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ETC3550/ETC5550

Applied forecasting

Ch9. ARIMA models

OTexts.org/fpp3/

Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

ARIMA models

- AR:** autoregressive (lagged observations as inputs)
- I:** integrated (differencing to make series stationary)
- MA:** moving average (lagged errors as inputs)

ARIMA models

AR: autoregressive (lagged observations as inputs)

I: integrated (differencing to make series stationary)

MA: moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

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Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s , the distribution of (y_t, \dots, y_{t+s}) does not depend on t .

Stationarity

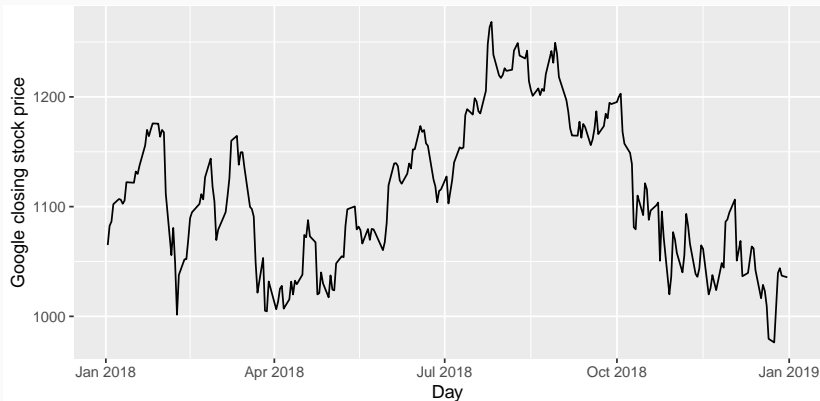
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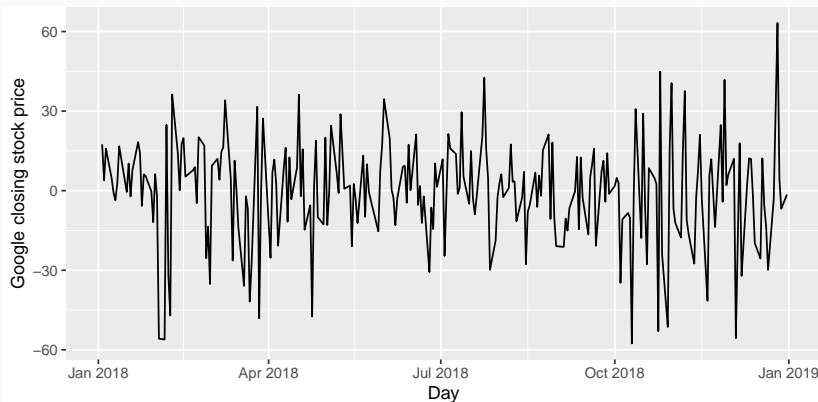
A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

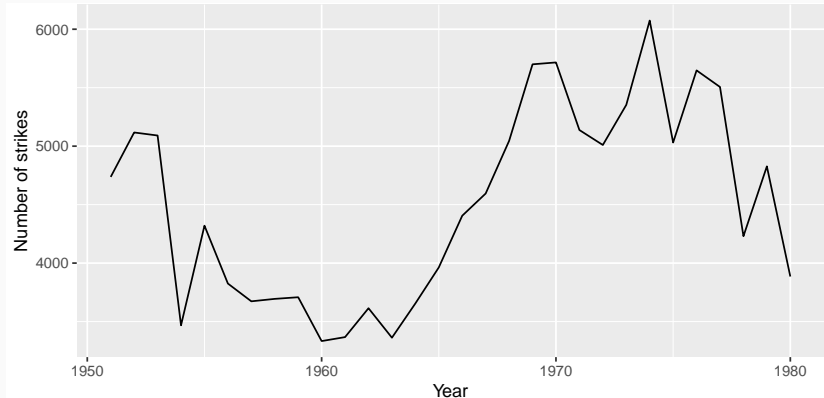
Stationary?



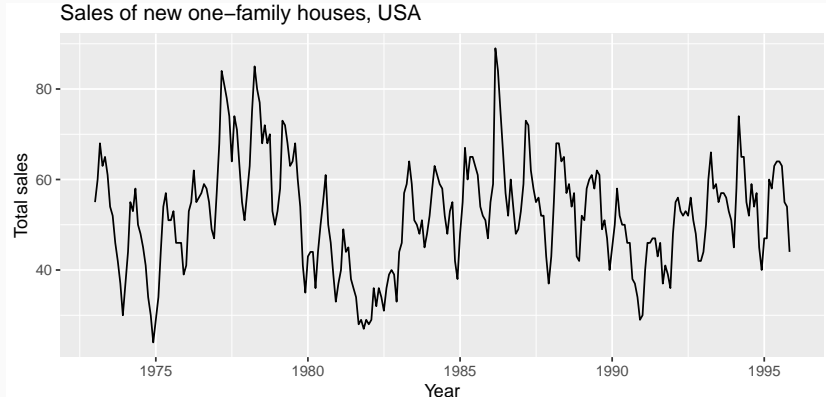
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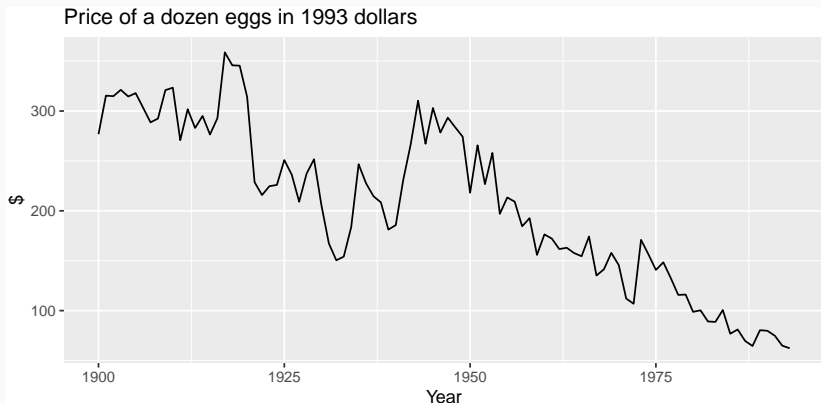
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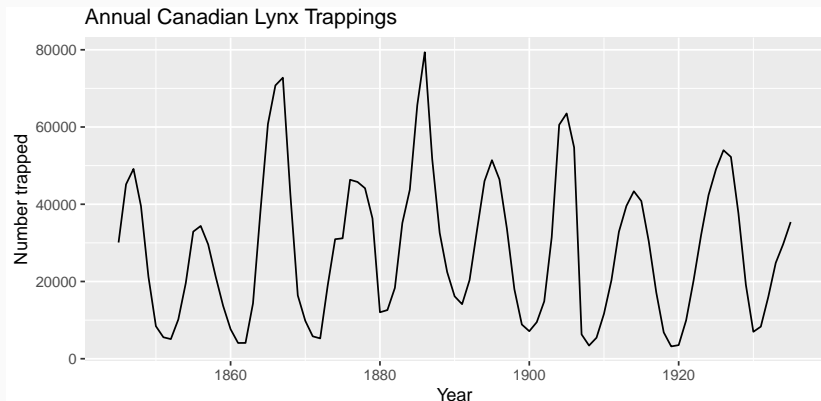
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Transformations help to **stabilize the variance**.

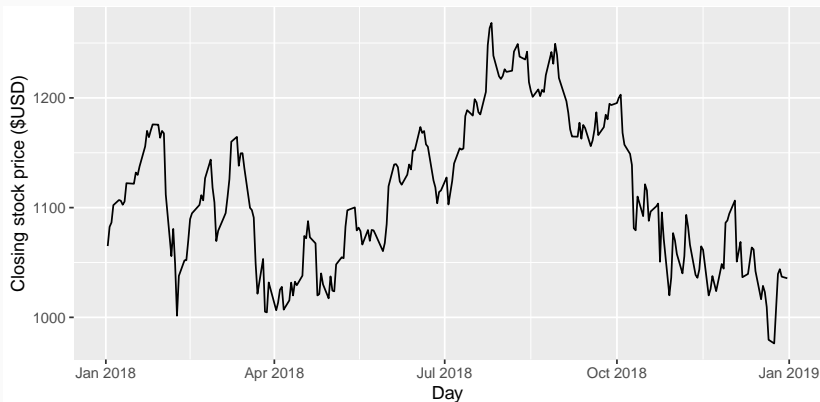
For ARIMA modelling, we also need to **stabilize the mean**.

Non-stationarity in the mean

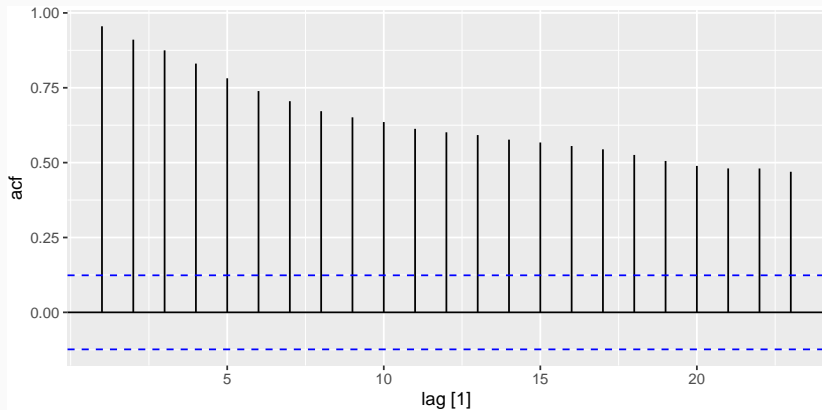
Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of r_1 is often large and positive.

