
Computer Graphics

9 - Orientation & Rotation

Yoonsang Lee
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Topics Covered

- Orientation vs. Rotation
- Degrees of Freedom
- 3D Orientation & Rotation Representations
 - Euler angles
 - Axis-angle (Rotation vector)
 - **Rotation matrices**
 - Unit quaternions
- 3D Orientation Interpolation

Orientation vs. Rotation, Degrees of Freedom

Orientation vs. Rotation

- Rotation

- Circular movement

회전하는 움직임 (움직이는 행위 자체)

- Orientation

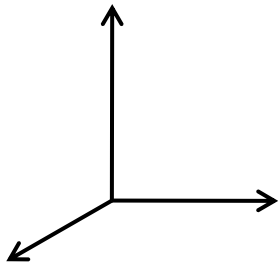
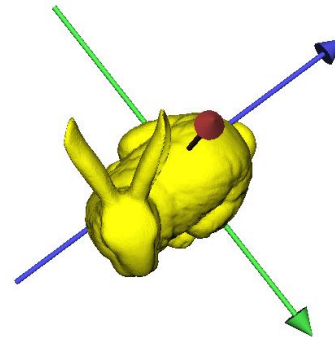
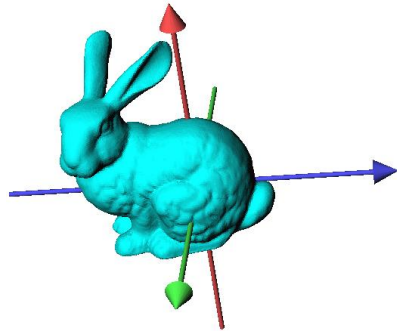
- The state of being oriented

방향을 나타내는 상태 (잡혀진 상태)

- Given a coordinate system, the orientation of an object can be represented **as a rotation from a reference orientation**

Analogy

- (point : vector) is similar to (orientation : rotation)
 - Both represent a sort of (state : movement)

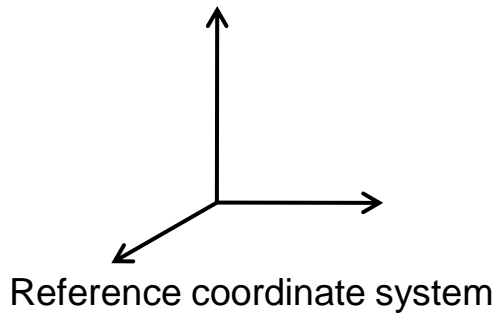
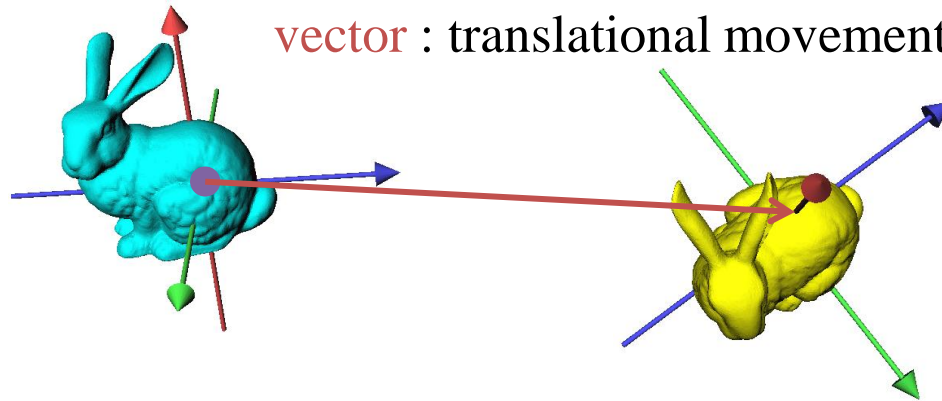


