MONASH University

MONASH BUSINESS SCHOOL

ETC3550: Applied forecasting for business and economics

Ch5. Regression models

OTexts.org/fpp2/

Outline

Multiple regression and forecasting

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + e_t.$$

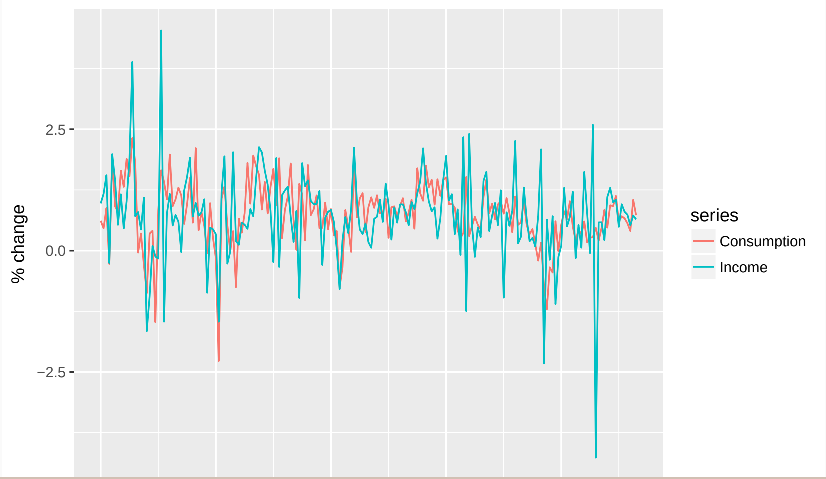
- y_t is the variable we want to predict: the “response” variable
- Each $x_{j,t}$ is numerical and is called a “predictor”. They are usually assumed to be known for all past and future times.
- The coefficients $\beta_1, \ldots, \beta_k$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.

That is, the coefficients measure the **marginal effects**.

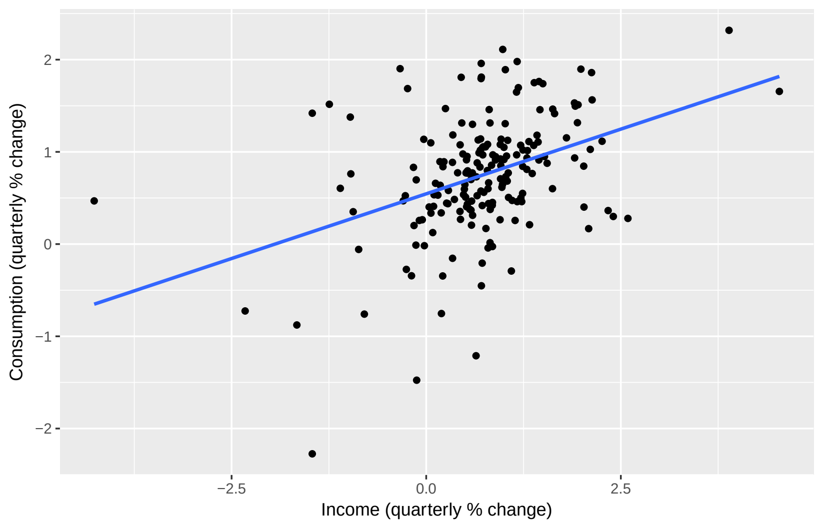
- e_t is a white noise error term

Example: US consumption expenditure

```
autoplot(uschange[,c("Consumption","Income")]) +
  ylab("% change") + xlab("Year")
```



Example: US consumption expenditure

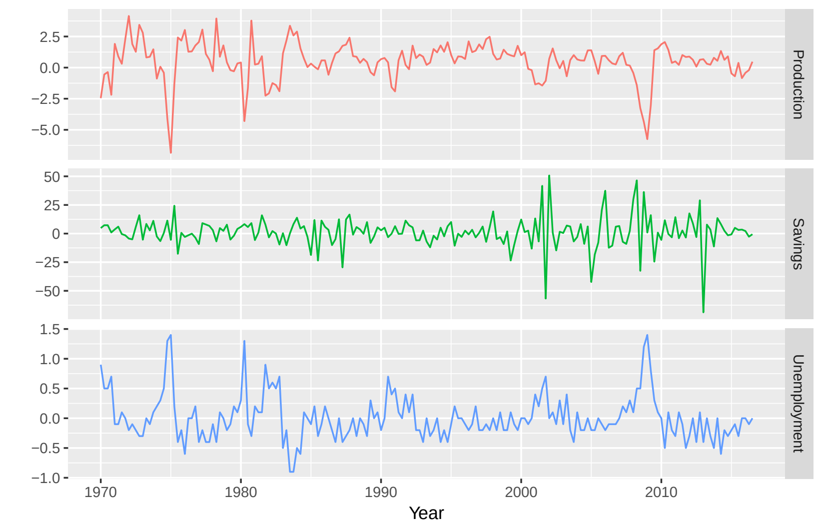


Example: US consumption expenditure

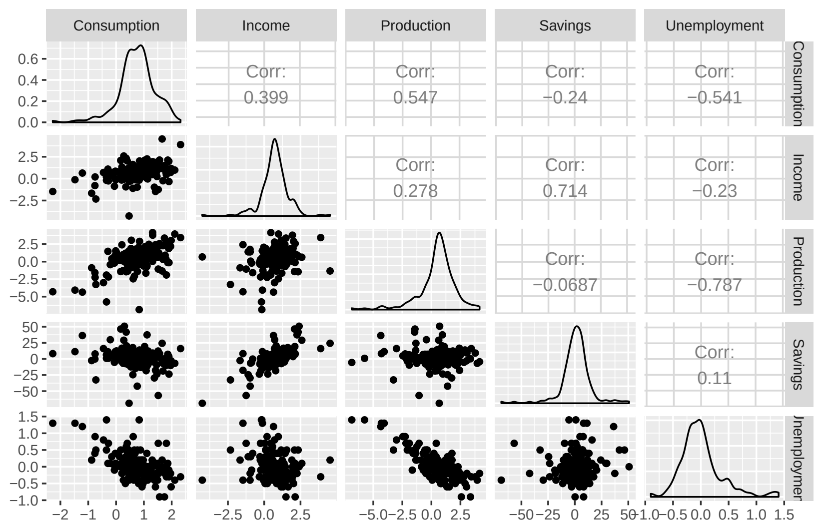
```
tslm(Consumption ~ Income, data=uschange) %>% summary
```

```
##
## Call:
## tslm(formula = Consumption ~ Income, data = uschange)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.40845 -0.31816  0.02558  0.29978  1.45157
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.54510    0.05569   9.789  < 2e-16 ***
## Income       0.28060    0.04744   5.915 1.58e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6026 on 185 degrees of freedom
## Multiple R-squared:  0.159,  Adjusted R-squared:  0.1545
## F-statistic: 34.98 on 1 and 185 DF,  p-value: 1.577e-08
```

