MONASH University

MONASH BUSINESS SCHOOL

# ETC3550 Applied forecasting for business and economics

## Ch9. ARIMA models

OTexts.org/fpp3/

# Outline


1. Stationarity and differencing
2. Non-seasonal ARIMA models
3. Estimation and order selection
4. ARIMA modelling in R
5. Forecasting
6. Seasonal ARIMA models
7. ARIMA vs ETS

# ARIMA models

| | |
|---|---|
| **AR**: | autoregressive (lagged observations as inputs) |
| **I**: | integrated (differencing to make series stationary) |
| **MA**: | moving average (lagged errors as inputs) |

# ARIMA models

**AR**: autoregressive (lagged observations as inputs)

**I**: integrated (differencing to make series stationary)

**MA**: moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

# Outline


1 Stationarity and differencing

2 Non-seasonal ARIMA models

3 Estimation and order selection

4 ARIMA modelling in R

5 Forecasting

6 Seasonal ARIMA models

7 ARIMA vs ETS

# Stationarity

**Definition**

If $\{y_t\}$ is a stationary time series, then for all $s$, the distribution of $(y_t, \ldots, y_{t+s})$ does not depend on $t$.

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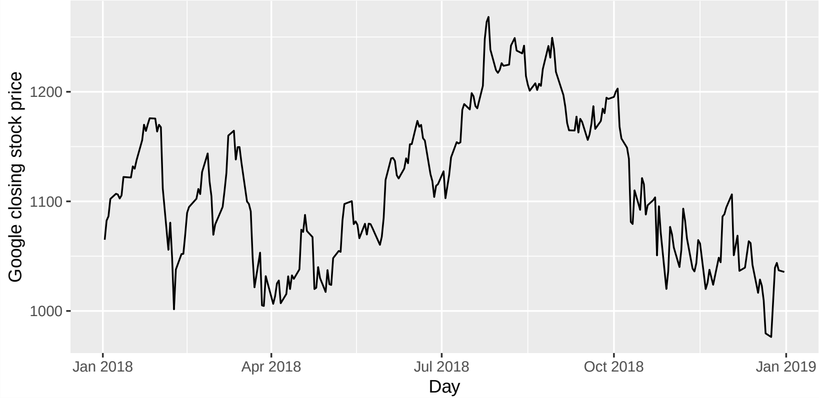
A **stationary series** is:

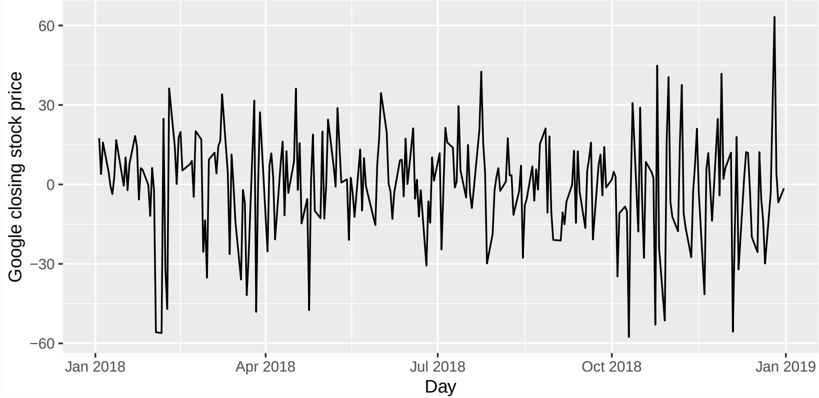
- roughly horizontal
- constant variance
- no patterns predictable in the long-term

# Stationary?



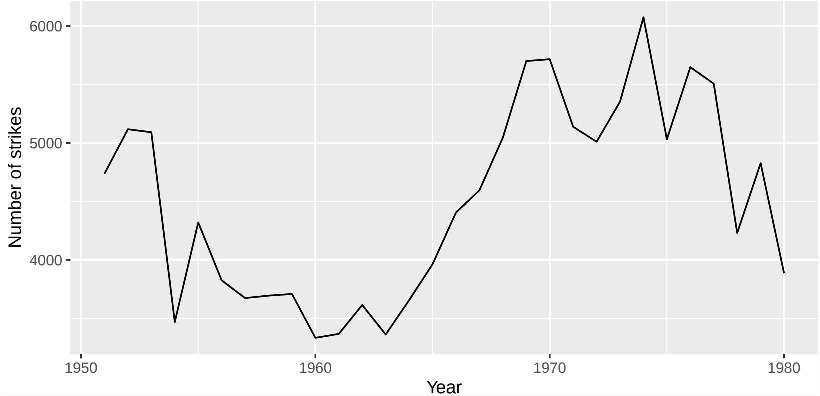
Google closing stock price
1200
1100
1000
Jan 2018
Apr 2018
Jul 2018
Oct 2018
Jan 2019
Day

# Stationary?



60
30
0
−30
−60
Google closing stock price
Jan 2018
Apr 2018
Jul 2018
Oct 2018
Jan 2019
Day

# Stationary?



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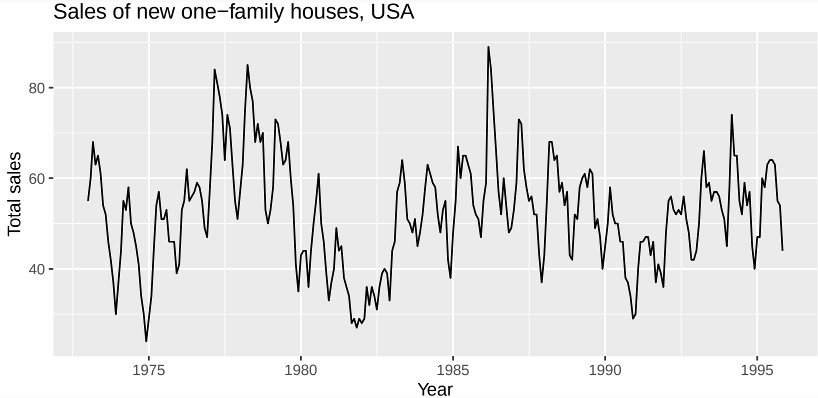
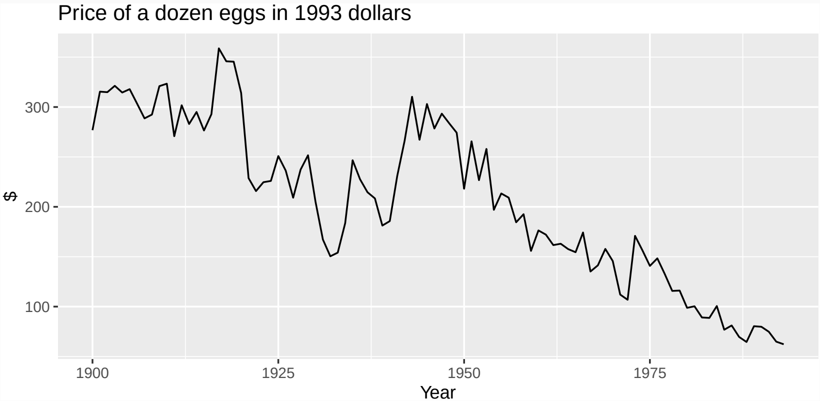


Sales of new one−family houses, USA
80
60
40
Total sales
1975
1980
1985
1990
1995
Year

# Stationary?

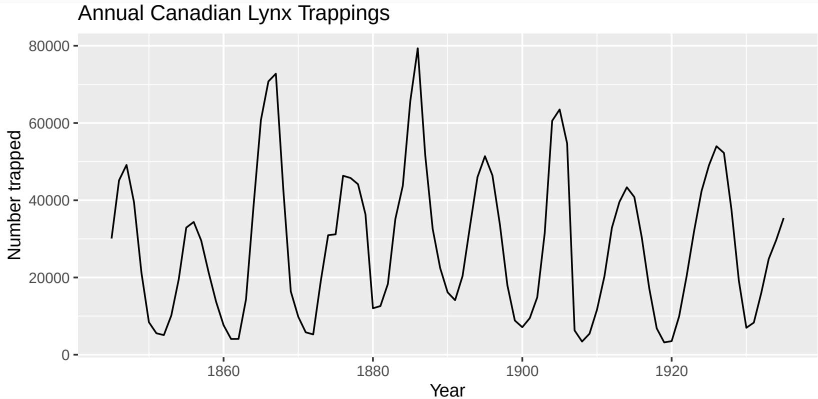


Price of a dozen eggs in 1993 dollars
300
200
100
$
1900
1925
1950
1975
Year

# Stationary?

Number of pigs slaughtered in Victoria
100
80
60
thousands
2010
2012
2014
2016
2018
Year

# Stationary?



Annual Canadian Lynx Trappings
80000
60000
40000
20000
0
Number trapped
1860
1880
1900
1920
Year

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Transformations help to **stabilize the variance**.

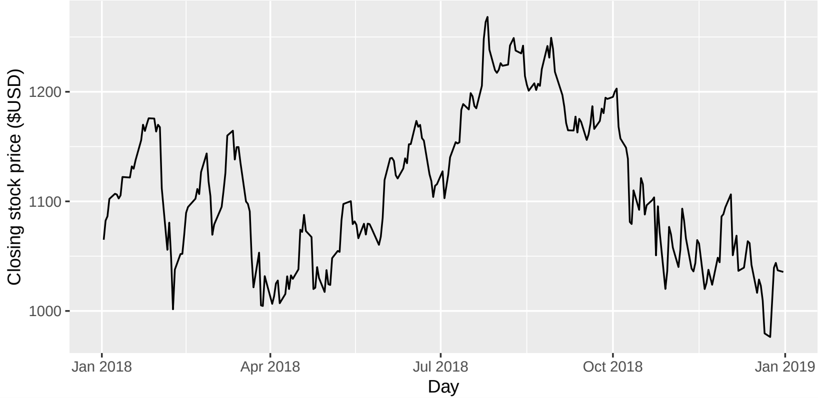
For ARIMA modelling, we also need to **stabilize the mean**.

# Non-stationarity in the mean

## Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of $r_1$ is often large and positive.

