

ETC3550/ETC5550

Applied forecasting

Ch10. Dynamic regression models

OTexts.org/fpp3/

Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

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Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Regression with ARIMA errors

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- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

where ε_t is white noise.

Residuals and errors

Example: $\eta_t = \text{ARIMA}(1,1,1)$

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

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Residuals and errors

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$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing η_t from ε_t .
- Only the errors ε_t are assumed to be white noise.
- In ordinary regression, η_t is assumed to be white noise and so $\eta_t = \varepsilon_t$.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

- 1 Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
- 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
- 3 p -values for coefficients usually too small (“spurious regression”).
- 4 AIC of fitted models misleading.

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 - 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
 - 3 p -values for coefficients usually too small (“spurious regression”).
 - 4 AIC of fitted models misleading.
- Minimizing $\sum \varepsilon_t^2$ avoids these problems.
 - Maximizing likelihood similar to minimizing $\sum \varepsilon_t^2$.

Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

where η_t is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

Stationarity

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

Stationarity

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t,$$
$$(1 - \phi_1 B)\eta'_t = (1 + \theta_1 B)\varepsilon_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $\eta'_t = \eta_t - \eta_{t-1}$.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

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Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t$$

$$\text{where } \phi(B)(1-B)^d \eta_t = \theta(B) \varepsilon_t$$

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$$\text{where } \phi(B)(1-B)^d \eta_t = \theta(B) \varepsilon_t$$

After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t$$

$$\text{where } \phi(B) \eta_t = \theta(B) \varepsilon_t$$

$$\text{and } y'_t = (1-B)^d y_t$$

Model selection

- Fit regression model with automatically selected ARIMA errors. (R will take care of differencing before estimation.)
- Check that ε_t series looks like white noise.

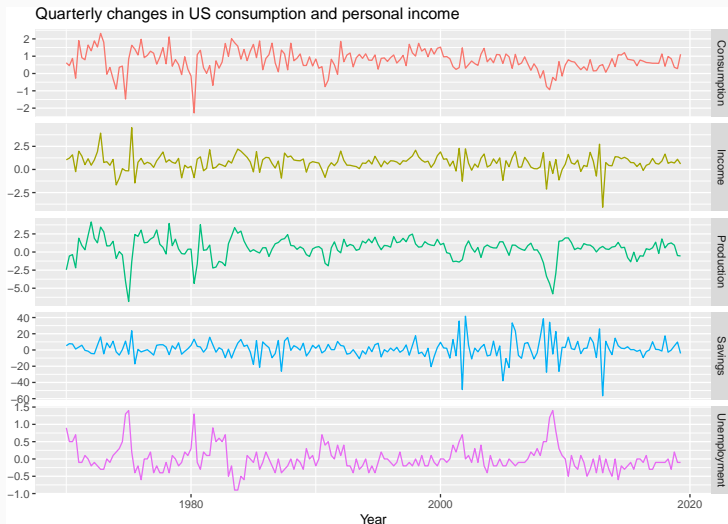
Model selection

- Fit regression model with automatically selected ARIMA errors. (R will take care of differencing before estimation.)
- Check that ε_t series looks like white noise.

Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

US personal consumption and income



US personal consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

US personal consumption and income

```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))
report(fit)
```

```
## Series: Consumption
## Model: LM w/ ARIMA(1,0,2) errors
##
## Coefficients:
##          ar1          ma1          ma2  Income  intercept
##          0.7070    -0.6172    0.2066    0.1976         0.5949
## s.e.    0.1068      0.1218    0.0741    0.0462         0.0850
##
## sigma^2 estimated as 0.3113:  log likelihood=-163
## AIC=338.1   AICc=338.5   BIC=357.8
```

US personal consumption and income

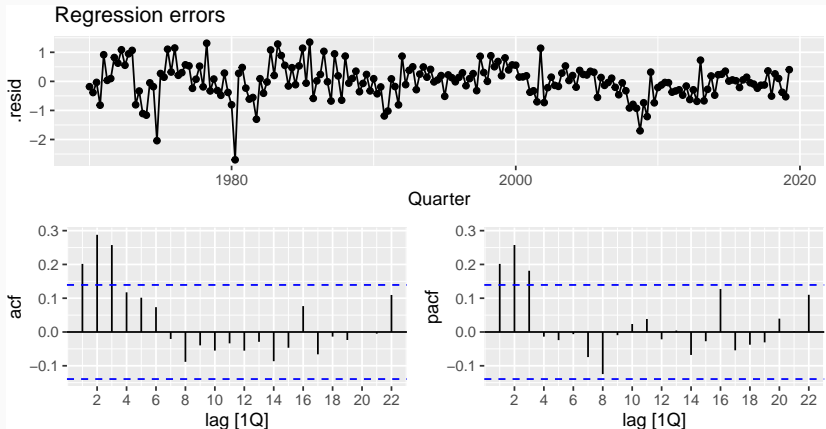
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fit <- us_change %>% model(ARIMA(Consumption ~ Income))  
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```

Write down the equations for the fitted model.

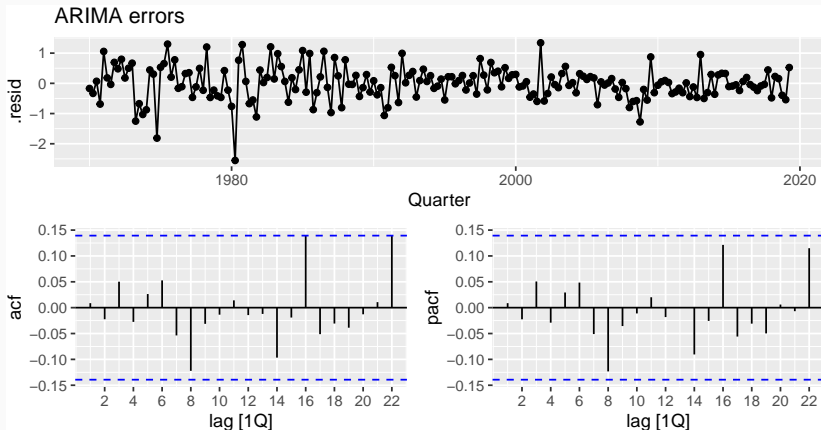
US personal consumption and income

```
residuals(fit, type='regression') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') +  
  ggtitle("Regression errors")
```