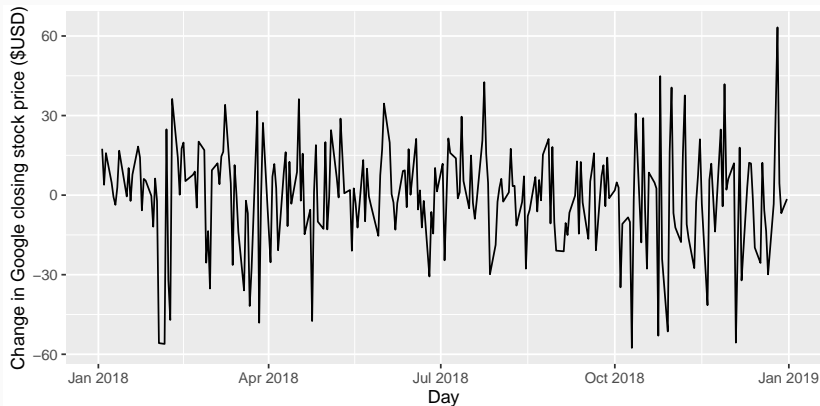
Example: Google stock price



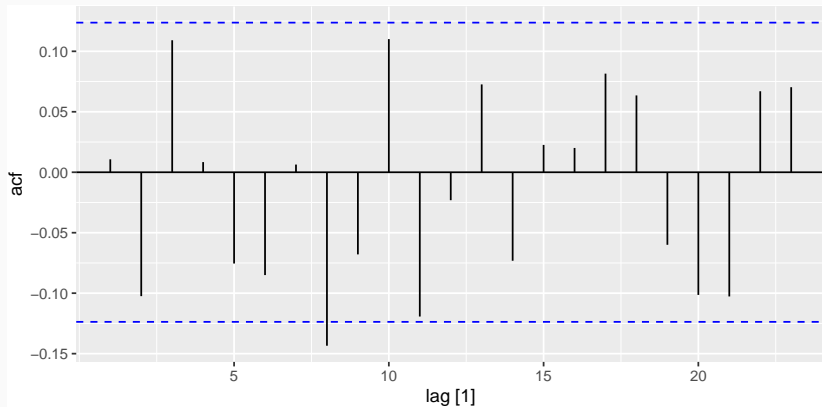
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Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series:

$$y'_t = y_t - y_{t-1}.$$

- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference y'_1 for the first observation.

Second-order differencing

Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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$$\begin{aligned}y_t'' &= y_t' - y_{t-1}' \\&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\&= y_t - 2y_{t-1} + y_{t-2}.\end{aligned}$$

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- y_t'' will have $T - 2$ values.
- In practice, it is almost never necessary to go beyond second-order differences.

Seasonal differencing

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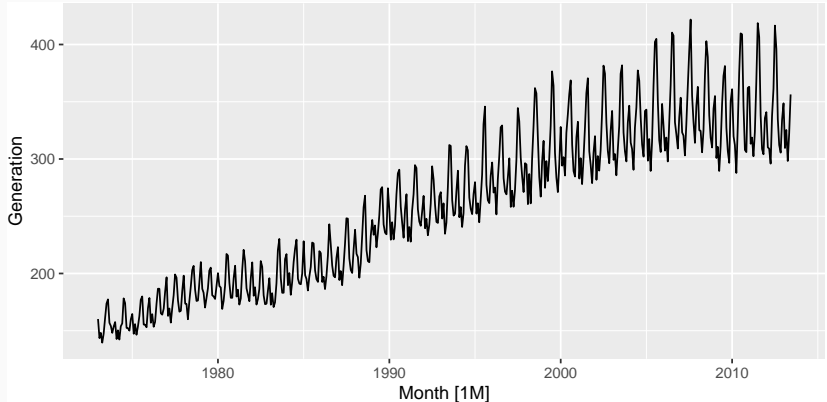
$$y'_t = y_t - y_{t-m}$$

where m = number of seasons.

- For monthly data $m = 12$.
- For quarterly data $m = 4$.

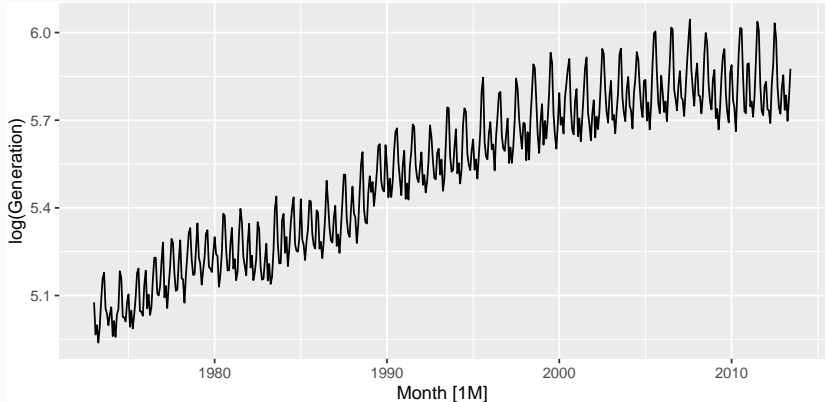
Electricity production

```
usmelec %>% autoplot(  
  Generation  
)
```



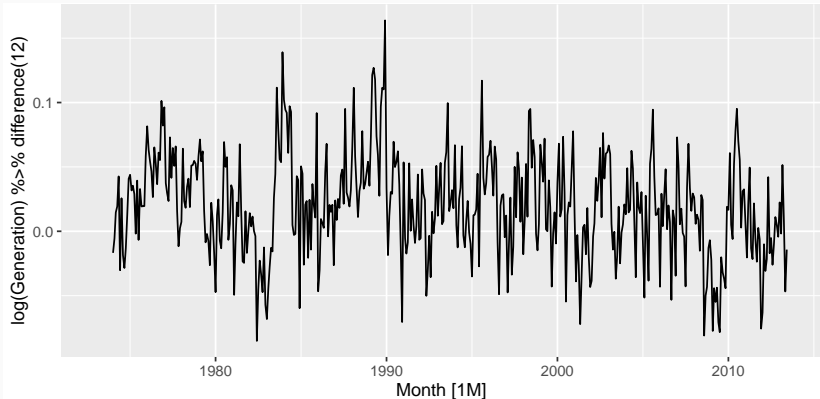
Electricity production

```
usmelec %>% autoplot(  
  log(Generation)  
)
```



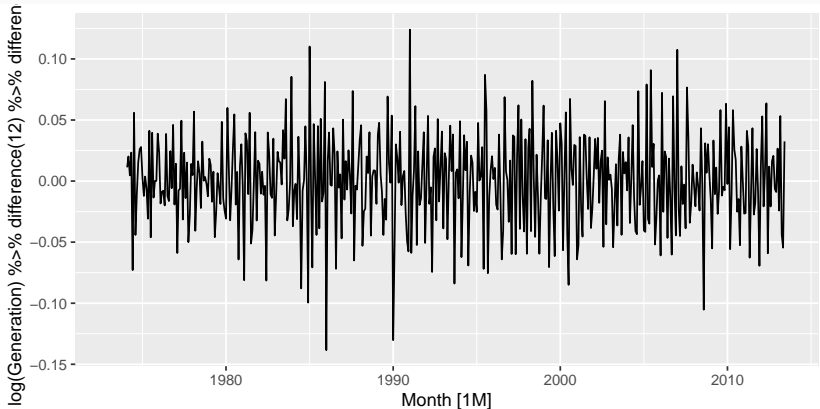
Electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12)  
)
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Electricity production

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usmelec %>% autoplot(  
  log(Generation) %>% difference(12) %>% difference()  
)
```



Electricity production

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If $y'_t = y_t - y_{t-12}$ denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned}y_t^* &= y'_t - y'_{t-1} \\&= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\&= y_t - y_{t-1} - y_{t-12} + y_{t-13} .\end{aligned}$$

Seasonal differencing

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- it makes no difference which is done first—the result will be the same.
- If seasonality is strong, we recommend that seasonal differencing be done first because sometimes the resulting series will be stationary and there will be no need for further first difference.

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It is important that if differencing is used, the differences are interpretable.

Interpretation of differencing

- first differences are the change between **one observation and the next**;
- seasonal differences are the change between **one year to the next**.

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But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

Unit root tests

Statistical tests to determine the required order of differencing.

- 1 Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
- 2 Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
- 3 Other tests available for seasonal data.

KPSS test

```
google_2018 %>%  
  features(Close, unitroot_kpss)
```

```
## # A tibble: 1 x 3  
##   Symbol kpss_stat kpss_pvalue  
##   <chr>      <dbl>      <dbl>  
## 1 GOOG      0.573      0.0252
```


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```

```
google_2018 %>%  
  features(Close, unitroot_ndiffs)
```

```
## # A tibble: 1 x 2  
##   Symbol ndiffs  
##   <chr>   <int>  
## 1 GOOG     1
```

Automatically selecting differences

STL decomposition: $y_t = T_t + S_t + R_t$

Seasonal strength $F_s = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right)$

If $F_s > 0.64$, do one seasonal difference.