Example: US consumption expenditure



Example: US consumption expenditure

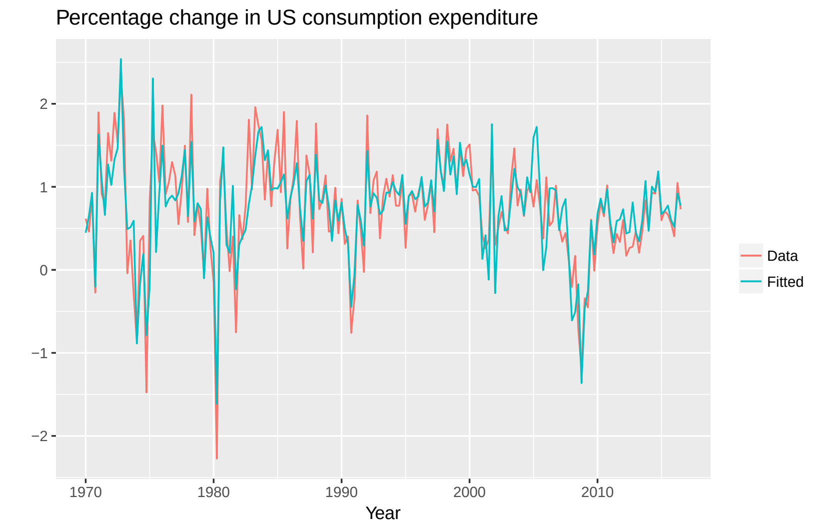


Example: US consumption expenditure

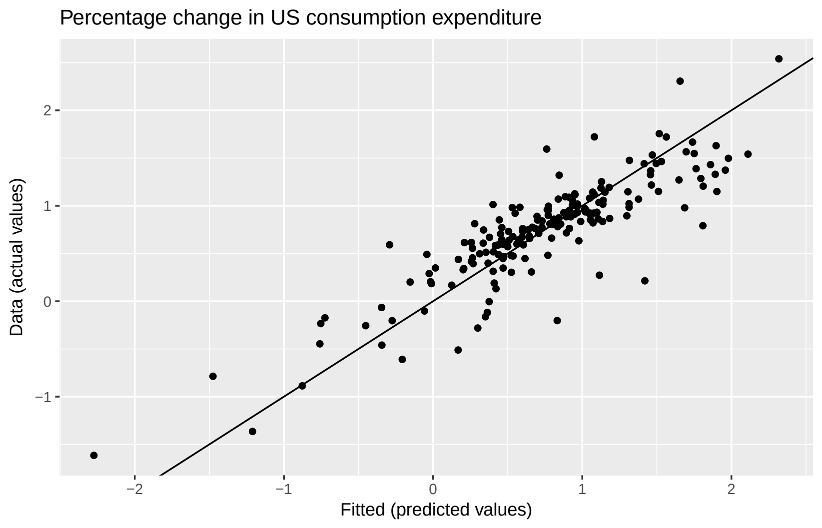
```r
fit.consMR <- tslm(Consumption ~
  Income + Production + Unemployment + Savings, data=uschange)
summary(fit.consMR)
```

```
##
## Call:
## tslm(formula = Consumption ~ Income + Production + Unemployment +
##     Savings, data = uschange)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.88296 -0.17638 -0.03679  0.15251  1.20553
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.26729    0.03721   7.184 1.68e-11 ***
## Income        0.71449    0.04219  16.934  < 2e-16 ***
## Production    0.04589    0.02588   1.773   0.0778 .
## Unemployment -0.20477    0.10550  -1.941   0.0538 .
## Savings      -0.04527    0.00278 -16.287  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3286 on 182 degrees of freedom
## Multiple R-squared:  0.754,  Adjusted R-squared:  0.7486
## F-statistic: 139.5 on 4 and 182 DF,  p-value: < 2.2e-16
```

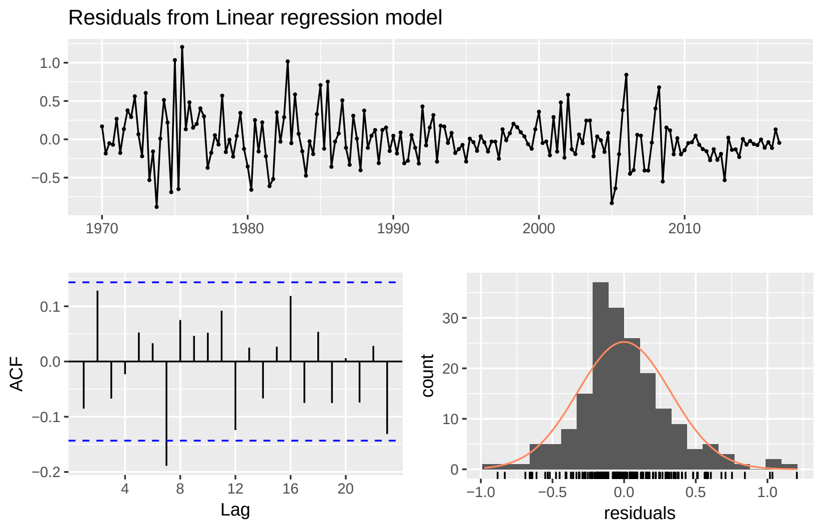
Example: US consumption expenditure



Example: US consumption expenditure



Example: US consumption expenditure



Outline

Trend

Linear trend

$$x_t = t$$

- $t = 1, 2, \ldots, T$
- Strong assumption that trend will continue.

Dummy variables

If a categorical variable takes only two values (e.g., ‘Yes’ or ‘No’), then an equivalent numerical variable can be constructed taking value 1 if yes and 0 if no. This is called a **dummy variable**.

	A	B
1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	No	0
7	Yes	1
8	Yes	1
9	No	0
10	No	0
11	No	0
12	No	0
13	Yes	1
14	No	0

Dummy variables

If there are more than two categories, then the variable can be coded using several dummy variables (one fewer than the total number of categories).