# Example: Google stock price



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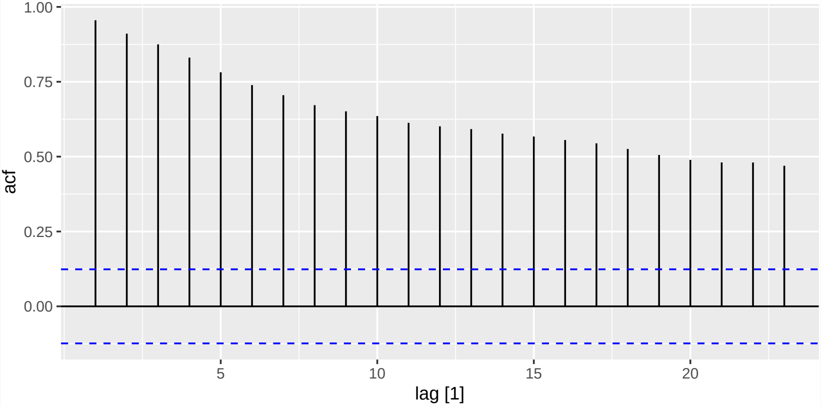

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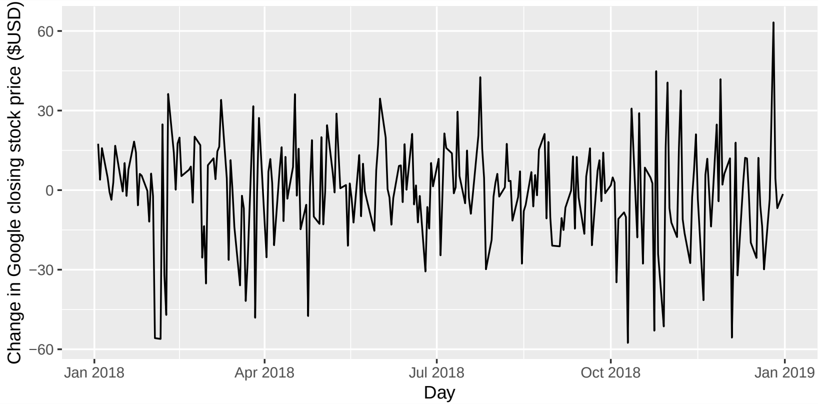

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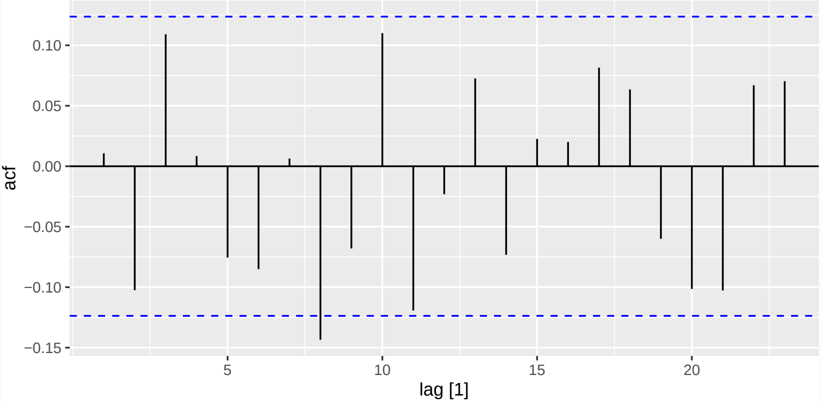

# Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series: $y'_t = y_t - y_{t-1}$.
- The differenced series will have only $T - 1$ values since it is not possible to calculate a difference $y'_1$ for the first observation.

# Second-order differencing

Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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$$\begin{aligned} y''_t &= y'_t - y'_{t-1} \\ &= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\ &= y_t - 2y_{t-1} + y_{t-2}. \end{aligned}$$

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- $y''_t$ will have $T - 2$ values.
- In practice, it is almost never necessary to go beyond second-order differences.

# Seasonal differencing

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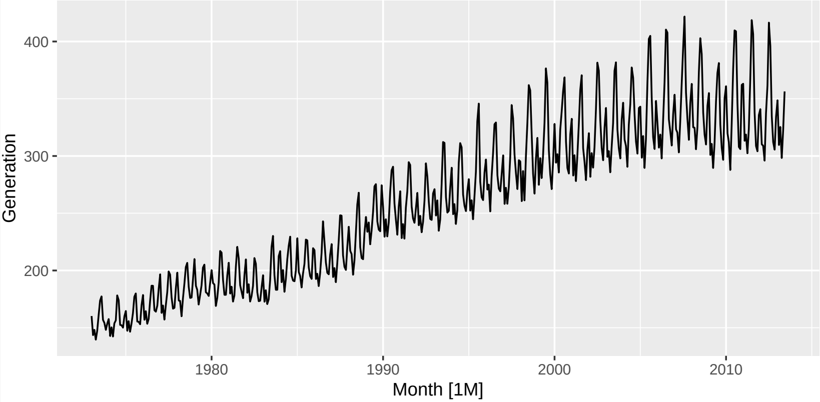
- For monthly data $m = 12$.
- For quarterly data $m = 4$.

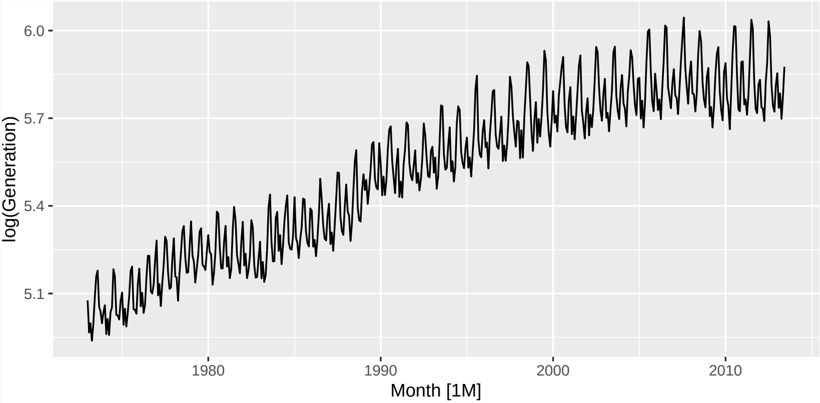
# Electricity production

```
usmelec %>% autoplot(
  Generation
)
```



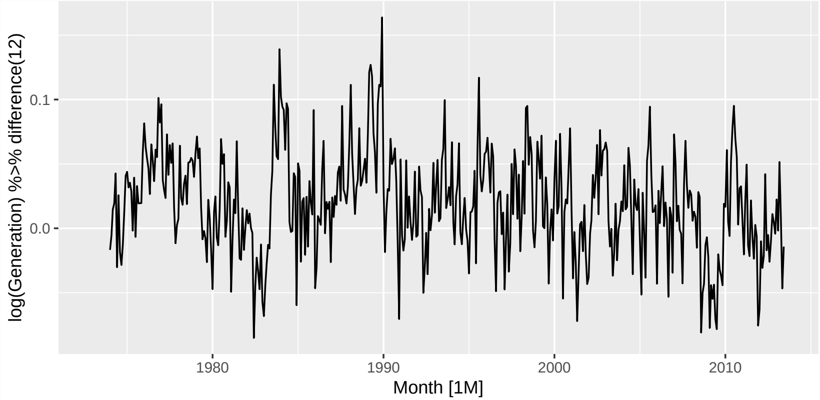
# Electricity production

```
usmelec %>% autoplot(
  log(Generation)
)
```



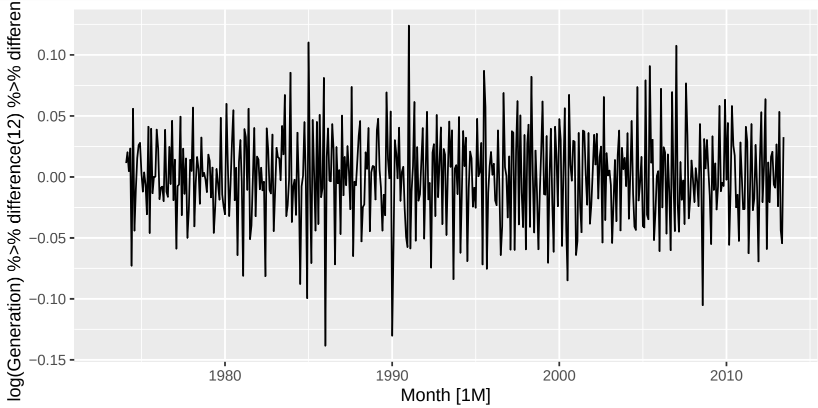
# Electricity production

```
usmelec %>% autoplot(
  log(Generation) %>% difference(12)
)
```



# Electricity production

```
usmelec %>% autoplot(
  log(Generation) %>% difference(12) %>% difference()
)
```



# Electricity production

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If $y'_t = y_t - y_{t-12}$ denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned} y^*_t &= y'_t - y'_{t-1} \\ &= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\ &= y_t - y_{t-1} - y_{t-12} + y_{t-13}\,. \end{aligned}$$

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It is important that if differencing is used, the differences are interpretable.

# Interpretation of differencing

- first differences are the change between **one observation and the next**;
- seasonal differences are the change between **one year to the next**.

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But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

# Unit root tests

**Statistical tests to determine the required order of differencing.**

1. Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
2. Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
3. Other tests available for seasonal data.

# KPSS test

```r
google_2018 %>%
  features(Close, unitroot_kpss)
```

```
## # A tibble: 1 x 3
##   Symbol kpss_stat kpss_pvalue
##   <chr>      <dbl>       <dbl>
## 1 GOOG       0.573      0.0252
```

# KPSS test

```r
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```

```r
google_2018 %>%
  features(Close, unitroot_ndiffs)
```

```
## # A tibble: 1 x 2
##   Symbol ndiffs
##   <chr>   <int>
## 1 GOOG        1
```

# Automatically selecting differences

STL decomposition: $y_t = T_t + S_t + R_t$

Seasonal strength $F_s = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right)$

If $F_s > 0.64$, do one seasonal difference.

```
usmelec %>% mutate(log_gen = log(Generation)) %>%
  features(log_gen, list(unitroot_nsdiffs, feat_stl))
```

```
## # A tibble: 1 x 10
##   nsdiffs trend_strength seasonal_streng~ seasonal_peak_y~
##     <int>          <dbl>            <dbl>            <dbl>
## 1       1          0.994            0.941                7
## # ... with 6 more variables: seasonal_trough_year <dbl>,
## #   spikiness <dbl>, linearity <dbl>, curvature <dbl>,
## #   stl_e_acf1 <dbl>, stl_e_acf10 <dbl>
```

# Automatically selecting differences

```
usmelec %>% mutate(log_gen = log(Generation)) %>%
  features(log_gen, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1
##   nsdiffs
##     <int>
## 1       1
```

```
usmelec %>% mutate(d_log_gen = difference(log(Generation), 12)) %>%
  features(d_log_gen, unitroot_ndiffs)
```

