

# **ETC3550**

## **Applied forecasting for business and economics**

Ch10. Dynamic regression models

[OTexts.org/fpp3/](https://OTexts.org/fpp3/)

# Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

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# Regression with ARIMA errors

## Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- $y_t$  modeled as function of  $k$  explanatory variables  $x_{1,t}, \dots, x_{k,t}$ .
- In regression, we assume that  $\varepsilon_t$  was WN.
- Now we want to allow  $\varepsilon_t$  to be autocorrelated.

# Regression with ARIMA errors

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- In regression, we assume that  $\varepsilon_t$  was WN.
- Now we want to allow  $\varepsilon_t$  to be autocorrelated.

## Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

where  $\varepsilon_t$  is white noise.

# Residuals and errors

**Example:  $\eta_t = \text{ARIMA}(1,1,1)$**

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

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# Residuals and errors

**Example:  $\eta_t = \text{ARIMA}(1,1,1)$**

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing  $\eta_t$  from  $\varepsilon_t$ .
- Only the errors  $\varepsilon_t$  are assumed to be white noise.
- In ordinary regression,  $\eta_t$  is assumed to be white noise and so  $\eta_t = \varepsilon_t$ .

# Estimation

If we minimize  $\sum \eta_t^2$  (by using ordinary regression):

- 1 Estimated coefficients  $\hat{\beta}_0, \dots, \hat{\beta}_k$  are no longer optimal as some information ignored;
- 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
- 3  $p$ -values for coefficients usually too small (“spurious regression”).
- 4 AIC of fitted models misleading.



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  - 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
  - 3  $p$ -values for coefficients usually too small (“spurious regression”).
  - 4 AIC of fitted models misleading.
- Minimizing  $\sum \varepsilon_t^2$  avoids these problems.
  - Maximizing likelihood similar to minimizing  $\sum \varepsilon_t^2$ .

# Stationarity

## Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

where  $\eta_t$  is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

# Stationarity

## Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
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# Stationarity

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## Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t,$$
$$(1 - \phi_1 B)\eta'_t = (1 + \theta_1 B)\varepsilon_t,$$

where  $y'_t = y_t - y_{t-1}$ ,  $x'_{t,i} = x_{t,i} - x_{t-1,i}$  and  $\eta'_t = \eta_t - \eta_{t-1}$ .

# Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

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## Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t$$

$$\text{where } \phi(B)(1-B)^d \eta_t = \theta(B) \varepsilon_t$$

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Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

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$$\text{where } \phi(B)(1-B)^d \eta_t = \theta(B) \varepsilon_t$$

## After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t.$$

$$\text{where } \phi(B) \eta_t = \theta(B) \varepsilon_t$$

$$\text{and } y'_t = (1-B)^d y_t$$

# Model selection

- Fit regression model with automatically selected ARIMA errors. (R will take care of differencing before estimation.)
- Check that  $\varepsilon_t$  series looks like white noise.



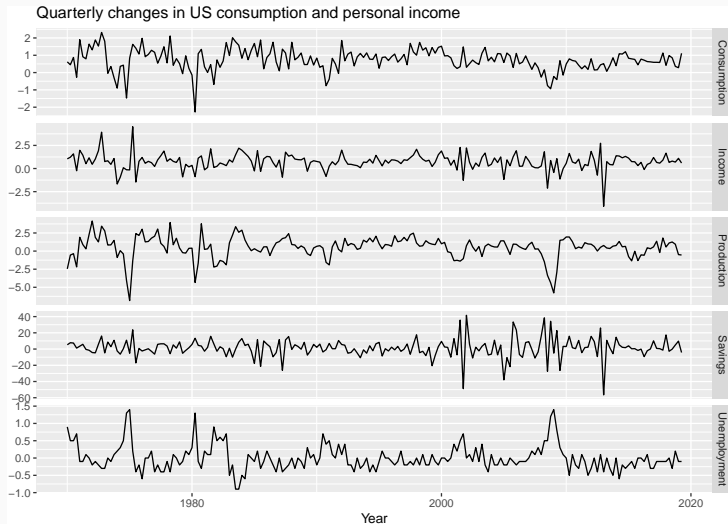
# Model selection

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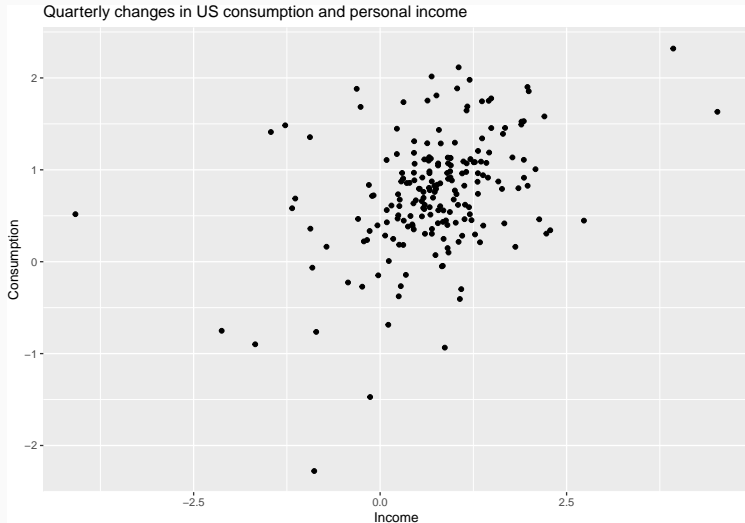
## Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

# US personal consumption and income



# US personal consumption and income



# US personal consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

# US personal consumption and income

```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))  
report(fit)
```

```
## Series: Consumption  
## Model: LM w/ ARIMA(1,0,2) errors  
##  
## Coefficients:  
##          ar1          ma1          ma2  Income  intercept  
##          0.7070    -0.6172    0.2066   0.1976         0.5949  
## s.e.    0.1068     0.1218    0.0741    0.0462         0.0850  
##  
## sigma^2 estimated as 0.3113:  log likelihood=-163  
## AIC=338.1   AICc=338.5   BIC=357.8
```

# US personal consumption and income

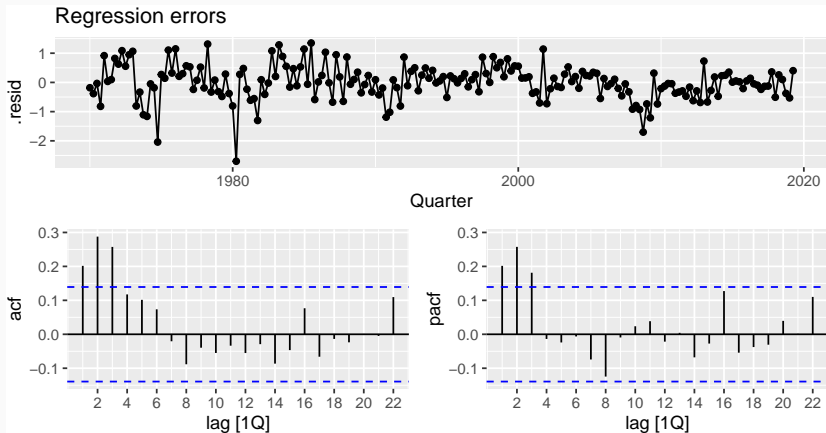
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```

Write down the equations for the fitted model.