	A	B	C	D	E
1	Monday	1	0	0	0
2	Tuesday	0	1	0	0
3	Wednesday	0	0	1	0
4	Thursday	0	0	0	1
5	Friday	0	0	0	0
6	Monday	1	0	0	0
7	Tuesday	0	1	0	0
8	Wednesday	0	0	1	0
9	Thursday	0	0	0	1
10	Friday	0	0	0	0
11	Monday	1	0	0	0
12	Tuesday	0	1	0	0
13	Wednesday	0	0	1	0
14	Thursday	0	0	0	1
15	Friday	0	0	0	0

Beware of the dummy variable trap!

- Using one dummy for each category gives too many dummy variables!
- The regression will then be singular and inestimable.
- Either omit the constant, or omit the dummy for one category.
- The coefficients of the dummies are relative to the omitted category.

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

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- For daily data: use 6 dummies
- What to do with weekly data?

Outliers

- If there is an outlier, you can use a dummy variable (taking value 1 for that observation and 0 elsewhere) to remove its effect.

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- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
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- What to do with weekly data?

Outliers

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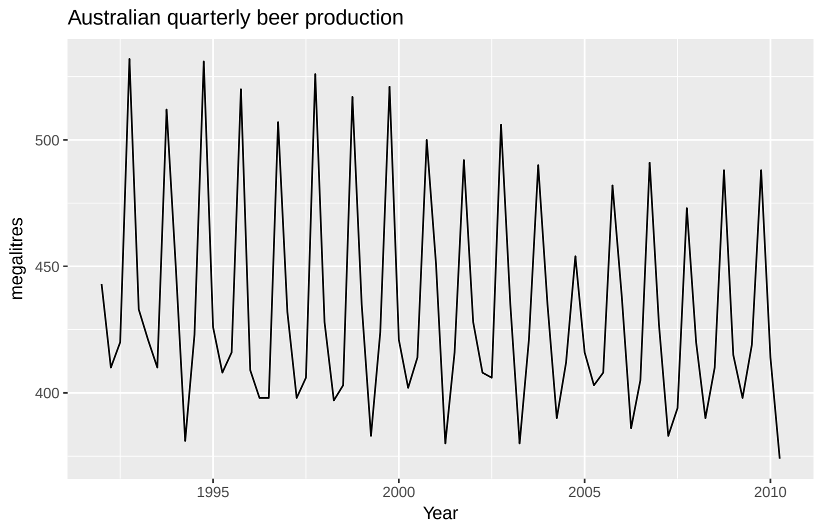
Public holidays

- For daily data: if it is a public holiday, dummy=1, otherwise dummy=0.

Beer production revisited



Beer production revisited

Regression model

$$y_t = \beta_0 + \beta_1 t + \beta_2 d_{1,t} + \beta_3 d_{2,t} + \beta_4 d_{3,t} + e_t$$

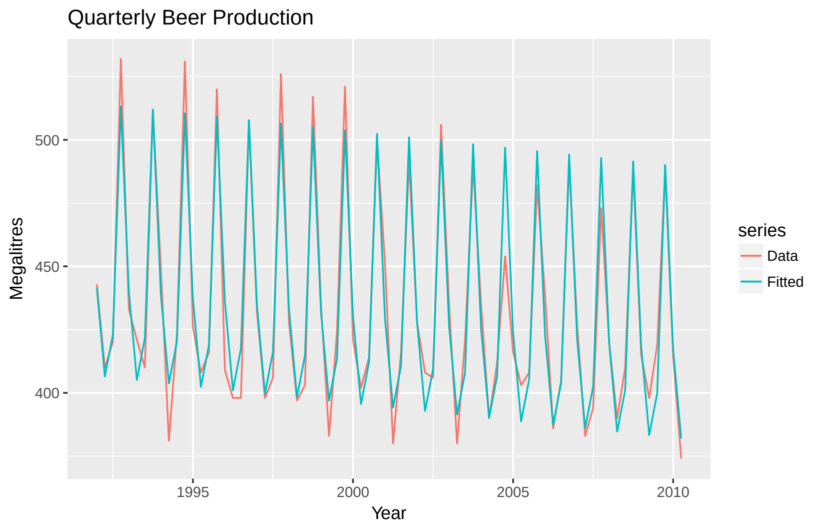
- $d_{i,t} = 1$ if t is quarter i and 0 otherwise.