```
usmelec %>% mutate(log_gen = log(Generation)) %>%  
  features(log_gen, list(unitroot_nsdiffs, feat_stl))
```

```
## # A tibble: 1 x 10  
##   nsdiffs trend_strength seasonal_streng~ seasonal_peak_y~  
##   <int>         <dbl>         <dbl>         <dbl>  
## 1         1         0.994         0.941         7  
## # ... with 6 more variables: seasonal_trough_year <dbl>,  
## #   spikiness <dbl>, linearity <dbl>, curvature <dbl>,  
## #   stl_e_acf1 <dbl>, stl_e_acf10 <dbl>
```

Automatically selecting differences

```
usmelec %>% mutate(log_gen = log(Generation)) %>%  
  features(log_gen, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1  
##   nsdiffs  
##   <int>  
## 1      1
```

```
usmelec %>% mutate(d_log_gen = difference(log(Generation), 12)) %>%  
  features(d_log_gen, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1  
##   nsdiffs  
##   <int>  
## 1      1
```

Your turn

For the `tourism` dataset, compute the total number of trips and find an appropriate differencing (after transformation if necessary) to obtain stationary data.

Backshift notation

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$$By_t = y_{t-1}$$

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$$B(By_t) = B^2y_t = y_{t-2}$$

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$$B(By_t) = B^2y_t = y_{t-2}$$

For monthly data, if we wish to shift attention to “the same month last year”, then B^{12} is used, and the notation is $B^{12}y_t = y_{t-12}$.

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Note that a first difference is represented by $(1 - B)$.

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Note that a first difference is represented by $(1 - B)$.

Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t$$

Backshift notation

- Second-order difference is denoted $(1 - B)^2$.
- *Second-order difference* is not the same as a *second difference*, which would be denoted $1 - B^2$;
- In general, a d th-order difference can be written as

$$(1 - B)^d y_t$$

- A seasonal difference followed by a first difference can be written as

$$(1 - B)(1 - B^m)y_t$$

Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

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For monthly data, $m = 12$ and we obtain the same result as earlier.

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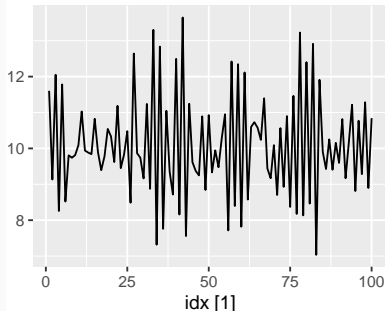
Autoregressive models

Autoregressive (AR) models:

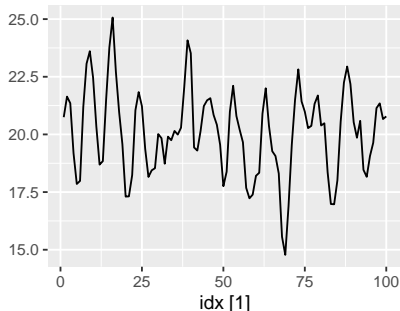
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.

AR(1)



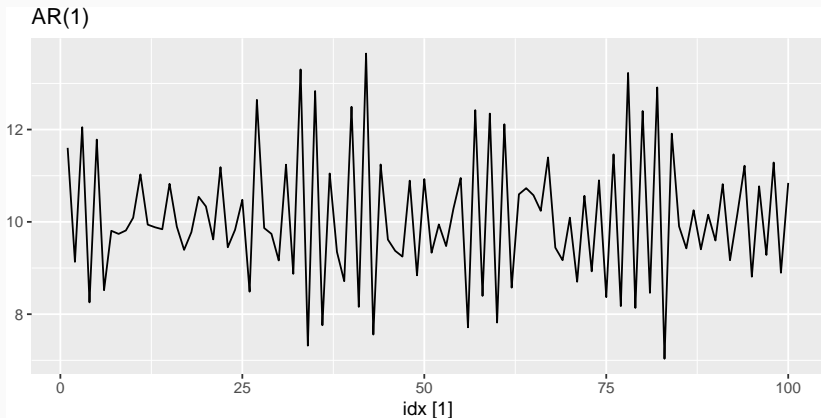
AR(2)



AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



AR(1) model

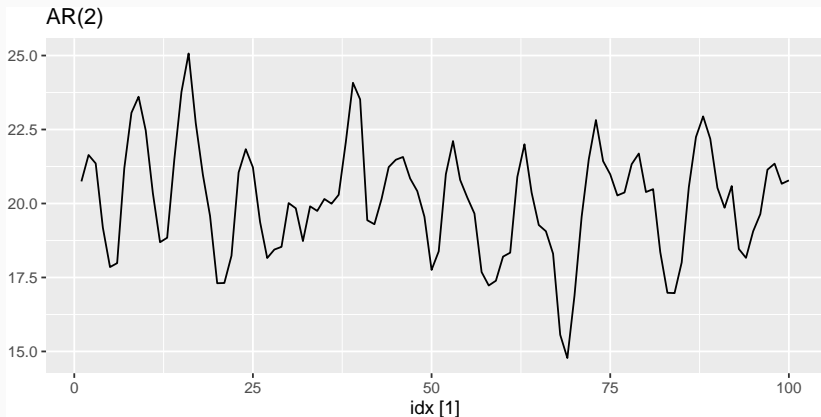
$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN**
- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values.**

AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
 $-1 < \phi_2 < 1$ $\phi_2 + \phi_1 < 1$ $\phi_2 - \phi_1 < 1$.
- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

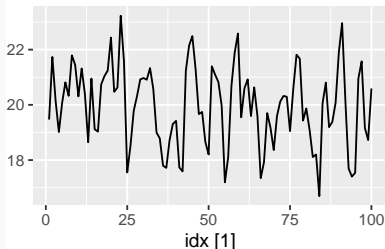
Moving Average (MA) models

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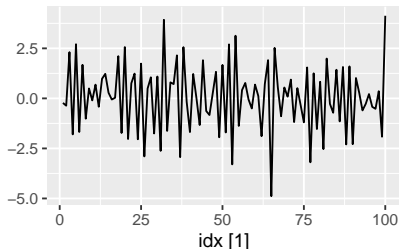
$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where ε_t is white noise. This is a multiple regression with **past errors** as predictors. *Don't confuse this with moving average smoothing!*

MA(1)



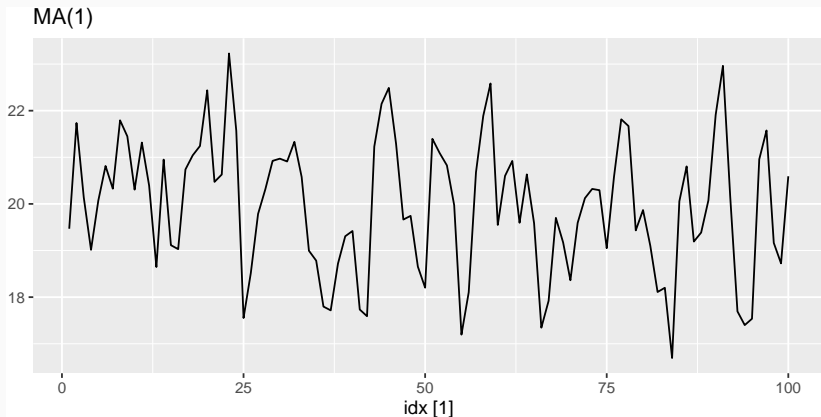
MA(2)



MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

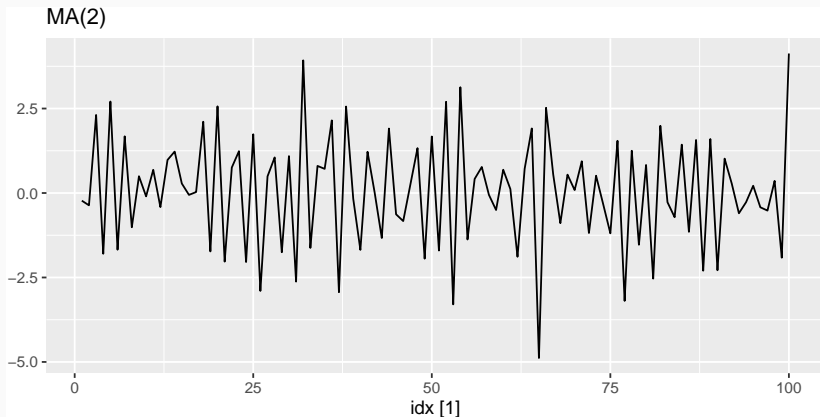
$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



MA(∞) models

It is possible to write any stationary AR(p) process as an MA(∞) process.

Example: AR(1)

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \varepsilon_t \\&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&\dots\end{aligned}$$

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Provided $-1 < \phi_1 < 1$:

$$y_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \dots$$

Invertibility

- Any $MA(q)$ process can be written as an $AR(\infty)$ process if we impose some constraints on the MA parameters.
- Then the MA model is called “invertible”.
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

Invertibility

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Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$ lie outside the unit circle on the complex plane.

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- For $q = 1$: $-1 < \theta_1 < 1$.
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 $-1 < \theta_2 < 1 \quad \theta_2 + \theta_1 > -1 \quad \theta_1 - \theta_2 < 1$.
- More complicated conditions hold for $q \geq 3$.
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ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

ARIMA models

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- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- $(1 - B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR: p = order of the autoregressive part

I: d = degree of first differencing involved

MA: q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA($p,0,0$)
- MA(q): ARIMA(0,0, q)

Backshift notation for ARIMA

■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

■ ARIMA(1,1,1) model:

$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR}(1) & \text{First} & & \text{MA}(1) \\ & \text{difference} & & \end{array}$$

Backshift notation for ARIMA

■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

■ ARIMA(1,1,1) model:

$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR}(1) & \text{First} & & \text{MA}(1) \\ & \text{difference} & & \end{array}$$

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

R model

Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

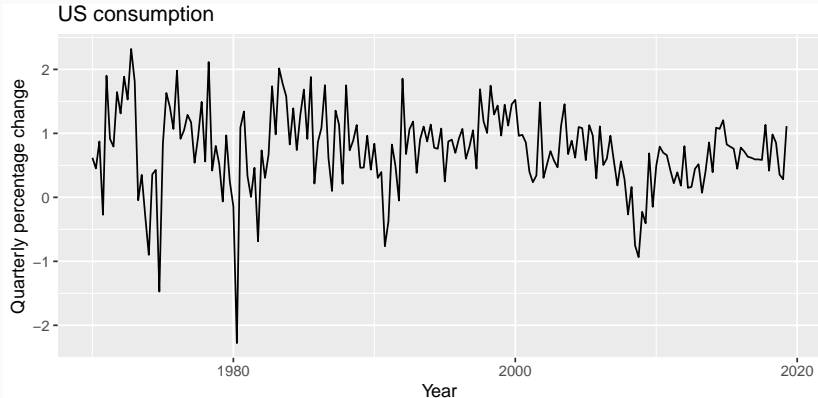
Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- μ is the mean of y'_t .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$.
- fable uses intercept form

US consumption expenditure