```
## # A tibble: 1 x 1
##   ndiffs
##    <int>
## 1      1
```

# Your turn

For the `tourism` dataset, compute the total number of trips and find an appropriate differencing (after transformation if necessary) to obtain stationary data.

# Backshift notation

A very useful notational device is the backward shift operator, $B$, which is used as follows:

$$By_t = y_{t-1}$$

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$$B(By_t) = B^2 y_t = y_{t-2}$$

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$$B(By_t) = B^2 y_t = y_{t-2}$$

For monthly data, if we wish to shift attention to "the same month last year", then $B^{12}$ is used, and the notation is $B^{12} y_t = y_{t-12}$.

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Note that a first difference is represented by $(1 - B)$.

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$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t$$

Note that a first difference is represented by $(1 - B)$.

Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t$$

# Backshift notation

- Second-order difference is denoted $(1 - B)^2$.
- *Second-order difference* is not the same as a *second difference*, which would be denoted $1 - B^2$;
- In general, a $d$th-order difference can be written as
$$(1 - B)^d y_t$$
- A seasonal difference followed by a first difference can be written as
$$(1 - B)(1 - B^m) y_t$$

# Backshift notation

The "backshift" notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1-B)(1-B^m)y_t &= (1-B-B^m+B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

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For monthly data, $m = 12$ and we obtain the same result as earlier.

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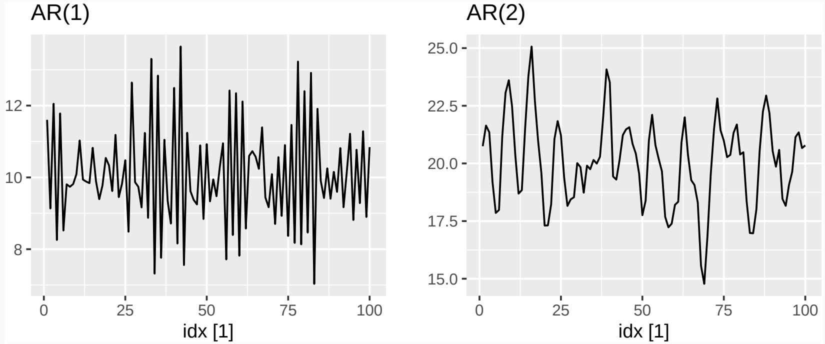
# Autoregressive models

## Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

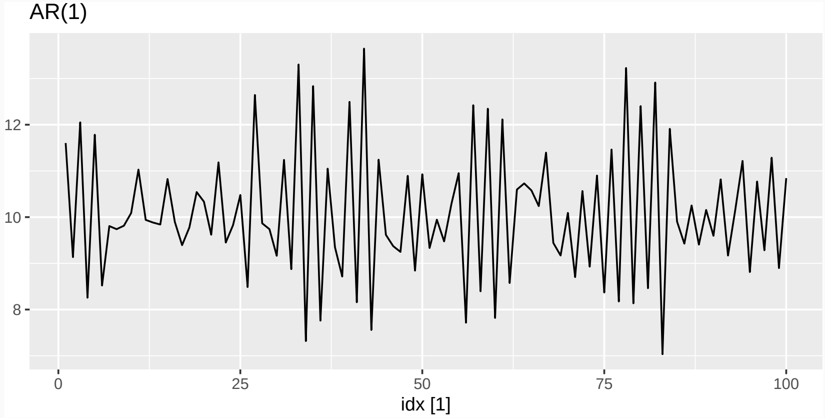
where $\varepsilon_t$ is white noise. This is a multiple regression with **lagged values** of $y_t$ as predictors.



# AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# AR(1) model

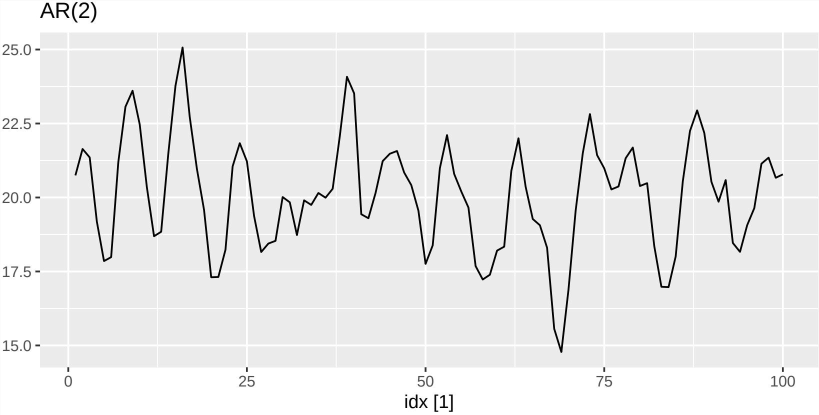
$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

- When $\phi_1 = 0$, $y_t$ is **equivalent to WN**
- When $\phi_1 = 1$ and $c = 0$, $y_t$ is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, $y_t$ is **equivalent to a RW with drift**
- When $\phi_1 < 0$, $y_t$ tends to **oscillate between positive and negative values**.

# AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \qquad T = 100.$$



# Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

## General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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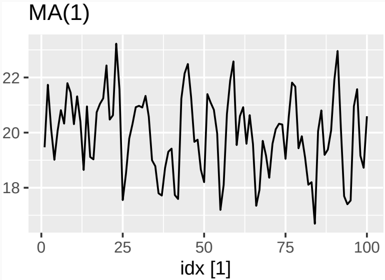
- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1$.
- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

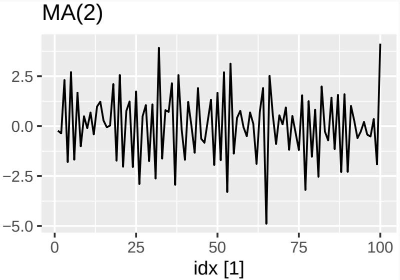
# Moving Average (MA) models

## Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q},$$

where $\varepsilon_t$ is white noise. This is a multiple regression with ***past errors*** as predictors. *Don't confuse this with moving average smoothing!*

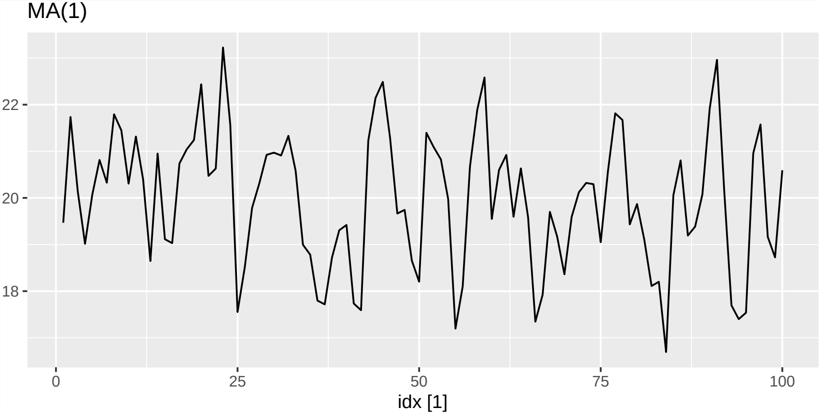


# MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

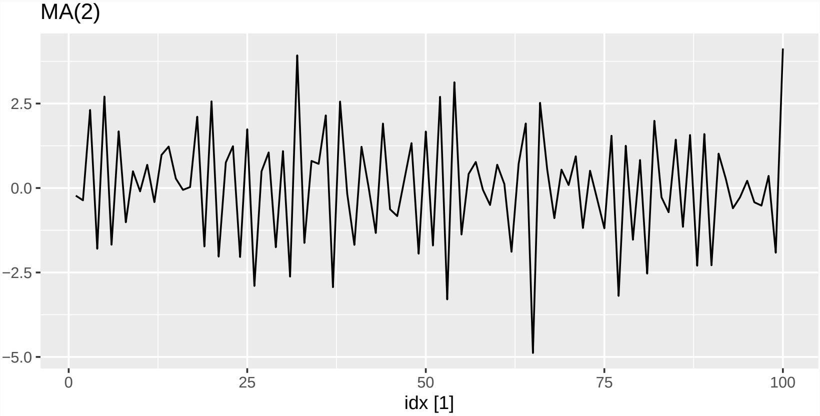
$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0,1), \quad T = 100.$$



# MA($\infty$) models

It is possible to write any stationary AR($p$) process as an MA($\infty$) process.

**Example: AR(1)**

$$\begin{aligned}
y_t &= \phi_1 y_{t-1} + \varepsilon_t \\
&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\
&\dots
\end{aligned}$$

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Provided $-1 < \phi_1 < 1$:

$$y_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \cdots$$

# Invertibility

- Any MA($q$) process can be written as an AR($\infty$) process if we impose some constraints on the MA parameters.
- Then the MA model is called "invertible".
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

# Invertibility

**General condition for invertibility**

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

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# ARIMA models

## Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

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## Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of $y_t$ and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

# ARIMA models

**Autoregressive Moving Average models:**

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of $y_t$ and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

**Autoregressive Integrated Moving Average models**

- Combine ARMA model with **differencing**.
- $(1 - B)^d y_t$ follows an ARMA model.

# ARIMA models

## Autoregressive Integrated Moving Average models

**ARIMA($p, d, q$) model**

AR: $p$ = order of the autoregressive part

I: $d$ = degree of first differencing involved

MA: $q$ = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR($p$): ARIMA($p$,0,0)
- MA($q$): ARIMA(0,0,$q$)

# Backshift notation for ARIMA

- ARMA model:

$$
\begin{aligned}
& y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t \\
\text{or} \quad & (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q) \varepsilon_t
\end{aligned}
$$

- ARIMA(1,1,1) model:

$$
\begin{array}{ccccc}
(1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\
\uparrow & \uparrow & & \uparrow \\
\text{AR(1)} & \text{First} & & \text{MA(1)} \\
 & \text{difference} & & 
\end{array}
$$

# Backshift notation for ARIMA

- ARMA model:

$$y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \cdots + \theta_q B^q \varepsilon_t$$

$$\text{or} \quad (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- ARIMA(1,1,1) model:

$$\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B)} y_t \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B)} \varepsilon_t$$

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

# R model

**Intercept form**

$$(1 - \phi_1 B - \cdots - \phi_p B^p) y_t' = c + (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

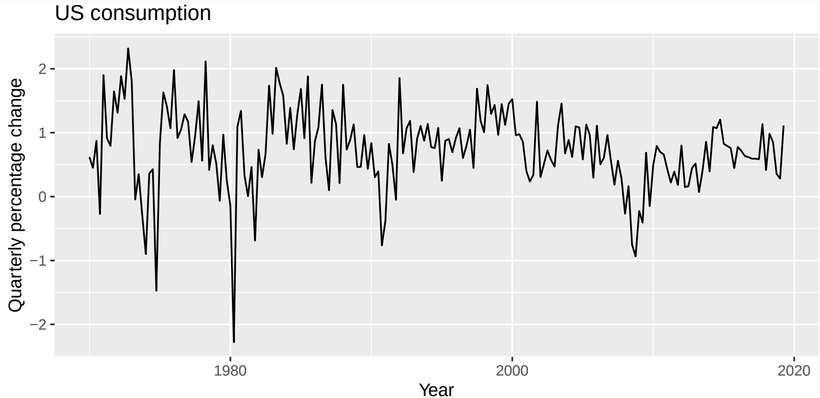
**Mean form**

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(y_t' - \mu) = (1 + \theta_1 B + \cdots + \theta_q B^q)\varepsilon_t$$

- $y_t' = (1 - B)^d y_t$
- $\mu$ is the mean of $y_t'$.
- $c = \mu(1 - \phi_1 - \cdots - \phi_p)$.
- fable uses intercept form

# US consumption expenditure

```
us_change %>% autoplot(Consumption) +
  xlab("Year") +
  ylab("Quarterly percentage change") +
  ggtitle("US consumption")
```



# US personal consumption