Beer production revisited

```r
fit.beer <- tslm(beer ~ trend + season)
summary(fit.beer)
```

```
##
## Call:
## tslm(formula = beer ~ trend + season)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -42.903  -7.599  -0.459   7.991  21.789
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 441.80044    3.73353 118.333  < 2e-16 ***
## trend        -0.34027    0.06657  -5.111 2.73e-06 ***
## season2     -34.65973    3.96832  -8.734 9.10e-13 ***
## season3     -17.82164    4.02249  -4.430 3.45e-05 ***
## season4      72.79641    4.02305  18.095  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

Beer production revisited



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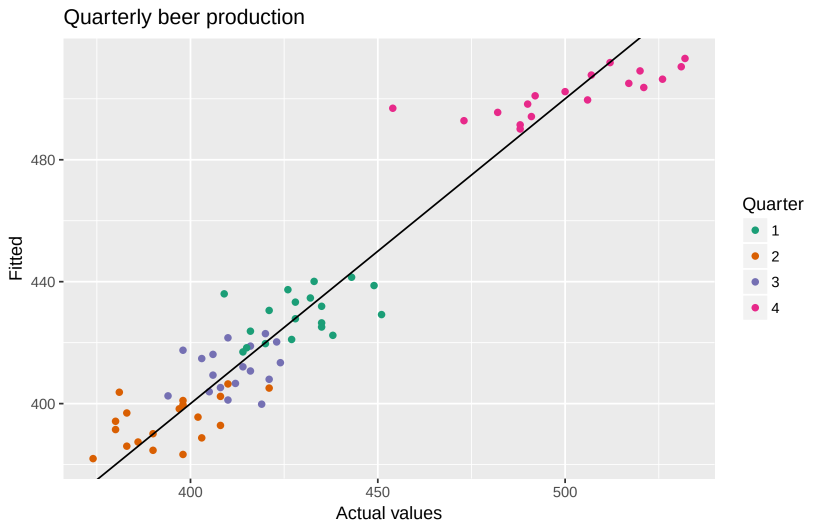
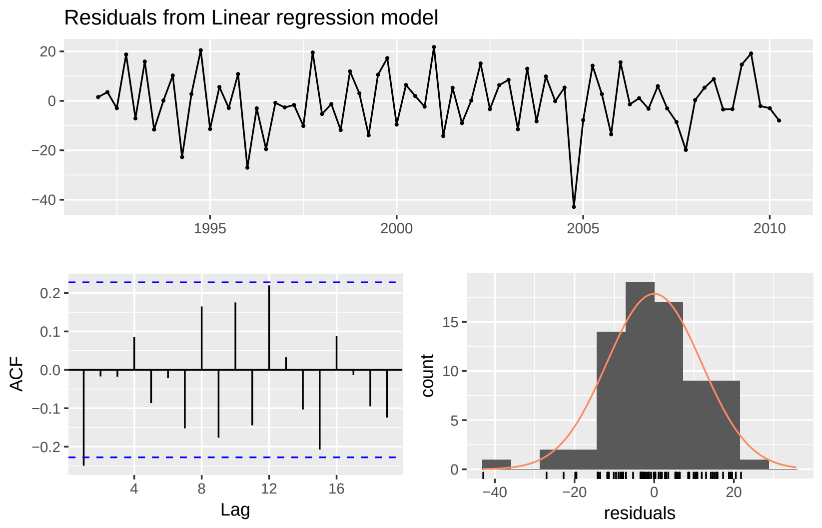
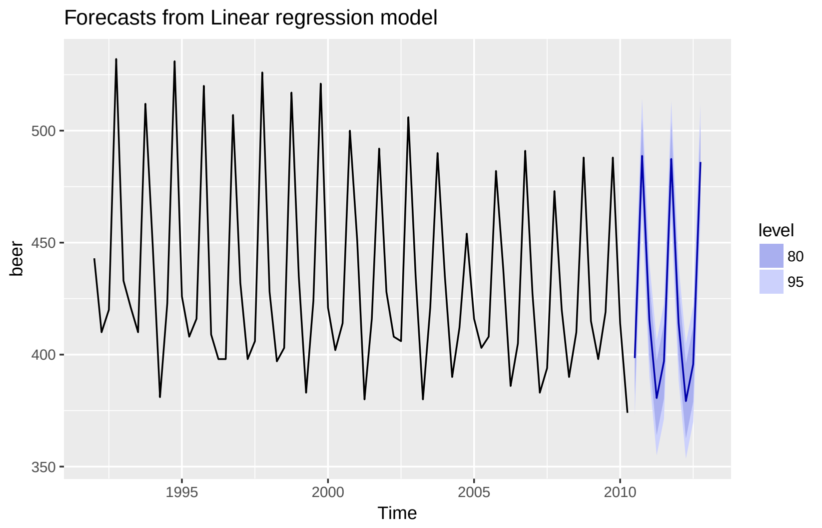


Beer production revisited



Beer production revisited



Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K} [\alpha_k s_k(t) + \beta_k c_k(t)] + e_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.
- Called "harmonic regression"

```
fit <- tslm(y ~ trend + fourier(y, K))
```

Harmonic regression: beer production

```r
fourier.beer <- tslm(beer ~ trend + fourier(beer, K=2))
summary(fourier.beer)
```

```
##
## Call:
## tslm(formula = beer ~ trend + fourier(beer, K = 2))
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -42.903  -7.599  -0.459   7.991  21.789
##
## Coefficients:
##                          Estimate Std. Error t value Pr(>|t|)
## (Intercept)             446.87920    2.87321 155.533  < 2e-16 ***
## trend                    -0.34027    0.06657  -5.111 2.73e-06 ***
## fourier(beer, K = 2)S1-4   8.91082    2.01125   4.430 3.45e-05 ***
## fourier(beer, K = 2)C1-4  53.72807    2.01125  26.714  < 2e-16 ***
## fourier(beer, K = 2)C2-4  13.98958    1.42256   9.834 9.26e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
```

Intervention variables

Spikes

- Equivalent to a dummy variable for handling an outlier.

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Steps

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Steps

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Change of slope

- Variables take values 0 before the intervention and values $\{1, 2, 3, \ldots\}$ afterwards.

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese new year similar.

Trading days

With monthly data, if the observations vary depending on how many different types of days in the month, then trading day predictors can be useful.

$$
\begin{aligned}
z_1 &= \text{\# Mondays in month;} \\
z_2 &= \text{\# Tuesdays in month;} \\
&\vdots \\
z_7 &= \text{\# Sundays in month.}
\end{aligned}
$$

Distributed lags

Lagged values of a predictor.

Example: x is advertising which has a delayed effect

$$
\begin{aligned}
x_1 &= \text{advertising for previous month;} \\
x_2 &= \text{advertising for two months previously;} \\
&\vdots \\
x_m &= \text{advertising for } m \text{ months previously.}
\end{aligned}
$$

Nonlinear trend

Piecewise linear trend with bend at τ

$$x_{1,t} = t$$

$$x_{2,t} = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

Nonlinear trend

Piecewise linear trend with bend at τ

$$x_{1,t} = t$$

$$x_{2,t} = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

Nonlinear trend

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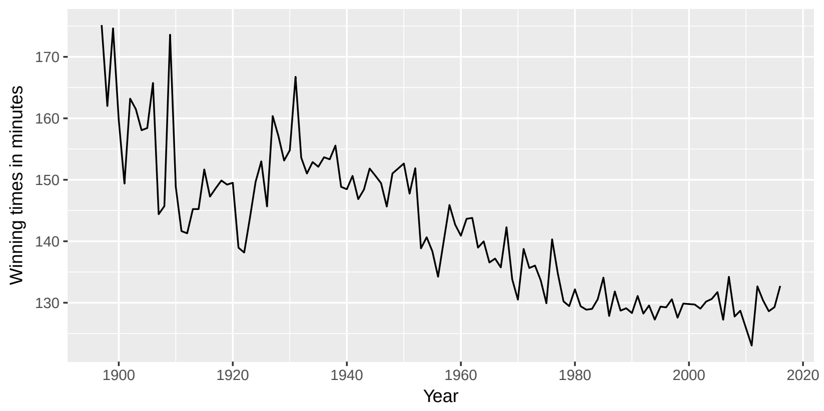
Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \ldots$$

NOT RECOMMENDED!

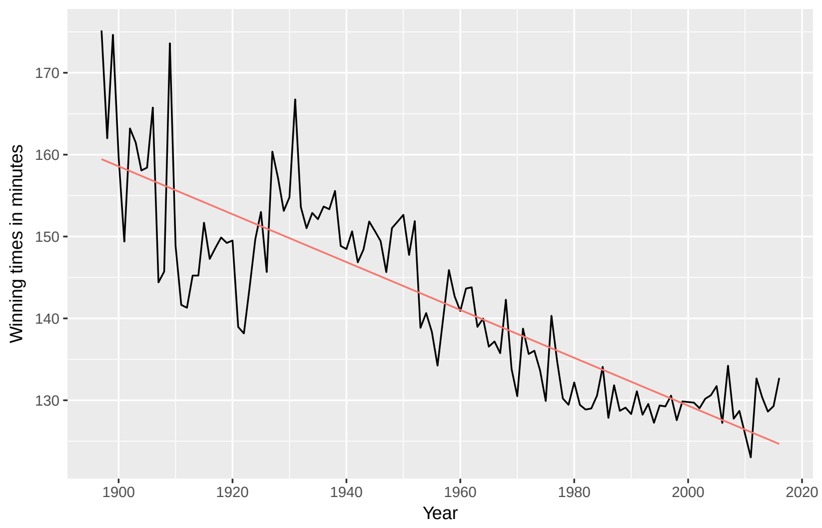
Example: Boston marathon winning times