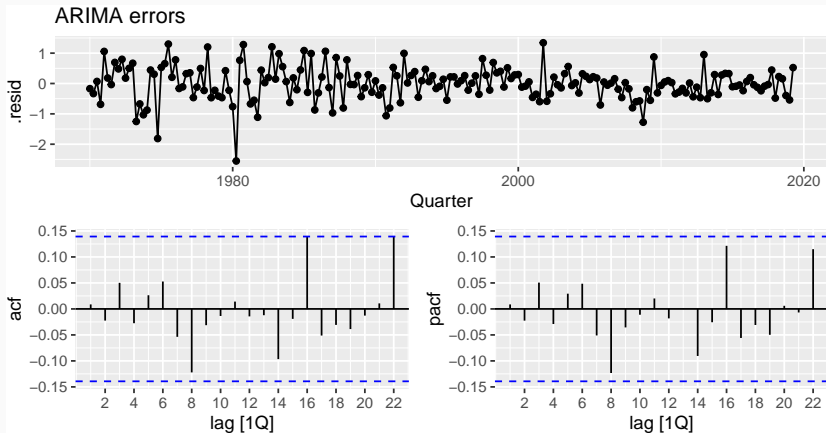
# US personal consumption and income

```
residuals(fit, type='regression') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') + ggtitle("Reg
```



# US personal consumption and income

```
residuals(fit, type='response') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') + ggtitle("ARI
```





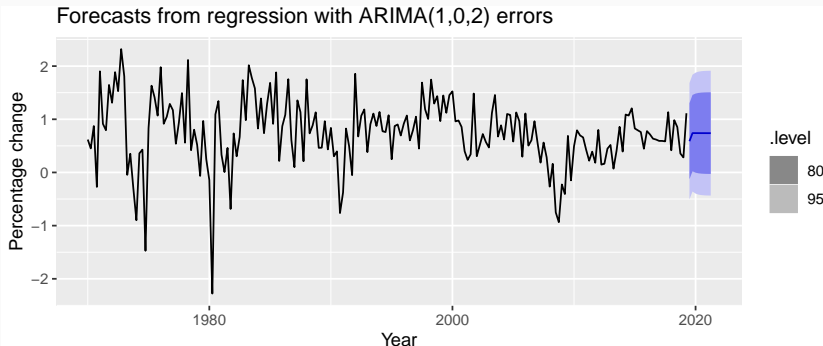
# US personal consumption and income

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 5, lag = 12)
```

```
## # A tibble: 1 x 3  
##   .model                lb_stat lb_pvalue  
##   <chr>                <dbl>    <dbl>  
## 1 ARIMA(Consumption ~ Income)    5.54    0.595
```

# US personal consumption and income

```
us_change_future <- new_data(us_change, 8) %>%  
  mutate(Income = mean(us_change$Income))  
forecast(fit, new_data = us_change_future) %>%  
  autoplot(us_change) +  
  labs(x = "Year", y = "Percentage change",  
        title = "Forecasts from regression with ARIMA(1,0,2) errors")
```



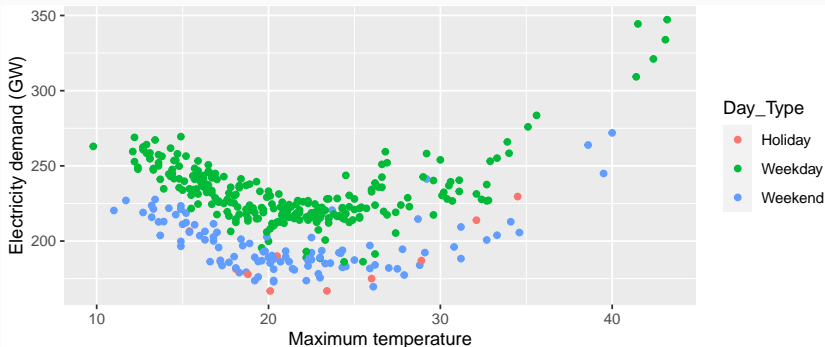
# Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

# Daily electricity demand

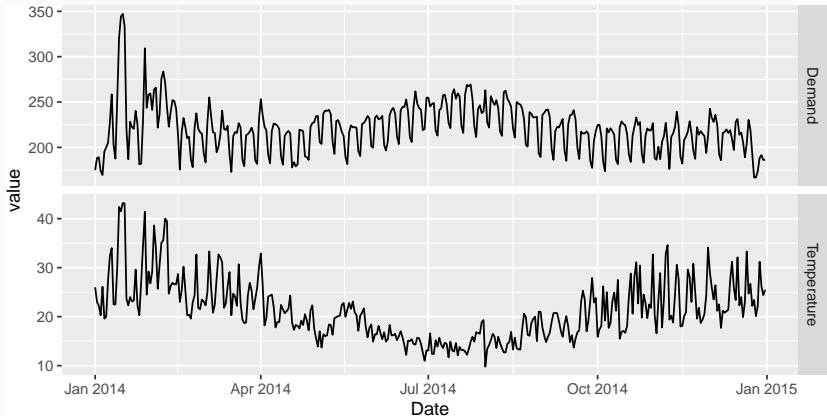
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
vic_elec_daily %>%  
  ggplot(aes(x=Temperature, y=Demand, colour=Day_Type)) +  
  geom_point() +  
  labs(x = "Maximum temperature", y = "Electricity demand (GW)")
```



# Daily electricity demand

```
vic_elec_daily %>%  
  gather("var", "value", Demand, Temperature) %>%  
  ggplot(aes(x = Date, y = value)) + geom_line() +  
  facet_grid(vars(var), scales = "free_y")
```