```
us_change %>% autoplot(Consumption) +  
  xlab("Year") +  
  ylab("Quarterly percentage change") +  
  ggtitle("US consumption")
```



US personal consumption

```
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))  
report(fit)
```

```
## Series: Consumption  
## Model: ARIMA(1,0,3) w/ mean  
##  
## Coefficients:  
##          ar1          ma1          ma2          ma3    constant  
##          0.5731   -0.3617   0.0925   0.1934     0.3160  
## s.e.    0.1503    0.1607   0.0787   0.0824     0.0371  
##  
## sigma^2 estimated as 0.3334:  log likelihood=-169.9  
## AIC=351.8   AICc=352.2   BIC=371.5
```

US personal consumption

```
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))  
report(fit)
```

```
## Series: Consumption  
## Model: ARIMA(1,0,3) w/ mean  
##  
## Coefficients:  
##          ar1          ma1          ma2          ma3    constant  
##          0.5731   -0.3617   0.0925   0.1934     0.3160  
## s.e.    0.1503    0.1607   0.0787   0.0824     0.0371  
##  
## sigma^2 estimated as 0.3334:  log likelihood=-169.9  
## AIC=351.8   AICc=352.2   BIC=371.5
```

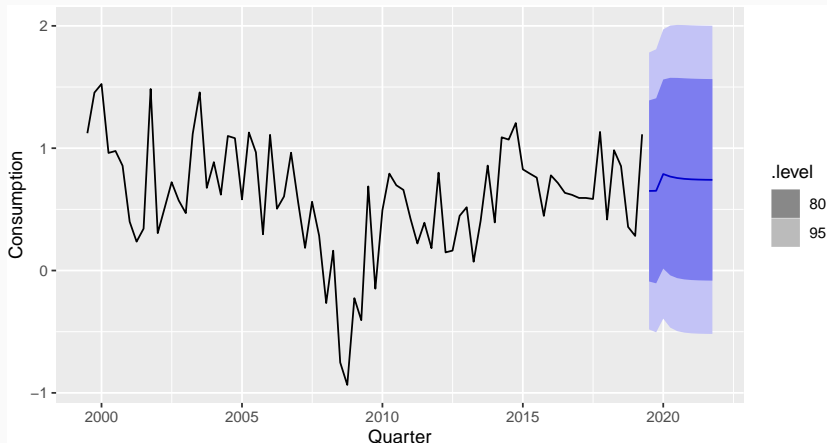
ARIMA(1,0,3) model:

$$y_t = 0.316 + 0.573y_{t-1} - 0.362\varepsilon_{t-1} + 0.0925\varepsilon_{t-2} + 0.193\varepsilon_{t-3} + \varepsilon_t,$$

where ε_t is white noise with a standard deviation of $0.577 = \sqrt{0.333}$.

US personal consumption

```
fit %>% forecast(h=10) %>%  
  autoplot(tail(us_change, 80))
```



Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

Understanding ARIMA models

Forecast variance and d

- The higher the value of d , the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Cyclic behaviour

- For cyclic forecasts, $p \geq 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length

$$(2\pi) / \left[\arccos(-\phi_1(1 - \phi_2)/(4\phi_2)) \right].$$

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- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection**
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Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

- MLE is very similar to least squares estimation obtained by minimizing

$$\sum_{t=1}^T e_t^2$$

- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k} , when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k} , when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

α_k = k th partial autocorrelation coefficient
= equal to the estimate of ϕ_k in regression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.$$

Partial autocorrelations

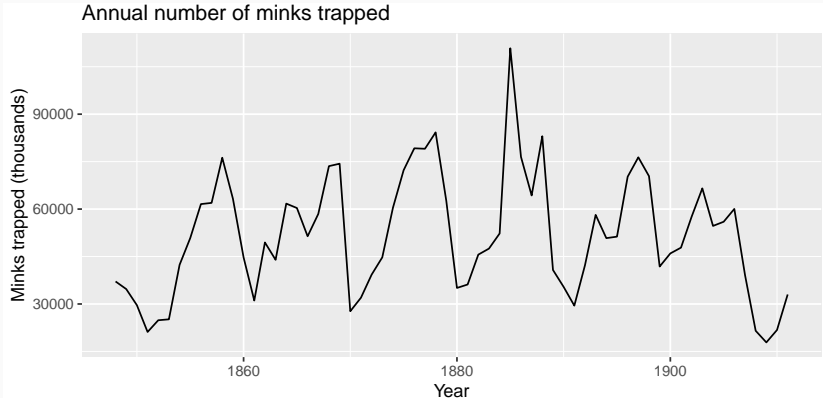
Partial autocorrelations measure relationship between y_t and y_{t-k} , when the effects of other time lags $-1, 2, 3, \dots, k-1$ are removed.

α_k = k th partial autocorrelation coefficient
= equal to the estimate of ϕ_k in regression:

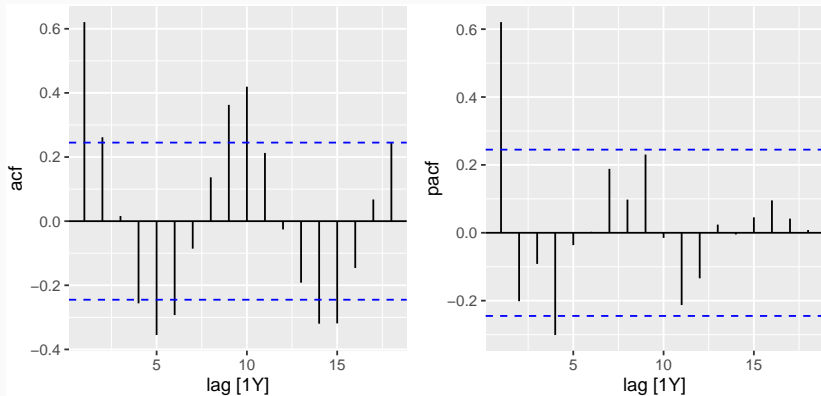
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.$$

- Varying number of terms on RHS gives α_k for different values of k .
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.
- Last significant α_k indicates the order of an AR model.

Example: Mink trapping

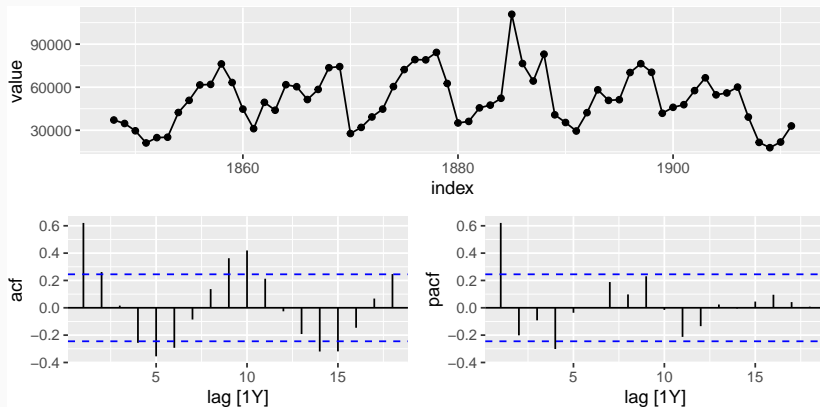


Example: Mink trapping



Example: Mink trapping

```
mink %>% gg_tsdisplay(value, plot_type='partial')
```



ACF and PACF interpretation

AR(1)

$$\begin{aligned}\rho_k &= \phi_1^k && \text{for } k = 1, 2, \dots; \\ \alpha_1 &= \phi_1 && \alpha_k = 0 \quad \text{for } k = 2, 3, \dots\end{aligned}$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

ACF and PACF interpretation

$AR(p)$

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the p th spike

So we have an $AR(p)$ model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag p in PACF, but none beyond p

ACF and PACF interpretation

MA(1)

$$\begin{aligned}\rho_1 &= \theta_1 & \rho_k &= 0 & \text{for } k = 2, 3, \dots; \\ \alpha_k &= -(-\theta_1)^k\end{aligned}$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

ACF and PACF interpretation

MA(q)

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the q th spike

So we have an MA(q) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag q in ACF, but none beyond q

Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2 \log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

Information criteria

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

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Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

Information criteria

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

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How does ARIMA() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via KPSS test and seasonal strength measure.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Step1: Select current model (with smallest AICc) from:

ARIMA(2, d , 2)

ARIMA(0, d , 0)

ARIMA(1, d , 0)

ARIMA(0, d , 1)

How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Step1: Select current model (with smallest AICc) from:

ARIMA(2, d , 2)

ARIMA(0, d , 0)

ARIMA(1, d , 0)

ARIMA(0, d , 1)

Step 2: Consider variations of current model:

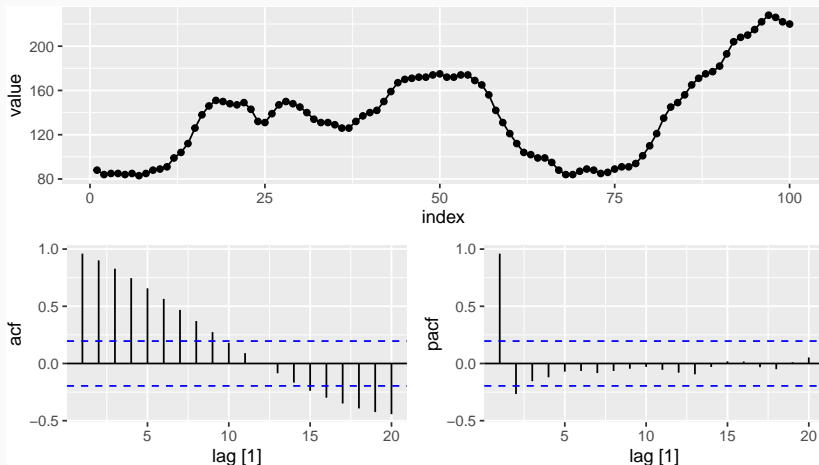
- vary one of p , q , from current model by ± 1 ;
- p , q both vary from current model by ± 1 ;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 2 until no lower AICc can be found.

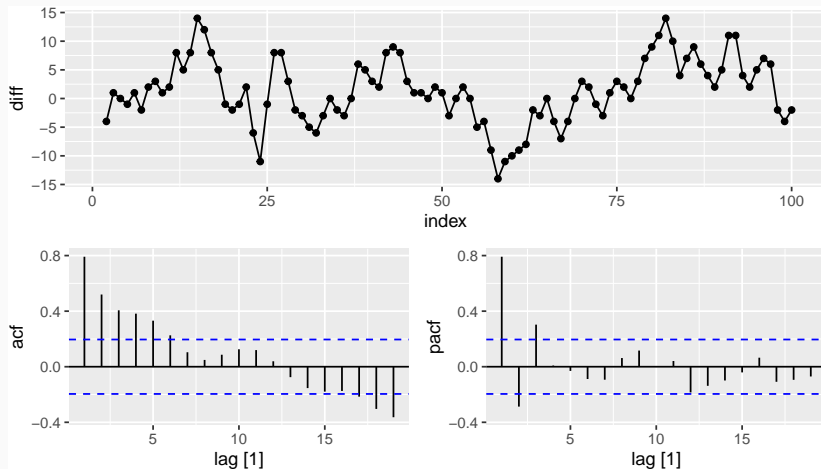
Choosing your own model