US personal consumption and income

```
residuals(fit, type='response') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') +  
  ggtitle("ARIMA errors")
```



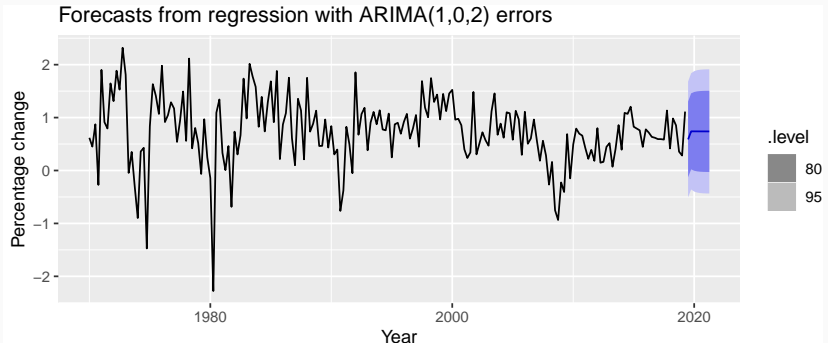
US personal consumption and income

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 5, lag = 12)
```

```
## # A tibble: 1 x 3  
##   .model                                lb_stat lb_pvalue  
##   <chr>                                <dbl>     <dbl>  
## 1 ARIMA(Consumption ~ Income)         5.54      0.595
```


US personal consumption and income

```
us_change_future <- new_data(us_change, 8) %>%  
  mutate(Income = mean(us_change$Income))  
forecast(fit, new_data = us_change_future) %>%  
  autoplot(us_change) +  
  labs(x = "Year", y = "Percentage change",  
       title = "Forecasts from regression with ARIMA(1,0,2) errors")
```



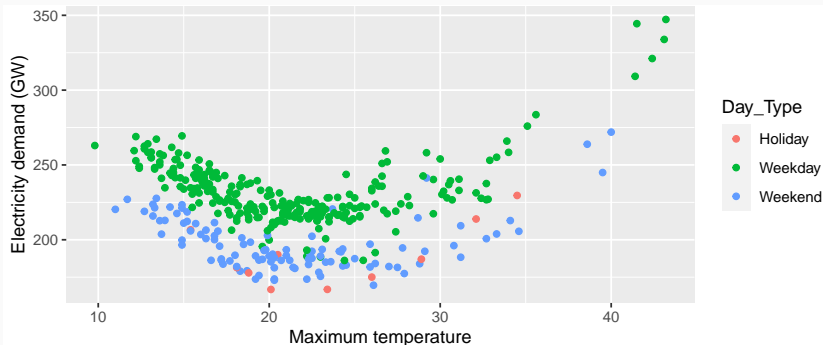
Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

Daily electricity demand

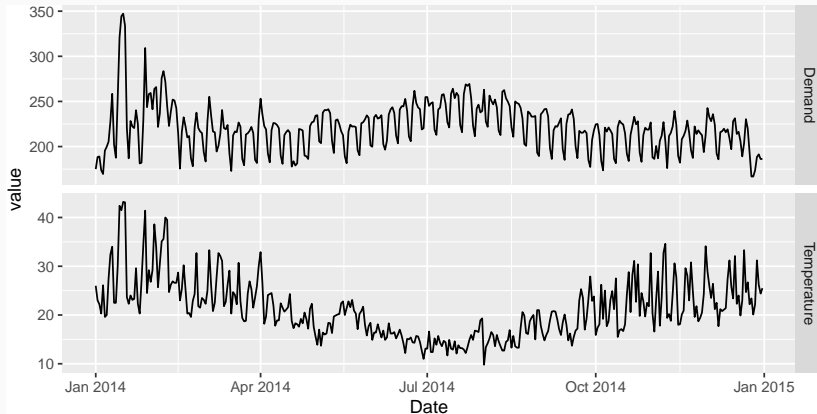
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
vic_elec_daily %>%  
  ggplot(aes(x=Temperature, y=Demand, colour=Day_Type)) +  
  geom_point() +  
  labs(x = "Maximum temperature", y = "Electricity demand (GW)")
```



Daily electricity demand

```
vic_elec_daily %>%  
  gather("var", "value", Demand, Temperature) %>%  
  ggplot(aes(x = Date, y = value)) + geom_line() +  
  facet_grid(vars(var), scales = "free_y")
```



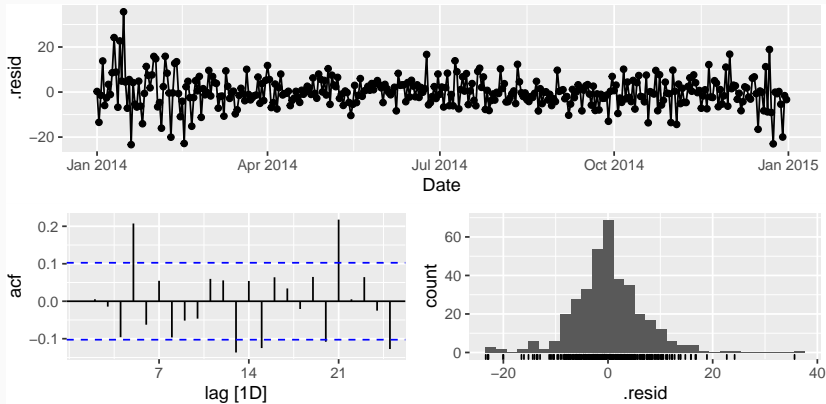
Daily electricity demand

```
fit <- vic_elec_daily %>%  
  model(ARIMA(Demand ~ Temperature + I(Temperature^2) +  
              (Day_Type=="Weekday")))  
report(fit)
```

```
## Series: Demand  
## Model: LM w/ ARIMA(2,1,2)(0,0,2)[7] errors  
##  
## Coefficients:  
##          ar1          ar2          ma1          ma2          sma1          sma2  
##          1.1521   -0.2750   -1.3851    0.4071    0.1589    0.3103  
## s.e.    0.6265    0.4812    0.6082    0.5804    0.0591    0.0538  
##          Temperature  I(Temperature^2)  
##              -7.947              0.1865  
## s.e.          0.492              0.0097  
##          Day_Type == "Weekday"TRUE  
##                      31.825  
## s.e.                      1.019  
##
```