Reference coordinate system

Analogy

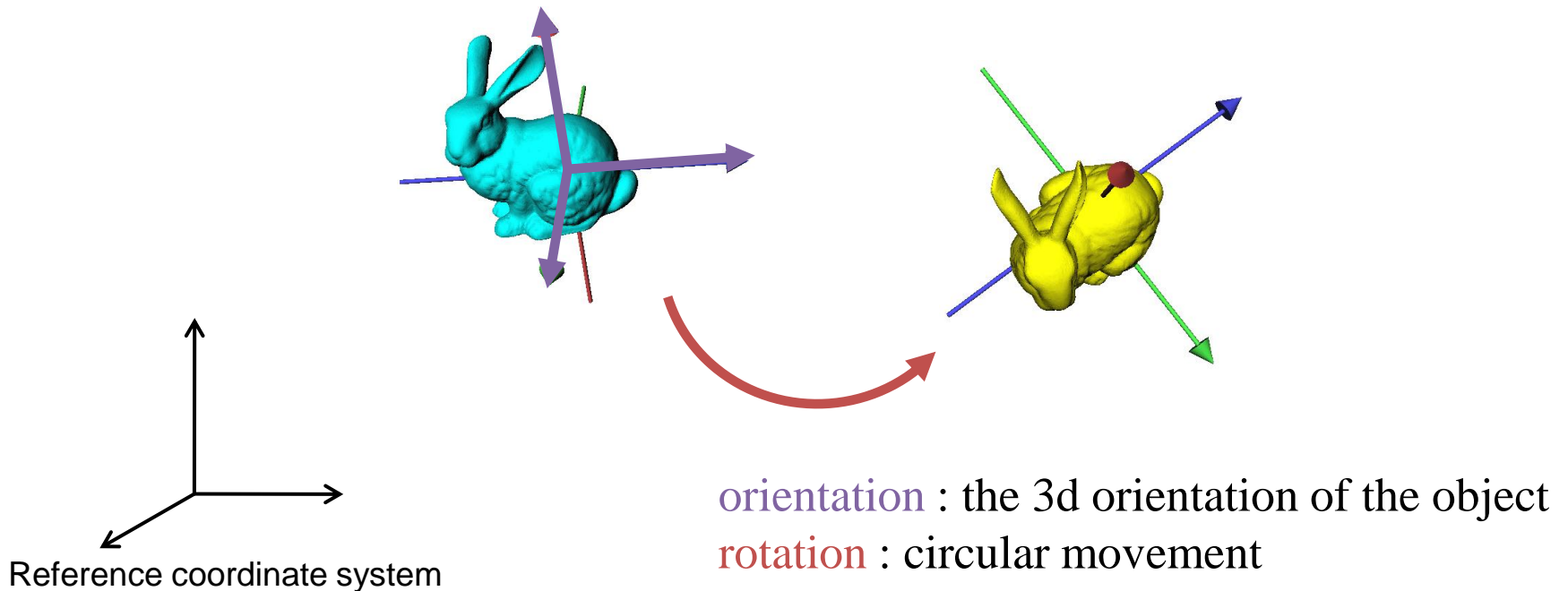
- (point : vector) is similar to (orientation : rotation)
 - Both represent a sort of (state : movement)

point : the 3d location of the object
vector : translational movement



Analogy

- (point : vector) is similar to (orientation : rotation)
 - Both represent a sort of (state : movement)



Analogy

- Point & vector

- $(\text{point}) + (\text{point}) \rightarrow (\text{UNDEFINED})$ *기준이 정해지지 않았으므로*
- $(\text{vector}) \pm (\text{vector}) \rightarrow (\text{vector})$ *차이를 더할 수 있음*
- $(\text{point}) \pm (\text{vector}) \rightarrow (\text{point})$
- $(\text{point}) - (\text{point}) \rightarrow (\text{vector})$

- Orientation & rotation

- $(\text{orientation}) (+) (\text{orientation}) \rightarrow (\text{UNDEFINED})$
- $(\text{rotation}) (\pm) (\text{rotation}) \rightarrow (\text{rotation})$
- $(\text{orientation}) (\pm) (\text{rotation}) \rightarrow (\text{orientation})$
- $(\text{orientation}) (-) (\text{orientation}) \rightarrow (\text{rotation})$

Matrix 곱 연산

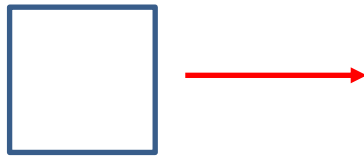
Not vector addition & subtraction

Degrees of Freedom (DOF)

자유도

- The number of independent parameters that define a unique configuration

어떤 움직임을 표현하려면 숫자 몇개가 필요한가.



