MONASH University

MONASH BUSINESS SCHOOL

ETC3550 Applied forecasting for business and economics

Ch10. Dynamic regression models

OTexts.org/fpp3/

Outline

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Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Regression with ARIMA errors

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- y_t modeled as function of k explanatory variables $x_{1,t}, \ldots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

where ε_t is white noise.

Residuals and errors

Example: η_t = ARIMA(1,1,1)

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Residuals and errors

Example: η_t = ARIMA(1,1,1)

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing η_t from ε_t.
- Only the errors ε_t are assumed to be white noise.
- In ordinary regression, η_t is assumed to be white noise and so $\eta_t = \varepsilon_t$.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

1. Estimated coefficients $\hat{\beta}_0, \ldots, \hat{\beta}_k$ are no longer optimal as some information ignored;
2. Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
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3. *p*-values for coefficients usually too small ("spurious regression").
4. AIC of fitted models misleading.

- Minimizing $\sum \varepsilon_t^2$ avoids these problems.
- Maximizing likelihood similar to minimizing $\sum \varepsilon_t^2$.

Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

where η_t is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

Stationarity

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
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Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t,$$
$$(1 - \phi_1 B)\eta'_t = (1 + \theta_1 B)\varepsilon_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $\eta'_t = \eta_t - \eta_{t-1}$.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

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Original data

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$$\text{where} \quad \phi(B)(1 - B)^d \eta_t = \theta(B)\varepsilon_t$$

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$$\text{where} \quad \phi(B)(1-B)^d \eta_t = \theta(B)\varepsilon_t$$

After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t.$$
$$\text{where} \quad \phi(B)\eta_t = \theta(B)\varepsilon_t$$
$$\text{and} \quad y'_t = (1-B)^d y_t$$

Model selection

- Fit regression model with automatically selected ARIMA errors. (R will take care of differencing before estimation.)
- Check that ε_t series looks like white noise.

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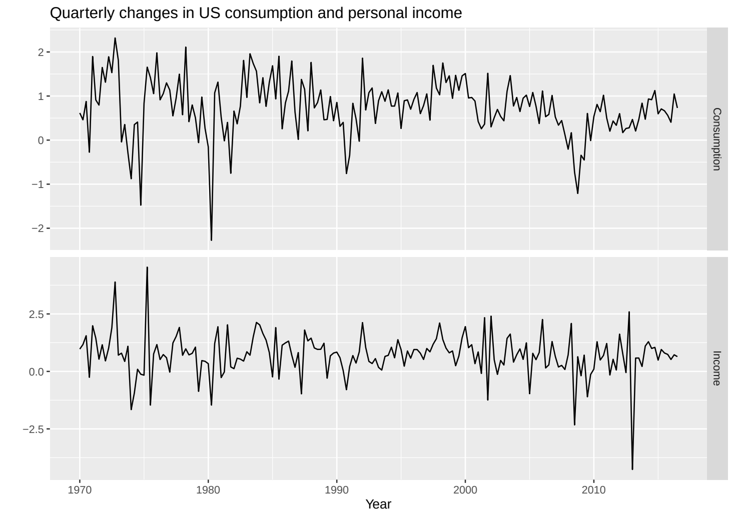
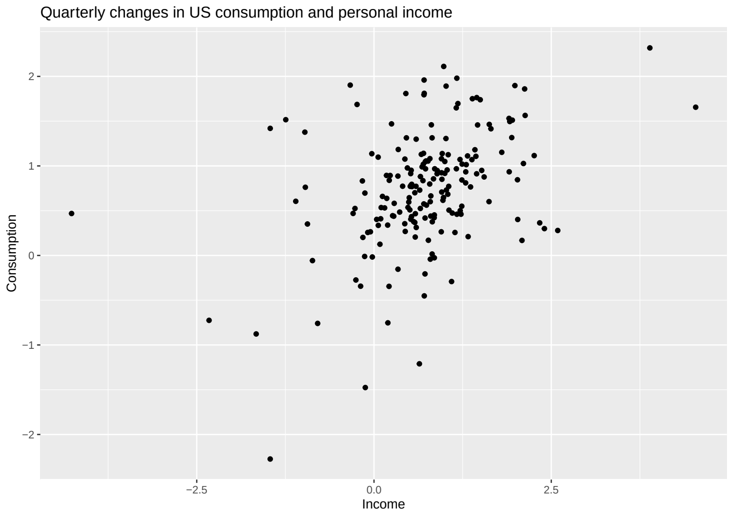
Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

US personal consumption and income



US personal consumption and income



US personal consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

US personal consumption and income

```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))
report(fit)
```

```
## Series: Consumption
## Model: LM w/ ARIMA(1,0,2) errors
##
## Coefficients:
##          ar1      ma1     ma2  Income  intercept
##       0.6922  -0.5758  0.1984  0.2028     0.5990
## s.e.  0.1159   0.1301  0.0756  0.0461     0.0884
##
## sigma^2 estimated as 0.3219:  log likelihood=-156.9
## AIC=325.9   AICc=326.4   BIC=345.3
```

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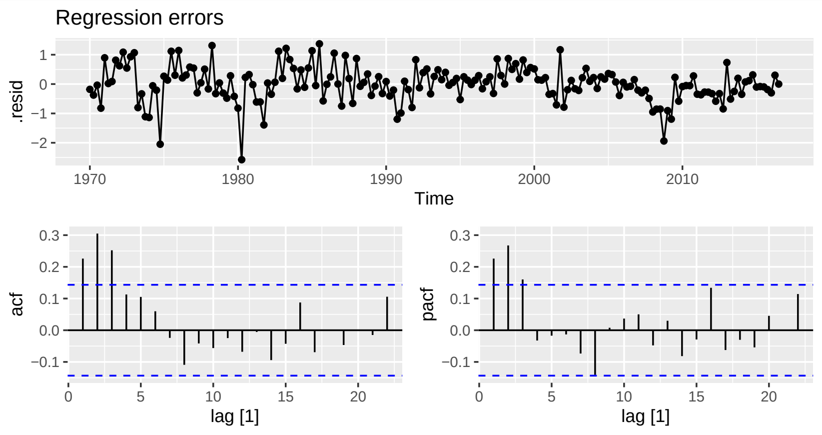
```

Write down the equations for the fitted model.

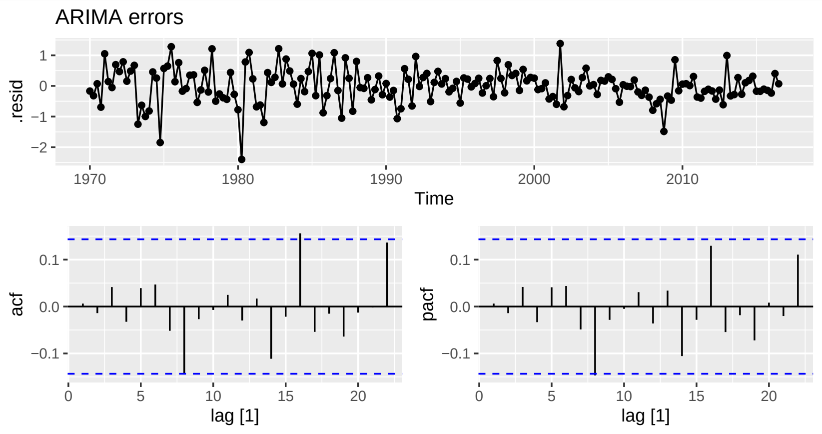
US personal consumption and income