Daily electricity demand

```
gg_tsresiduals(fit)
```



Daily electricity demand

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 9, lag = 14)
```

```
## # A tibble: 1 x 3  
##   .model                                lb_stat lb_pvalue  
##   <chr>                                <dbl>     <dbl>  
## 1 "ARIMA(Demand ~ Temperature + I(Temper~ 38.1     3.54e-7
```

Daily electricity demand

```
# Forecast one day ahead
vic_next_day <- new_data(vic_elec_daily, 1) %>%
  mutate(Temperature = 26, Day_Type = "Holiday")
forecast(fit, vic_next_day)
```

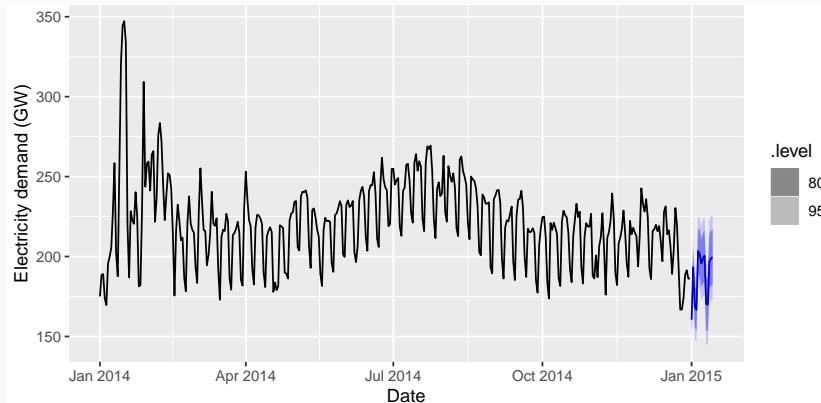
```
## # A tibble: 1 x 6 [1D]
## # Key:   .model [1]
##   .model Date          Demand .distribution Temperature
##   <chr>  <date>          <dbl> <dist>          <dbl>
## 1 "ARIM~ 2015-01-01    161. N(161, 49)          26
## # ... with 1 more variable: Day_Type <chr>
```


Daily electricity demand

```
vic_elec_future <- new_data(vic_elec_daily, 14) %>%  
  mutate(  
    Temperature = 26,  
    Holiday = c(TRUE, rep(FALSE, 13)),  
    Day_Type = case_when(  
      Holiday ~ "Holiday",  
      wday(Date) %in% 2:6 ~ "Weekday",  
      TRUE ~ "Weekend"  
    )  
  )
```

Daily electricity demand

```
forecast(fit, new_data=vic_elec_future) %>%  
  autoplot(vic_elec_daily) + ylab("Electricity demand (GW)")
```



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Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

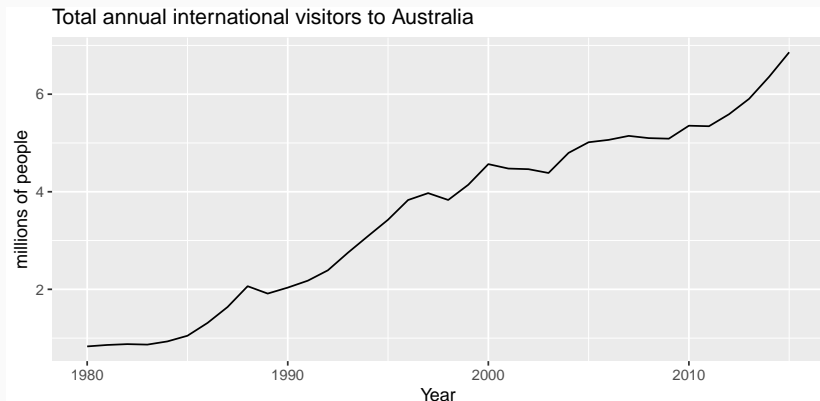
where η_t is ARIMA process with $d \geq 1$.

Difference both sides until η_t is stationary:

$$y'_t = \beta_1 + \eta'_t$$

where η'_t is ARMA process.

International visitors



International visitors

Deterministic trend

```
fit_deterministic <- aus_visitors %>%  
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: value  
## Model: LM w/ ARIMA(2,0,0) errors  
##  
## Coefficients:  
##          ar1          ar2  trend()  intercept  
##          1.113   -0.3805    0.1710     0.4156  
## s.e.    0.160     0.1585    0.0088     0.1897  
##  
## sigma^2 estimated as 0.02979:  log likelihood=13.6  
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```


International visitors

Deterministic trend

```
fit_deterministic <- aus_visitors %>%  
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: value  
## Model: LM w/ ARIMA(2,0,0) errors  
##  
## Coefficients:  
##          ar1      ar2  trend()  intercept  
##          1.113 -0.3805   0.1710    0.4156  
## s.e.    0.160   0.1585   0.0088    0.1897  
##  
## sigma^2 estimated as 0.02979:  log likelihood=13.6  
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

$$y_t = 0.42 + 0.17t + \eta_t$$

$$\eta_t = 1.11\eta_{t-1} - 0.38\eta_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim \text{NID}(0, 0.0298).$$

International visitors

Stochastic trend

```
fit_stochastic <- aus_visitors %>%  
  model(Stochastic = ARIMA(value ~ pdq(d=1)))  
report(fit_stochastic)
```

```
## Series: value  
## Model: ARIMA(0,1,1) w/ drift  
##  
## Coefficients:  
##           ma1    constant  
##      0.3006      0.1735  
## s.e.  0.1647      0.0390  
##  
## sigma^2 estimated as 0.03376:  log likelihood=10.62  
## AIC=-15.24   AICc=-14.46   BIC=-10.57
```

International visitors

Stochastic trend

