different from a zero set.

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Box-Pierce test

$$Q = T \sum_{k=1}^{h} r_k^2$$

where h is max lag being considered and T is number of observations.

- If each r_k close to zero, Q will be **small**.
- If some r_k values large (positive or negative), Q will be **large**.

Portmanteau tests

Consider a *whole set* of r_k values, and develop a test to see whether the set is significantly different from a zero set.

Ljung-Box test

$$Q^* = T(T+2)\sum_{k=1}^{h}(T-k)^{-1}r_k^2$$

where h is max lag being considered and T is number of observations.

- My preferences: $h = 10$ for non-seasonal data, $h = 2m$ for seasonal data.
- Better performance, especially in small samples.

Portmanteau tests

- If data are WN, Q^* has χ^2 distribution with $(h - K)$ degrees of freedom where K = no. parameters in model.
- When applied to raw data, set $K = 0$.
- For the Google example:

```r
# lag=h and fitdf=K
Box.test(residuals(fit)$.resid, lag=10, fitdf=0, type="Lj")
```

```
##
##  Box-Ljung test
##
## data:  residuals(fit)$.resid
## X-squared = 11.031, df = 10, p-value =
## 0.3551
```

checkresiduals function

```r
checkresiduals(naive(goog200))
```

Your turn

Compute seasonal naïve forecasts for quarterly Australian beer production from 1992.

```
beer <- ausbeer %>% filter(year(Time) >= 1992)
fit <- beer %>% model(snaive(Production))
fit %>% forecast() %>% autoplot(beer)
```

Test if the residuals are white noise.

```
fit %>% residuals %>% checkresiduals(.resid)
```

What do you conclude?

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2 Box-Cox transformations
3 Residual diagnostics
4 Evaluating forecast accuracy
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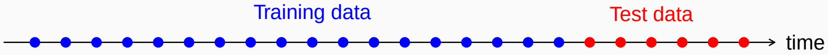
Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

Forecast errors

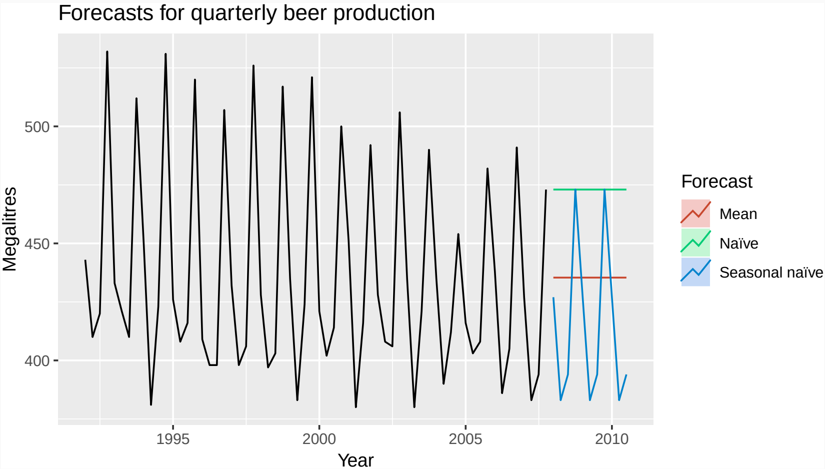
Forecast "error": the difference between an observed value and its forecast.

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.
- These are *true* forecast errors as the test data is not used in computing $\hat{y}_{T+h|T}$.

Measures of forecast accuracy



Measures of forecast accuracy

$$
\begin{aligned}
y_{T+h} &= (T+h)\text{th observation, } h = 1, \dots, H \\
\hat{y}_{T+h|T} &= \text{its forecast based on data up to time } T. \\
e_{T+h} &= y_{T+h} - \hat{y}_{T+h|T}
\end{aligned}
$$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e^2_{T+h}) \qquad \text{RMSE} = \sqrt{\text{mean}(e^2_{T+h})}$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