```r
autoplot(marathon) +
  xlab("Year") +  ylab("Winning times in minutes")
```



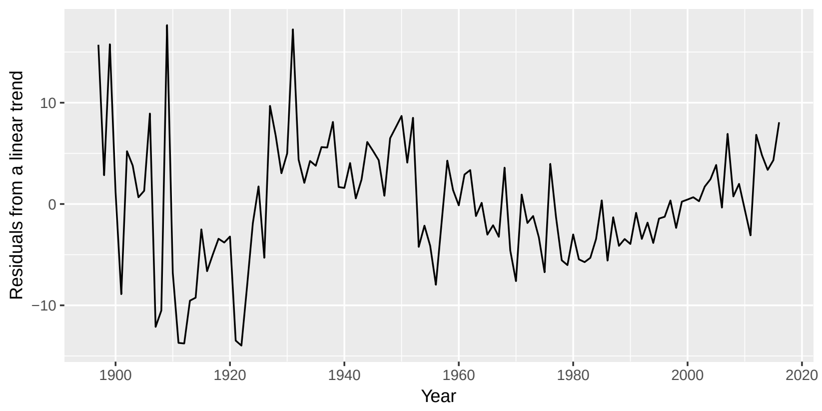

```r
fit.lin <- tslm(marathon ~ trend)
```

Example: Boston marathon winning times



Example: Boston marathon winning times

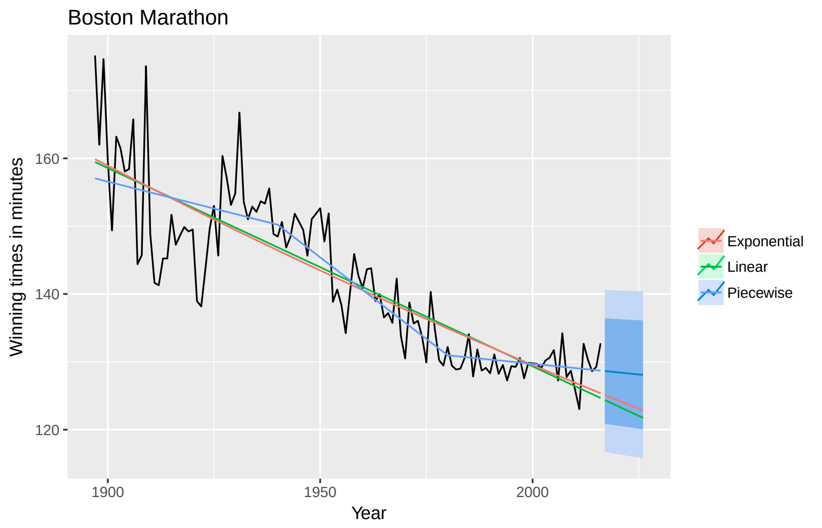
```
autoplot(residuals(fit.lin)) +
    xlab("Year") + ylab("Residuals from a linear trend")
```



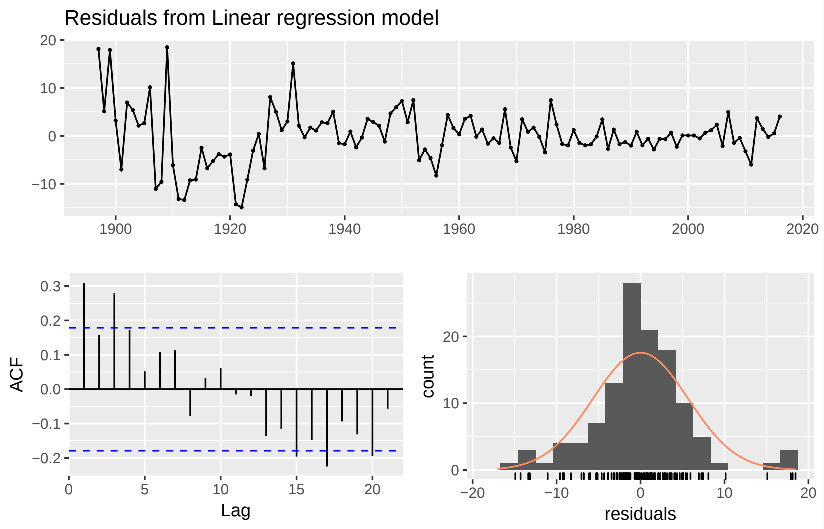
Example: Boston marathon winning times

```r
# Linear trend
fit.lin <- tslm(marathon ~ trend)
fcasts.lin <- forecast(fit.lin, h=10)