```
fit_stochastic <- aus_visitors %>%  
  model(Stochastic = ARIMA(value ~ pdq(d=1)))  
report(fit_stochastic)
```

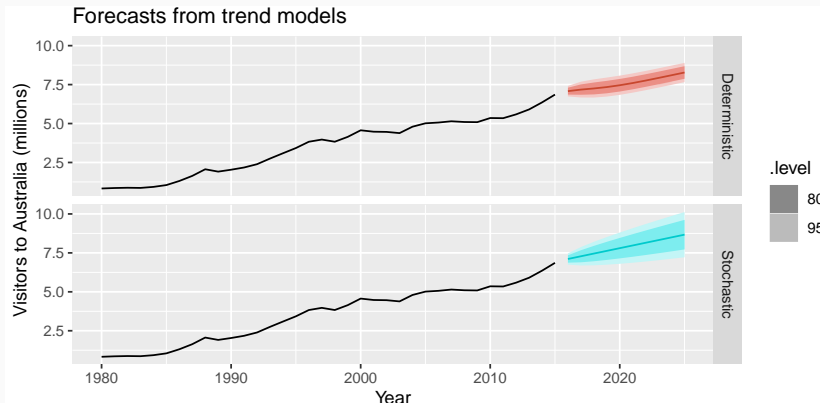
```
## Series: value  
## Model: ARIMA(0,1,1) w/ drift  
##  
## Coefficients:  
##           ma1  constant  
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## s.e.  0.1647    0.0390  
##  
## sigma^2 estimated as 0.03376:  log likelihood=10.62  
## AIC=-15.24  AICc=-14.46  BIC=-10.57
```

$$y_t - y_{t-1} = 0.17 + \varepsilon_t$$

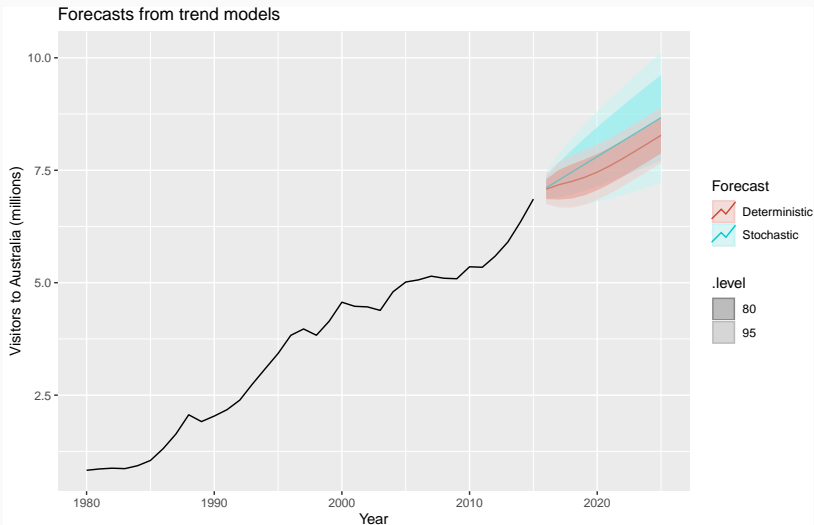
$$y_t = y_0 + 0.17t + \eta_t$$

$$\eta_t = \eta_{t-1} + 0.30\varepsilon_{t-1} + \varepsilon_t$$

International visitors



International visitors



Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

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Dynamic harmonic regression

Combine Fourier terms with ARIMA errors

Advantages

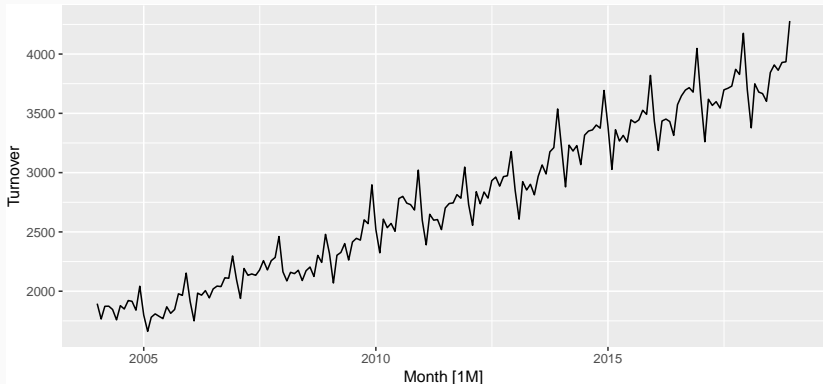
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of K (but more wiggly seasonality can be handled by increasing K);
- the short-term dynamics are easily handled with a simple ARMA error.

Disadvantages

- seasonality is assumed to be fixed

Eating-out expenditure