```r
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))
report(fit)
```

```
## Series: Consumption
## Model: ARIMA(1,0,3) w/ mean
##
## Coefficients:
##          ar1      ma1     ma2     ma3  constant
##       0.5731  -0.3617  0.0925  0.1934    0.3160
## s.e.  0.1503   0.1607  0.0787  0.0824    0.0371
##
## sigma^2 estimated as 0.3334:  log likelihood=-169.9
## AIC=351.8   AICc=352.2   BIC=371.5
```

# US personal consumption

```r
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))
report(fit)
```

```
## Series: Consumption
## Model: ARIMA(1,0,3) w/ mean
##
## Coefficients:
##          ar1      ma1     ma2     ma3  constant
##       0.5731  -0.3617  0.0925  0.1934    0.3160
## s.e.  0.1503   0.1607  0.0787  0.0824    0.0371
##
## sigma^2 estimated as 0.3334:  log likelihood=-169.9
## AIC=351.8   AICc=352.2   BIC=371.5
```

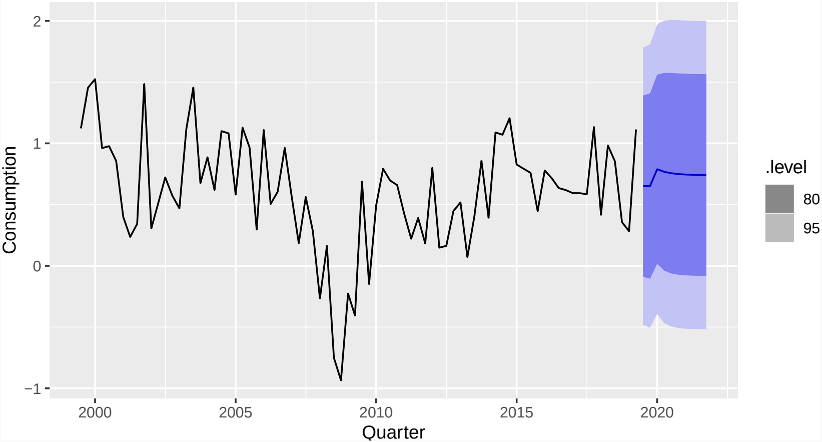
## ARIMA(1,0,3) model:

$$y_t = 0.316 + 0.573 y_{t-1} - 0.362\varepsilon_{t-1} + 0.0925\varepsilon_{t-2} + 0.193\varepsilon_{t-3} + \varepsilon_t,$$

where $\varepsilon_t$ is white noise with a standard deviation of $0.577 = \sqrt{0.333}$.

# US personal consumption

```
fit %>% forecast(h=10) %>%
  autoplot(tail(us_change, 80))
```



# Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

# Understanding ARIMA models

## Forecast variance and $d$

- The higher the value of $d$, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

## Cyclic behaviour

- For cyclic forecasts, $p \geq 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length
$$(2\pi)/\left[\text{arc cos}(-\phi_1(1-\phi_2)/(4\phi_2))\right].$$

# Outline

1 Stationarity and differencing

2 Non-seasonal ARIMA models

3 Estimation and order selection

4 ARIMA modelling in R

5 Forecasting

6 Seasonal ARIMA models

7 ARIMA vs ETS

# Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c$, $\phi_1, \dots, \phi_p$, $\theta_1, \dots, \theta_q$.

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- MLE is very similar to least squares estimation obtained by minimizing
$$\sum_{t-1}^{T} e_t^2$$
- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

# Partial autocorrelations

**Partial autocorrelations** measure relationship between $y_t$ and $y_{t-k}$, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

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$$\begin{aligned}\alpha_k &= k\text{th partial autocorrelation coefficient}\\ &= \text{equal to the estimate of } \phi_k \text{ in regression:}\\ &\qquad y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.\end{aligned}$$

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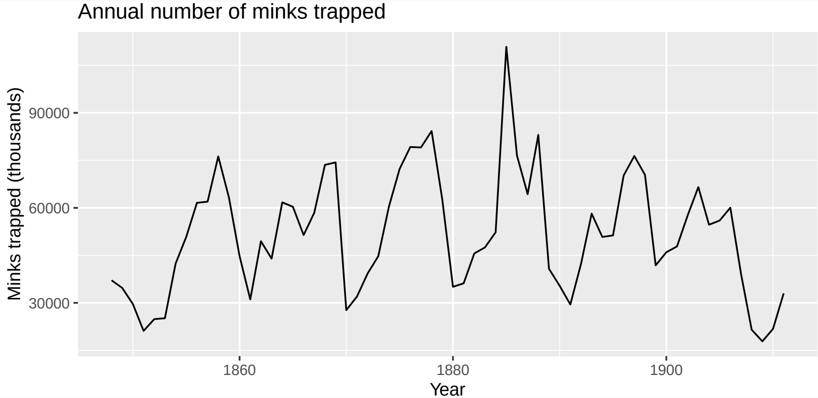
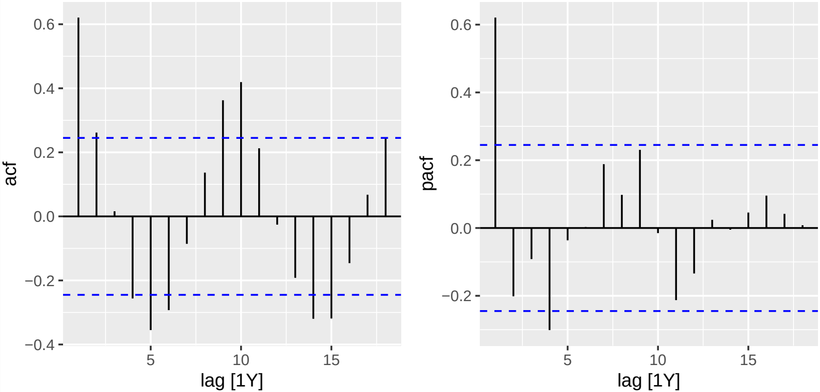
- Varying number of terms on RHS gives $\alpha_k$ for different values of $k$.
- $\alpha_1 = \rho_1$
- same critical values of $\pm 1.96/\sqrt{T}$ as for ACF.
- Last significant $\alpha_k$ indicates the order of an AR model.

# Example: Mink trapping

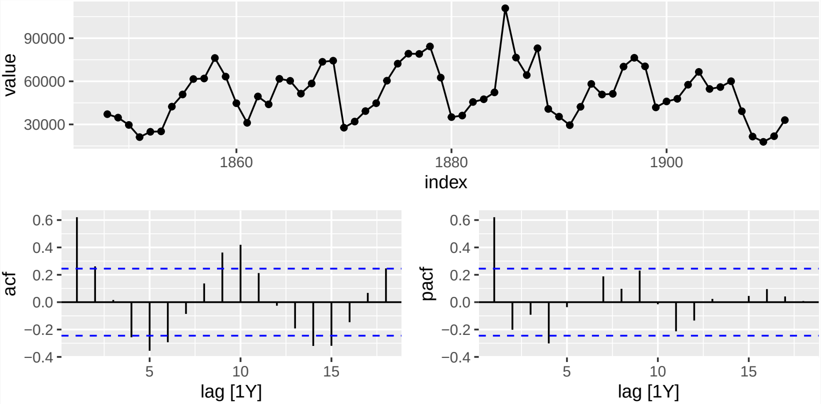


# Example: Mink trapping



# Example: Mink trapping

```r
mink %>% gg_tsdisplay(value, plot_type='partial')
```



# ACF and PACF interpretation

**AR(1)**

$$\rho_k = \phi_1^k \qquad \text{for } k = 1, 2, \ldots;$$

$$\alpha_1 = \phi_1 \qquad \alpha_k = 0 \qquad \text{for } k = 2, 3, \ldots.$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

# ACF and PACF interpretation

**AR($p$)**

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the $p$th spike

So we have an AR($p$) model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag $p$ in PACF, but none beyond $p$

# ACF and PACF interpretation

**MA(1)**

$$\rho_1 = \theta_1 \qquad \rho_k = 0 \qquad \text{for } k = 2, 3, \dots;$$
$$\alpha_k = -(-\theta_1)^k$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

# ACF and PACF interpretation

**MA($q$)**

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the $q$th spike

So we have an MA($q$) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag $q$ in ACF, but none beyond $q$

# Information criteria

**Akaike's Information Criterion (AIC):**

$$\text{AIC} = -2\log(L) + 2(p+q+k+1),$$

where $L$ is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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**Corrected AIC:**

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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**Bayesian Information Criterion:**

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

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**Bayesian Information Criterion:**

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p+q+k+1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

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# How does ARIMA() work?

**A non-seasonal ARIMA process**

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: $p, q, d$

**Hyndman and Khandakar (JSS, 2008) algorithm:**

- Select no. differences $d$ and $D$ via KPSS test and seasonal strength measure.
- Select $p, q$ by minimising AICc.
- Use stepwise search to traverse model space.

# How does ARIMA() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where $L$ is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

# How does ARIMA() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where $L$ is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

**Step1:** Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

# How does ARIMA() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where $L$ is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

**Step1:** Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

**Step 2:** Consider variations of current model:

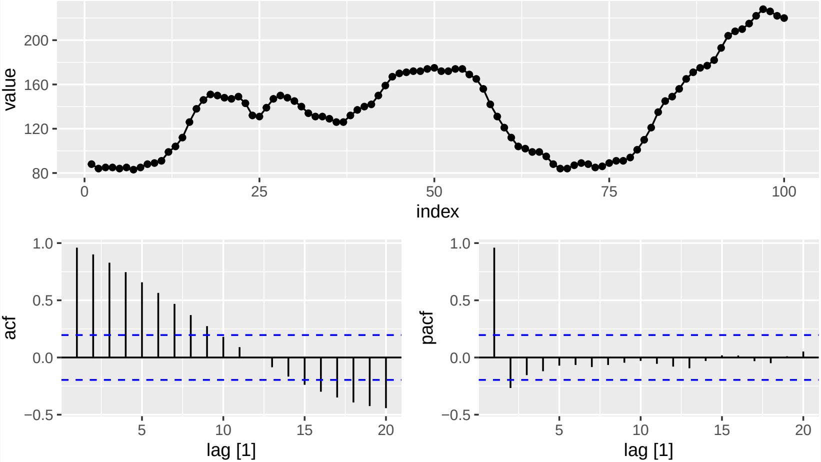
- vary one of $p, q$, from current model by $\pm 1$;
- $p, q$ both vary from current model by $\pm 1$;
- Include/exclude $c$ from current model.

Model with lowest AICc becomes current model.

**Repeat Step 2 until no lower AICc can be found.**

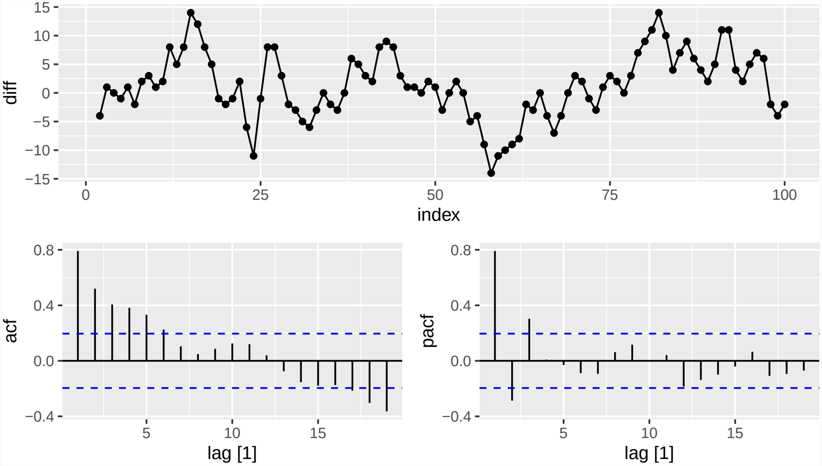
# Choosing your own model