```
web_usage <- as_tsibble(WWWusage)
web_usage %>% gg_tsdisplay(value, plot_type = 'partial')
```



Choosing your own model

```
web_usage %>% mutate(diff = difference(value)) %>%  
  gg_tsdisplay(diff, plot_type = 'partial')
```



Choosing your own model

```
fit <- web_usage %>%  
  model(arima = ARIMA(value ~ pdq(3, 1, 0)))  
report(fit)
```

```
## Series: value  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1          ar2          ar3  
##          1.151   -0.6612   0.3407  
## s.e.    0.095    0.1353   0.0941  
##  
## sigma^2 estimated as 9.656:  log likelihood=-252  
## AIC=512    AICc=512.4    BIC=522.4
```

Choosing your own model

```
web_usage %>%  
  model(ARIMA(value ~ pdq(d=1))) %>%  
  report()
```

```
## Series: value  
## Model: ARIMA(1,1,1)  
##  
## Coefficients:  
##          ar1      ma1  
##      0.6504  0.5256  
## s.e.  0.0842  0.0896  
##  
## sigma^2 estimated as 9.995:  log likelihood=-254.2  
## AIC=514.3   AICc=514.5   BIC=522.1
```

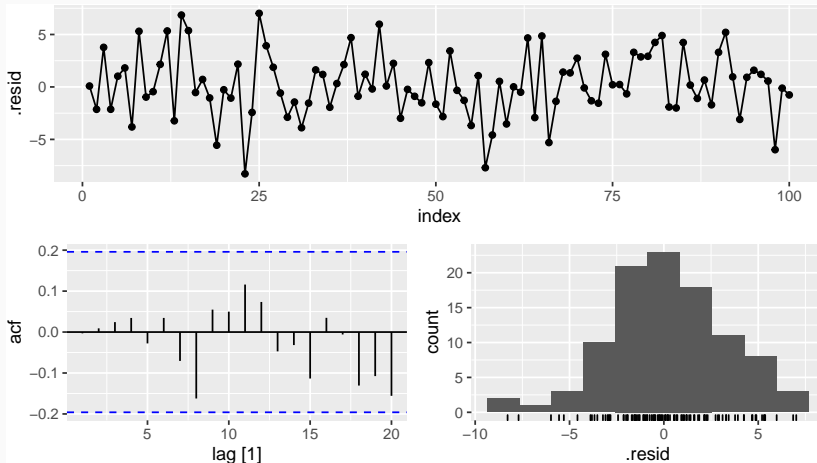
Choosing your own model

```
web_usage %>%  
  model(ARIMA(value ~ pdq(d=1),  
    stepwise = FALSE, approximation = FALSE)) %>%  
  report()
```

```
## Series: value  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1          ar2          ar3  
##          1.151    -0.6612    0.3407  
## s.e.    0.095     0.1353    0.0941  
##  
## sigma^2 estimated as 9.656:  log likelihood=-252  
## AIC=512    AICc=512.4    BIC=522.4
```

Choosing your own model

```
gg_tsresiduals(fit)
```



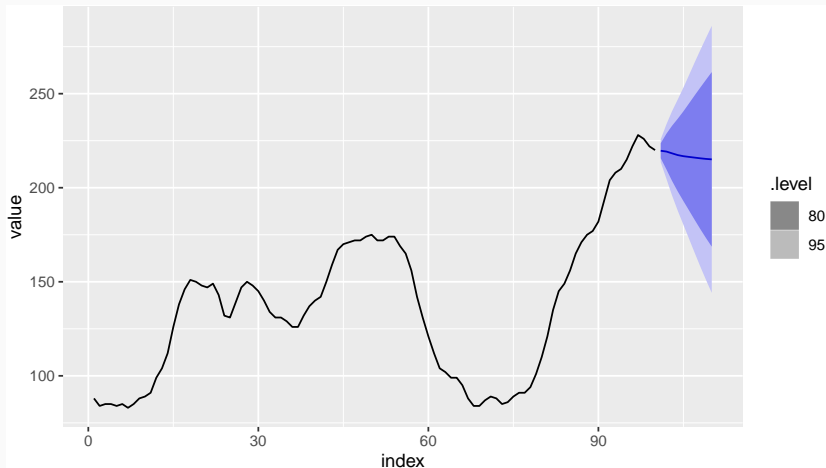
Choosing your own model

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 10, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 arima      4.49      0.722
```


Choosing your own model

```
fit %>% forecast(h = 10) %>%  
  autoplot(web_usage)
```



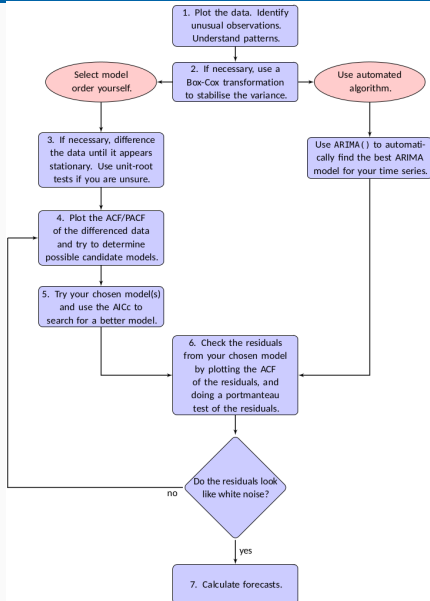
Modelling procedure with ARIMA ()

- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 If the data are non-stationary: take first differences of the data until the data are stationary.
- 4 Examine the ACF/PACF: Is an $AR(p)$ or $MA(q)$ model appropriate?
- 5 Try your chosen model(s), and use the AICc to search for a better model.
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

Automatic modelling procedure with ARIMA ()

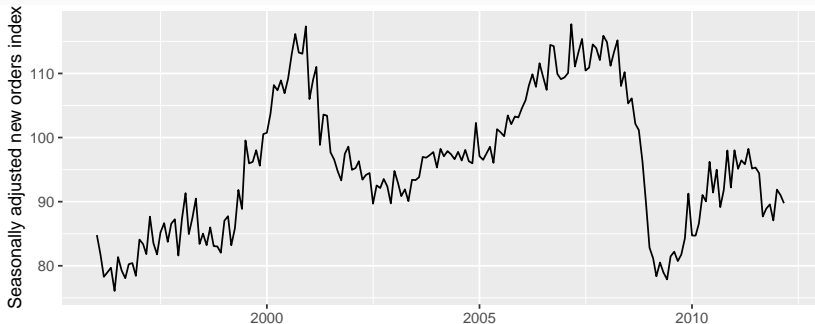
- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 Use ARIMA to automatically select a model.
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

Modelling procedure



Seasonally adjusted electrical equipment

```
elecequip <- as_tsibble(fpp2::elecequip)
dcmp <- elecequip %>%
  model(STL(value ~ season(window = "periodic"))) %>%
  components() %>% select(-.model)
dcmp %>% as_tsibble %>%
  autoplot(season_adjust) + xlab("Year") +
  ylab("Seasonally adjusted new orders index")
```

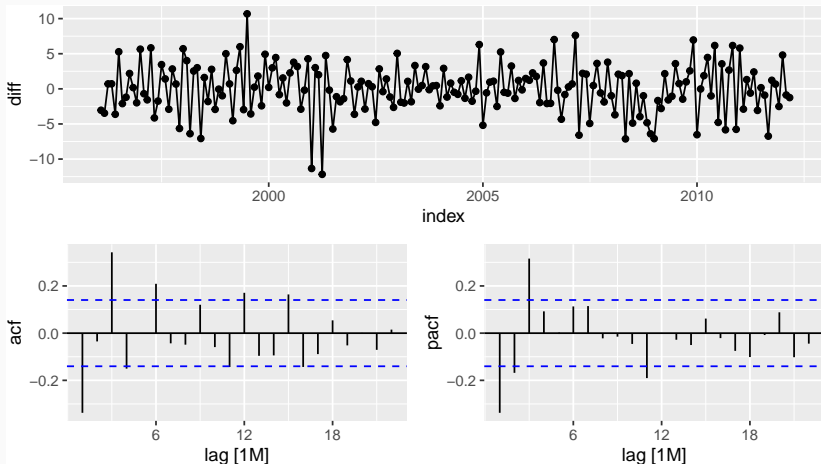


Seasonally adjusted electrical equipment

- 1 Time plot shows sudden changes, particularly big drop in 2008/2009 due to global economic environment. Otherwise nothing unusual and no need for data adjustments.
- 2 No evidence of changing variance, so no Box-Cox transformation.
- 3 Data are clearly non-stationary, so we take first differences.