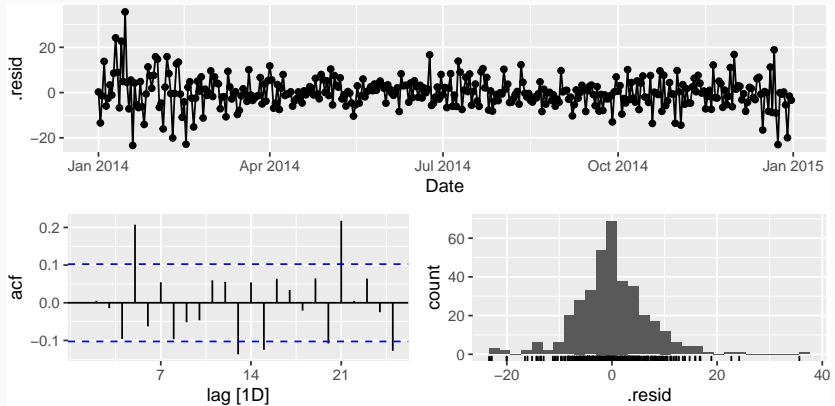


# Daily electricity demand

```
fit <- vic_elec_daily %>%  
  model(ARIMA(Demand ~ Temperature + I(Temperature^2) +  
             (Day_Type=="Weekday")))  
report(fit)  
  
## Series: Demand  
## Model: LM w/ ARIMA(2,1,2)(0,0,2)[7] errors  
##  
## Coefficients:  
##          ar1          ar2          ma1          ma2          sma1          sma2  
##          1.1521   -0.2750   -1.3851    0.4071    0.1589    0.3103  
## s.e.         0.6265    0.4812    0.6082    0.5804    0.0591    0.0538  
##          Temperature  I(Temperature^2)  
##          -7.947                0.1865  
## s.e.         0.492                0.0097  
##          Day_Type == "Weekday"TRUE  
##                      31.825  
## s.e.                      1.019  
##
```

# Daily electricity demand

```
gg_tsresiduals(fit)
```



# Daily electricity demand

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 8, lag = 14)
```

```
## # A tibble: 1 x 3  
##   .model                                lb_stat lb_pvalue  
##   <chr>                                <dbl>     <dbl>  
## 1 "ARIMA(Demand ~ Temperature + I(Tempera~ 38.1 1.05e-6
```



# Daily electricity demand

```
# Forecast one day ahead
```

```
vic_next_day <- new_data(vic_elec_daily, 1) %>%  
  mutate(Temperature = 26, Day_Type = "Holiday")  
forecast(fit, vic_next_day)
```

```
## # A tibble: 1 x 6 [1D]
```

```
## # Key:   .model [1]
```

```
##   .model Date      Demand .distribution Temperature
```

```
##   <chr>  <date>      <dbl> <dist>          <dbl>
```

```
## 1 "ARIM~ 2015-01-01  161. N(161, 49)          26
```

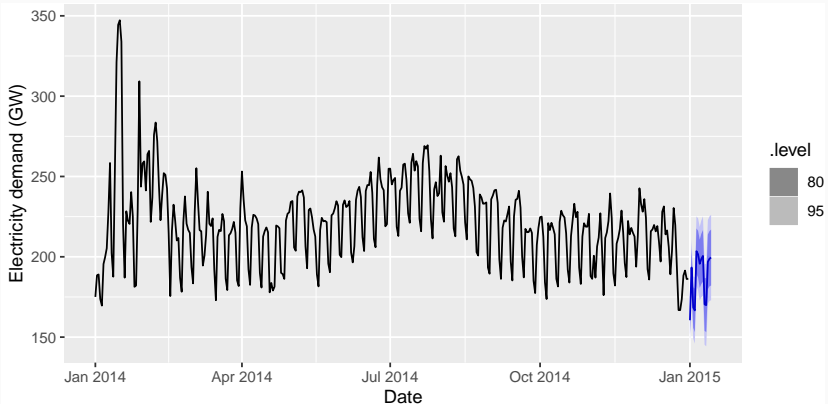
```
## # ... with 1 more variable: Day_Type <chr>
```

# Daily electricity demand

```
vic_elec_future <- new_data(vic_elec_daily, 14) %>%  
  mutate(  
    Temperature = 26,  
    Holiday = c(TRUE, rep(FALSE, 13)),  
    Day_Type = case_when(  
      Holiday ~ "Holiday",  
      wday(Date) %in% 2:6 ~ "Weekday",  
      TRUE ~ "Weekend"  
    )  
  )
```

# Daily electricity demand

```
forecast(fit, new_data=vic_elec_future) %>%  
  autoplot(vic_elec_daily) + ylab("Electricity demand (GW)")
```



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# Stochastic & deterministic trends

## Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where  $\eta_t$  is ARMA process.

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## Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where  $\eta_t$  is ARIMA process with  $d \geq 1$ .

# Stochastic & deterministic trends

## Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where  $\eta_t$  is ARMA process.

## Stochastic trend

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where  $\eta_t$  is ARIMA process with  $d \geq 1$ .

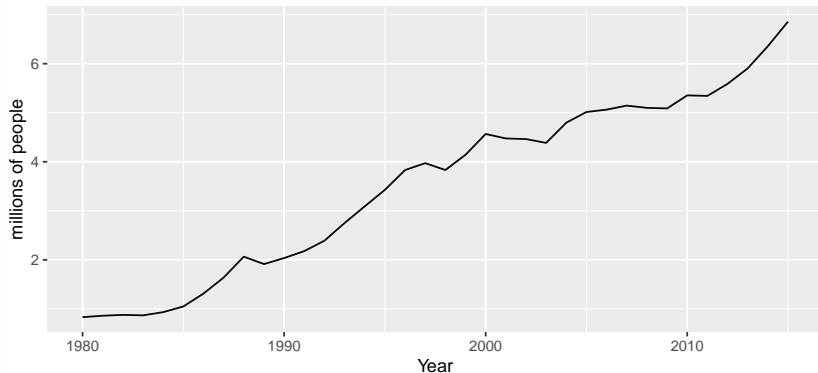
Difference both sides until  $\eta_t$  is stationary:

$$y'_t = \beta_1 + \eta'_t$$

where  $\eta'_t$  is ARMA process.

# International visitors

Total annual international visitors to Australia





# International visitors

## Deterministic trend