```
web_usage <- as_tsibble(WWWusage)
web_usage %>% gg_tsdisplay(value, plot_type = 'partial')
```



# Choosing your own model

```
web_usage %>% mutate(diff = difference(value)) %>%
  gg_tsdisplay(diff, plot_type = 'partial')
```



# Choosing your own model

```r
fit <- web_usage %>%
  model(arima = ARIMA(value ~ pdq(3, 1, 0)))
report(fit)
```

```
## Series: value
## Model: ARIMA(3,1,0)
##
## Coefficients:
##          ar1      ar2     ar3
##        1.151  -0.6612  0.3407
## s.e.   0.095   0.1353  0.0941
##
## sigma^2 estimated as 9.656:  log likelihood=-252
## AIC=512   AICc=512.4   BIC=522.4
```

# Choosing your own model

```
web_usage %>%
  model(ARIMA(value ~ pdq(d=1))) %>%
report()
```

```
## Series: value
## Model: ARIMA(1,1,1)
##
## Coefficients:
##          ar1     ma1
##       0.6504  0.5256
## s.e.  0.0842  0.0896
##
## sigma^2 estimated as 9.995:  log likelihood=-254.2
## AIC=514.3   AICc=514.5   BIC=522.1
```

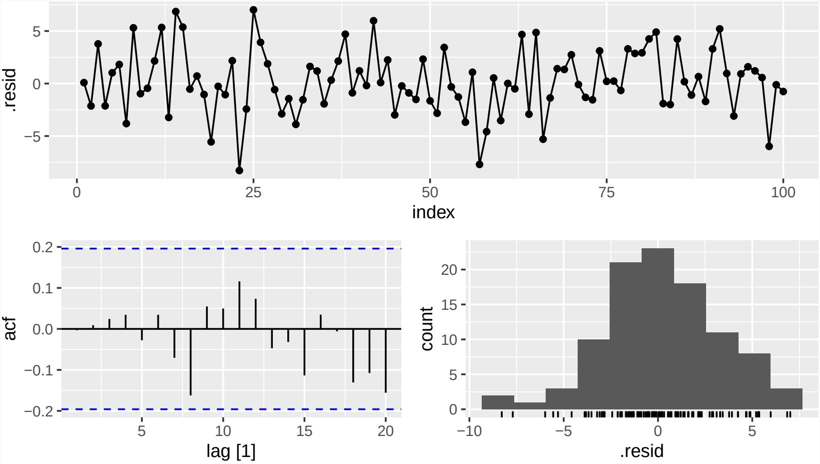
# Choosing your own model

```r
web_usage %>%
  model(ARIMA(value ~ pdq(d=1),
    stepwise = FALSE, approximation = FALSE)) %>%
  report()
```

```
## Series: value
## Model: ARIMA(3,1,0)
##
## Coefficients:
##          ar1      ar2     ar3
##        1.151  -0.6612  0.3407
## s.e.   0.095   0.1353  0.0941
##
## sigma^2 estimated as 9.656:  log likelihood=-252
## AIC=512   AICc=512.4   BIC=522.4
```

# Choosing your own model

```
gg_tsresiduals(fit)
```



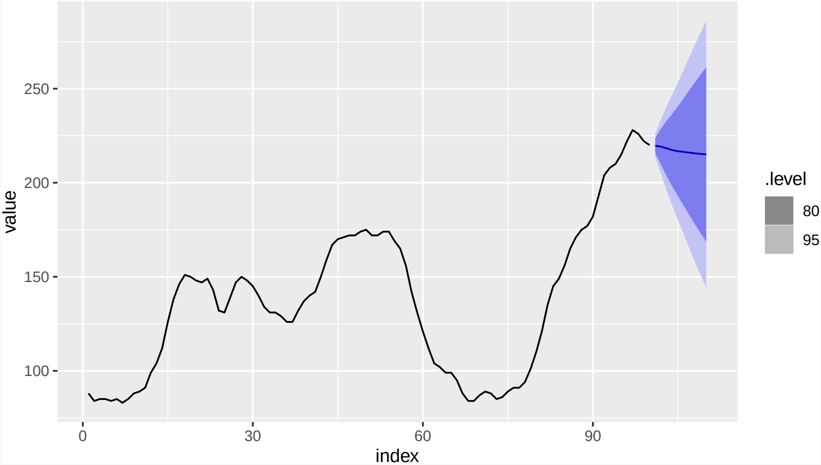
# Choosing your own model

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 10, dof = 3)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 arima     4.49     0.722
```

# Choosing your own model

```
fit %>% forecast(h = 10) %>%
  autoplot(web_usage)
```



# Modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. If the data are non-stationary: take first differences of the data until the data are stationary.
4. Examine the ACF/PACF: Is an AR($p$) or MA($q$) model appropriate?
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

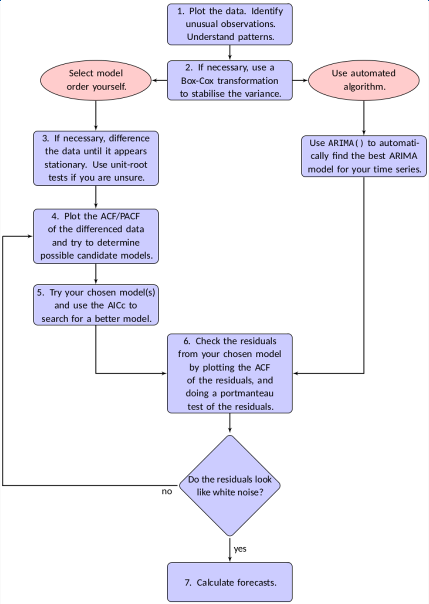
# Automatic modelling procedure with `ARIMA()`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.

3. Use `ARIMA` to automatically select a model.

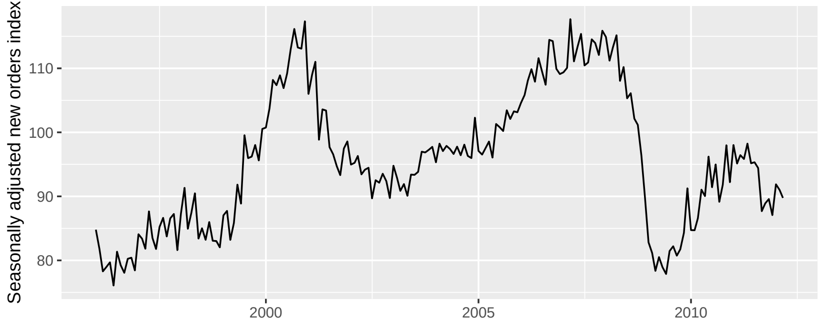
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

# Modelling procedure



# Seasonally adjusted electrical equipment

```
elecequip <- as_tsibble(fpp2::elecequip)
dcmp <- elecequip %>%
  model(STL(value ~ season(window = "periodic"))) %>%
  components() %>% select(-.model)
dcmp %>% as_tsibble %>%
  autoplot(season_adjust) + xlab("Year") +
  ylab("Seasonally adjusted new orders index")
```

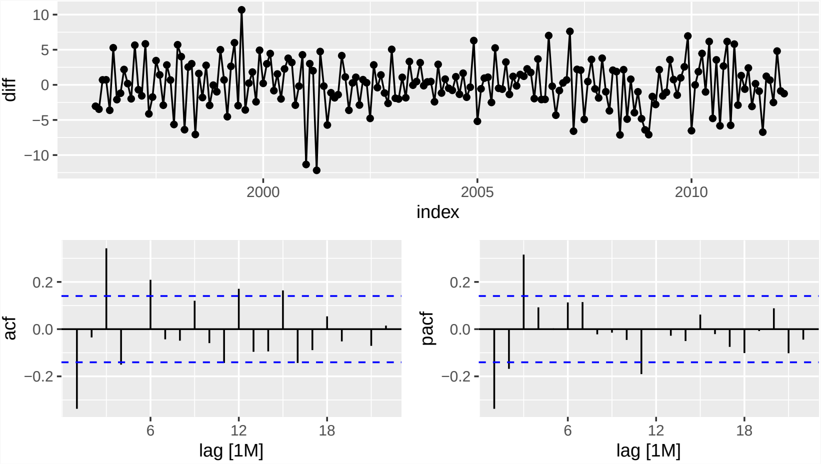


# Seasonally adjusted electrical equipment

1. Time plot shows sudden changes, particularly big drop in 2008/2009 due to global economic environment. Otherwise nothing unusual and no need for data adjustments.
2. No evidence of changing variance, so no Box-Cox transformation.
3. Data are clearly non-stationary, so we take first differences.

# Seasonally adjusted electrical equipment

```
dcmp %>% mutate(diff = difference(season_adjust)) %>%
  gg_tsdisplay(diff, plot_type = 'partial')
```



# Seasonally adjusted electrical equipment

4. PACF is suggestive of AR(3). So initial candidate model is ARIMA(3,1,0). No other obvious candidates.
5. Fit ARIMA(3,1,0) model along with variations: ARIMA(4,1,0), ARIMA(2,1,0), ARIMA(3,1,1), etc. ARIMA(3,1,1) has smallest AICc value.