```
aus_cafe <- aus_retail %>% filter(  
  Industry == "Cafes, restaurants and takeaway food services",  
  year(Month) %in% 2004:2018  
) %>% summarise(Turnover = sum(Turnover))  
aus_cafe %>% autoplot(Turnover)
```

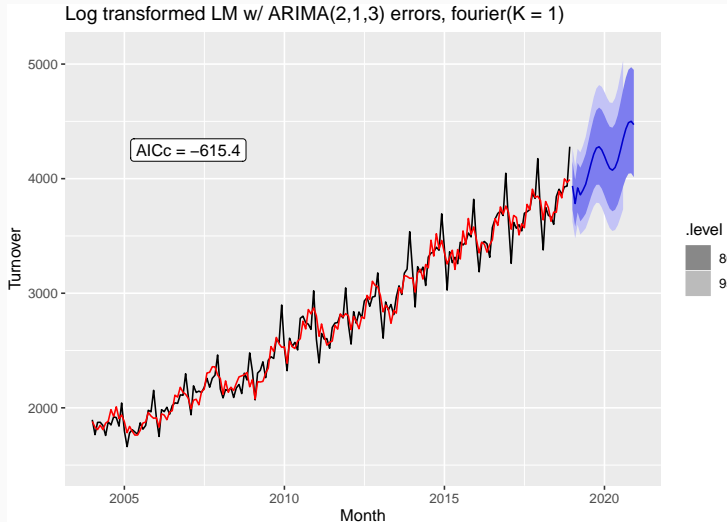


Eating-out expenditure

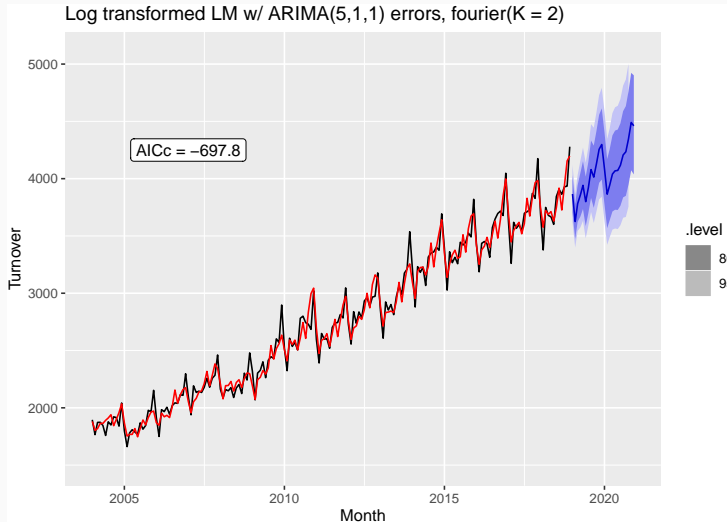
```
fit <- aus_cafe %>% model(  
  `K = 1` = ARIMA(log(Turnover) ~ fourier(K = 1) + PDQ(0,0,0)),  
  `K = 2` = ARIMA(log(Turnover) ~ fourier(K = 2) + PDQ(0,0,0)),  
  `K = 3` = ARIMA(log(Turnover) ~ fourier(K = 3) + PDQ(0,0,0)),  
  `K = 4` = ARIMA(log(Turnover) ~ fourier(K = 4) + PDQ(0,0,0)),  
  `K = 5` = ARIMA(log(Turnover) ~ fourier(K = 5) + PDQ(0,0,0)),  
  `K = 6` = ARIMA(log(Turnover) ~ fourier(K = 6) + PDQ(0,0,0)))  
glance(fit)
```

.model	sigma2	log_lik	AIC	AICc	BIC
K = 1	0.0017	317.2	-616.5	-615.4	-587.8
K = 2	0.0011	361.9	-699.7	-697.8	-661.5
K = 3	0.0008	393.6	-763.2	-761.3	-725.0
K = 4	0.0005	426.8	-821.6	-818.2	-770.6
K = 5	0.0003	473.7	-919.5	-916.9	-874.8
K = 6	0.0003	474.0	-920.1	-917.5	-875.4

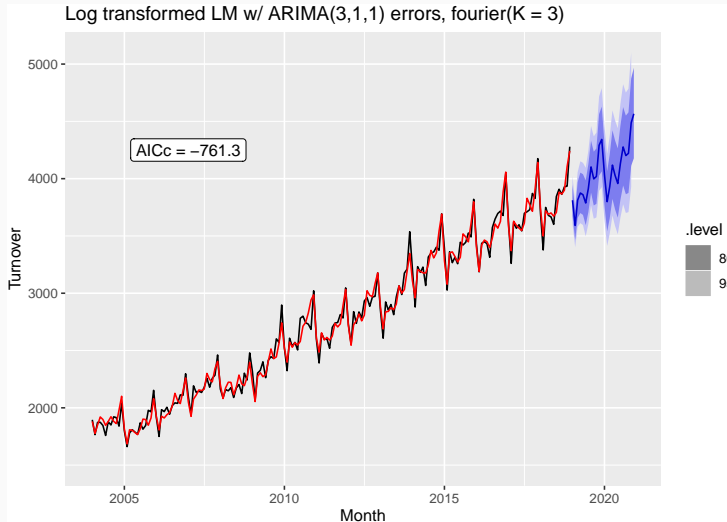
Eating-out expenditure



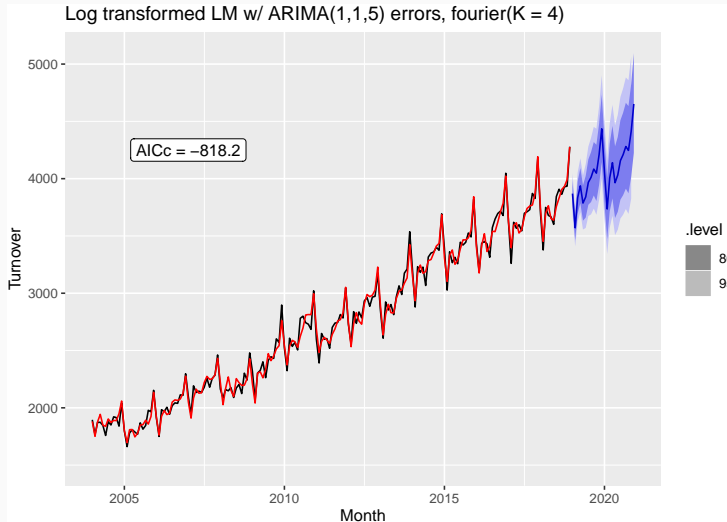
Eating-out expenditure



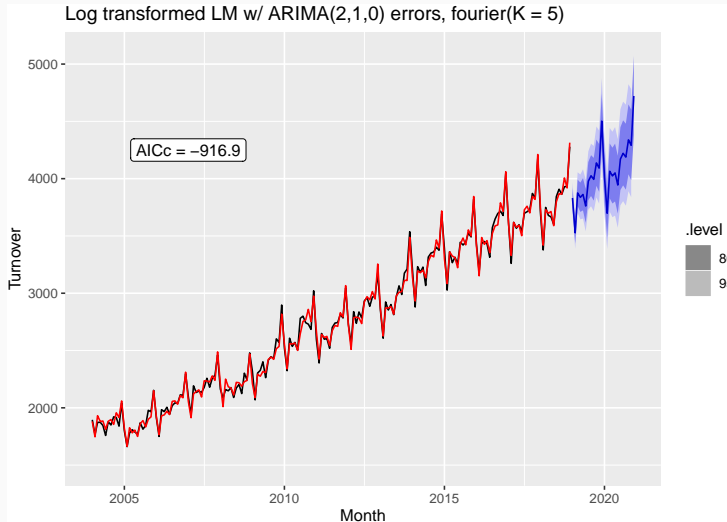
Eating-out expenditure



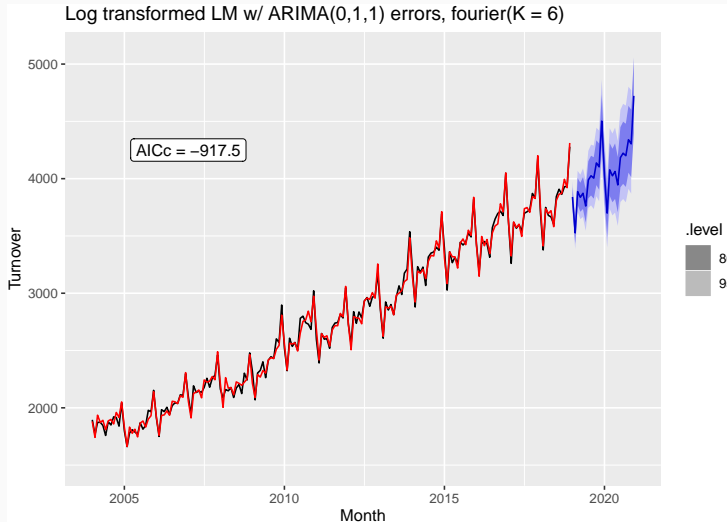
Eating-out expenditure



Eating-out expenditure



Eating-out expenditure



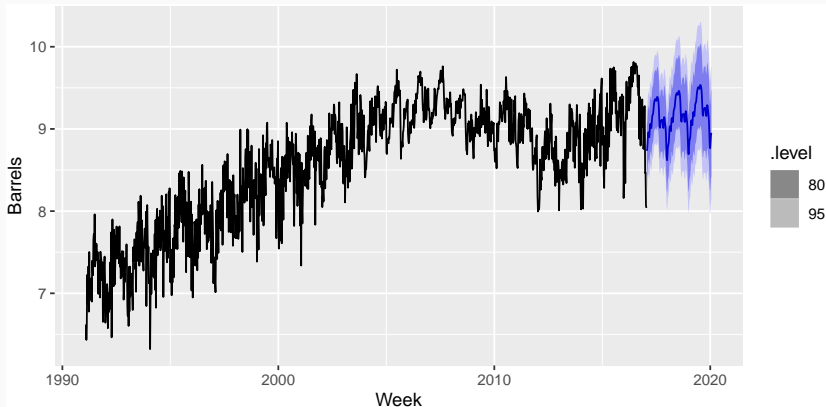
Example: weekly gasoline products