```
fit_deterministic <- aus_visitors %>%  
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: value  
## Model: LM w/ ARIMA(2,0,0) errors  
##  
## Coefficients:  
##          ar1          ar2  trend()  intercept  
##          1.113   -0.3805   0.1710     0.4156  
## s.e.    0.160    0.1585   0.0088     0.1897  
##  
## sigma^2 estimated as 0.02979:  log likelihood=13.6  
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

# International visitors

## Deterministic trend

```
fit_deterministic <- aus_visitors %>%  
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: value  
## Model: LM w/ ARIMA(2,0,0) errors  
##  
## Coefficients:  
##          ar1          ar2  trend()  intercept  
##          1.113   -0.3805    0.1710     0.4156  
## s.e.    0.160    0.1585    0.0088     0.1897  
##  
## sigma^2 estimated as 0.02979:  log likelihood=13.6  
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

$$y_t = 0.42 + 0.17t + \eta_t$$

$$\eta_t = 1.11\eta_{t-1} - 0.38\eta_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim \text{NID}(0, 0.0298).$$

# International visitors

## Stochastic trend

```
fit_stochastic <- aus_visitors %>%  
  model(Stochastic = ARIMA(value ~ pdq(d=1)))  
report(fit_stochastic)
```

```
## Series: value  
## Model: ARIMA(0,1,1) w/ drift  
##  
## Coefficients:  
##           ma1    constant  
##      0.3006      0.1735  
## s.e.  0.1647      0.0390  
##  
## sigma^2 estimated as 0.03376:  log likelihood=10.62  
## AIC=-15.24   AICc=-14.46   BIC=-10.57
```

# International visitors

## Stochastic trend

```
fit_stochastic <- aus_visitors %>%  
  model(Stochastic = ARIMA(value ~ pdq(d=1)))  
report(fit_stochastic)
```

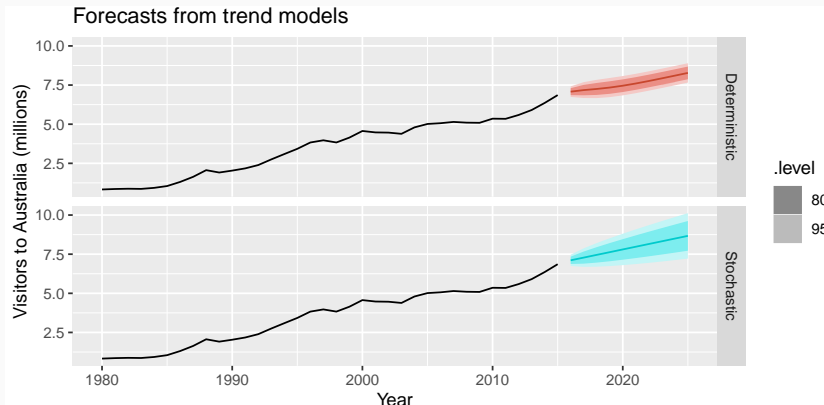
```
## Series: value  
## Model: ARIMA(0,1,1) w/ drift  
##  
## Coefficients:  
##          ma1  constant  
##      0.3006    0.1735  
## s.e.  0.1647    0.0390  
##  
## sigma^2 estimated as 0.03376:  log likelihood=10.62  
## AIC=-15.24  AICc=-14.46  BIC=-10.57
```

$$y_t - y_{t-1} = 0.17 + \varepsilon_t$$

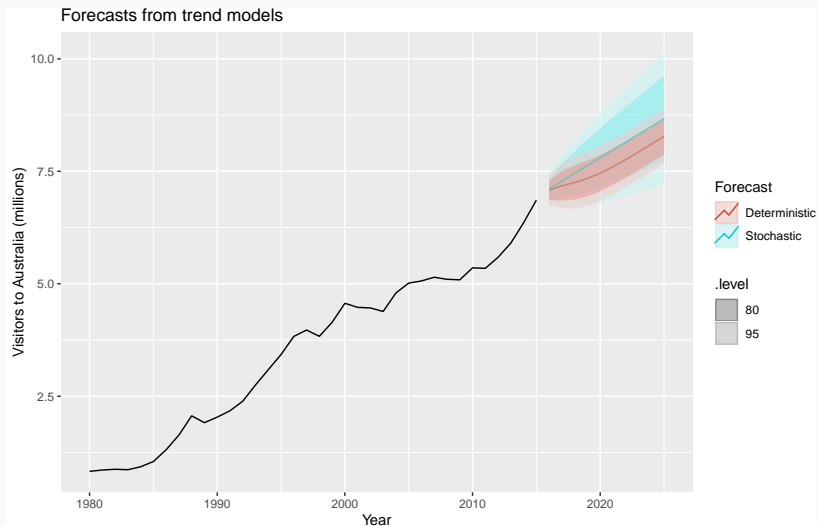
$$y_t = y_0 + 0.17t + \eta_t$$

$$\eta_t = \eta_{t-1} + 0.30\varepsilon_{t-1} + \varepsilon_t$$

# International visitors



# International visitors



# Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

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# Dynamic harmonic regression

## Combine Fourier terms with ARIMA errors

### Advantages

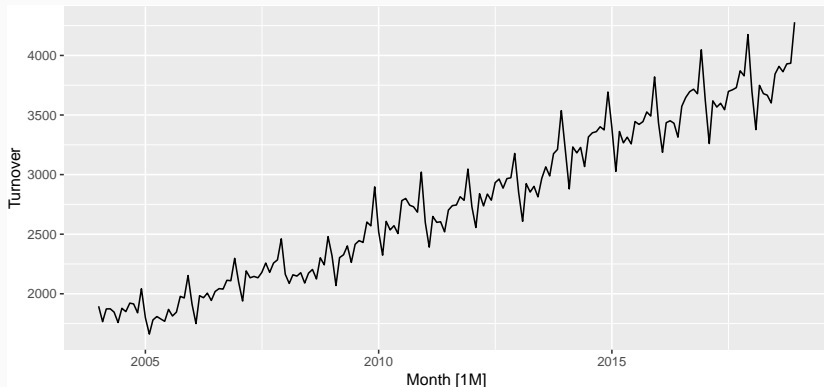
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of  $K$  (but more wiggly seasonality can be handled by increasing  $K$ );
- the short-term dynamics are easily handled with a simple ARMA error.

### Disadvantages

- seasonality is assumed to be fixed

# Eating-out expenditure