# Seasonally adjusted electrical equipment

```r
fit <- dcmp %>%
  model(arima = ARIMA(season_adjust))
report(fit)
```

```
## Series: season_adjust
## Model: ARIMA(3,1,0)
##
## Coefficients:
##           ar1      ar2     ar3
##       -0.3418  -0.0426  0.3185
## s.e.   0.0681   0.0725  0.0682
##
## sigma^2 estimated as 9.639:  log likelihood=-493.8
## AIC=995.6   AICc=995.8   BIC=1009
```

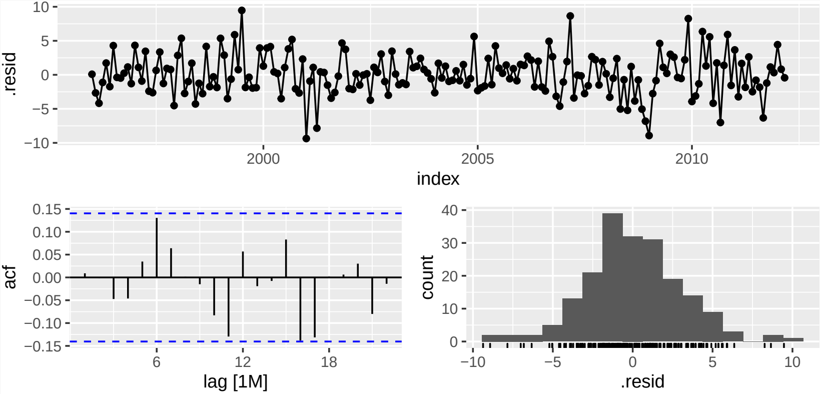
# Seasonally adjusted electrical equipment

```r
fit <- dcmp %>%
  model(arima = ARIMA(season_adjust, approximation=FALSE))
report(fit)
```

```
## Series: season_adjust
## Model: ARIMA(3,1,1)
##
## Coefficients:
##          ar1     ar2     ar3      ma1
##       0.0044  0.0916  0.3698  -0.3921
## s.e.  0.2201  0.0984  0.0669   0.2426
##
## sigma^2 estimated as 9.577:  log likelihood=-492.7
## AIC=995.4   AICc=995.7   BIC=1012
```

# Seasonally adjusted electrical equipment

6. ACF plot of residuals from ARIMA(3,1,1) model look like white noise.

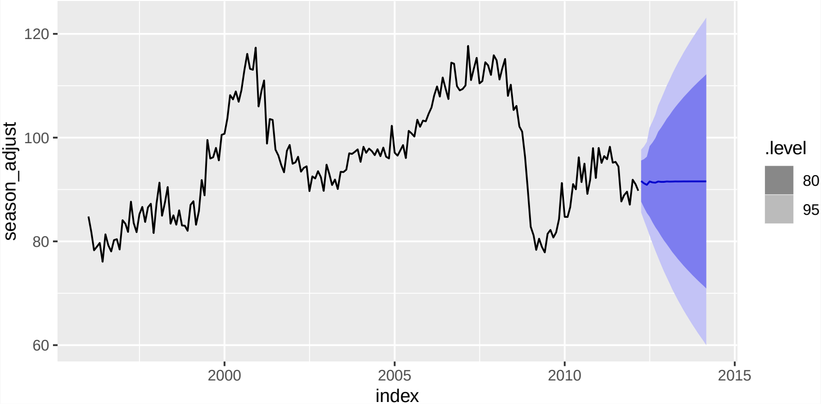


# Seasonally adjusted electrical equipment

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 arima     24.0     0.241
```

# Seasonally adjusted electrical equipment

```r
fit %>% forecast %>% autoplot(dcmp)
```



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1. Stationarity and differencing
2. Non-seasonal ARIMA models
3. Estimation and order selection
4. ARIMA modelling in R
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6. Seasonal ARIMA models
7. ARIMA vs ETS

# Point forecasts

1. Rearrange ARIMA equation so $y_t$ is on LHS.
2. Rewrite equation by replacing $t$ by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \ldots$.

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B)\varepsilon_t,$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1-\phi_1B-\phi_2B^2-\phi_3B^3)(1-B)y_t = (1+\theta_1B)\varepsilon_t,$$

$$\begin{aligned}\left[1-(1+\phi_1)B+(\phi_1-\phi_2)B^2+(\phi_2-\phi_3)B^3+\phi_3B^4\right]y_t \\ = (1+\theta_1B)\varepsilon_t,\end{aligned}$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned}\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B)\varepsilon_t,\end{aligned}$$

$$\begin{aligned}y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}.\end{aligned}$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B) y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\left[1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4\right] y_t \\ = (1 + \theta_1 B)\varepsilon_t,$$

$$y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

# Point forecasts (h=1)

$$\begin{aligned}y_t = (1+\phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.\end{aligned}$$

# Point forecasts (h=1)

$$
\begin{aligned}
y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3} \\
- \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.
\end{aligned}
$$

**ARIMA(3,1,1) forecasts: Step 2**

$$
\begin{aligned}
y_{T+1} = (1+\phi_1)y_T - (\phi_1-\phi_2)y_{T-1} - (\phi_2-\phi_3)y_{T-2} \\
- \phi_3 y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.
\end{aligned}
$$

# Point forecasts (h=1)

$$\begin{aligned} y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

## ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned} y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + \varepsilon_{T+1} + \theta_1 \varepsilon_T. \end{aligned}$$

## ARIMA(3,1,1) forecasts: Step 3

$$\begin{aligned} \hat{y}_{T+1|T} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3 y_{T-3} + \theta_1 e_T. \end{aligned}$$

# Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}.$$

# Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

**ARIMA(3,1,1) forecasts: Step 2**

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_{T} - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3 y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

# Point forecasts (h=2)

$$\begin{aligned}y_t = (1+\phi_1)y_{t-1} - (\phi_1-\phi_2)y_{t-2} - (\phi_2-\phi_3)y_{t-3}\\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.\end{aligned}$$

## ARIMA(3,1,1) forecasts: Step 2

$$\begin{aligned}y_{T+2} = (1+\phi_1)y_{T+1} - (\phi_1-\phi_2)y_{T} - (\phi_2-\phi_3)y_{T-1}\\ - \phi_3 y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.\end{aligned}$$

## ARIMA(3,1,1) forecasts: Step 3

$$\begin{aligned}\hat{y}_{T+2|T} = (1+\phi_1)\hat{y}_{T+1|T} - (\phi_1-\phi_2)y_{T} - (\phi_2-\phi_3)y_{T-1}\\ - \phi_3 y_{T-2}.\end{aligned}$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,$q$):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \dots.$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,$q$):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \ldots.$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,$q$):

$$y_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \ldots.$$

- AR(1): Rewrite as MA($\infty$) and use above result.
- Other models beyond scope of this subject.

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
  - the uncertainty in the parameter estimates has not been accounted for.
  - the ARIMA model assumes historical patterns will not change during the forecast period.
  - the ARIMA model assumes uncorrelated future errors

# Your turn

For the GDP data (from `global_economy`):

- fit a suitable ARIMA model to the logged data for all countries
- check the residual diagnostics for Australia;
- produce forecasts of your fitted model for Australia.

# Outline

1 Stationarity and differencing

2 Non-seasonal ARIMA models

3 Estimation and order selection

4 ARIMA modelling in R

5 Forecasting

6 Seasonal ARIMA models

7 ARIMA vs ETS

# Seasonal ARIMA models

| | | |
|---|---|---|
| ARIMA | $\underbrace{(p, d, q)}$ | $\underbrace{(P, D, Q)_m}$ |
| | $\uparrow$ | $\uparrow$ |
| | Non-seasonal part of the model | Seasonal part of of the model |

where $m$ = number of observations per year.

# Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

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$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

# Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

(Non-seasonal AR(1))

(Seasonal AR(1))

(Non-seasonal difference)

(Seasonal difference)

(Non-seasonal MA(1))

(Seasonal MA(1))

# Seasonal ARIMA models

E.g., ARIMA$(1,1,1)(1,1,1)_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t = (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned}
y_t = {} & (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\
& - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\
& - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\
& + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}.
\end{aligned}$$

# Common ARIMA models

The US Census Bureau uses the following models most often:

| | |
|---|---|
| ARIMA$(0,1,1)(0,1,1)_m$ | with log transformation |
| ARIMA$(0,1,2)(0,1,1)_m$ | with log transformation |
| ARIMA$(2,1,0)(0,1,1)_m$ | with log transformation |
| ARIMA$(0,2,2)(0,1,1)_m$ | with log transformation |
| ARIMA$(2,1,2)(0,1,1)_m$ | with no transformation |

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)$_{12}$ will show:**

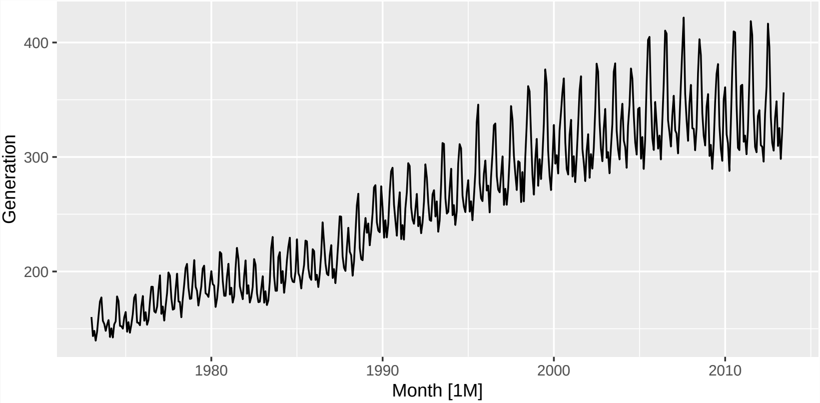
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, . . . .

**ARIMA(0,0,0)(1,0,0)$_{12}$ will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

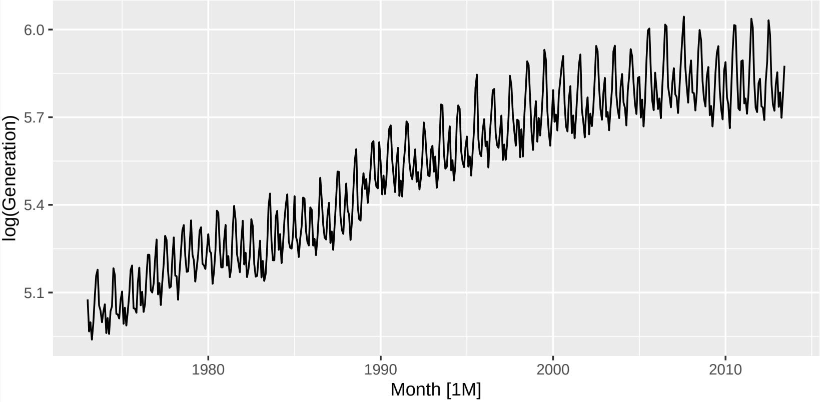
# Example: US electricity production

```r
usmelec %>% autoplot(
  Generation
)
```



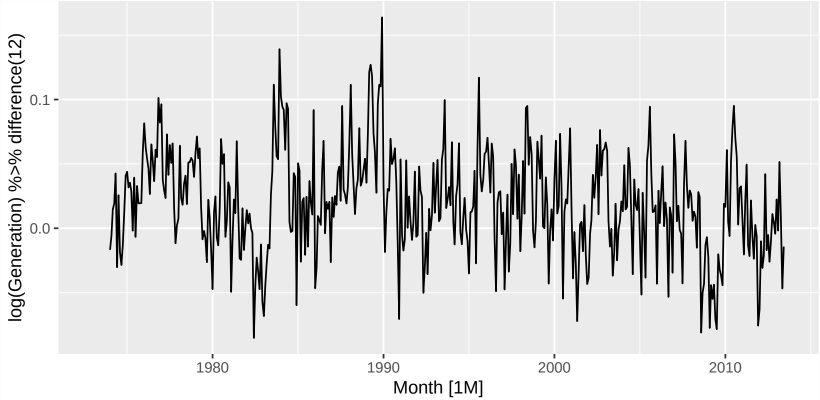
# Example: US electricity production