# Exponential trend
fit.exp <- tslm(marathon ~ trend, lambda = 0)
fcasts.exp <- forecast(fit.exp, h=10)

# Piecewise linear trend
t.break1 <- 1940
t.break2 <- 1980
t <- time(marathon)
t1 <- ts(pmax(0, t-t.break1), start=1897)
t2 <- ts(pmax(0, t-t.break2), start=1897)
fit.pw <- tslm(marathon ~ t + t1 + t2)
t.new <- t[length(t)] + seq(10)
t1.new <- t1[length(t1)] + seq(10)
t2.new <- t2[length(t2)] + seq(10)
newdata <- cbind(t=t.new, t1=t1.new, t2=t2.new) %>%
  as.data.frame
fcasts.pw <- forecast(fit.pw, newdata = newdata)
```

Example: Boston marathon winning times

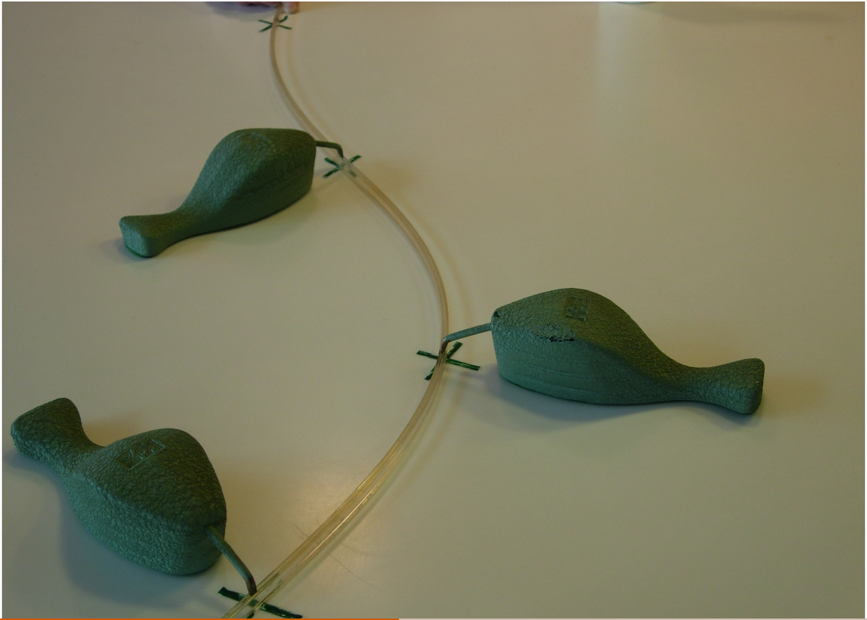


Example: Boston marathon winning times



##

Interpolating splines



Interpolating splines

Interpolating splines

Interpolating splines

A spline is a continuous function $f(x)$ interpolating all points (κ_j, y_j) for $j = 1, \ldots, K$ and consisting of polynomials between each consecutive pair of 'knots' κ_j and κ_{j+1}.

Interpolating splines

A spline is a continuous function $f(x)$ interpolating all points (κ_j, y_j) for $j = 1, \dots, K$ and consisting of polynomials between each consecutive pair of 'knots' κ_j and κ_{j+1}.

- Parameters constrained so that $f(x)$ is continuous.
- Further constraints imposed to give continuous derivatives.

General linear regression splines

- Let $\kappa_1 < \kappa_2 < \cdots < \kappa_K$ be "knots" in interval (a, b).
- Let $x_1 = x$, $x_j = (x - \kappa_{j-1})_+$ for $j = 2, \ldots, K + 1$.
- Then the regression is piecewise linear with bends at the knots.

General cubic splines

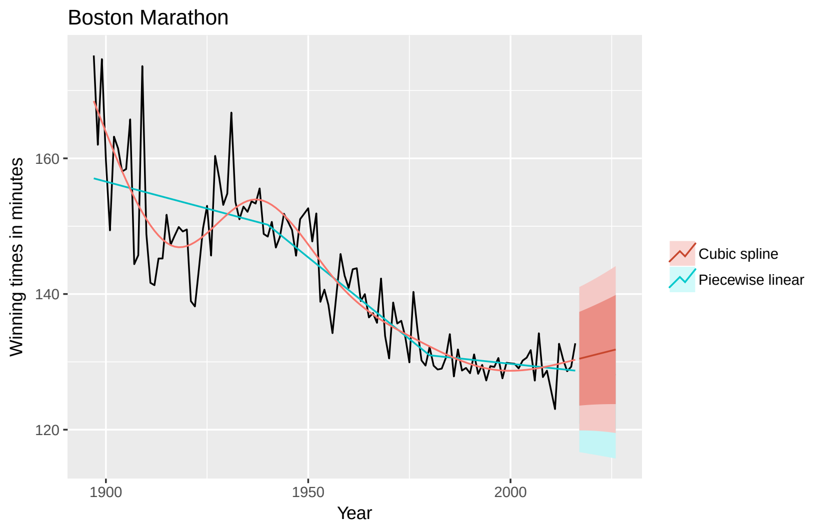
- Let $x_1 = x$, $x_2 = x^2$, $x_3 = x^3$, $x_j = (x - \kappa_{j-3})^3_+$ for $j = 4, \dots, K + 3$.
- Then the regression is piecewise cubic, but smooth at the knots.
- Choice of knots can be difficult and arbitrary.
- Automatic knot selection algorithms very slow.

Example: Boston marathon winning times

```r
# Spline trend
library(splines)
t <- time(marathon)
fit.splines <- lm(marathon ~ ns(t, df=6))
summary(fit.splines)
```

```
##
## Call:
## lm(formula = marathon ~ ns(t, df = 6))
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -13.0028  -2.5722   0.0122   2.1242  21.5681
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)     168.447      2.086  80.743  < 2e-16 ***
## ns(t, df = 6)1   -6.948      2.688  -2.584    0.011 *
## ns(t, df = 6)2  -28.856      3.416  -8.448 1.16e-13 ***
## ns(t, df = 6)3  -35.081      3.045 -11.522  < 2e-16 ***
## ns(t, df = 6)4  -32.563      2.652 -12.279  < 2e-16 ***
## ns(t, df = 6)5  -64.847      5.322 -12.184  < 2e-16 ***
## ns(t, df = 6)6  -21.002      2.403  -8.741 2.46e-14 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.834 on 113 degrees of freedom
```

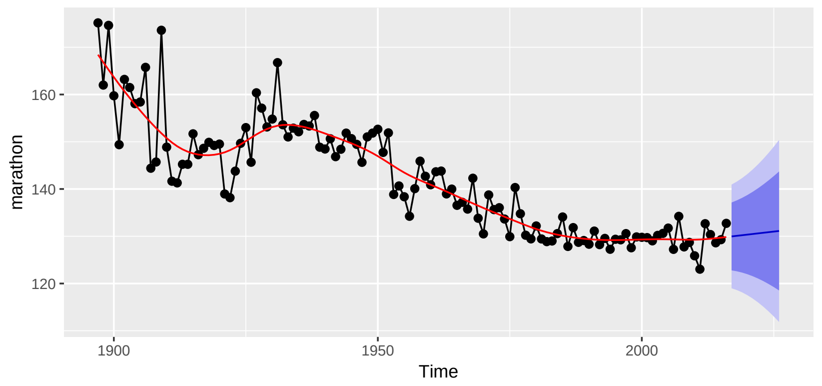
Example: Boston marathon winning times



splinef

A slightly different type of spline is provided by `splinef`