```
aus_cafe <- aus_retail %>% filter(  
  Industry == "Cafes, restaurants and takeaway food services",  
  year(Month) %in% 2004:2018  
) %>% summarise(Turnover = sum(Turnover))  
aus_cafe %>% autoplot(Turnover)
```

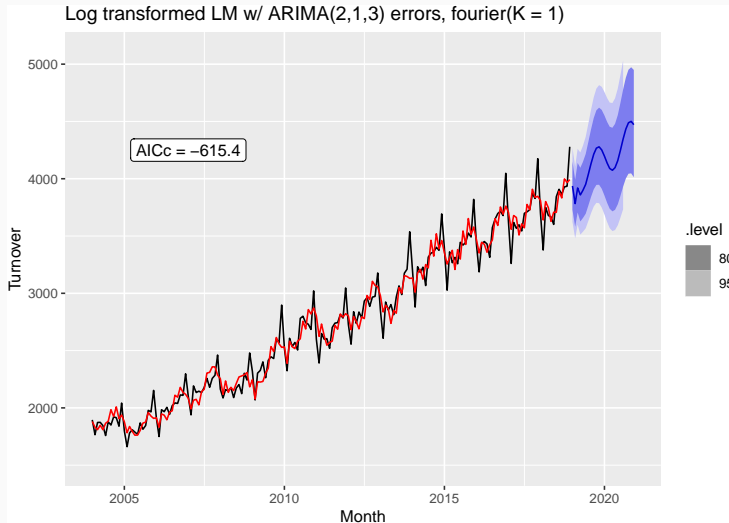


# Eating-out expenditure

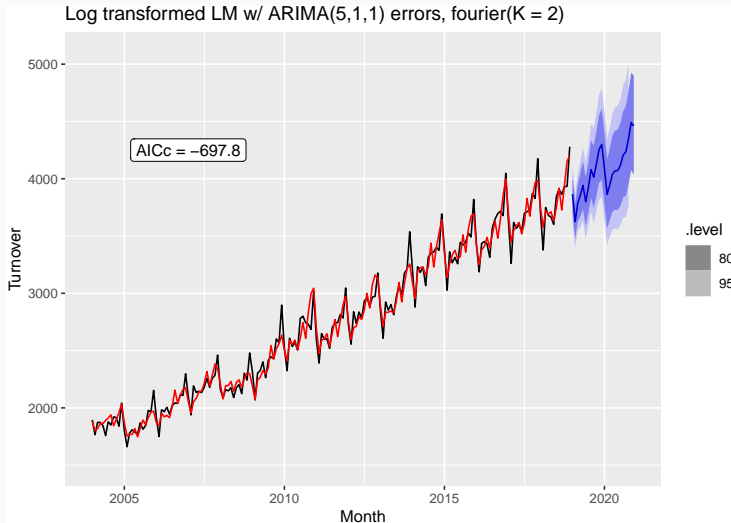
```
fit <- aus_cafe %>% model(  
  K = 1 = ARIMA(log(Turnover) ~ fourier(K = 1) + PDQ(0,0,0)),  
  K = 2 = ARIMA(log(Turnover) ~ fourier(K = 2) + PDQ(0,0,0)),  
  K = 3 = ARIMA(log(Turnover) ~ fourier(K = 3) + PDQ(0,0,0)),  
  K = 4 = ARIMA(log(Turnover) ~ fourier(K = 4) + PDQ(0,0,0)),  
  K = 5 = ARIMA(log(Turnover) ~ fourier(K = 5) + PDQ(0,0,0)),  
  K = 6 = ARIMA(log(Turnover) ~ fourier(K = 6) + PDQ(0,0,0)))  
glance(fit)
```

.model	sigma2	log_lik	AIC	AICc	BIC
K = 1	0.0017	317.2	-616.5	-615.4	-587.8
K = 2	0.0011	361.9	-699.7	-697.8	-661.5
K = 3	0.0008	393.6	-763.2	-761.3	-725.0
K = 4	0.0005	426.8	-821.6	-818.2	-770.6
K = 5	0.0003	473.7	-919.5	-916.9	-874.8
K = 6	0.0003	474.0	-920.1	-917.5	-875.4

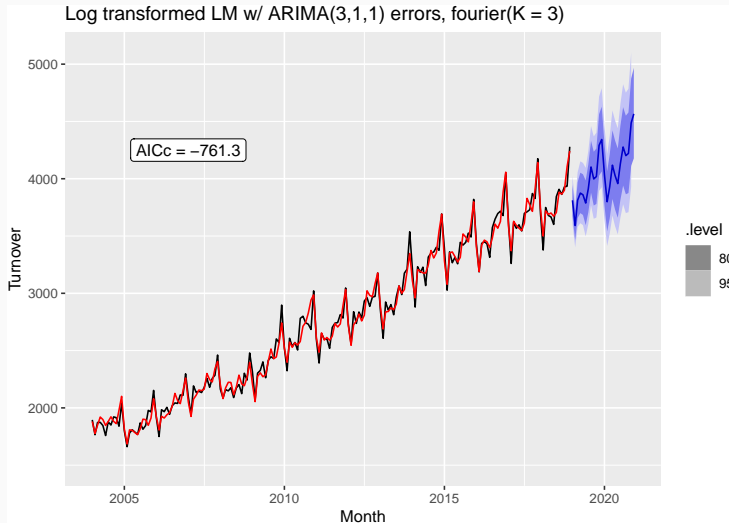
# Eating-out expenditure



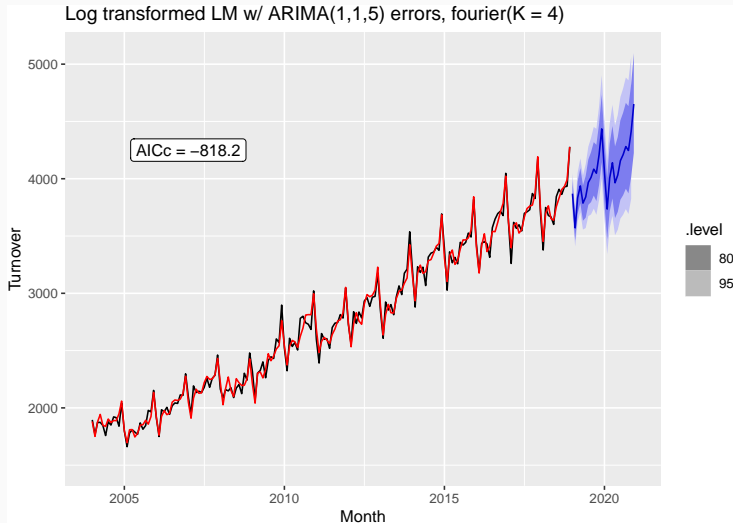
# Eating-out expenditure



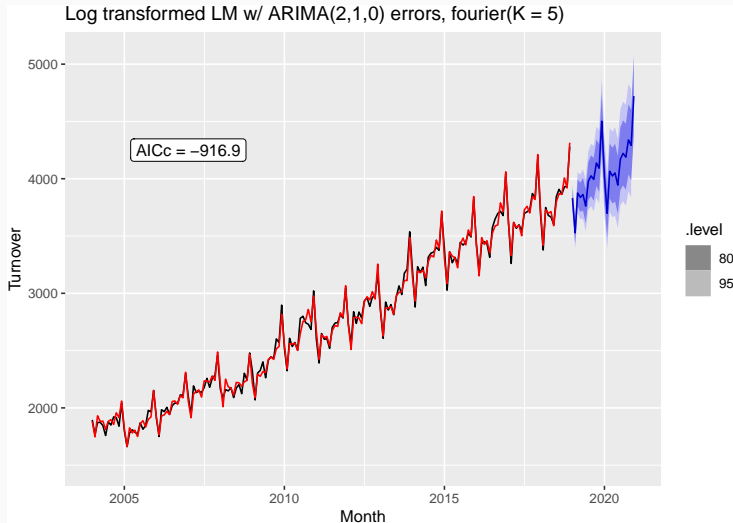
# Eating-out expenditure



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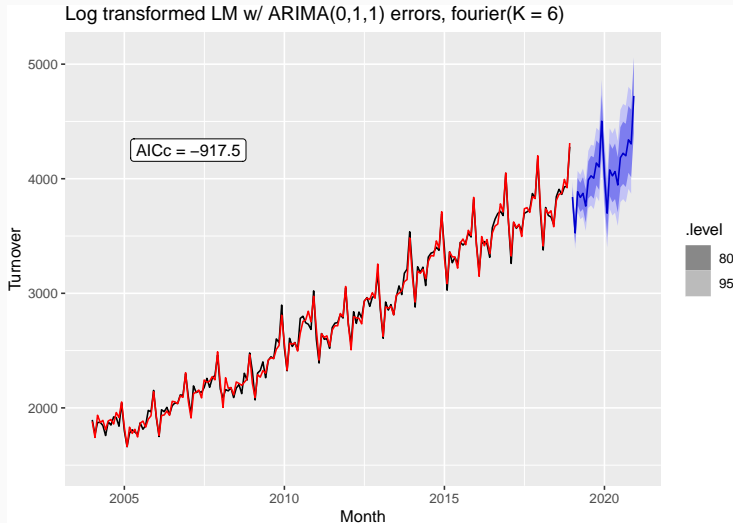


# Eating-out expenditure





# Eating-out expenditure



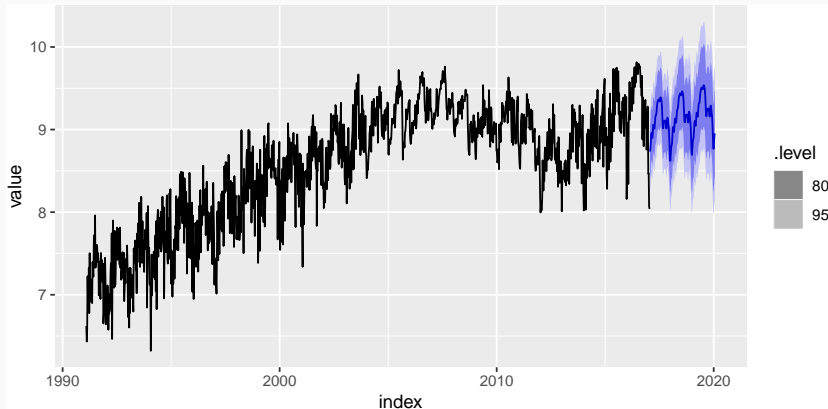
# Example: weekly gasoline products