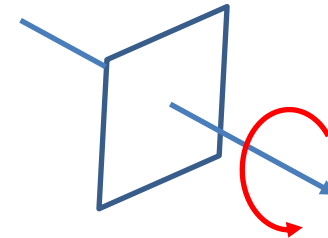
Translation along one
direction

: 1 DOF

한 방향으로만 움직임

↓
숫자 하나로 표현됨

: 1 자유도 movement

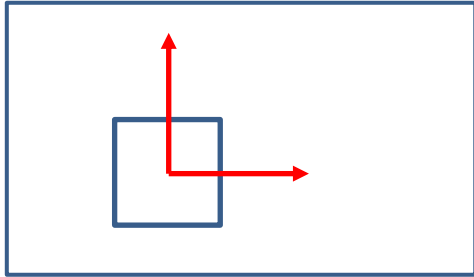


Rotation about an axis

: 1 DOF

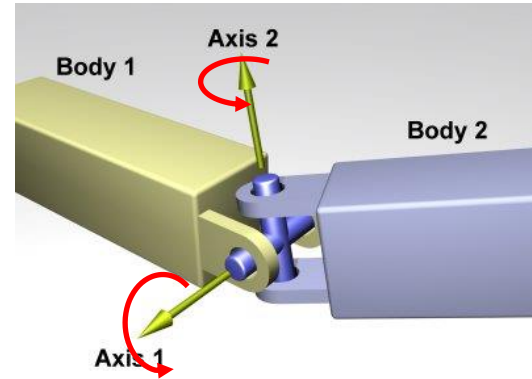
회전 축에 대한
회전

(rotation angle)