```r
residuals(fit, type='regression') %>%
  gg_tsdisplay(.resid, plot_type = 'partial') + ggtitle("Reg
```



US personal consumption and income

```r
residuals(fit, type='response') %>%
  gg_tsdisplay(.resid, plot_type = 'partial') + ggtitle("ARI
```



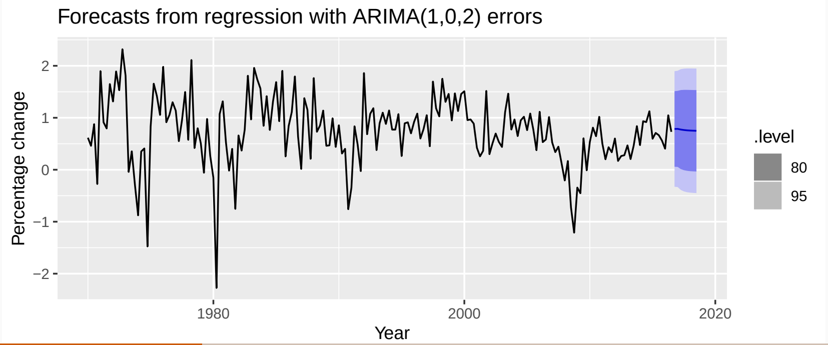
US personal consumption and income

```r
augment(fit) %>%
  features(.resid, ljung_box, dof = 5, lag = 12)
```

```
## # A tibble: 1 x 3
##   .model                          lb_stat lb_pvalue
##   <chr>                             <dbl>     <dbl>
## 1 ARIMA(Consumption ~ Income)        6.35     0.500
```

US personal consumption and income

```
us_change_future <- new_data(us_change, 8) %>%
  mutate(Income = mean(us_change$Income))
forecast(fit, new_data = us_change_future) %>%
  autoplot(us_change) +
  labs(x = "Year", y = "Percentage change",
       title = "Forecasts from regression with ARIMA(1,0,2) errors")
```



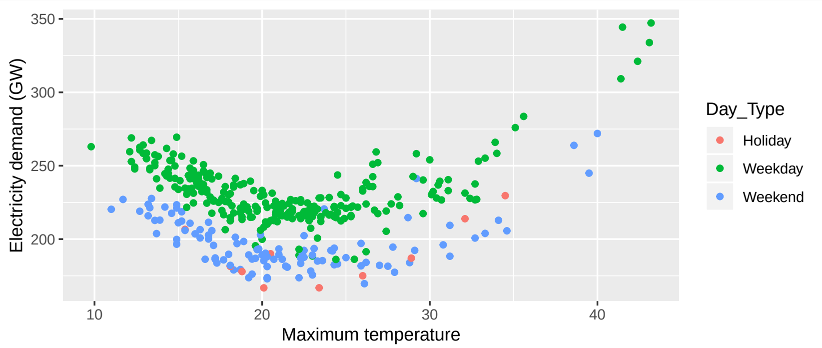
Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

Daily electricity demand

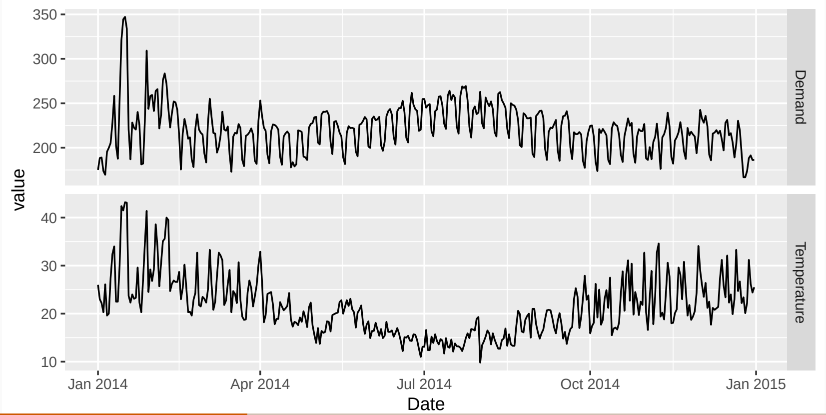
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
vic_elec_daily %>%
  ggplot(aes(x=Temperature, y=Demand, colour=Day_Type)) +
  geom_point() +
  labs(x = "Maximum temperature", y = "Electricity demand (GW)")
```



Daily electricity demand

```r
vic_elec_daily %>%
  gather("var", "value", Demand, Temperature) %>%
  ggplot(aes(x = Date, y = value)) + geom_line() +
  facet_grid(vars(var), scales = "free_y")
```