```
gasoline <- as_tsibble(fpp2::gasoline)
fit <- gasoline %>% model(ARIMA(value ~ fourier(K = 13) + PDQ(0,0,0)))
report(fit)
```

```
## Series: value
## Model: LM w/ ARIMA(0,1,1) errors
##
## Coefficients:
##          ma1  fourier(K = 13)C1_52  fourier(K = 13)S1_52
##          -0.8934                -0.1121                -0.2300
## s.e.       0.0132                0.0123                0.0122
##          fourier(K = 13)C2_52  fourier(K = 13)S2_52
##                      0.0420                0.0317
## s.e.              0.0099                0.0099
##          fourier(K = 13)C3_52  fourier(K = 13)S3_52
##                      0.0832                0.0346
## s.e.              0.0094                0.0094
##          fourier(K = 13)C4_52  fourier(K = 13)S4_52
##                      0.0185                0.0398
## s.e.              0.0092                0.0092
##          fourier(K = 13)C5_52  fourier(K = 13)S5_52
##                      -0.0315                0.0009
## s.e.              0.0091                0.0091
##          fourier(K = 13)C6_52  fourier(K = 13)S6_52
```

# Example: weekly gasoline products

```
forecast(fit, h = "3 years") %>%  
  autoplot(gasoline)
```



# 5-minute call centre volume

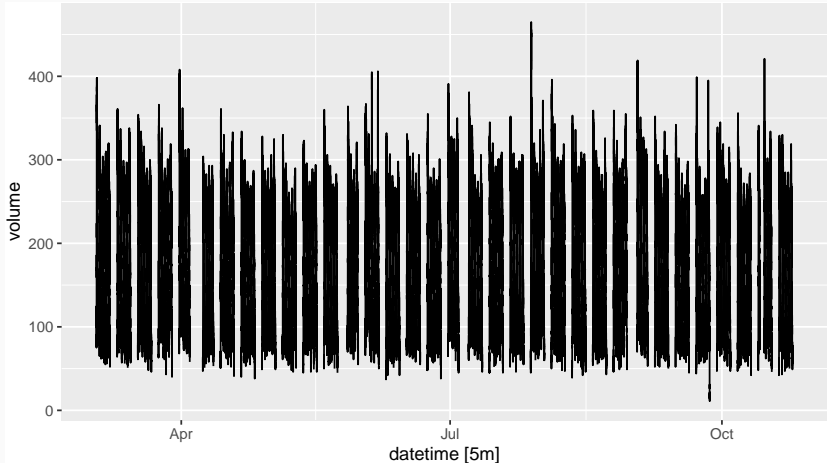
```
(calls <- read_tsv("http://robjhyndman.com/data/callcenter.txt")) %>%  
  gather("date", "volume", -X1) %>% transmute(  
    time = X1, date = as.Date(date, format = "%d/%m/%Y"),  
    datetime = as_datetime(date) + time, volume) %>%  
  as_tsibble(index = datetime))
```