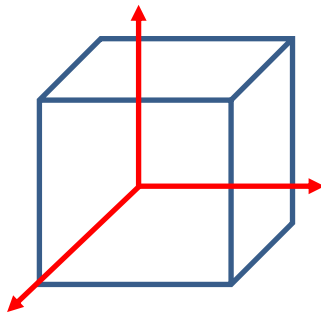
Translation on a plane

: 2 DOFs



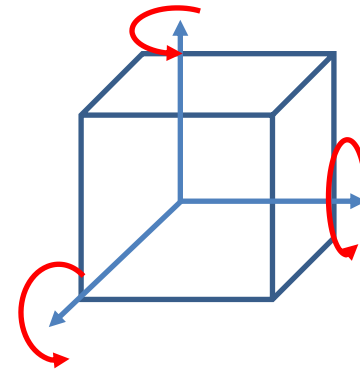
Rotation about two axes

: 2 DOFs



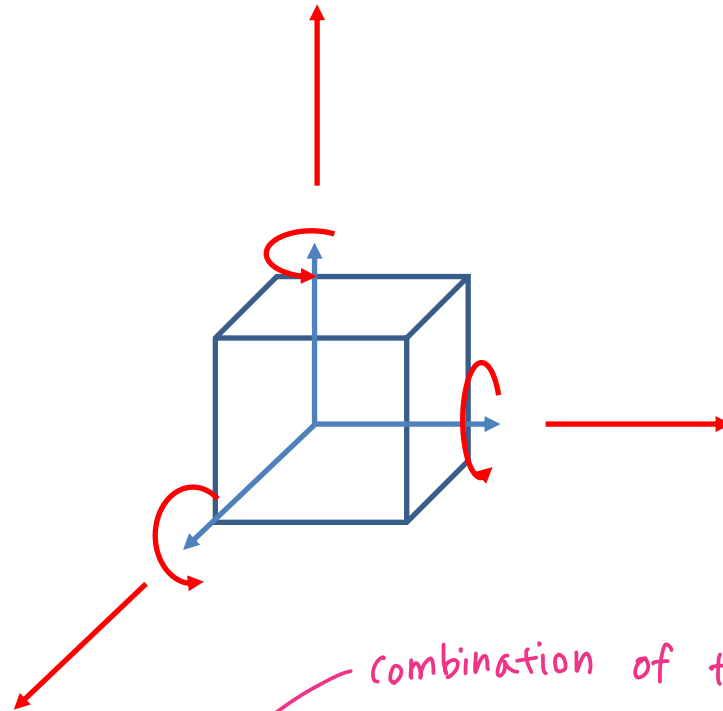
Translation in 3D space

: 3 DOFs



Rotation in 3D space

: 3 DOFs



combination of translation & rotation

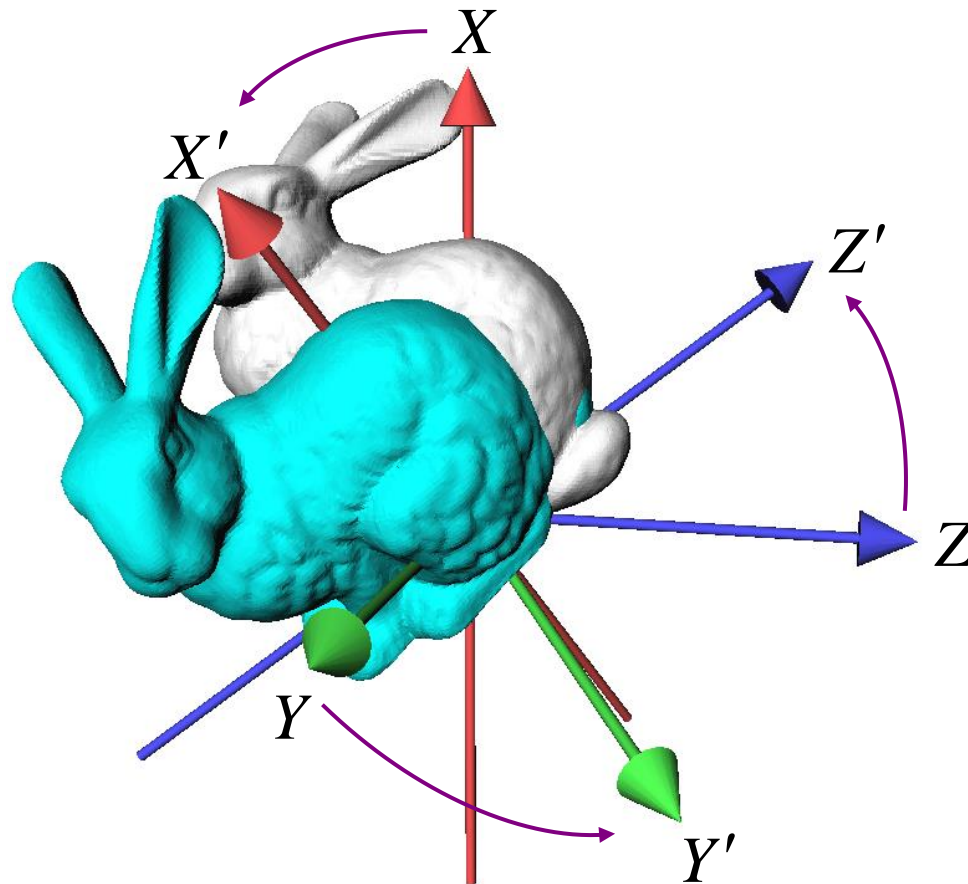
Any rigid motion in 3D
space

: 6 DOF

3D Orientation & Rotation Representations

3D Rotation

- Given two arbitrary orientations of a rigid object,



3D Rotation

- We can always find a fixed axis of rotation and an angle about the axis