Measures of forecast accuracy

$$\begin{aligned} y_{T+h} &= (T+h)\text{th observation}, h = 1, \dots, H \\ \hat{y}_{T+h|T} &= \text{its forecast based on data up to time } T. \\ e_{T+h} &= y_{T+h} - \hat{y}_{T+h|T} \end{aligned}$$

$$\begin{aligned} \text{MAE} &= \text{mean}(|e_{T+h}|) \\ \text{MSE} &= \text{mean}(e_{T+h}^2) \qquad \text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)} \\ \text{MAPE} &= 100\text{mean}(|e_{T+h}|/|y_{T+h}|) \end{aligned}$$

- MAE, MSE, RMSE are all scale dependent.
- MAPE is scale independent but is only sensible if $y_t \gg 0$ for all t, and y has a natural zero.

Measures of forecast accuracy

Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

where Q is a stable measure of the scale of the time series $\{y_t\}$.

Proposed by Hyndman and Koehler (IJF, 2006).

For non-seasonal time series,

$$Q = (T-1)^{-1}\sum_{t=2}^{T}|y_t - y_{t-1}|$$

works well. Then MASE is equivalent to MAE relative to a naïve method.

Measures of forecast accuracy

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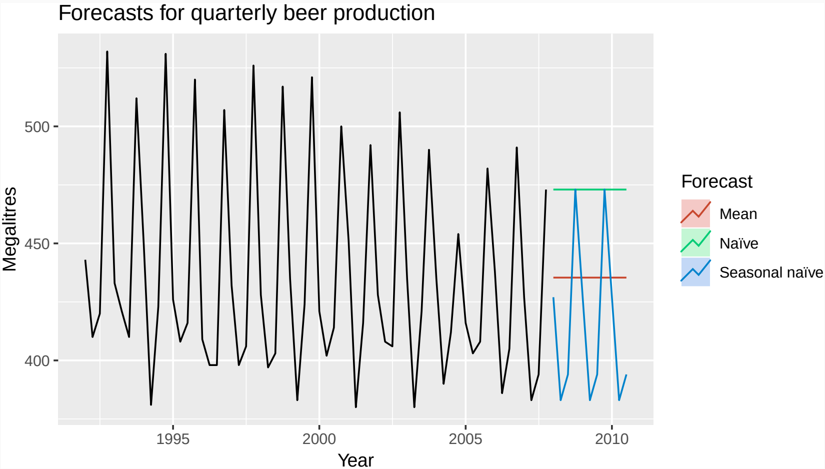
For seasonal time series,

$$Q = (T - m)^{-1} \sum_{t=m+1}^{T} |y_t - y_{t-m}|$$

works well. Then MASE is equivalent to MAE relative to a seasonal naïve method.

Measures of forecast accuracy



Measures of forecast accuracy

```
beer2 <- ausbeer %>% filter(between(year(Time), 1992, 2007))
beer3 <- ausbeer %>% filter(year(Time) > 2007)
fc <- beer2 %>%
  model(
    Mean = MEAN(Production),
    Naïve = NAIVE(Production),
    Seasonal naïve = SNAIVE(Production)
  ) %>%
  forecast(h = 10)
accuracy(fc, beer3)
```

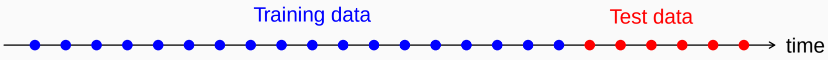
	RMSE	MAE	MAPE
Mean	38.44724	34.825	8.283390
Naïve	62.69290	57.400	14.184424
Seasonal naïve	14.31084	13.400	3.168503

Poll: true or false?

1. Good forecast methods should have normally distributed residuals.
2. A model with small residuals will give good forecasts.
3. The best measure of forecast accuracy is MAPE.
4. If your model doesn’t forecast well, you should make it more complicated.
5. Always choose the model with the best forecast accuracy as measured on the test set.