Seasonally adjusted electrical equipment

```
dcmp %>% mutate(diff = difference(season_adjust)) %>%  
  gg_tsdisplay(diff, plot_type = 'partial')
```



Seasonally adjusted electrical equipment

- 4 PACF is suggestive of AR(3). So initial candidate model is ARIMA(3,1,0). No other obvious candidates.
- 5 Fit ARIMA(3,1,0) model along with variations: ARIMA(4,1,0), ARIMA(2,1,0), ARIMA(3,1,1), etc. ARIMA(3,1,1) has smallest AICc value.

Seasonally adjusted electrical equipment

```
fit <- dcmp %>%  
  model(arima = ARIMA(season_adjust))  
report(fit)
```

```
## Series: season_adjust  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1      ar2      ar3  
##      -0.3418  -0.0426  0.3185  
## s.e.   0.0681   0.0725  0.0682  
##  
## sigma^2 estimated as 9.639:  log likelihood=-493.8  
## AIC=995.6   AICc=995.8   BIC=1009
```

Seasonally adjusted electrical equipment

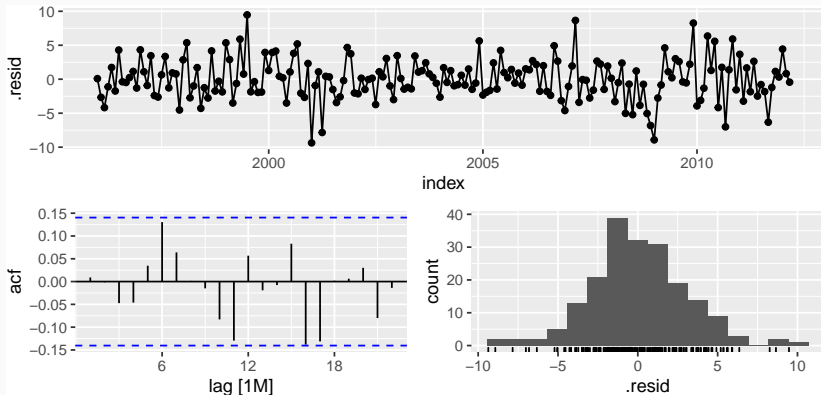
```
fit <- dcmp %>%  
  model(arima = ARIMA(season_adjust, approximation=FALSE))  
report(fit)
```

```
## Series: season_adjust  
## Model: ARIMA(3,1,1)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ma1  
##      0.0044  0.0916  0.3698 -0.3921  
## s.e.  0.2201  0.0984  0.0669  0.2426  
##  
## sigma^2 estimated as 9.577:  log likelihood=-492.7  
## AIC=995.4   AICc=995.7   BIC=1012
```

Seasonally adjusted electrical equipment

6

ACF plot of residuals from ARIMA(3,1,1) model look like white noise.

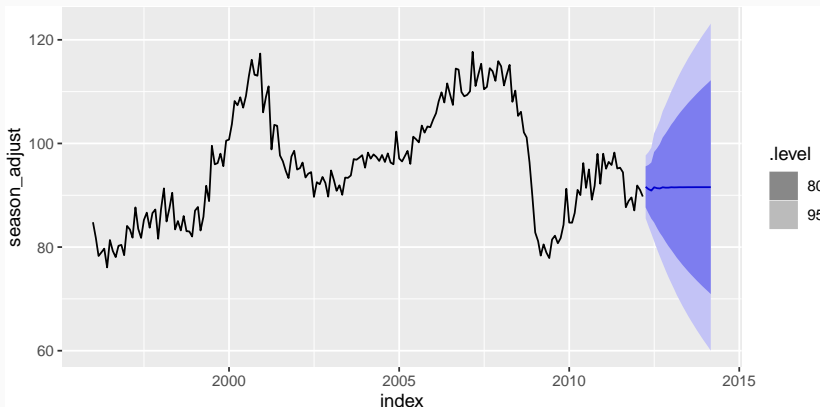


Seasonally adjusted electrical equipment

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>    <dbl>
## 1 arima    24.0      0.241
```

Seasonally adjusted electrical equipment

```
fit %>% forecast %>% autoplot(dcmp)
```



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Point forecasts

- 1 Rearrange ARIMA equation so y_t is on LHS.
- 2 Rewrite equation by replacing t by $T + h$.
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \dots$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

$$\begin{aligned} y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \theta_1e_T.$$

Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1 + \phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2}.$$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96 \sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

Prediction intervals

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96 \sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

Prediction intervals

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where $v_{T+h|T}$ is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

- AR(1): Rewrite as MA(∞) and use above result.
- Other models beyond scope of this subject.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
 - ▶ the uncertainty in the parameter estimates has not been accounted for.
 - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
 - ▶ the ARIMA model assumes uncorrelated future errors₁₀₀

Your turn

For the GDP data (from `global_economy`):

- fit a suitable ARIMA model to the logged data for all countries
- check the residual diagnostics for Australia;
- produce forecasts of your fitted model for Australia.

Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

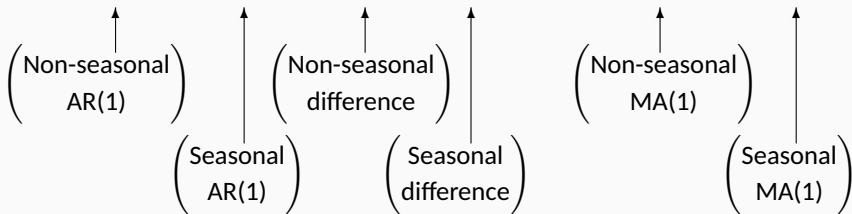
E.g., $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$ model (without constant)

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)₄ model (without constant)
$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

Seasonal ARIMA models

E.g., $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$ model (without constant)
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$



Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)₄ model (without constant)
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1) _m	with log transformation
ARIMA(0,1,2)(0,1,1) _m	with log transformation
ARIMA(2,1,0)(0,1,1) _m	with log transformation
ARIMA(0,2,2)(0,1,1) _m	with log transformation
ARIMA(2,1,2)(0,1,1) _m	with no transformation

Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

ARIMA(0,0,0)(0,0,1)₁₂ will show:

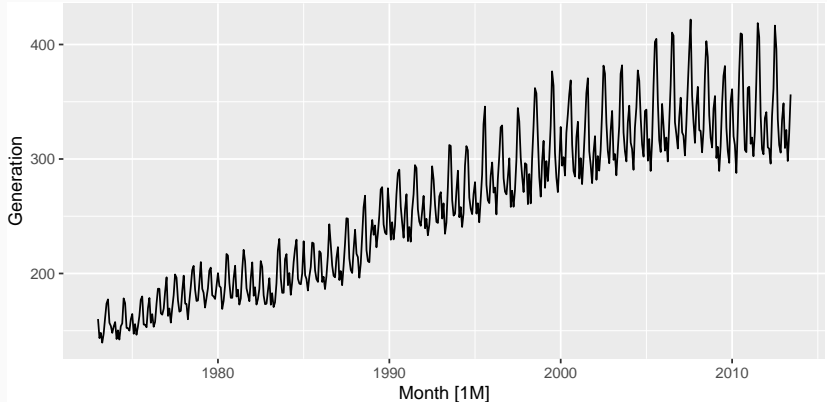
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36,

ARIMA(0,0,0)(1,0,0)₁₂ will show:

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

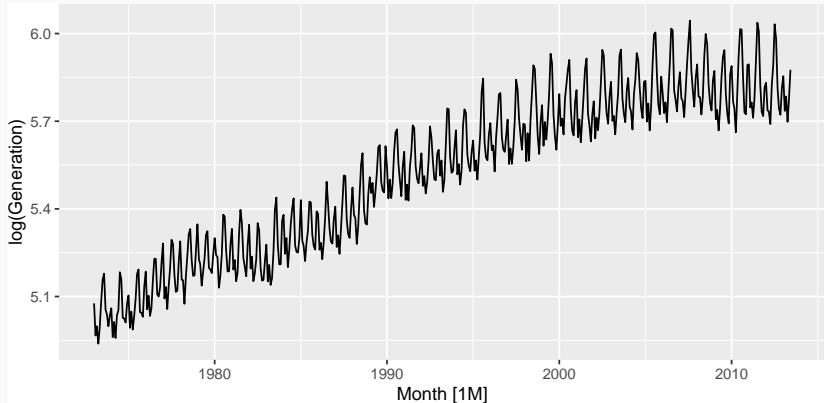
Example: US electricity production

```
usmelec %>% autoplot(  
  Generation  
)
```



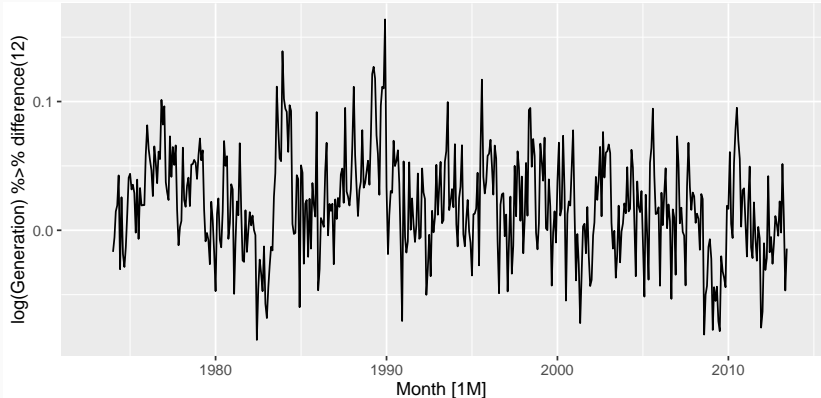
Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation)  
)
```