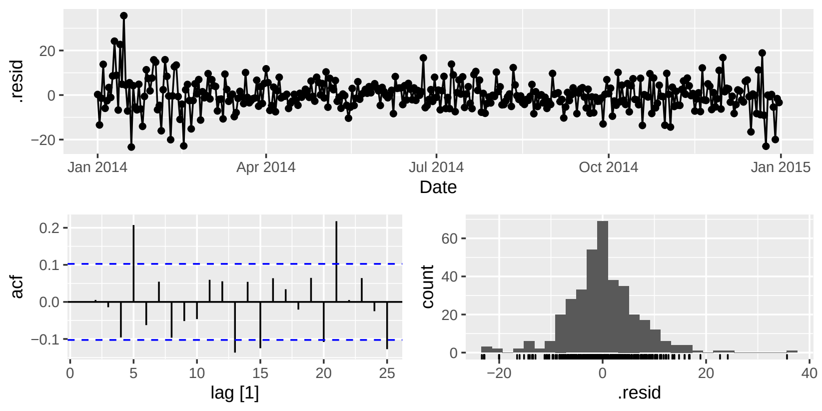
Daily electricity demand

```
fit <- vic_elec_daily %>%
  model(ARIMA(Demand ~ Temperature + I(Temperature^2) +
                (Day_Type=="Weekday")))
report(fit)
```

```
## Series: Demand
## Model: LM w/ ARIMA(2,1,2)(0,0,2)[7] errors
##
## Coefficients:
##          ar1      ar2      ma1     ma2    sma1    sma2
##       1.1521  -0.2750  -1.3851  0.4071  0.1589  0.3103
## s.e.  0.6265   0.4812   0.6082  0.5805  0.0591  0.0538
##       Temperature  I(Temperature^2)
##            -7.947            0.1865
## s.e.        0.492            0.0097
##       Day_Type == "Weekday"TRUE
##                          31.825
## s.e.                      1.019
##
```

Daily electricity demand

```r
augment(fit) %>%
  gg_tsdisplay(.resid, plot_type = "histogram")
```



Daily electricity demand

```r
augment(fit) %>%
  features(.resid, ljung_box, dof = 8, lag = 14)
```

```
## # A tibble: 1 x 3
##   .model                                              lb_stat lb_pvalue
##   <chr>                                                 <dbl>     <dbl>
## 1 "ARIMA(Demand ~ Temperature + I(Tempera...    38.1   1.05e-6
```

Daily electricity demand

```r
# Forecast one day ahead
vic_next_day <- new_data(vic_elec_daily, 1) %>%
  mutate(Temperature = 26, Day_Type = "Holiday")
forecast(fit, vic_next_day)
```

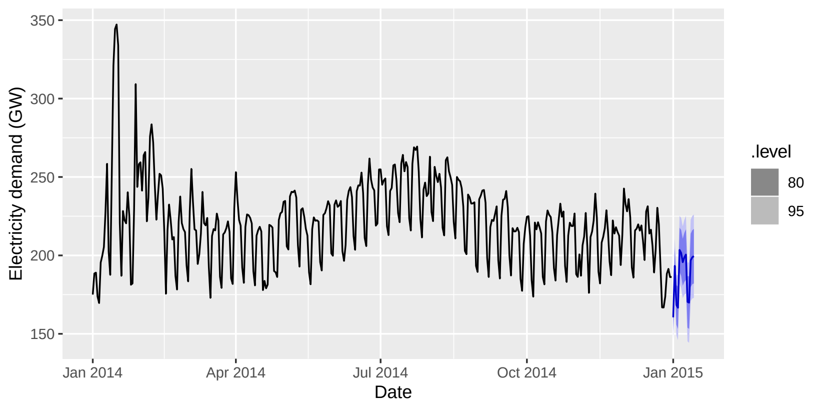
```
## # A fable: 1 x 6 [1D]
## # Key:     .model [1]
##   .model Date       Demand .distribution Temperature
##   <chr>  <date>      <dbl> <dist>              <dbl>
## 1 "ARIM... 2015-01-01   161. N(161, 49)            26
## # ... with 1 more variable: Day_Type <chr>
```

Daily electricity demand

```r
vic_elec_future <- new_data(vic_elec_daily, 14) %>%
  mutate(
    Temperature = 26,
    Holiday = c(TRUE, rep(FALSE, 13)),
    Day_Type = case_when(
      Holiday ~ "Holiday",
      wday(Date) %in% 2:6 ~ "Weekday",
      TRUE ~ "Weekend"
    )
  )
```

Daily electricity demand

```r
forecast(fit, vic_elec_future) %>%
  autoplot(vic_elec_daily) + ylab("Electricity demand (GW)")
```



Outline

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

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Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

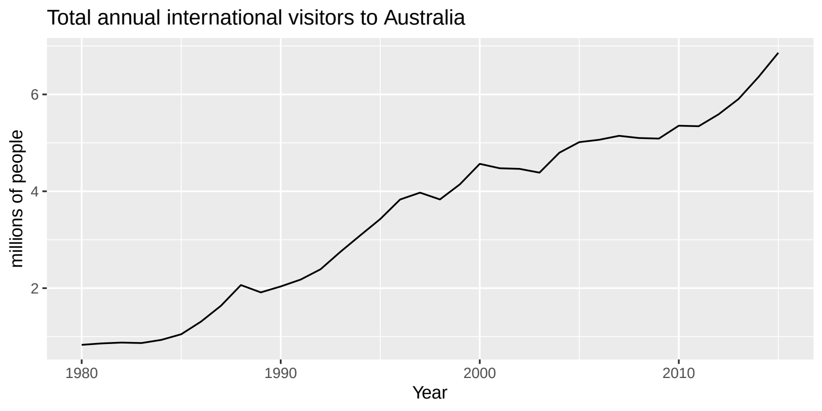
where η_t is ARIMA process with $d \geq 1$.

Difference both sides until η_t is stationary:

$$y'_t = \beta_1 + \eta'_t$$

where η'_t is ARMA process.

International visitors



International visitors

Deterministic trend

```r
fit_deterministic <- aus_visitors %>%
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))
report(fit_deterministic)
```

```
## Series: value
## Model: LM w/ ARIMA(2,0,0) errors
##
## Coefficients:
##          ar1      ar2  trend()  intercept
##        1.113  -0.3805   0.1710     0.4156
## s.e.   0.160   0.1585   0.0088     0.1897
##
## sigma^2 estimated as 0.02979:  log likelihood=13.6
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

International visitors

Deterministic trend

```
fit_deterministic <- aus_visitors %>%
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))
report(fit_deterministic)
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## Coefficients:
##          ar1      ar2  trend()  intercept
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## sigma^2 estimated as 0.02979:  log likelihood=13.6
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

$$y_t = 0.42 + 0.17t + \eta_t$$
$$\eta_t = 1.11\eta_{t-1} - 0.38\eta_{t-2} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 0.0298).$$