```
fit <- us_gasoline %>%  
  model(ARIMA(Barrels ~ fourier(K = 13) + PDQ(0,0,0)))  
report(fit)
```

```
## Series: Barrels  
## Model: LM w/ ARIMA(0,1,1) errors  
##  
## Coefficients:  
##          ma1  fourier(K = 13)C1_52  fourier(K = 13)S1_52  
##          -0.8934                -0.1121                -0.2300  
## s.e.       0.0132                0.0123                0.0122  
##          fourier(K = 13)C2_52  fourier(K = 13)S2_52  
##                      0.0420                0.0317  
## s.e.              0.0099                0.0099  
##          fourier(K = 13)C3_52  fourier(K = 13)S3_52  
##                      0.0832                0.0346  
## s.e.              0.0094                0.0094  
##          fourier(K = 13)C4_52  fourier(K = 13)S4_52  
##                      0.0185                0.0398  
## s.e.              0.0092                0.0092  
##          fourier(K = 13)C5_52  fourier(K = 13)S5_52  
##                      -0.0315                0.0009  
## s.e.              0.0091                0.0091  
##          fourier(K = 13)C6_52  fourier(K = 13)S6_52
```

Example: weekly gasoline products

```
forecast(fit, h = "3 years") %>%  
  autoplot(us_gasoline)
```



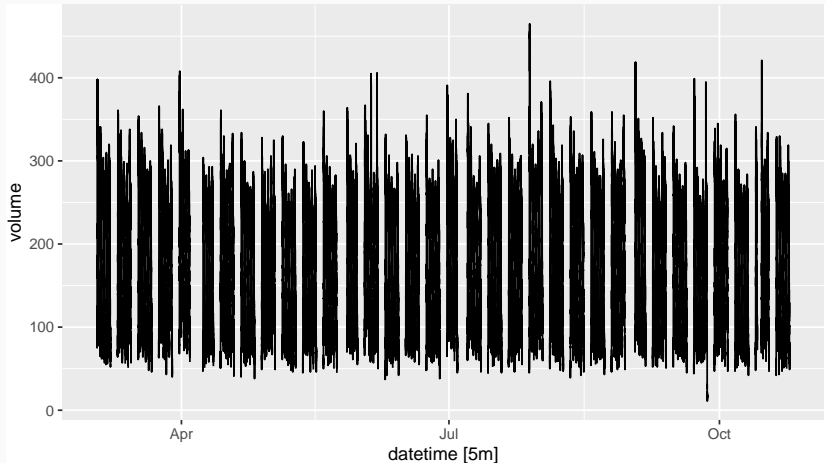
5-minute call centre volume

```
(calls <- readr::read_tsv("http://robjhyndman.com/data/callcenter.txt") %>%  
  rename(time = X1) %>%  
  pivot_longer(-time, names_to = "date", values_to = "volume") %>%  
  mutate(  
    date = as.Date(date, format = "%d/%m/%Y"),  
    datetime = as_datetime(date) + time  
  ) %>%  
  as_tsibble(index = datetime))
```

```
## # A tsibble: 27,716 x 4 [5m] <UTC>  
##   time   date      volume datetime  
##   <time> <date>      <dbl> <dtm>  
## 1 07:00 2003-03-03      111 2003-03-03 07:00:00  
## 2 07:05 2003-03-03      113 2003-03-03 07:05:00  
## 3 07:10 2003-03-03       76 2003-03-03 07:10:00  
## 4 07:15 2003-03-03       82 2003-03-03 07:15:00  
## 5 07:20 2003-03-03       91 2003-03-03 07:20:00  
## 6 07:25 2003-03-03       87 2003-03-03 07:25:00  
## 7 07:30 2003-03-03       75 2003-03-03 07:30:00
```

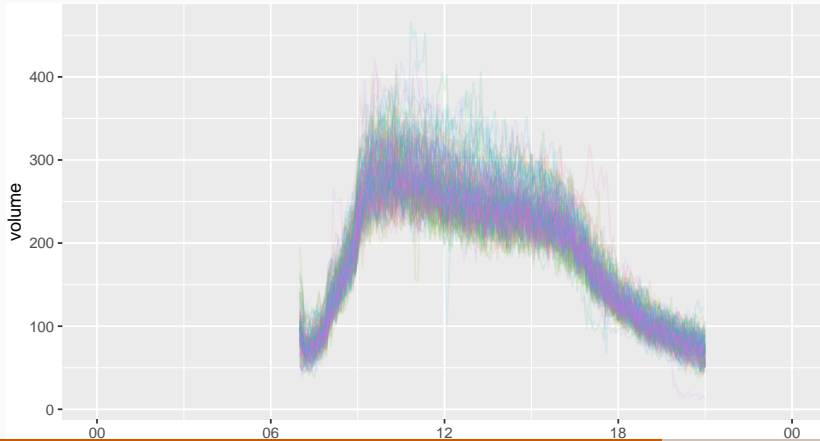
5-minute call centre volume