```
fc <- splinef(marathon)
autoplot(fc)
```



splinef

- Cubic **smoothing** splines (rather than cubic regression splines).
- Still piecewise cubic, but with many more knots (one at each observation).
- Coefficients constrained to prevent the curve becoming too “wiggly”.
- Degrees of freedom selected automatically.
- Equivalent to ARIMA(0,2,2) and Holt’s method.

Outline

Multiple regression and forecasting

For forecasting purposes, we require the following assumptions:

- e_t are uncorrelated and zero mean
- e_t are uncorrelated with each $x_{j,t}$.

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- e_t are uncorrelated with each $x_{j,t}$.

It is **useful** to also have $e_t \sim \mathrm{N}(0, \sigma^2)$ when producing prediction intervals or doing statistical tests.

Residual plots

Useful for spotting outliers and whether the linear model was appropriate.

- Scatterplot of residuals e_t against each predictor $x_{j,t}$.
- Scatterplot residuals against the fitted values $\hat{y}_t$
- Expect to see scatterplots resembling a horizontal band with no values too far from the band and no patterns such as curvature or increasing spread.

Residual patterns

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

Breusch-Godfrey test

OLS regression:

$$y_t = \beta_0 + \beta_1 x_{t,1} + \cdots + \beta_k x_{t,k} + u_t$$

Auxiliary regression:

$$\hat{u}_t = \beta_0 + \beta_1 x_{t,1} + \cdots + \beta_k x_{t,k} + \rho_1 \hat{u}_{t-1} + \cdots + \rho_p \hat{u}_{t-p} + \varepsilon_t$$

If R^2 statistic is calculated for this model, then

$$(T - p)R^2 \sim \chi^2_p,$$

when there is no serial correlation up to lag p, and T = length of series.

- Ljung-Box test not recommended for regression models.

Beer production again

```
##
##  Breusch-Godfrey test for serial correlation of order up to 8
##
## data:  Residuals from Linear regression model
## LM test = 9.3083, df = 8, p-value = 0.317
```

If the model fails the Breusch-Godfrey test . . .

- The forecasts are not wrong, but have higher variance than they need to.
- There is information in the residuals that we should exploit.
- This is done with a regression model with ARMA errors which will be covered in week 12.

US consumption again