International visitors

Stochastic trend

```r
fit_stochastic <- aus_visitors %>%
  model(Stochastic = ARIMA(value ~ pdq(d=1)))
report(fit_stochastic)
```

```
## Series: value
## Model: ARIMA(0,1,1) w/ drift
##
## Coefficients:
##          ma1  constant
##       0.3006    0.1735
## s.e.  0.1647    0.0390
##
## sigma^2 estimated as 0.03376:  log likelihood=10.62
## AIC=-15.24   AICc=-14.46   BIC=-10.57
```

International visitors

Stochastic trend

```
fit_stochastic <- aus_visitors %>%
  model(Stochastic = ARIMA(value ~ pdq(d=1)))
report(fit_stochastic)
```

```
## Series: value
## Model: ARIMA(0,1,1) w/ drift
##
## Coefficients:
##          ma1  constant
##       0.3006    0.1735
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##
## sigma^2 estimated as 0.03376:  log likelihood=10.62
## AIC=-15.24   AICc=-14.46   BIC=-10.57
```
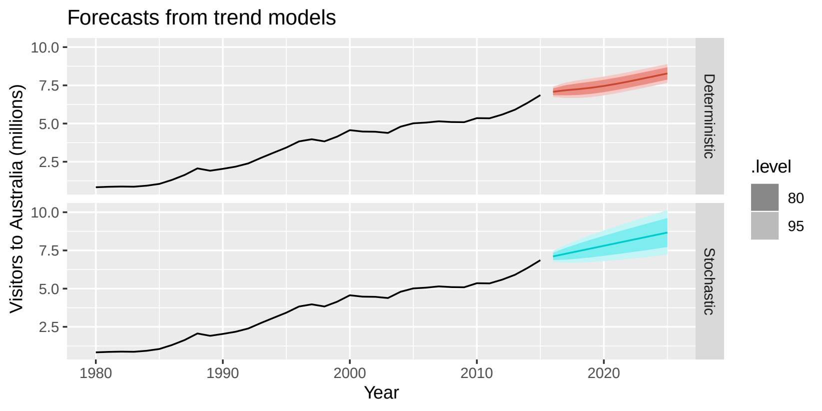
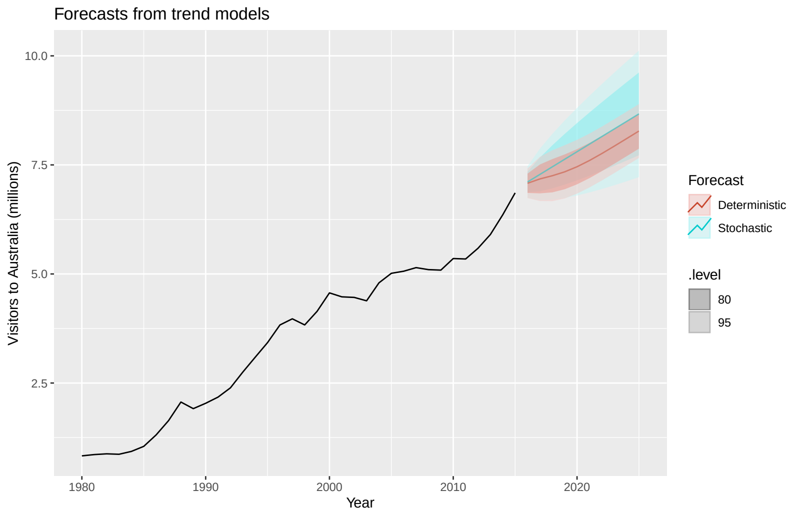

$$
\begin{aligned}
y_t - y_{t-1} &= 0.17 + \varepsilon_t \\
y_t &= y_0 + 0.17t + \eta_t \\
\eta_t &= \eta_{t-1} + 0.30\varepsilon_{t-1} + \varepsilon_t
\end{aligned}
$$

International visitors



International visitors



Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

Outline

Dynamic harmonic regression

Combine Fourier terms with ARIMA errors

Advantages

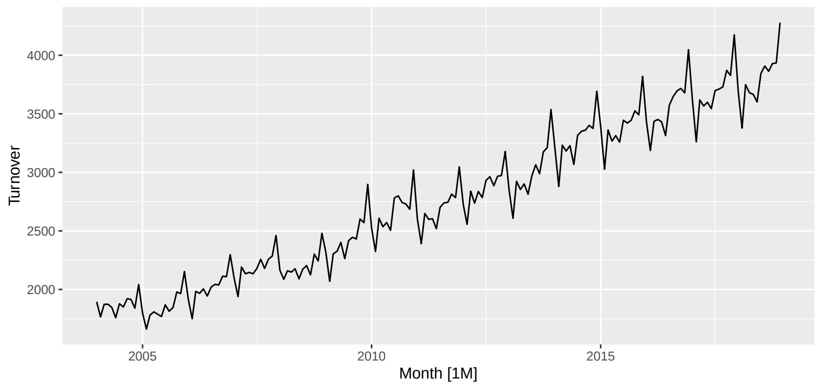
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of K (but more wiggly seasonality can be handled by increasing K);
- the short-term dynamics are easily handled with a simple ARMA error.

Disadvantages

- seasonality is assumed to be fixed

Eating-out expenditure

```
aus_cafe <- aus_retail %>% filter(
    Industry == "Cafes, restaurants and takeaway food services",
    year(Month) %in% 2004:2018
  ) %>% summarise(Turnover = sum(Turnover))
aus_cafe %>% autoplot(Turnover)
```