Time series cross-validation

Traditional evaluation



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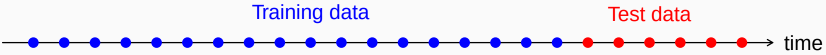

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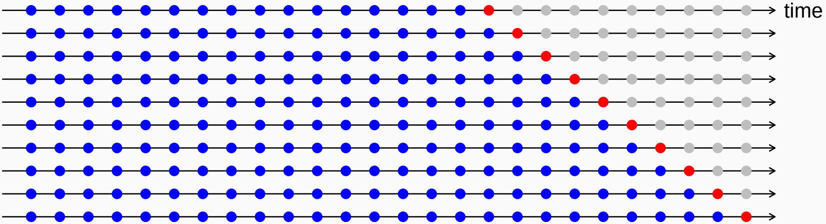
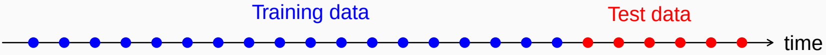
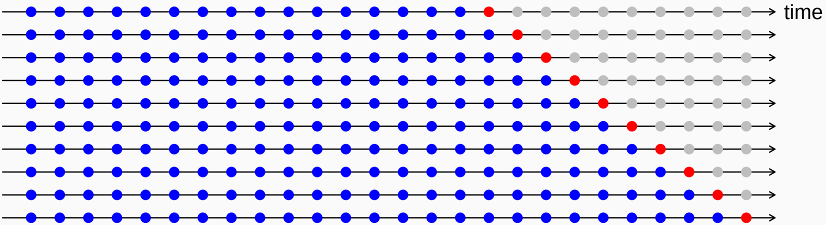


Time series cross-validation

Traditional evaluation



Time series cross-validation



- Forecast accuracy averaged over test sets.
- Also known as "evaluation on a rolling forecasting origin"

tsCV function

```r
e <- tsCV(goog200, rwf, drift=TRUE, h=1)
sqrt(mean(e^2, na.rm=TRUE))
sqrt(mean(residuals(rwf(goog200, drift=TRUE))^2,
                                         na.rm=TRUE))
```

A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.

Pipe function

%TODO: better example of pipe utility

Ugly code:

```
e <- tsCV(goog200, rwf, drift=TRUE, h=1)
sqrt(mean(e^2, na.rm=TRUE))
sqrt(mean(residuals(rwf(goog200, drift=TRUE))^2,
                                        na.rm=TRUE))
```

Better with a pipe:

```
goog200 %>%
  tsCV(forecastfunction=rwf, drift=TRUE, h=1) -> e
e^2 %>% mean(na.rm=TRUE) %>% sqrt
goog200 %>% rwf(drift=TRUE) %>% residuals -> res
```

Outline

Prediction intervals

- A forecast $\hat{y}_{T+h|T}$ is (usually) the mean of the conditional distribution $y_{T+h} \mid y_1, \dots, y_T$.
- A prediction interval gives a region within which we expect y_{T+h} to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

  where $\hat{\sigma}_h$ is the st dev of the h-step distribution.
- When $h = 1$, $\hat{\sigma}_h$ can be estimated from the residuals.

Prediction intervals

Naive forecast with prediction interval:

```
res_sd <- sqrt(mean(res^2, na.rm=TRUE))
c(tail(goog200,1)) + 1.96 * res_sd * c(-1,1)
```

```
naive(goog200, level=95)
```

Prediction intervals

- Point forecasts are often useless without prediction intervals.
- Prediction intervals require a stochastic model (with random errors, etc).
- Multi-step forecasts for time series require a more sophisticated approach (with PI getting wider as the forecast horizon increases).

Prediction intervals

Assume residuals are normal, uncorrelated, sd = $\hat{\sigma}$:

Mean forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{1 + 1/T}$
Naïve forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{h}$
Seasonal naïve forecasts	$\hat{\sigma}_h = \hat{\sigma}\sqrt{k + 1}$
Drift forecasts:	$\hat{\sigma}_h = \hat{\sigma}\sqrt{h(1 + h/T)}.$

where k is the integer part of $(h - 1)/m$.

Note that when $h = 1$ and T is large, these all give the same approximate value $\hat{\sigma}$.

Prediction intervals

- Computed automatically using: `naive()`, `snaive()`, `rwf()`, `meanf()`, etc.
- Use `level` argument to control coverage.
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.