```
##
##  Breusch-Godfrey test for serial correlation of order up to 8
##
## data:  Residuals from Linear regression model
## LM test = 14.874, df = 8, p-value = 0.06163
```

Outline

Selecting predictors

- When there are many predictors, how should we choose which ones to use?
- We need a way of comparing two competing models.

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- We need a way of comparing two competing models.

What not to do!

- Plot y against a particular predictor (x_j) and if it shows no noticeable relationship, drop it.
- Do a multiple linear regression on all the predictors and disregard all variables whose p values are greater than 0.05.
- Maximize R^2 or minimize MSE

Comparing regression models

Computer output for regression will always give the R^2 value. This is a useful summary of the model.

- It is equal to the square of the correlation between y and $\hat{y}$.
- It is often called the "coefficient of determination".
- It can also be calculated as follows:

$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$

- It is the proportion of variance accounted for (explained) by the predictors.

Comparing regression models

However …

- R^2 does not allow for "degrees of freedom".
- Adding *any* variable tends to increase the value of R^2, even if that variable is irrelevant.

Comparing regression models

However …

- R^2 does not allow for "degrees of freedom".
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To overcome this problem, we can use *adjusted* R^2:

$$\bar{R}^2 = 1 - (1 - R^2)\frac{T - 1}{T - k - 1}$$

where k = no. predictors and T = no. observations.

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Maximizing $\bar{R}^2$ is equivalent to minimizing $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{1}{T-k-1}\sum_{t=1}^{T} e_t^2$$

Cross-validation

Cross-validation for regression

(Assuming future predictors are known)

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

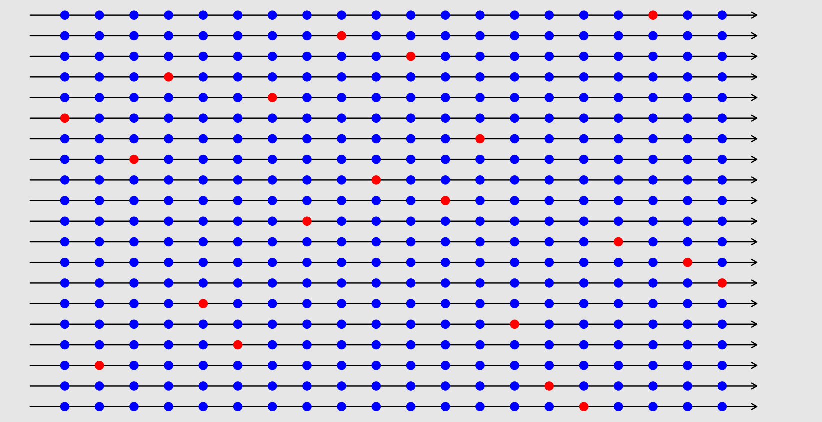
Cross-validation

Traditional evaluation

Cross-validation

Traditional evaluation

Leave-one-out cross-validation



Cross-validation

Leave-one-out cross-validation for regression can be carried out using the following steps.

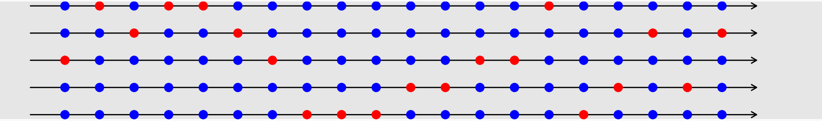
- Remove observation t from the data set, and fit the model using the remaining data. Then compute the error ($e_t^* = y_t - \hat{y}_t$) for the omitted observation.
- Repeat step 1 for $t = 1, \dots, T$.
- Compute the MSE from $\{e_1^*, \dots, e_T^*\}$. We shall call this the CV.

The best model is the one with minimum CV.

Cross-validation

Five-fold cross-validation

- 20 observations. 4 test observations per fold



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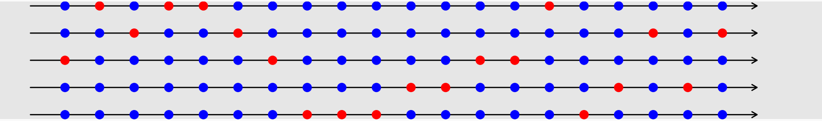
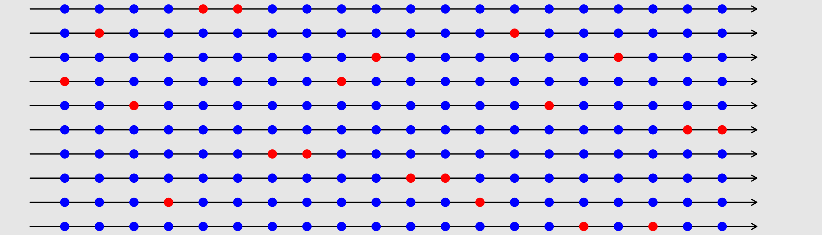


Ten-fold cross-validation

- 20 observations. 2 test observations per fold



Cross-validation

Ten-fold cross-validation

- Randomly split data into 10 parts.
- Select one part for test set, and use remaining parts as training set. Compute accuracy measures on test observations.
- Repeat for each of 10 parts
- Average over all measures.

Akaike's Information Criterion

$$\text{AIC} = -2\log(L) + 2(k+1)$$

where L is the likelihood and k is the number of predictors in the model.

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where L is the likelihood and k is the number of predictors in the model.

- This is a *penalized likelihood* approach.
- *Minimizing* the AIC gives the best model for prediction.
- AIC penalizes terms more heavily than $\bar{R}^2$.
- Minimizing the AIC is asymptotically equivalent to minimizing MSE via leave-one-out cross-validation.

Corrected AIC

For small values of T, the AIC tends to select too many predictors, and so a bias-corrected version of the AIC has been developed.

$$\text{AIC}_{\text{C}} = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-1}$$

As with the AIC, the AIC_{C} should be minimized.

Bayesian Information Criterion

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- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = T[1 - 1/(log(T) - 1)]$.

Choosing regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

Choosing regression variables

Backwards stepwise regression

- Start with a model containing all variables.
- Try subtracting one variable at a time. Keep the model if it has lower CV or AICc.
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Notes

- Stepwise regression is not guaranteed to lead to the best possible model.
- Inference on coefficients of final model will be wrong.

Outline

Ex-ante versus ex-post forecasts

- *Ex ante forecasts* are made using only information available in advance.
  - require forecasts of predictors
- *Ex post forecasts* are made using later information on the predictors.
  - useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

Scenario based forecasting

- Assumes possible scenarios for the predictor variables
- Prediction intervals for scenario based forecasts do not include the uncertainty associated with the future values of the predictor variables.

Building a predictive regression model

- If getting forecasts of predictors is difficult, you can use lagged predictors instead.
$$y_t = \beta_0 + \beta_1 x_{1,t-h} + \cdots + \beta_k x_{k,t-h} + \varepsilon_t$$
- A different model for each forecast horizon h.

Outline

Matrix formulation

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + e_t.$$

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Let $\boldsymbol{y} = (y_1, \dots, y_T)'$, $\boldsymbol{e} = (e_1, \dots, e_T)'$,
$\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$ and

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \dots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \dots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,T} & x_{2,T} & \dots & x_{k,T} \end{bmatrix}.$$

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Then

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}.$$

Matrix formulation

Least squares estimation

Minimize: $(\boldsymbol{y} - \boldsymbol{X}\beta)'(\boldsymbol{y} - \boldsymbol{X}\beta)$

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$$\hat{\sigma}^2 = \frac{1}{T-k-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})$$

Note: If you fall for the dummy variable trap, $(\boldsymbol{X}'\boldsymbol{X})$ is a singular matrix.

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If the errors are iid and normally distributed, then

$$\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X}\beta, \sigma^2\boldsymbol{I}).$$

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So **MLE = OLS**.

Multiple regression forecasts

Optimal forecasts

$$\hat{y}^* = \text{E}(y^*|\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{x}^*) = \boldsymbol{x}^*\hat{\beta} = \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$

where $\boldsymbol{x}^*$ is a row vector containing the values of the regressors for the forecasts (in the same format as $\boldsymbol{X}$).

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- This ignores any errors in $\boldsymbol{x}^*$.
- 95% prediction intervals assuming normal errors:

$$\hat{y}^* \pm 1.96\sqrt{\mathrm{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*)}.$$

Multiple regression forecasts

Fitted values

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\beta} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{H}\boldsymbol{y}$$

where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is the "hat matrix".

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Leave-one-out residuals

Let $h_1, \dots, h_T$ be the diagonal values of $\boldsymbol{H}$, then the cross-validation statistic is

$$\text{CV} = \frac{1}{T}\sum_{t=1}^{T}[e_t/(1-h_t)]^2,$$

where e_t is the residual obtained from fitting the model to all T observations.

Outline

Correlation is not causation

- When x is useful for predicting y, it is not necessarily causing y.
- e.g., predict number of drownings y using number of ice-creams sold x.
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature x and people z to predict drownings y).

Multicollinearity

In regression analysis, multicollinearity occurs when:

- Two predictors are highly correlated (i.e., the correlation between them is close to ± 1).
- A linear combination of some of the predictors is highly correlated with another predictor.
- A linear combination of one subset of predictors is highly correlated with a linear combination of another subset of predictors.

Multicollinearity

If multicollinearity exists. . .

- the numerical estimates of coefficients may be wrong (worse in Excel than in a statistics package)
- don't rely on the *p*-values to determine significance.
- there is no problem with model *predictions* provided the regressors used for forecasting are within the range used for fitting.
- omitting variables can help.
- combining variables can help.

Outliers and influential observations

Things to watch for

- *Outliers*: observations that produce large residuals.
- *Influential observations*: removing them would markedly change the coefficients. (Often outliers in the x variable).
- *Lurking variable*: a predictor not included in the regression but which has an important effect on the response.
- Points should not normally be removed without a good explanation of why they are different.