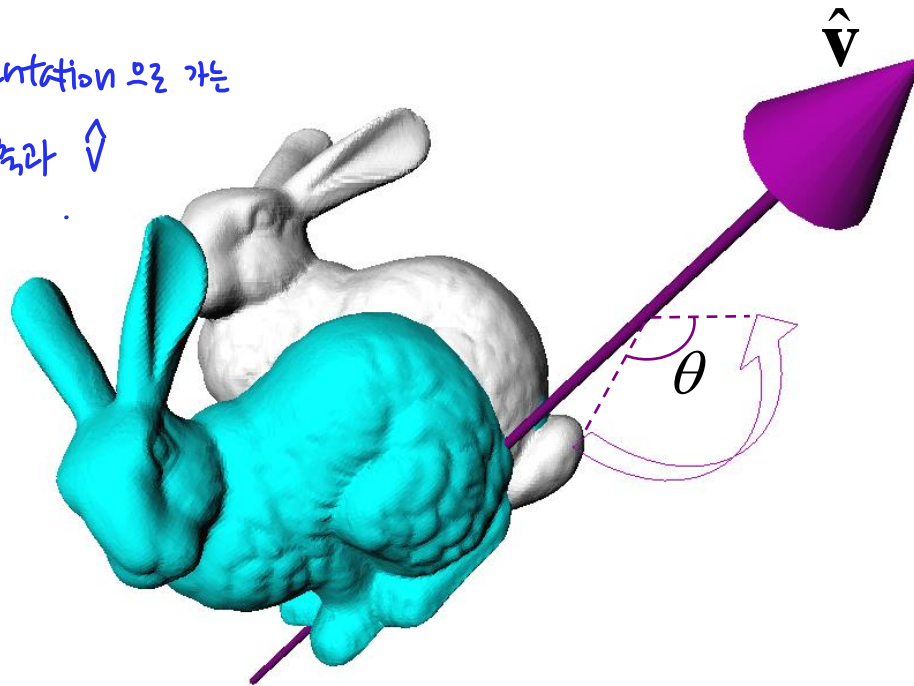
하나의 orientation 에서 다른 orientation 으로 가는

임의의 rotation 은 항상 한 회전축과 \hat{v}

그 회전축에 대한 회전 각으로 θ

표현할 수 있다

\Rightarrow Euler의 rotation Theorem



Euler's Rotation Theorem

The general displacement of a rigid body with one point fixed is a rotation about some axis

Leonhard Euler (1707-1783)

In other words,

- Arbitrary 3D rotation equals to one rotation around an axis
- Any 3D rotation leaves one vector unchanged
→ 회전축 벡터

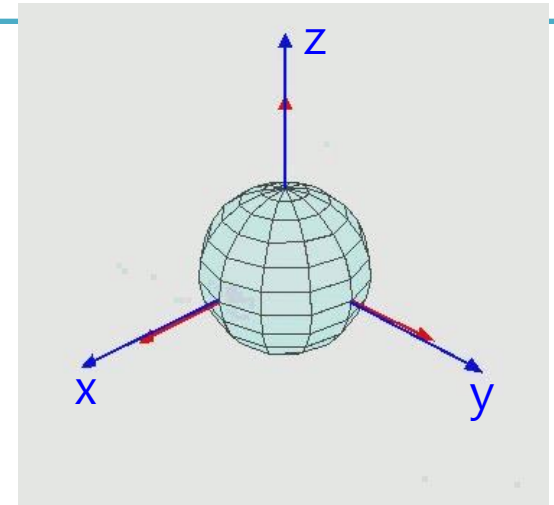
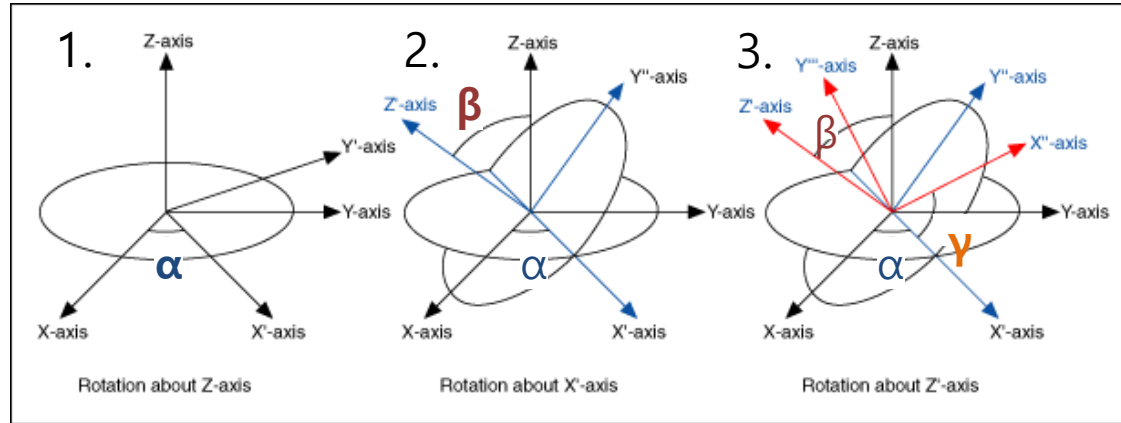
Describing 3D Rotation & Orientation