```r
usmelec %>% autoplot(
  log(Generation)
)
```



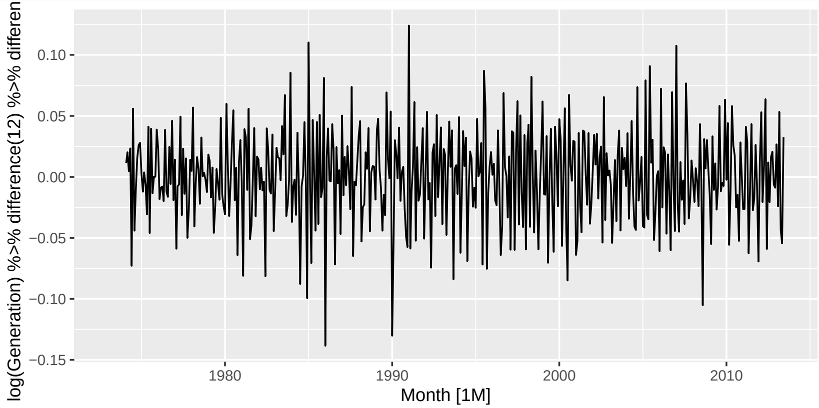
# Example: US electricity production

```
usmelec %>% autoplot(
  log(Generation) %>% difference(12)
)
```



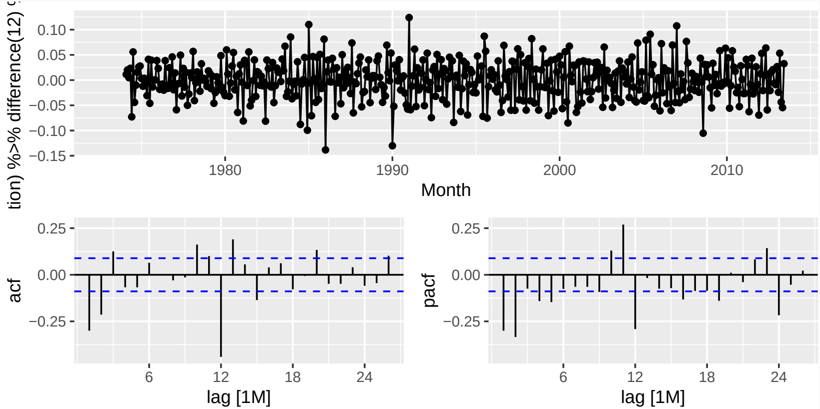
# Example: US electricity production

```
usmelec %>% autoplot(
  log(Generation) %>% difference(12) %>% difference()
)
```



# Example: US electricity production

```
usmelec %>% gg_tsdisplay(
  log(Generation) %>% difference(12) %>% difference(),
  plot_type = "partial")
```



# Example: US electricity production

- $d = 1$ and $D = 1$ seems necessary
- $P = 0$ and $Q = 1$ suggested by seasonal lags
- $p = 0$ and $q = 3$ suggested by non-seasonal lags.

# Example: US electricity production

```r
usmelec %>%
  model(arima = ARIMA(log(Generation) ~ pdq(0,1,3) + PDQ(0,1,1))) %>%
  report()
```

```
## Series: Generation
## Model: ARIMA(0,1,3)(0,1,1)[12]
## Transformation: log(.x)
##
## Coefficients:
##           ma1      ma2      ma3     sma1
##       -0.4266  -0.2496  -0.0439  -0.8358
## s.e.   0.0462   0.0516   0.0430   0.0262
##
## sigma^2 estimated as 0.0006904:  log likelihood=1045
## AIC=-2080   AICc=-2080   BIC=-2059
```

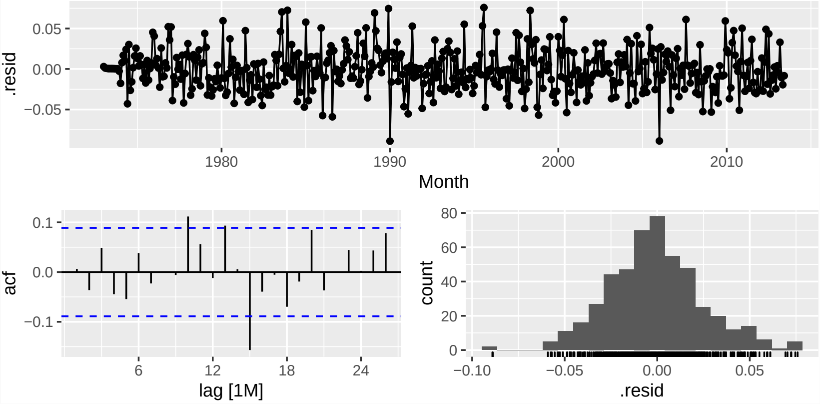
# Example: US electricity production

```
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  report()
```

```
## Series: Generation
## Model: ARIMA(1,1,1)(2,1,1)[12]
## Transformation: log(.x)
##
## Coefficients:
##          ar1      ma1    sar1     sar2     sma1
##       0.4116  -0.8483  0.0100  -0.1017  -0.8204
## s.e.  0.0617   0.0348  0.0561   0.0529   0.0357
##
## sigma^2 estimated as 0.0006841:  log likelihood=1047
## AIC=-2082   AICc=-2082   BIC=-2057
```

# Example: US electricity production

```r
fit <- usmelec %>%
  model(arima = ARIMA(log(Generation)))
gg_tsresiduals(fit)
```



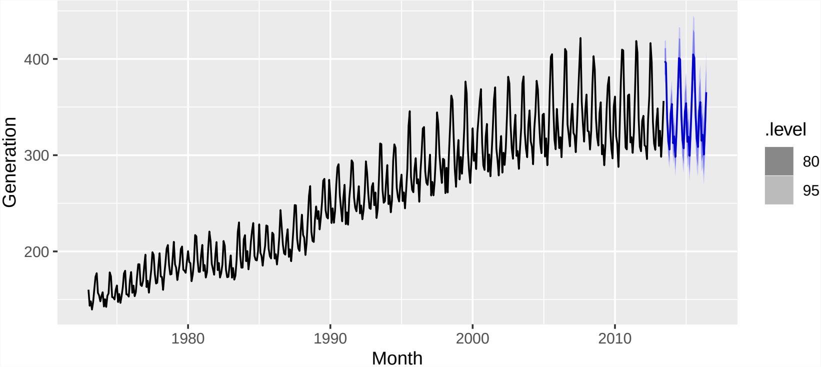
# Example: US electricity production

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 24, dof = 5)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 arima     38.7   0.00484
```

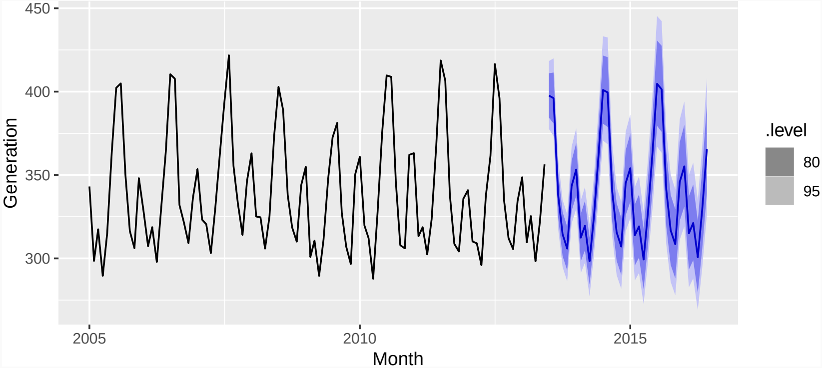
# Example: US electricity production

```r
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  forecast(h = "3 years") %>%
  autoplot(usmelec)
```

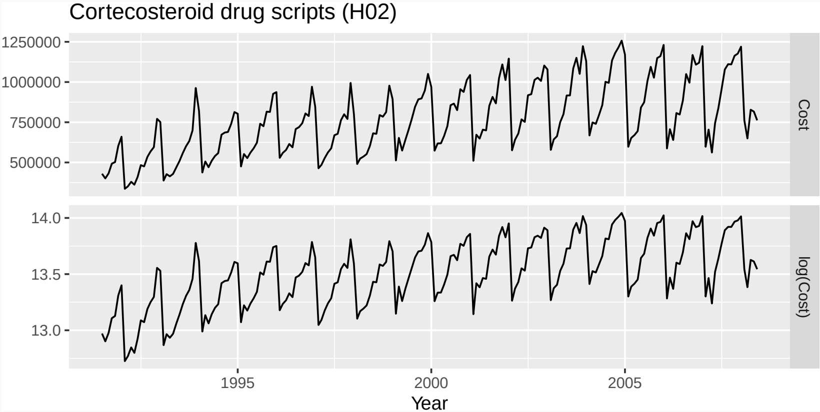


# Example: US electricity production

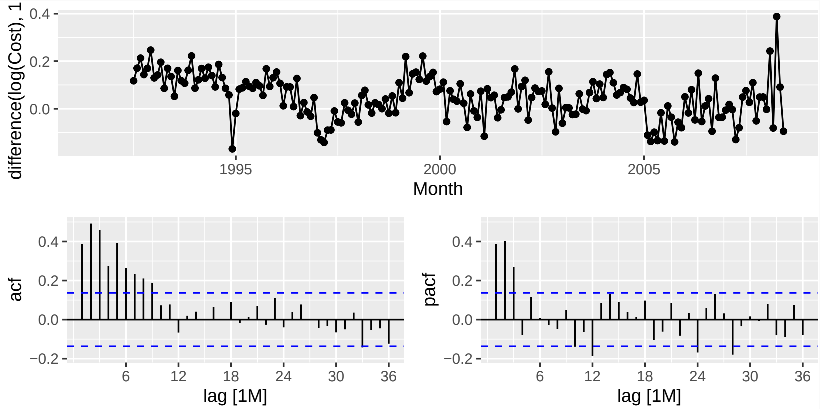
```r
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  forecast(h = "3 years") %>%
  autoplot(filter(usmelec, year(Month) >= 2005))
```



# Cortecosteroid drug sales



# Cortecosteroid drug sales



# Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA$(3,0,0)(2,1,0)_{12}$.