```
calls %>% fill_gaps() %>% autoplot(volume)
```



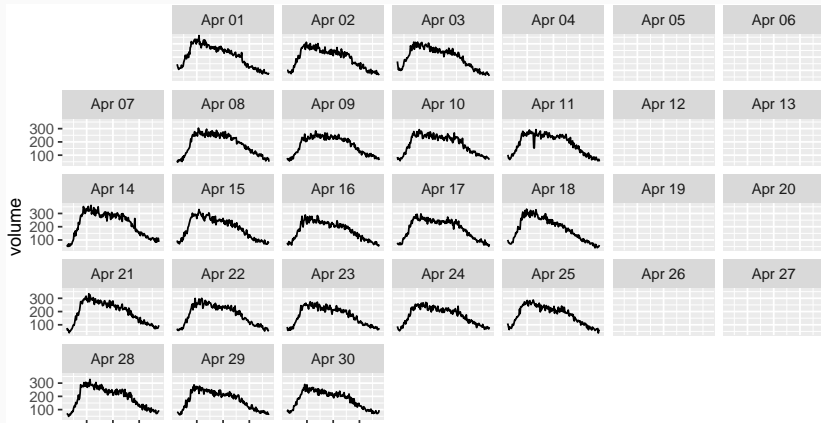
5-minute call centre volume

```
calls %>% fill_gaps() %>%  
  gg_season(volume, period = "day", alpha = 0.1) +  
  guides(colour = FALSE)
```



5-minute call centre volume

```
library(sugrrants)
calls %>% filter(month(date, label = TRUE) == "Apr") %>%
  ggplot(aes(x = time, y = volume)) +
  geom_line() + facet_calendar(date)
```



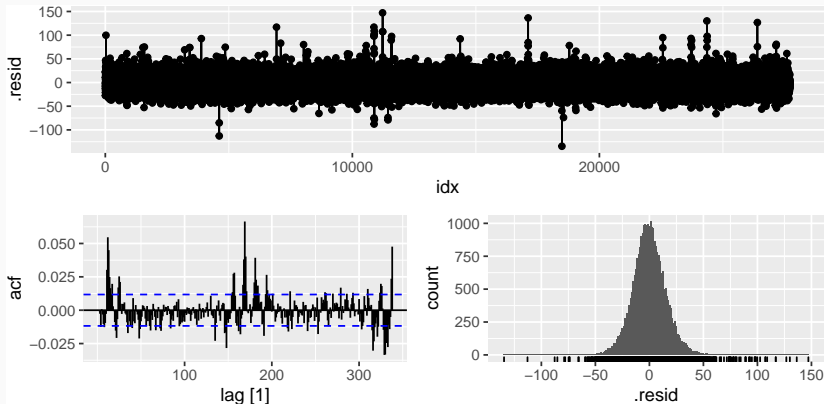
5-minute call centre volume

```
calls_md1 <- calls %>%  
  mutate(idx = row_number()) %>%  
  update_tsibble(index = idx)  
fit <- calls_md1 %>%  
  model(ARIMA(volume ~ fourier(169, K = 10) + pdq(d=0) + PDQ(0,0,0)))  
report(fit)
```

```
## Series: volume  
## Model: LM w/ ARIMA(1,0,3) errors  
##  
## Coefficients:  
##          ar1          ma1          ma2          ma3  
##          0.9894   -0.7383   -0.0333   -0.0282  
## s.e.    0.0010    0.0061    0.0075    0.0060  
##          fourier(169, K = 10)C1_169  
##                               -79.0702  
## s.e.                               0.7001  
##          fourier(169, K = 10)S1_169  
##                               55.2985  
## s.e.                               0.7006  
##          fourier(169, K = 10)C2_169  
##                               -32.3615  
## s.e.                               0.3784  
##          fourier(169, K = 10)S2_169
```

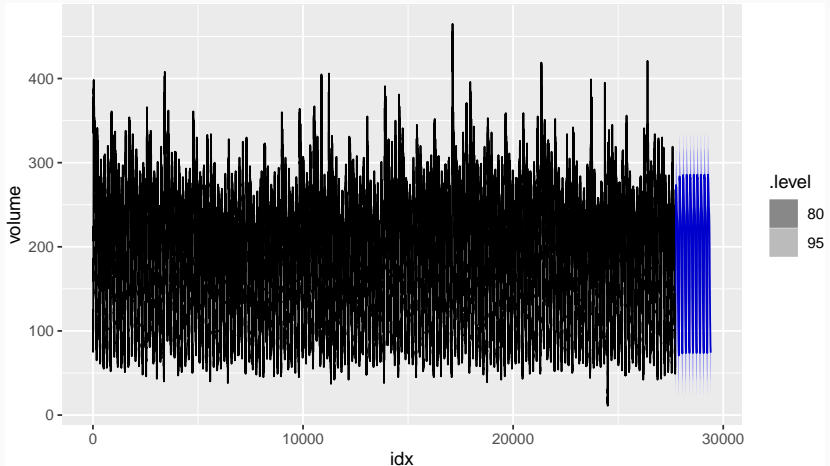
5-minute call centre volume

```
gg_tsresiduals(fit, lag = 338)
```



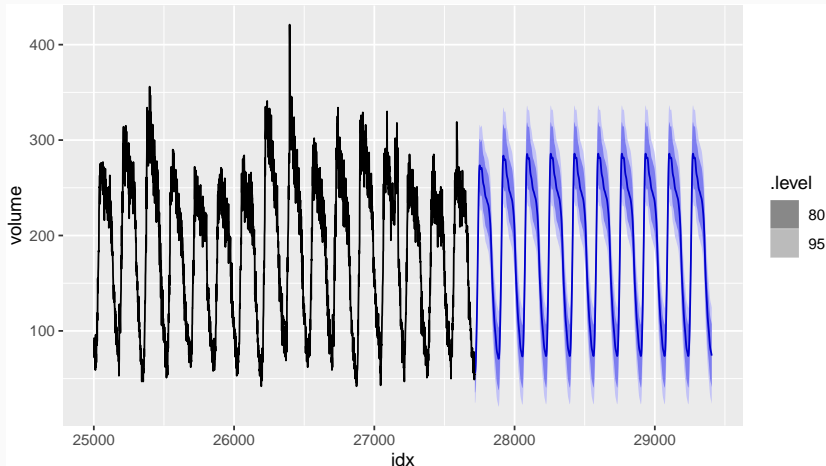
5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%  
  autoplot(calls_mdl)
```