- Several ways to describe 3D rotation and orientation
 - Euler angles
 - Rotation vector (Axis-angle)
 - Rotation matrices
 - Unit quaternions

Euler Angles

- Express any arbitrary 3D rotation using **three rotation angles about three principle axes**
 - x, y, z axes

Example: ZXZ Euler Angles



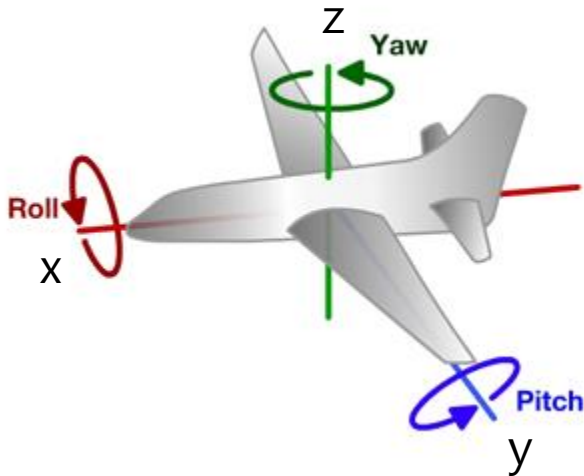
- 1. Rotate about Z-axis by α
- 2. Rotate about X-axis of the new frame by β
- 3. Rotate about Z-axis of the new frame by γ

<https://commons.wikimedia.org/wiki/File:Euler2a.gif>

$$R = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_z(\alpha) R_x(\beta) R_z(\gamma)$$

Example: Yaw-Pitch-Roll Convention (ZYX Euler Angles)



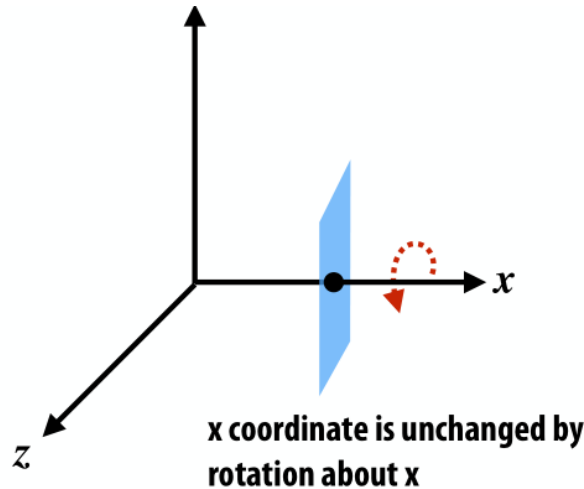
- Common for describing the orientation of aircrafts
- 1. Rotate about Z-axis by **yaw** angle
- 2. Rotate about Y-axis of the new frame by **pitch** angle
- 3. Rotate about X-axis of the new frame by **roll** angle

$$R = R_z(\text{yaw}) R_y(\text{pitch}) R_x(\text{roll})$$

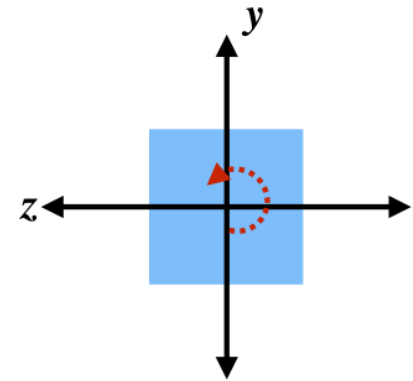
Recall: Rotation Matrix in 3D

Rotation about x axis:

$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$



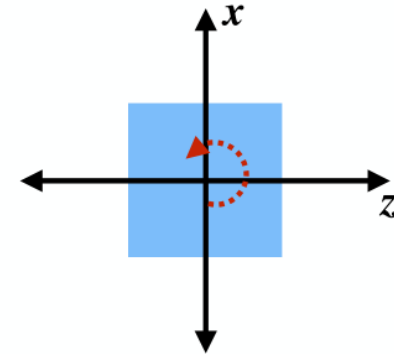
View looking down -x axis:



Rotation about y axis:

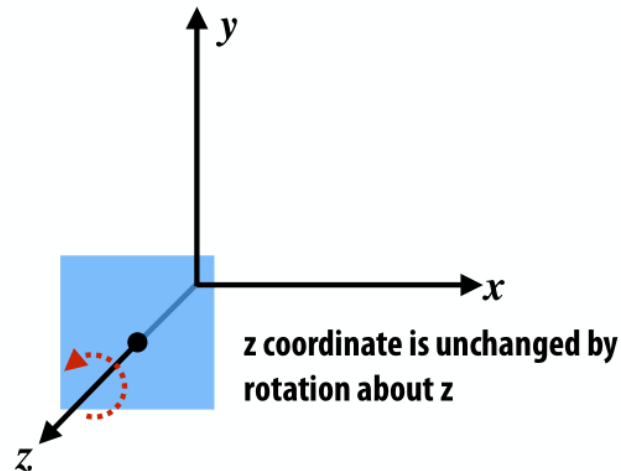
$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \boxed{-}\sin \theta & 0 & \cos \theta \end{bmatrix}$$

View looking down -y axis:



Rotation about z axis:

$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Euler Angles

- Possible 12 combinations

- XYZ, XYX, XZY, XZX

- YZX, YZY, YXZ, YXY

- ZXY, ZXZ, ZYX, ZYZ

- (Combination is possible as long as the same axis does not appear consecutively.)

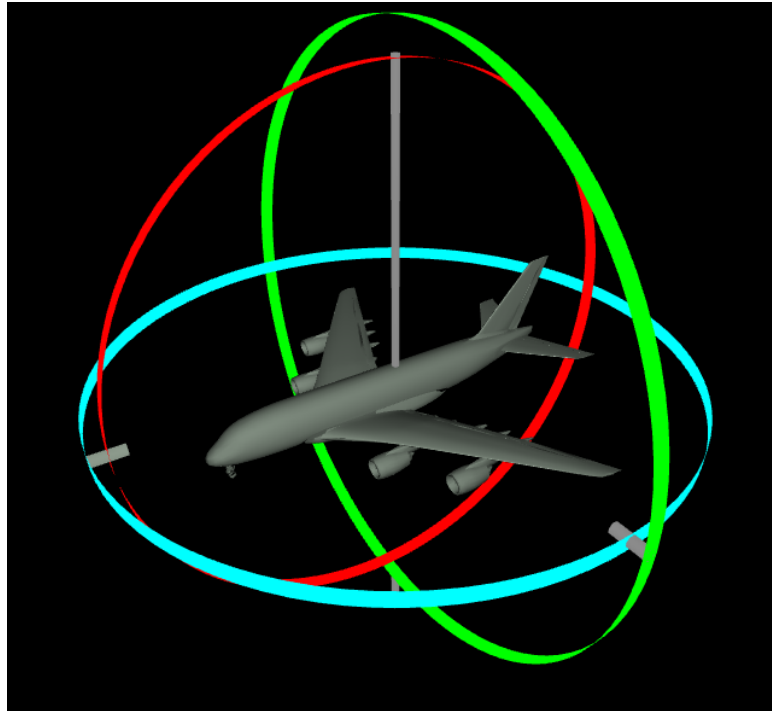
XXY 연속으로 같은 두 축

이런 combination은 불가능

→ 두 정보밖에 가지고 있지않음

3차원공간상의 회전은 세가지 정보 필요.