# Cortecosteroid drug sales

| .model | AICc |
|---|---|
| ARIMA(3,0,1)(0,1,2)[12] | -485.5 |
| ARIMA(3,0,1)(1,1,1)[12] | -484.3 |
| ARIMA(3,0,1)(0,1,1)[12] | -483.7 |
| ARIMA(3,0,1)(2,1,0)[12] | -476.3 |
| ARIMA(3,0,0)(2,1,0)[12] | -475.1 |
| ARIMA(3,0,2)(2,1,0)[12] | -474.9 |
| ARIMA(3,0,1)(1,1,0)[12] | -463.4 |

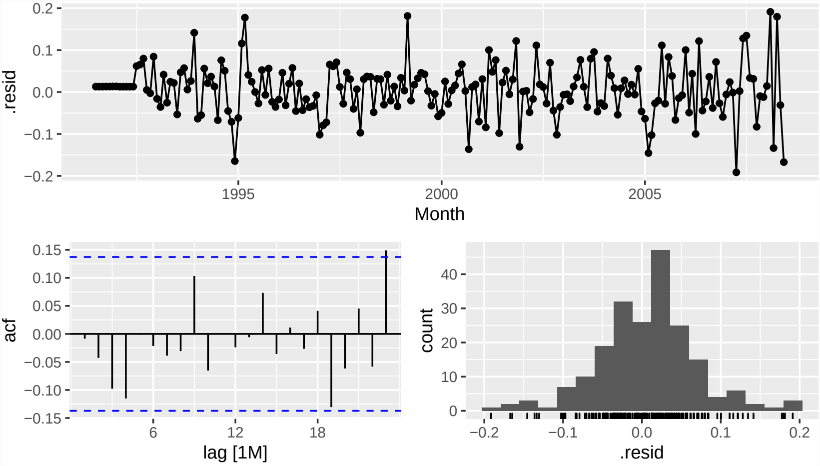
# Cortecosteroid drug sales

```r
fit <- h02 %>%
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(3,0,1)(0,1,2)[12]
## Transformation: log(.x)
##
## Coefficients:
##           ar1     ar2     ar3     ma1     sma1     sma2
##       -0.1602  0.5481  0.5678  0.3826  -0.5222  -0.1769
## s.e.   0.1636  0.0878  0.0942  0.1895   0.0861   0.0872
##
## sigma^2 estimated as 0.004289:  log likelihood=250.1
## AIC=-486.1   AICc=-485.5   BIC=-463.3
```

# Cortecosteroid drug sales

```r
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 best      50.5    0.0109
```

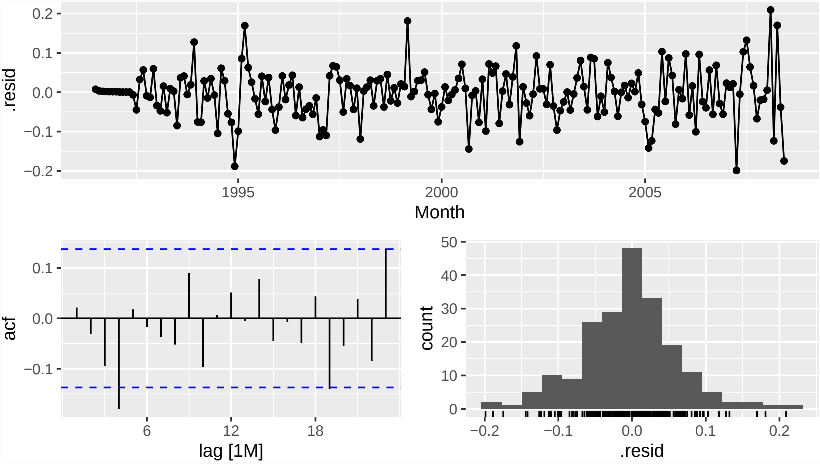
# Cortecosteroid drug sales

```r
fit <- h02 %>% model(auto = ARIMA(log(Cost)))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(2,1,0)(0,1,1)[12]
## Transformation: log(.x)
##
## Coefficients:
##           ar1      ar2     sma1
##       -0.8491  -0.4207  -0.6401
## s.e.   0.0712   0.0714   0.0694
##
## sigma^2 estimated as 0.004399:  log likelihood=245.4
## AIC=-482.8   AICc=-482.6   BIC=-469.8
```

# Cortecosteroid drug sales

```r
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 3)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 auto      57.5   0.00513
```

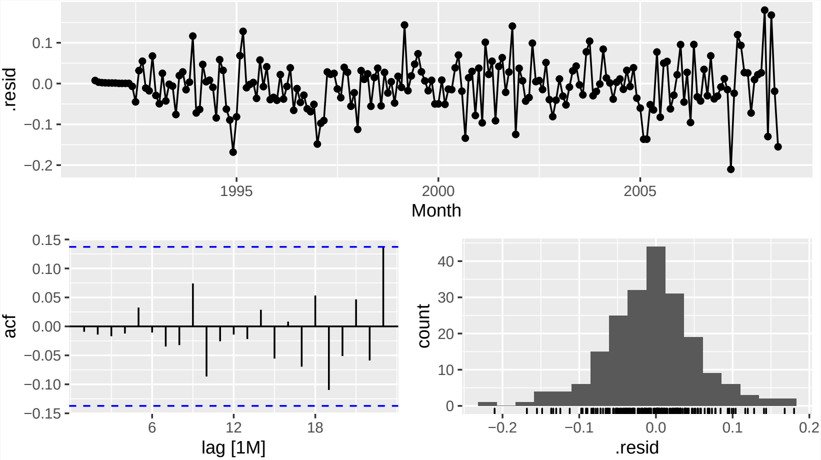
# Cortecosteroid drug sales

```r
fit <- h02 %>%
  model(best = ARIMA(log(Cost), stepwise = FALSE,
                     approximation = FALSE,
                     order_constraint = p + q + P + Q <= 9))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(4,1,1)(2,1,2)[12]
## Transformation: log(.x)
##
## Coefficients:
##           ar1     ar2     ar3      ar4      ma1    sar1
##       -0.0426  0.2097  0.2016  -0.2273  -0.7423  0.6213
## s.e.   0.2167  0.1814  0.1144   0.0810   0.2075  0.2421
##          sar2     sma1    sma2
##       -0.3832  -1.2018  0.4958
## s.e.   0.1185   0.2492  0.2136
##
## sigma^2 estimated as 0.004061:  log likelihood=254.3
## AIC=-488.6   AICc=-487.4   BIC=-456.1
```

# Cortecosteroid drug sales

```
gg_tsresiduals(fit)
```



# Cortecosteroid drug sales

```
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 best      35.1     0.136
```

# Cortecosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 %>%
  filter_index(~ "2006 Jun") %>%
  model(
    ARIMA(log(Cost) ~ pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit %>%
  forecast(h = "2 years") %>%
  accuracy(h02)
```

# Cortecosteroid drug sales

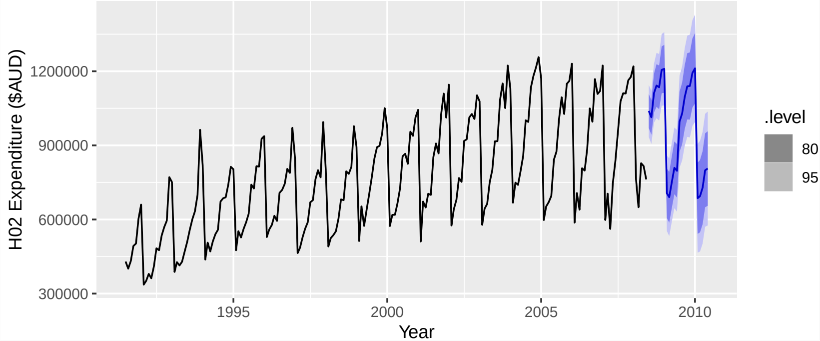
| .model | RMSE |
|---|---|
| ARIMA(3,0,1)(1,1,1)[12] | 61878 |
| ARIMA(3,0,1)(0,1,2)[12] | 62142 |
| ARIMA(3,0,1)(0,1,1)[12] | 62947 |
| ARIMA(2,1,0)(0,1,1)[12] | 62984 |
| ARIMA(4,1,1)(2,1,2)[12] | 63114 |
| ARIMA(3,0,2)(2,1,0)[12] | 65146 |
| ARIMA(3,0,1)(2,1,0)[12] | 65270 |
| ARIMA(3,0,1)(1,1,0)[12] | 66644 |
| ARIMA(3,0,0)(2,1,0)[12] | 66816 |

# Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

# Cortecosteroid drug sales

```
fit <- h02 %>%
  model(ARIMA(Cost ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))
fit %>% forecast %>% autoplot(h02) +
  ylab("H02 Expenditure ($AUD)") + xlab("Year")
```



# Outline


1 Stationarity and differencing

2 Non-seasonal ARIMA models

3 Estimation and order selection

4 ARIMA modelling in R

5 Forecasting

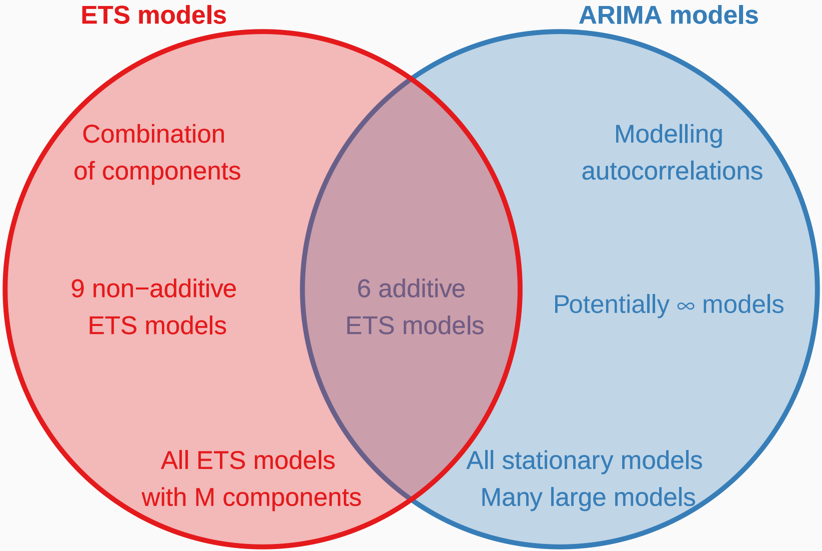
6 Seasonal ARIMA models

7 ARIMA vs ETS

# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS


ETS models
ARIMA models
Combination of components
Modelling autocorrelations
9 non−additive ETS models
6 additive ETS models
Potentially ∞ models
All ETS models with M components
All stationary models
Many large models

# Equivalences

| ETS model | ARIMA model | Parameters |
|---|---|---|
| ETS(A,N,N) | ARIMA(0,1,1) | $\theta_1 = \alpha - 1$ |
| ETS(A,A,N) | ARIMA(0,2,2) | $\theta_1 = \alpha + \beta - 2$ |
| | | $\theta_2 = 1 - \alpha$ |
| ETS(A,$\text{A}_\text{d}$,N) | ARIMA(1,1,2) | $\phi_1 = \phi$ |
| | | $\theta_1 = \alpha + \phi\beta - 1 - \phi$ |
| | | $\theta_2 = (1 - \alpha)\phi$ |
| ETS(A,N,A) | ARIMA(0,0,$m$)(0,1,0)$_m$ | |
| ETS(A,A,A) | ARIMA(0,1,$m+1$)(0,1,0)$_m$ | |
| ETS(A,$\text{A}_\text{d}$,A) | ARIMA(1,0,$m+1$)(0,1,0)$_m$ | |