```
## # A tsibble: 27,716 x 4 [5m] <UTC>
```

##	time	date	datetime	volume
##	<time>	<date>	<dtm>	<dbl>
##	1 07:00	2003-03-03	2003-03-03 07:00:00	111
##	2 07:05	2003-03-03	2003-03-03 07:05:00	113
##	3 07:10	2003-03-03	2003-03-03 07:10:00	76
##	4 07:15	2003-03-03	2003-03-03 07:15:00	82
##	5 07:20	2003-03-03	2003-03-03 07:20:00	91
##	6 07:25	2003-03-03	2003-03-03 07:25:00	87
##	7 07:30	2003-03-03	2003-03-03 07:30:00	75
##	8 07:35	2003-03-03	2003-03-03 07:35:00	89
##	9 07:40	2003-03-03	2003-03-03 07:40:00	99
##	10 07:45	2003-03-03	2003-03-03 07:45:00	125

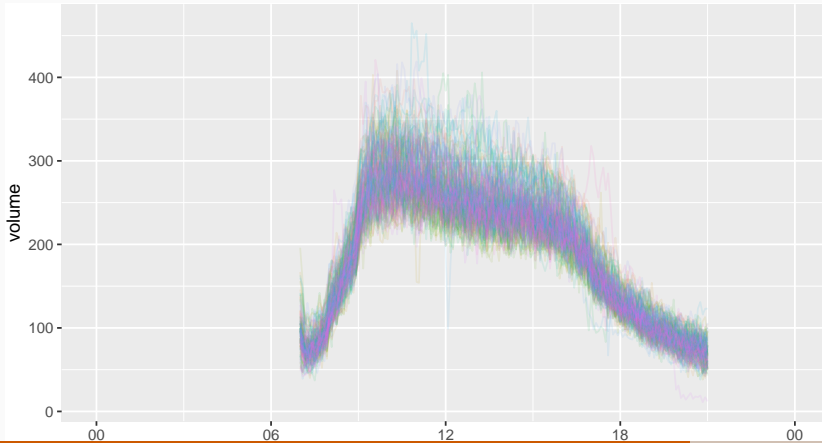
# 5-minute call centre volume

```
calls %>% fill_gaps() %>% autoplot(volume)
```



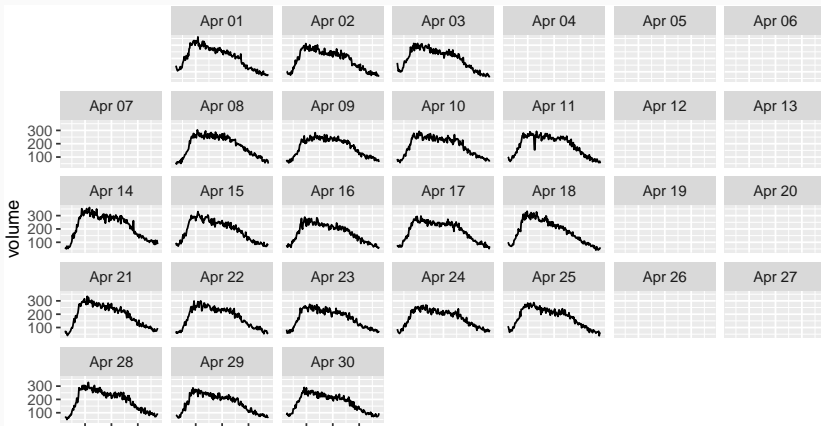
# 5-minute call centre volume

```
calls %>% fill_gaps() %>%  
  gg_season(volume, period = "day", alpha = 0.1) +  
  guides(colour = FALSE)
```



# 5-minute call centre volume

```
library(sugrants)
calls %>% filter(month(date, label = TRUE) == "Apr") %>%
  ggplot(aes(x = time, y = volume)) +
  geom_line() + facet_calendar(date)
```



# 5-minute call centre volume

```
calls_md1 <- calls %>%  
  mutate(idx = row_number()) %>%  
  update_tsibble(index = idx)  
fit <- calls_md1 %>%  
  model(ARIMA(volume ~ fourier(169, K = 10) + pdq(d=0) + PDQ(0,0,0)))  
report(fit)
```

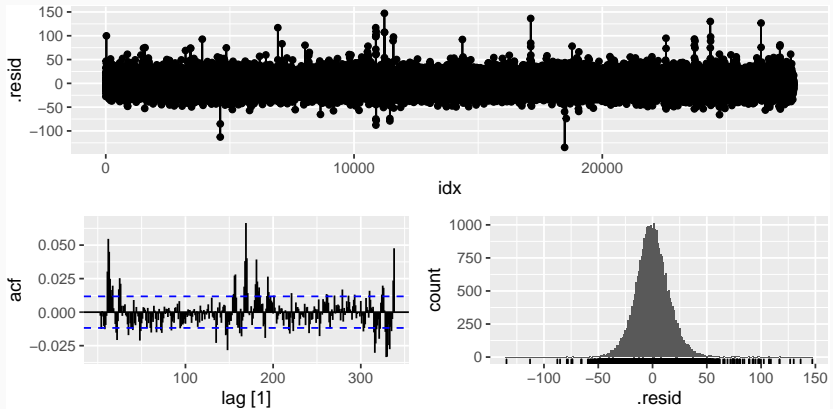
```
## Series: volume  
## Model: LM w/ ARIMA(1,0,3) errors  
##  
## Coefficients:  
##          ar1          ma1          ma2          ma3  
##          0.9894   -0.7383   -0.0333   -0.0282  
## s.e.    0.0010    0.0061    0.0075    0.0060  
##          fourier(169, K = 10)C1_169  
##                               -79.0702  
## s.e.                               0.7001  
##          fourier(169, K = 10)S1_169  
##                               55.2985  
## s.e.                               0.7006  
##          fourier(169, K = 10)C2_169  
##                               -32.3615  
## s.e.                               0.3784  
##          fourier(169, K = 10)S2_169
```



# 5-minute call centre volume

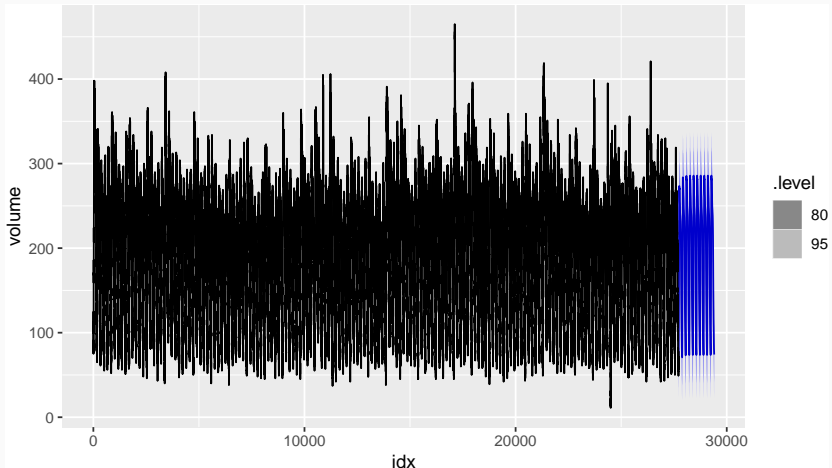
```
augment(fit) %>%
```