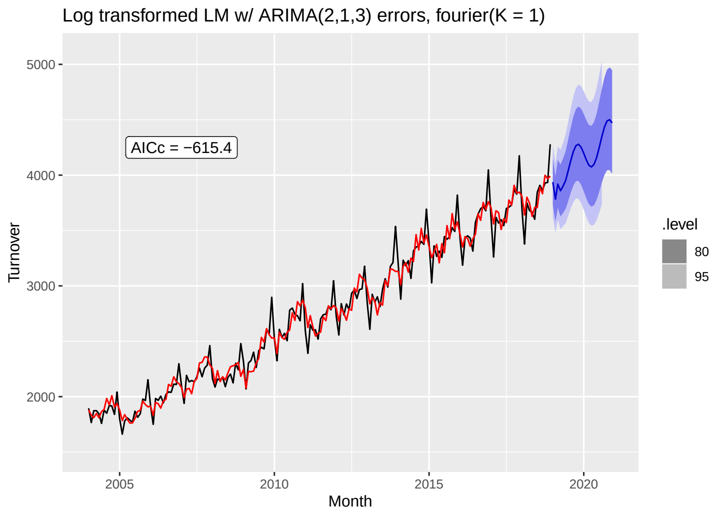


Eating-out expenditure

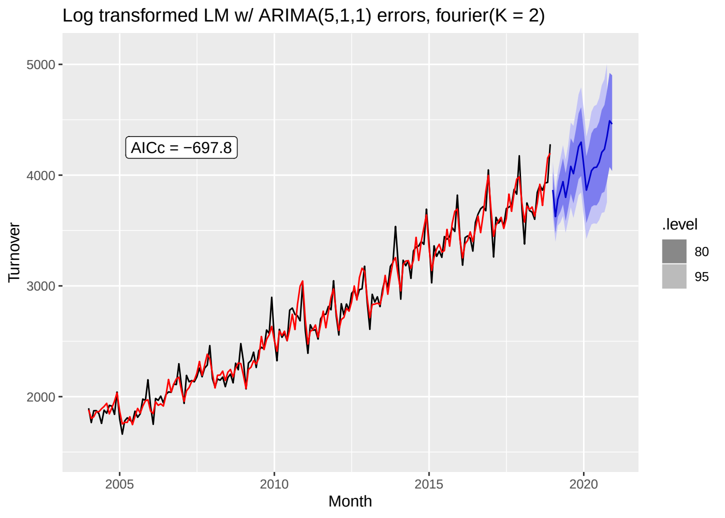
```r
fit <- aus_cafe %>% model(
    K = 1 = ARIMA(log(Turnover) ~ fourier(K = 1) + PDQ(0,0,0)),
    K = 2 = ARIMA(log(Turnover) ~ fourier(K = 2) + PDQ(0,0,0)),
    K = 3 = ARIMA(log(Turnover) ~ fourier(K = 3) + PDQ(0,0,0)),
    K = 4 = ARIMA(log(Turnover) ~ fourier(K = 4) + PDQ(0,0,0)),
    K = 5 = ARIMA(log(Turnover) ~ fourier(K = 5) + PDQ(0,0,0)),
    K = 6 = ARIMA(log(Turnover) ~ fourier(K = 6) + PDQ(0,0,0)))
glance(fit)
```

.model	sigma2	log_lik	AIC	AICc	BIC
K = 1	0.0017	317.2	-616.5	-615.4	-587.8
K = 2	0.0011	361.9	-699.7	-697.8	-661.5
K = 3	0.0008	393.6	-763.2	-761.3	-725.0
K = 4	0.0005	426.8	-821.6	-818.2	-770.6
K = 5	0.0003	473.7	-919.5	-916.9	-874.8
K = 6	0.0003	474.0	-920.1	-917.5	-875.4

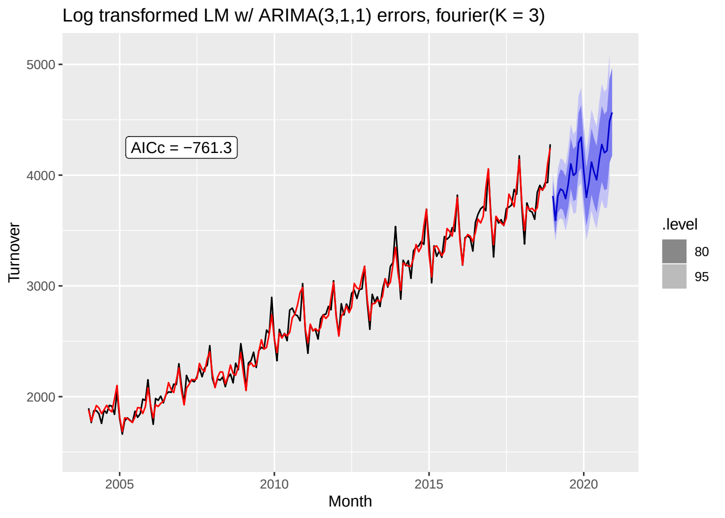
Eating-out expenditure



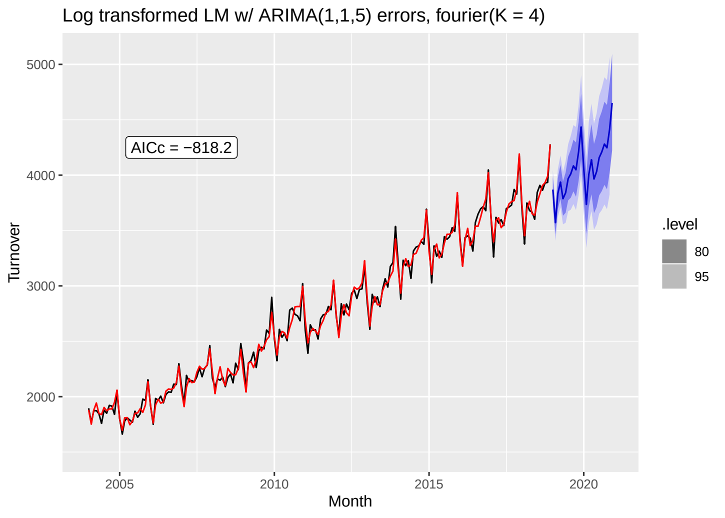
Eating-out expenditure



Eating-out expenditure



Eating-out expenditure



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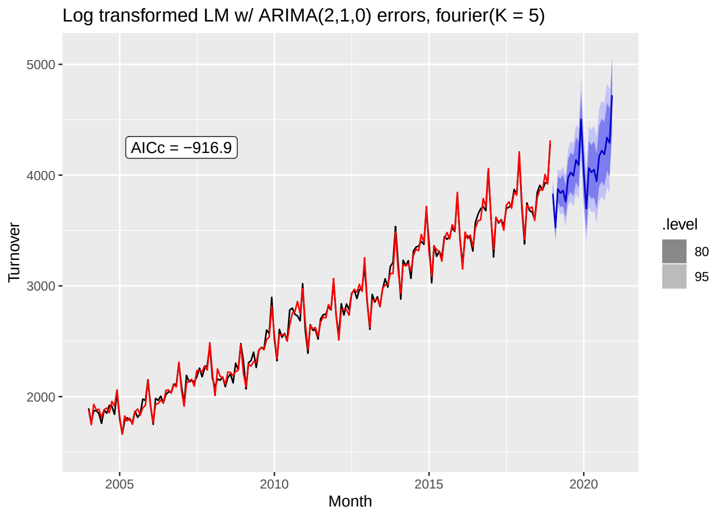