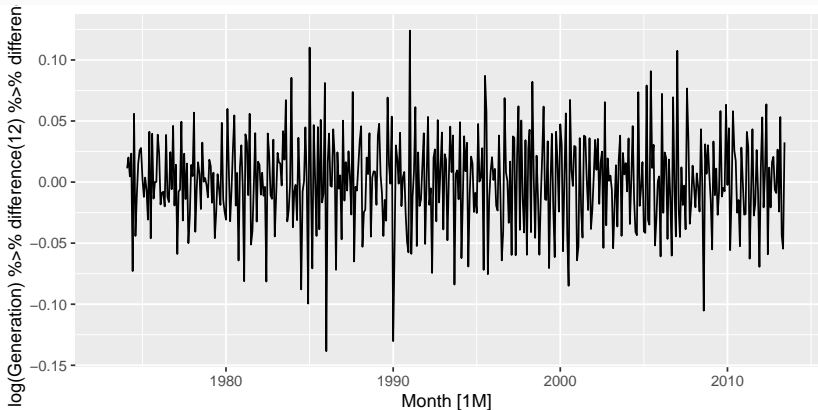
Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12)  
)
```



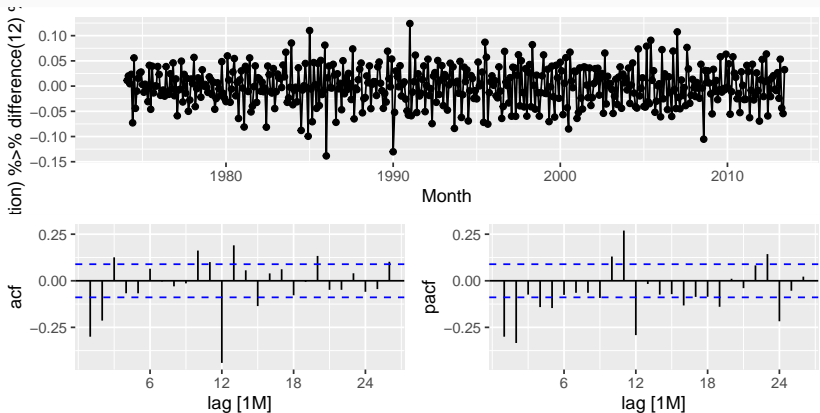
Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12) %>% difference()  
)
```



Example: US electricity production

```
usmelec %>% gg_tsdisplay(  
  log(Generation) %>% difference(12) %>% difference(),  
  plot_type = "partial")
```



Example: US electricity production

- $d = 1$ and $D = 1$ seems necessary
- $P = 0$ and $Q = 1$ suggested by seasonal lags
- $p = 0$ and $q = 3$ suggested by non-seasonal lags.

Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation) ~ pdq(0,1,3) + PDQ(0,1,1))) %>%  
  report()
```

```
## Series: Generation  
## Model: ARIMA(0,1,3)(0,1,1)[12]  
## Transformation: log(.x)  
##  
## Coefficients:  
##          ma1          ma2          ma3          sma1  
##      -0.4266  -0.2496  -0.0439  -0.8358  
## s.e.   0.0462   0.0516   0.0430   0.0262  
##  
## sigma^2 estimated as 0.0006904:  log likelihood=1045  
## AIC=-2080   AICc=-2080   BIC=-2059
```

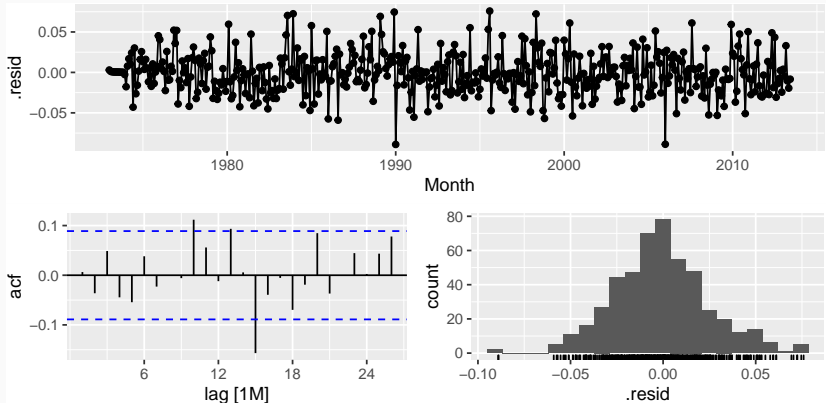
Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  report()
```

```
## Series: Generation  
## Model: ARIMA(1,1,1)(2,1,1)[12]  
## Transformation: log(.x)  
##  
## Coefficients:  
##          ar1          ma1          sar1          sar2          sma1  
##          0.4116   -0.8483   0.0100   -0.1017   -0.8204  
## s.e.    0.0617    0.0348   0.0561    0.0529    0.0357  
##  
## sigma^2 estimated as 0.0006841:  log likelihood=1047  
## AIC=-2082    AICc=-2082    BIC=-2057
```

Example: US electricity production

```
fit <- usmelec %>%  
  model(arima = ARIMA(log(Generation)))  
gg_tsresiduals(fit)
```



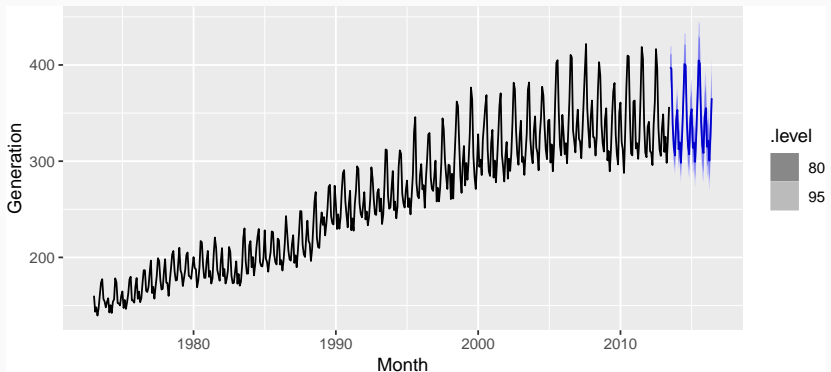
Example: US electricity production

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 24, dof = 5)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 arima      38.7    0.00484
```

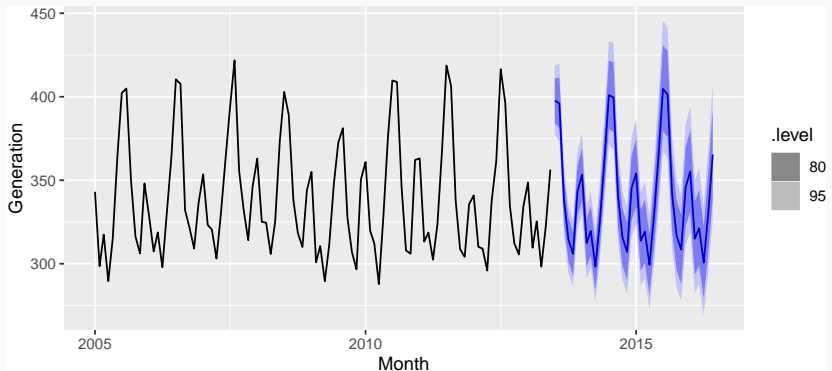
Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  forecast(h = "3 years") %>%  
  autoplot(usmelec)
```

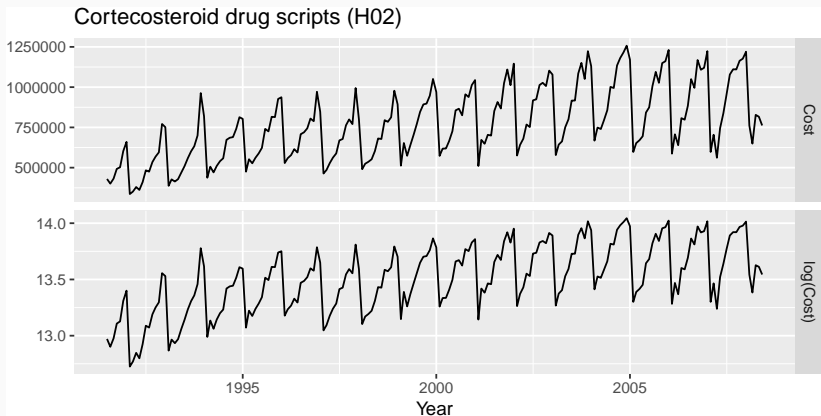


Example: US electricity production

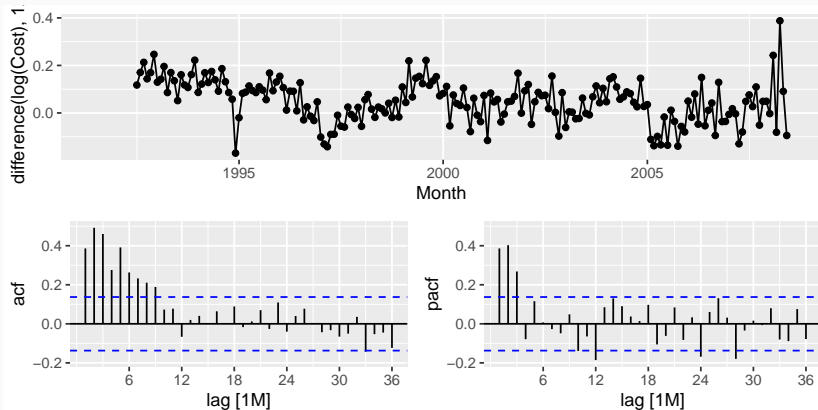
```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  forecast(h = "3 years") %>%  
  autoplot(filter(usmelec, year(Month) >= 2005))
```



Corticosteroid drug sales



Corticosteroid drug sales



Corticosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: $\text{ARIMA}(3,0,0)(2,1,0)_{12}$.

Corticosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485.5
ARIMA(3,0,1)(1,1,1)[12]	-484.3
ARIMA(3,0,1)(0,1,1)[12]	-483.7
ARIMA(3,0,1)(2,1,0)[12]	-476.3
ARIMA(3,0,0)(2,1,0)[12]	-475.1
ARIMA(3,0,2)(2,1,0)[12]	-474.9
ARIMA(3,0,1)(1,1,0)[12]	-463.4

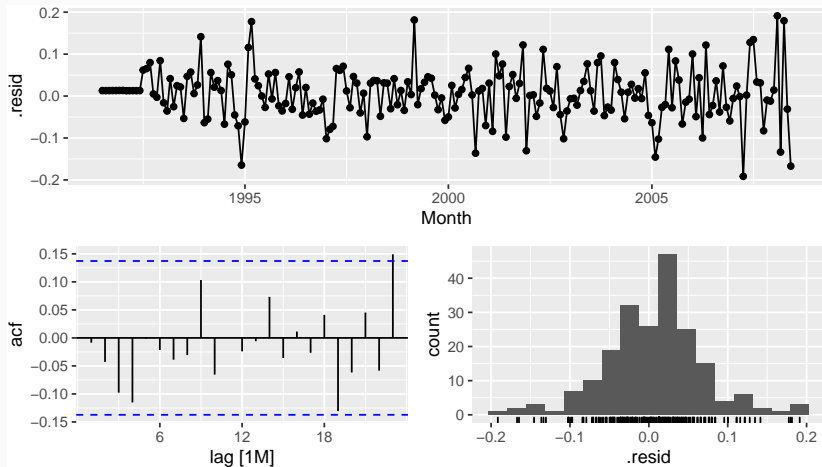
Corticosteroid drug sales

```
fit <- h02 %>%  
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(3,0,1)(0,1,2)[12]  
## Transformation: log(.x)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ma1      sma1      sma2  
##      -0.1602  0.5481  0.5678  0.3826 -0.5222 -0.1769  
## s.e.   0.1636  0.0878  0.0942  0.1895  0.0861  0.0872  
##  
## sigma^2 estimated as 0.004289: log likelihood=250.1  
## AIC=-486.1   AICc=-485.5   BIC=-463.3
```


Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      50.5      0.0109
```

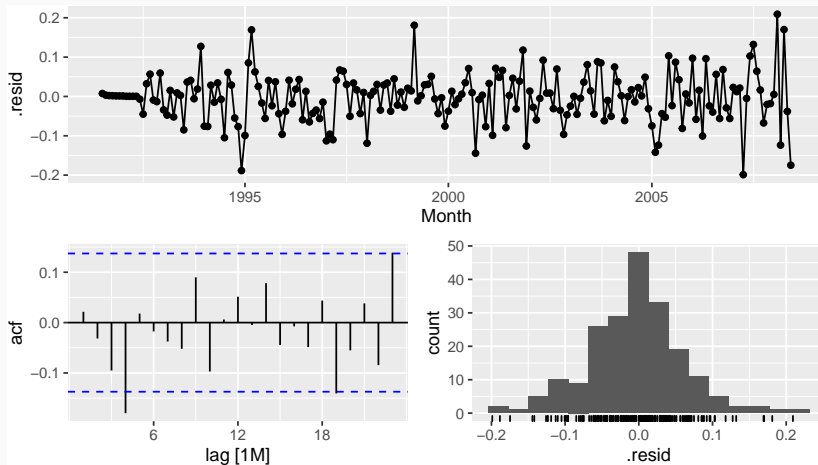
Corticosteroid drug sales

```
fit <- h02 %>% model(auto = ARIMA(log(Cost)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(2,1,0)(0,1,1)[12]  
## Transformation: log(.x)  
##  
## Coefficients:  
##           ar1      ar2      sma1  
##      -0.8491  -0.4207  -0.6401  
## s.e.   0.0712   0.0714   0.0694  
##  
## sigma^2 estimated as 0.004399:  log likelihood=245.4  
## AIC=-482.8   AICc=-482.6   BIC=-469.8
```

Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 auto      57.5    0.00513
```

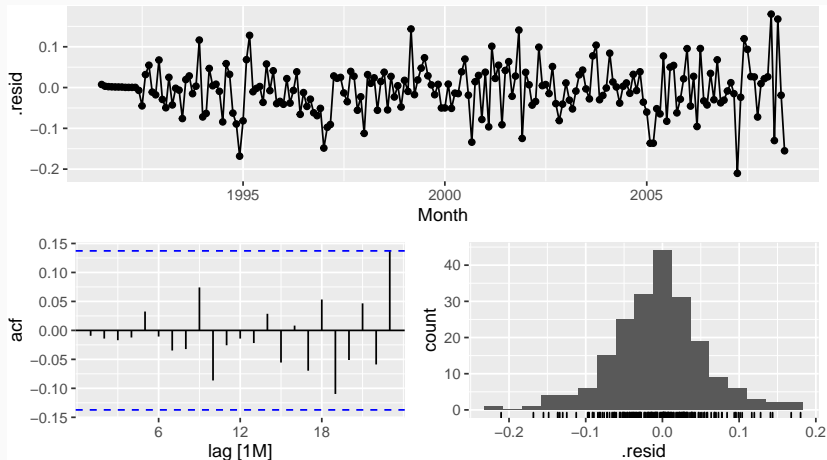
Corticosteroid drug sales

```
fit <- h02 %>%  
  model(best = ARIMA(log(Cost), stepwise = FALSE,  
    approximation = FALSE,  
    order_constraint = p + q + P + Q <= 9))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(4,1,1)(2,1,2)[12]  
## Transformation: log(.x)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ar4      ma1      sar1  
##      -0.0426  0.2097  0.2016 -0.2273 -0.7423  0.6213  
## s.e.   0.2167  0.1814  0.1144  0.0810  0.2075  0.2421  
##          sar2      sma1      sma2  
##      -0.3832 -1.2018  0.4958  
## s.e.   0.1185  0.2492  0.2136  
##  
## sigma^2 estimated as 0.004061: log likelihood=254.3  
## AIC=-488.6   AICc=-487.4   BIC=-456.1
```

Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      35.1      0.136
```


Corticosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 %>%  
  filter_index(~ "2006 Jun") %>%  
  model(  
    ARIMA(log(Cost) ~ pdq(3, 0, 0) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 2) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(1, 1, 0))  
    # ... #  
  )  
  
fit %>%  
  forecast(h = "2 years") %>%  
  accuracy(h02)
```

Corticosteroid drug sales

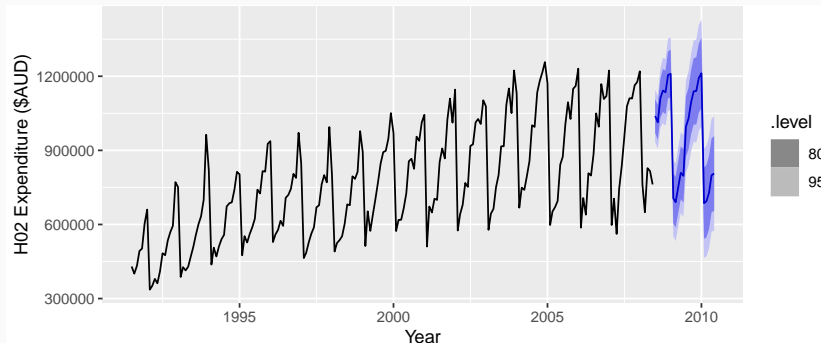
.model	RMSE
ARIMA(3,0,1)(1,1,1)[12]	61878
ARIMA(3,0,1)(0,1,2)[12]	62142
ARIMA(3,0,1)(0,1,1)[12]	62947
ARIMA(2,1,0)(0,1,1)[12]	62984
ARIMA(4,1,1)(2,1,2)[12]	63114
ARIMA(3,0,2)(2,1,0)[12]	65146
ARIMA(3,0,1)(2,1,0)[12]	65270
ARIMA(3,0,1)(1,1,0)[12]	66644
ARIMA(3,0,0)(2,1,0)[12]	66816

Corticosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

Corticosteroid drug sales

```
fit <- h02 %>%  
  model(ARIMA(Cost ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))  
fit %>% forecast %>% autoplot(h02) +  
  ylab("H02 Expenditure ($AUD)") + xlab("Year")
```



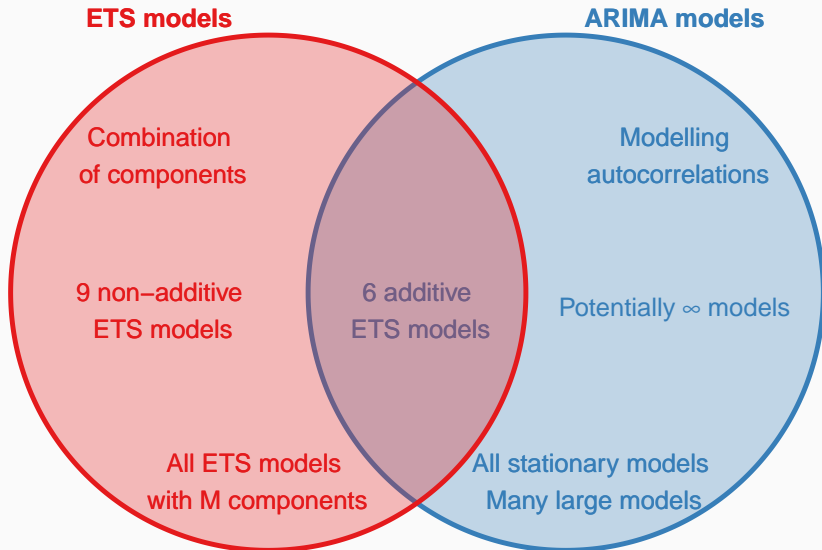
Outline

- 1 Stationarity and differencing
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ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

ARIMA vs ETS



Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$
ETS(A,A _d ,N)	ARIMA(1,1,2)	$\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0) _m	
ETS(A,A,A)	ARIMA(0,1,m + 1)(0,1,0) _m	
ETS(A,A _d ,A)	ARIMA(1,0,m + 1)(0,1,0) _m	