```
gg_tsdisplay(.resid, plot_type = "histogram", lag_max = 338)
```



# 5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%  
  autoplot(calls_mdl)
```



# Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

# Lagged predictors

Sometimes a change in  $x_t$  does not affect  $y_t$  instantaneously

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- $y_t$  = sales,  $x_t$  = advertising.
- $y_t$  = stream flow,  $x_t$  = rainfall.
- $y_t$  = size of herd,  $x_t$  = breeding stock.

# Lagged predictors

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  - $y_t$  = stream flow,  $x_t$  = rainfall.
  - $y_t$  = size of herd,  $x_t$  = breeding stock.
- 
- These are dynamic systems with input ( $x_t$ ) and output ( $y_t$ ).
  - $x_t$  is often a leading indicator.
  - There can be multiple predictors.

# Lagged predictors

The model include present and past values of predictor:  $x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \dots + \nu_k x_{t-k} + \eta_t$$

where  $\eta_t$  is an ARIMA process.

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where  $\eta_t$  is an ARIMA process.

**Rewrite model as**

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \dots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$



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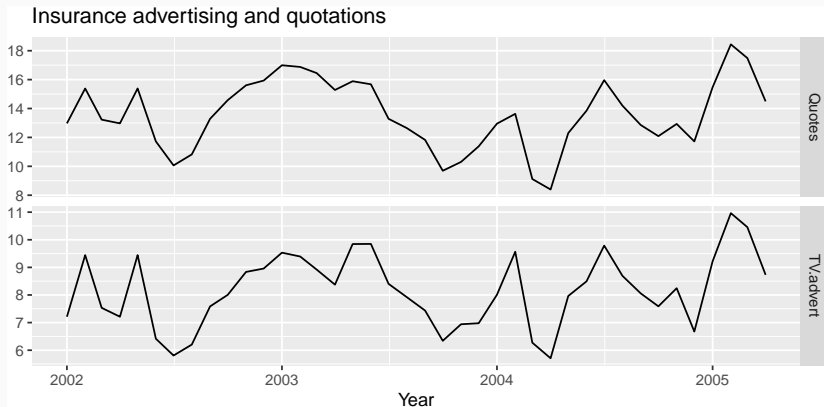
where  $\eta_t$  is an ARIMA process.

**Rewrite model as**

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \dots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

- $\nu(B)$  is called a *transfer function* since it describes how change in  $x_t$  is transferred to  $y_t$ .
- $x$  can influence  $y$ , but  $y$  is not allowed to influence  $x$ .

# Example: Insurance quotes and TV adverts



# Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  # Restrict data so models use same fitting period  
  mutate(Quotes = c(NA,NA,NA,Quotes[4:40])) %>%  
  # Estimate models  
  model(  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert)),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +  
      lag(TV.advert, 2)),  
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +  
      lag(TV.advert, 2) + lag(TV.advert, 3))  
  )
```

# Example: Insurance quotes and TV adverts

```
glance(fit)
```

Lag order	sigma2	log_lik	AIC	AICc	BIC
0	0.2650	-28.28	66.56	68.33	75.01
1	0.2094	-24.04	58.09	59.85	66.53
2	0.2150	-24.02	60.03	62.58	70.17
3	0.2056	-22.16	60.31	64.96	73.83

# Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))  
report(fit)
```

```
## Series: Quotes  
## Model: LM w/ ARIMA(3,0,0) errors  
##  
## Coefficients:  
##          ar1          ar2          ar3  TV.advert  lag(TV.advert)  
##          1.4117   -0.9317   0.3591      1.2564         0.1625  
## s.e.    0.1698    0.2545   0.1592      0.0667         0.0591  
##      intercept  
##           2.0393  
## s.e.       0.9931  
##  
## sigma^2 estimated as 0.2165:  log likelihood=-23.89  
## AIC=61.78   AICc=65.28   BIC=73.6
```

# Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))  
report(fit)
```

```
## Series: Quotes  
## Model: LM w/ ARIMA(3,0,0) errors  
##  
## Coefficients:  
##          ar1      ar2      ar3  TV.advert  lag(TV.advert)  
##          1.4117 -0.9317  0.3591    1.2564      0.1625  
## s.e.    0.1698   0.2545  0.1592    0.0667      0.0591  
##      intercept  
##           2.0393  
## s.e.      0.9931  
##  
## sigma^2 estimated as 0.2165:  log likelihood=-23.89  
## AIC=61.78   AICc=65.28   BIC=73.6
```

$$y_t = 2.04 + 1.26x_t + 0.16x_{t-1} + \eta_t,$$
$$\eta_t = 1.41\eta_{t-1} - 0.93\eta_{t-2} + 0.36\eta_{t-3} + \varepsilon_t,$$

# Example: Insurance quotes and TV adverts

```
advert_a <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 10)  
forecast(fit, advert_a) %>% autoplot(insurance)
```



# Example: Insurance quotes and TV adverts

```
advert_b <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 8)  
forecast(fit, advert_b) %>% autoplot(insurance)
```





# Example: Insurance quotes and TV adverts

```
advert_c <- new_data(insurance, 20) %>%  
  mutate(TV.advert = 6)  
forecast(fit, advert_c) %>% autoplot(insurance)
```