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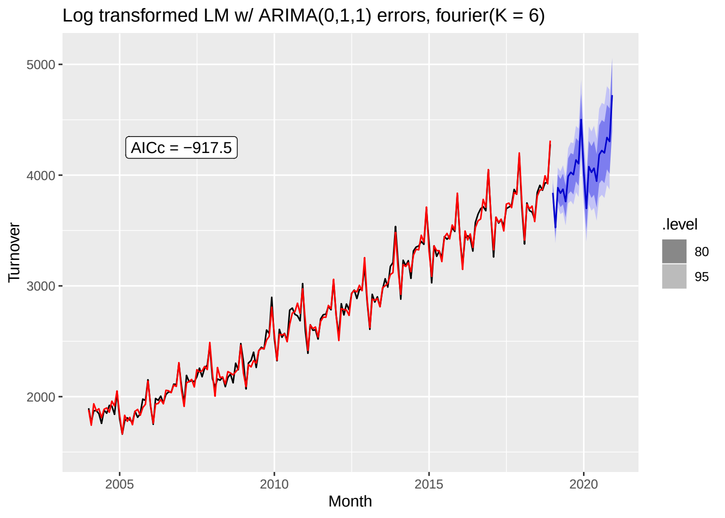


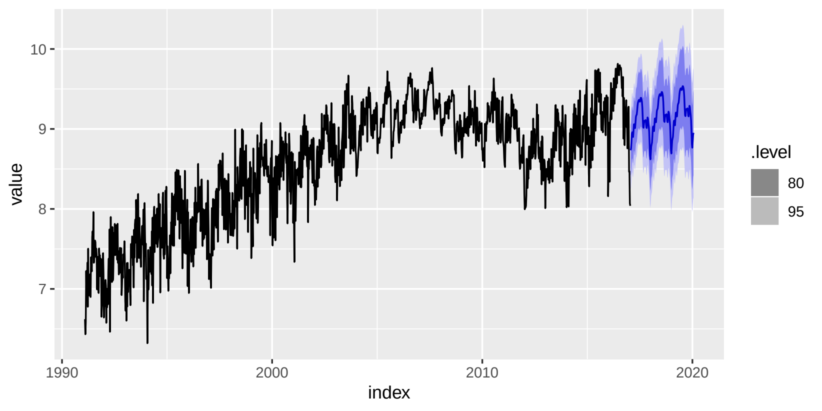
Example: weekly gasoline products

```r
gasoline <- as_tsibble(fpp2::gasoline)
fit <- gasoline %>% model(ARIMA(value ~ fourier(K = 13) + PDQ(0,0,0)))
report(fit)
```

```
## Series: value
## Model: LM w/ ARIMA(0,1,1) errors
##
## Coefficients:
##             ma1  fourier(K = 13)C1_52  fourier(K = 13)S1_52
##         -0.8934               -0.1121               -0.2300
## s.e.    0.0132                0.0123                0.0122
##         fourier(K = 13)C2_52  fourier(K = 13)S2_52
##                       0.0420                0.0317
## s.e.                  0.0099                0.0099
##         fourier(K = 13)C3_52  fourier(K = 13)S3_52
##                       0.0832                0.0346
## s.e.                  0.0094                0.0094
##         fourier(K = 13)C4_52  fourier(K = 13)S4_52
##                       0.0185                0.0398
## s.e.                  0.0092                0.0092
##         fourier(K = 13)C5_52  fourier(K = 13)S5_52
##                      -0.0315                0.0009
## s.e.                  0.0091                0.0091
##         fourier(K = 13)C6_52  fourier(K = 13)S6_52
```

Example: weekly gasoline products

```
forecast(fit, h = "3 years") %>%
  autoplot(gasoline)
```



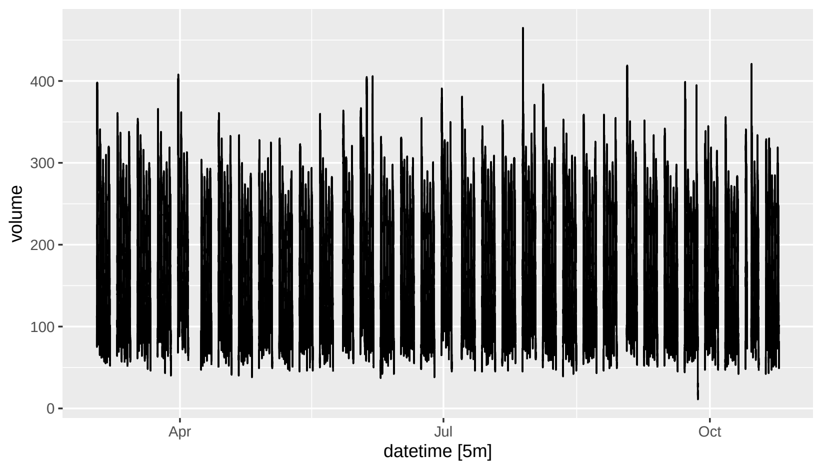
5-minute call centre volume

```
(calls <- read_tsv("http://robjhyndman.com/data/callcenter.txt") %>%
  gather("date", "volume", -X1) %>% transmute(
    time = X1, date = as.Date(date, format = "%d/%m/%Y"),
    datetime = as_datetime(date) + time, volume) %>%
  as_tsibble(index = datetime))
```

```
## # A tsibble: 27,716 x 4 [5m] <UTC>
##     time   date       datetime            volume
##     <time> <date>     <dttm>               <dbl>
##   1 07:00  2003-03-03 2003-03-03 07:00:00    111
##   2 07:05  2003-03-03 2003-03-03 07:05:00    113
##   3 07:10  2003-03-03 2003-03-03 07:10:00     76
##   4 07:15  2003-03-03 2003-03-03 07:15:00     82
##   5 07:20  2003-03-03 2003-03-03 07:20:00     91
##   6 07:25  2003-03-03 2003-03-03 07:25:00     87
##   7 07:30  2003-03-03 2003-03-03 07:30:00     75
##   8 07:35  2003-03-03 2003-03-03 07:35:00     89
##   9 07:40  2003-03-03 2003-03-03 07:40:00     99
##  10 07:45  2003-03-03 2003-03-03 07:45:00    125
```