5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%  
  autoplot(filter(calls_mdl, idx > 25000))
```



Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

- y_t = sales, x_t = advertising.
 - y_t = stream flow, x_t = rainfall.
 - y_t = size of herd, x_t = breeding stock.
-
- These are dynamic systems with input (x_t) and output (y_t).
 - x_t is often a leading indicator.
 - There can be multiple predictors.

Lagged predictors

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \dots + \nu_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

Lagged predictors

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$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \dots + \nu_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \dots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

Lagged predictors

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \dots + \nu_k x_{t-k} + \eta_t$$

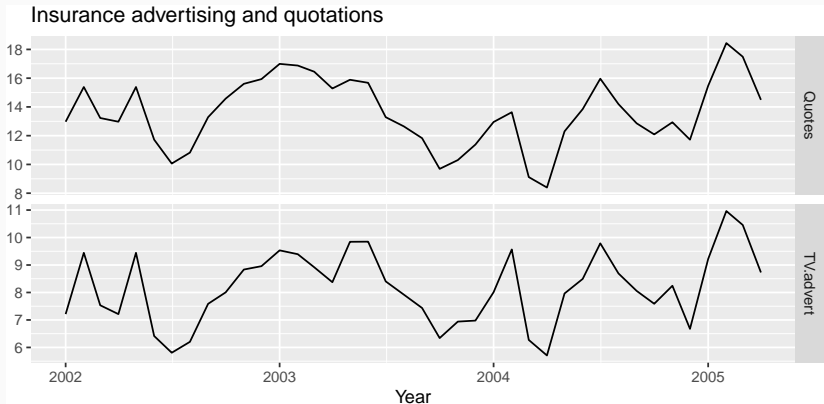
where η_t is an ARIMA process.

Rewrite model as

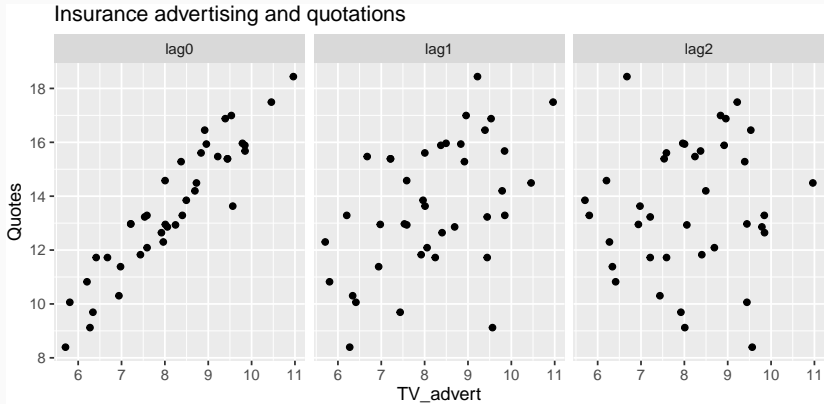
$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \dots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

- $\nu(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t .
- x can influence y , but y is not allowed to influence x .

Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  # Restrict data so models use same fitting period  
  mutate(Quotes = c(NA,NA,NA,Quotes[4:40])) %>%  
  # Estimate models  
  model(  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert)),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +  
      lag(TV.advert, 2)),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +  
      lag(TV.advert, 2) + lag(TV.advert, 3))  
  )
```

Example: Insurance quotes and TV adverts

```
glance(fit)
```

Lag order	sigma2	log_lik	AIC	AICc	BIC
0	0.2650	-28.28	66.56	68.33	75.01
1	0.2094	-24.04	58.09	59.85	66.53
2	0.2150	-24.02	60.03	62.58	70.17
3	0.2056	-22.16	60.31	64.96	73.83

Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))  
report(fit)
```

```
## Series: Quotes  
## Model: LM w/ ARIMA(3,0,0) errors  
##  
## Coefficients:  
##          ar1      ar2      ar3 TV.advert lag(TV.advert)  
##          1.4117 -0.9317  0.3591    1.2564         0.1625  
## s.e.    0.1698   0.2545  0.1592    0.0667         0.0591  
##      intercept  
##           2.0393  
## s.e.       0.9931  
##  
## sigma^2 estimated as 0.2165: log likelihood=-23.89  
## AIC=61.78   AICc=65.28   BIC=73.6
```

Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))  
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```

```
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##      intercept  
##           2.0393  
## s.e.       0.9931  
##  
## sigma^2 estimated as 0.2165: log likelihood=-23.89  
## AIC=61.78   AICc=65.28   BIC=73.6
```

$$y_t = 2.04 + 1.26x_t + 0.16x_{t-1} + \eta_t,$$

$$\eta_t = 1.41\eta_{t-1} - 0.932\eta_{t-2} + 0.359\eta_{t-3} + \varepsilon_t$$

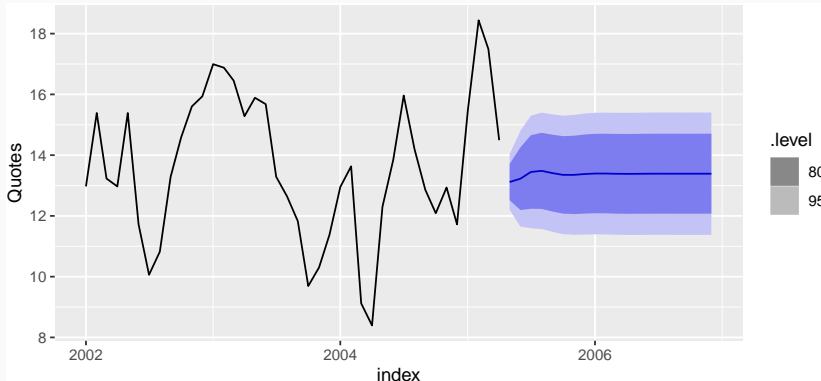
Example: Insurance quotes and TV adverts

```
advert_a <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 10)  
forecast(fit, advert_a) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_b <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 8)  
forecast(fit, advert_b) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_c <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 6)  
forecast(fit, advert_c) %>% autoplot(insurance)
```