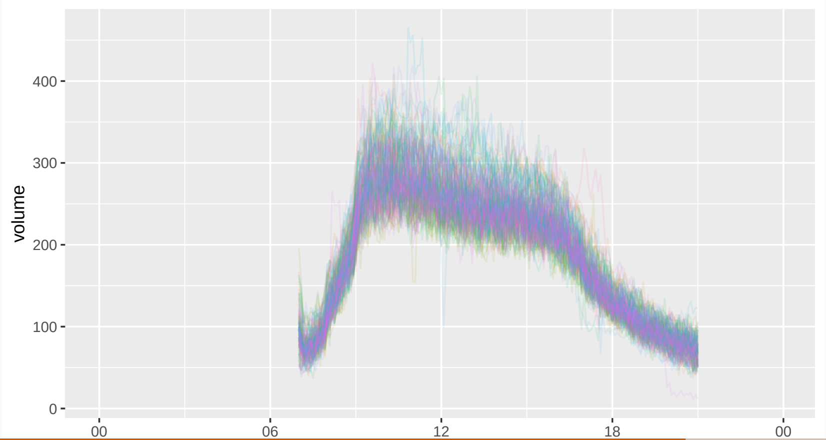
5-minute call centre volume

```
calls %>% fill_gaps() %>% autoplot(volume)
```



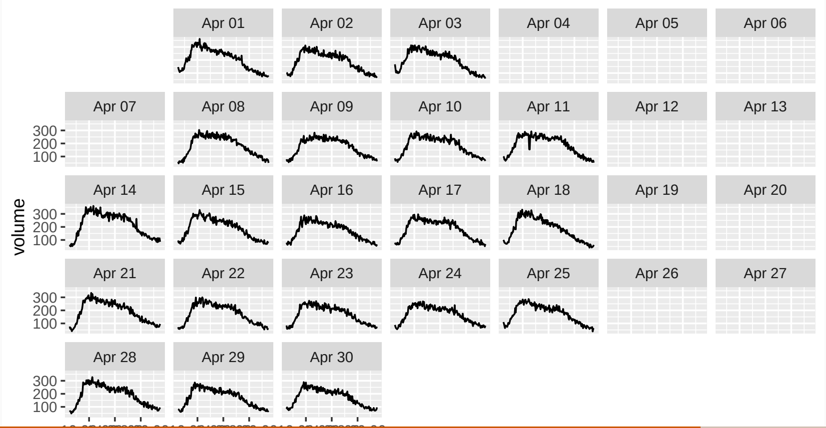
5-minute call centre volume

```
calls %>% fill_gaps() %>%
  gg_season(volume, period = "day", alpha = 0.1) +
  guides(colour = FALSE)
```



5-minute call centre volume

```r
library(sugrrants)
calls %>% filter(month(date, label = TRUE) == "Apr") %>%
  ggplot(aes(x = time, y = volume)) +
  geom_line() + facet_calendar(date)
```



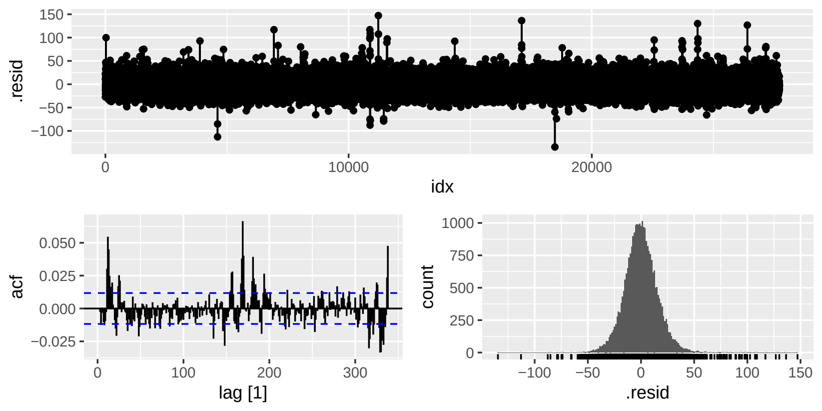
5-minute call centre volume

```r
calls_mdl <- calls %>%
  mutate(idx = row_number()) %>%
  update_tsibble(index = idx)
fit <- calls_mdl %>%
  model(ARIMA(volume ~ fourier(169, K = 10) + pdq(d=0) + PDQ(0,0,0)))
report(fit)
```

```
## Series: volume
## Model: LM w/ ARIMA(1,0,3) errors
##
## Coefficients:
##          ar1      ma1      ma2      ma3
##       0.9894  -0.7383  -0.0333  -0.0282
## s.e.  0.0010   0.0061   0.0075   0.0060
##       fourier(169, K = 10)C1_169
##                         -79.0702
## s.e.                      0.7001
##       fourier(169, K = 10)S1_169
##                          55.2985
## s.e.                      0.7007
##       fourier(169, K = 10)C2_169
##                         -32.3615
## s.e.                      0.3784
##       fourier(169, K = 10)S2_169
```

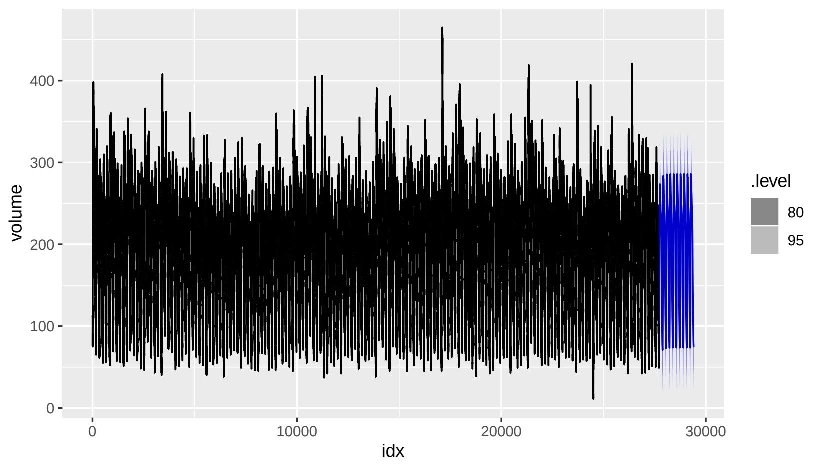
5-minute call centre volume

```r
augment(fit) %>%
  gg_tsdisplay(.resid, plot_type = "histogram", lag_max = 338)
```



5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%
  autoplot(calls_mdl)
```



Outline

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

- These are dynamic systems with input (x_t) and output (y_t).
- x_t is often a leading indicator.
- There can be multiple predictors.

Lagged predictors

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots$.

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \cdots + \nu_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

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where η_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

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The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots$.

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \cdots + \nu_k x_{t-k} + \eta_t$$

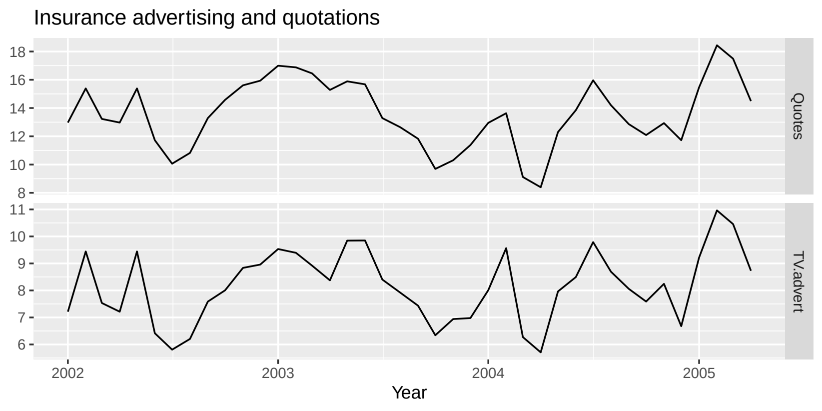
where η_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

- $\nu(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t.
- x can influence y, but y is not allowed to influence x.

Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts

```
fit <- insurance %>%
  # Restrict data so models use same fitting period
  mutate(Quotes = c(NA,NA,NA,Quotes[4:40])) %>%
  # Estimate models
  model(
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert)),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +
            lag(TV.advert, 2)),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +
            lag(TV.advert, 2) + lag(TV.advert, 3))
  )
```

Example: Insurance quotes and TV adverts

```r
glance(fit)
```

Lag order	sigma2	log_lik	AIC	AICc	BIC
0	0.2650	-28.28	66.56	68.33	75.01
1	0.2094	-24.04	58.09	59.85	66.53
2	0.2150	-24.02	60.03	62.58	70.17
3	0.2056	-22.16	60.31	64.96	73.83

Example: Insurance quotes and TV adverts

```r
fit <- insurance %>%
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))
report(fit)
```

```
## Series: Quotes
## Model: LM w/ ARIMA(3,0,0) errors
##
## Coefficients:
##          ar1      ar2     ar3  TV.advert  lag(TV.advert)
##       1.4117  -0.9317  0.3591     1.2564          0.1625
## s.e.  0.1698   0.2545  0.1592     0.0667          0.0591
##       intercept
##          2.0393
## s.e.     0.9931
##
## sigma^2 estimated as 0.2165:  log likelihood=-23.89
## AIC=61.78   AICc=65.28   BIC=73.6
```

Example: Insurance quotes and TV adverts

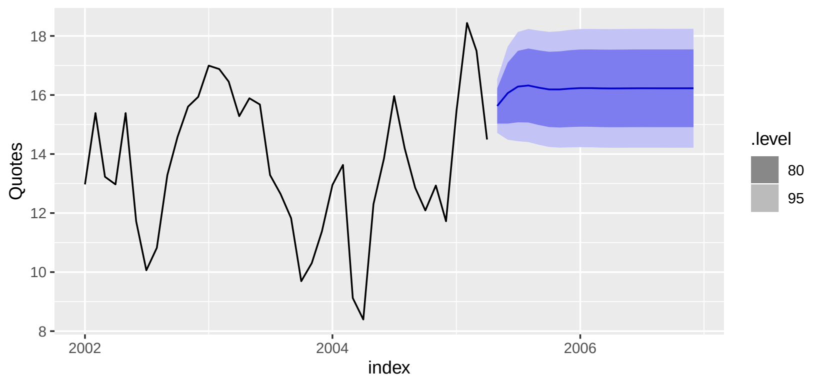
```
fit <- insurance %>%
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))
report(fit)
```

```
## Series: Quotes
## Model: LM w/ ARIMA(3,0,0) errors
##
## Coefficients:
##          ar1      ar2     ar3  TV.advert  lag(TV.advert)
##       1.4117  -0.9317  0.3591     1.2564          0.1625
## s.e.  0.1698   0.2545  0.1592     0.0667          0.0591
##       intercept
##          2.0393
## s.e.     0.9931
##
## sigma^2 estimated as 0.2165:  log likelihood=-23.89
## AIC=61.78   AICc=65.28   BIC=73.6
```

$$
\begin{aligned}
y_t &= 2.04 + 1.26x_t + 0.16x_{t-1} + \eta_t, \\
\eta_t &= 1.41\eta_{t-1} - 0.93\eta_{t-2} + 0.36\eta_{t-3} + \varepsilon_t,
\end{aligned}
$$

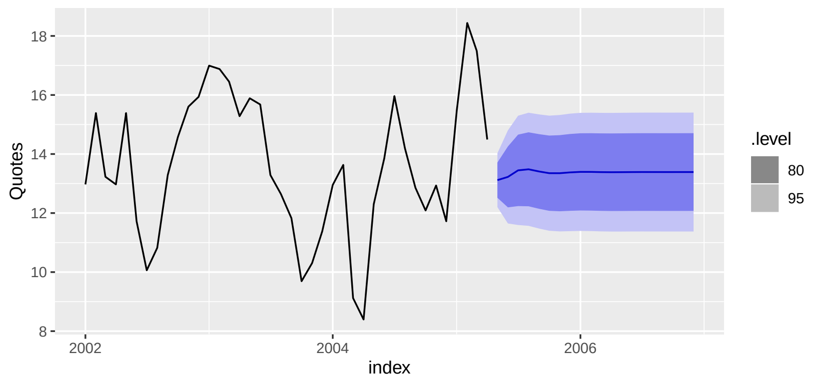
Example: Insurance quotes and TV adverts

```
advert_a <- new_data(insurance, 20) %>%
  mutate(TV.advert = 10)
forecast(fit, advert_a) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_b <- new_data(insurance, 20) %>%
  mutate(TV.advert = 8)
forecast(fit, advert_b) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```r
advert_c <- new_data(insurance, 20) %>%
  mutate(TV.advert = 6)
forecast(fit, advert_c) %>% autoplot(insurance)
```