[Practice] Euler Angles Online Demo

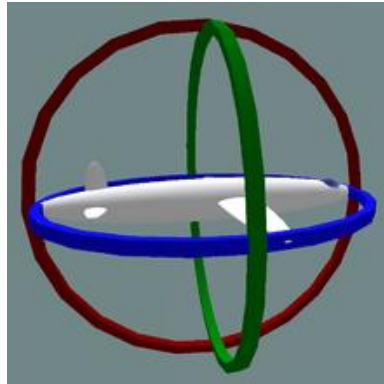


<http://www.ctr.alie.com/Teaching/COMPS/CI290/Materials/EulerAnglesViz/>

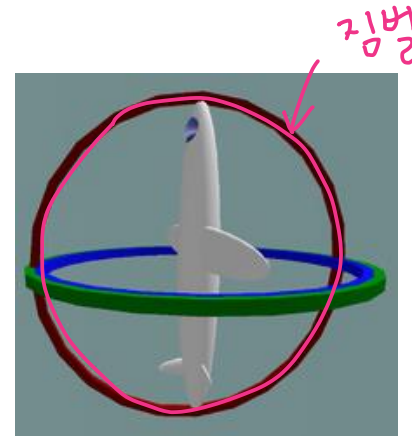
- Try to change yaw, pitch, roll angles

Gimbal Lock

- One potential problem that Euler angles can suffer from is 'gimbal lock'
- This results when two axes effectively line up, resulting in a temporary **loss of a degree of freedom**



Normal situation.
The plane can rotate
in any directions



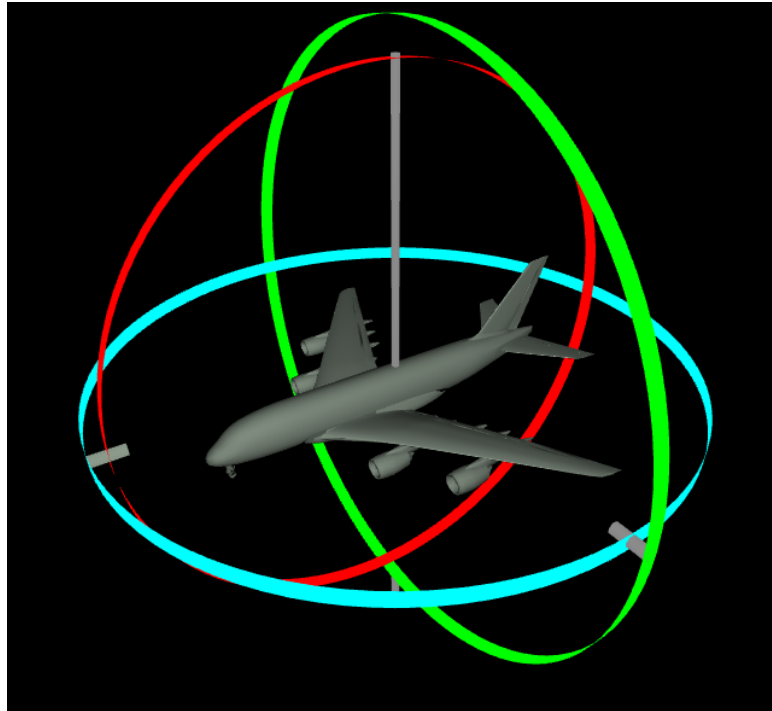
Gimbal lock:

(two out of the three
gimbals are in the same
plane, one DoF is lost)

→ 특정 방향으로 회전 X, 자유도를 하나 잃게 됨

- Euler angles have **singularities**, i.e., it loses DoFs (can't move in a certain direction) at some configurations

[Practice] Gimbal Lock



<http://www.ctr.alie.com/Teaching/COMPS/CI290/Materials/EulerAnglesViz/>

- Make gimbal lock by aligning two of three rotation axes
 - Set pitch to 90 degrees

[Practice] Euler Angles in OpenGL

- Start with the practice code from the lecture “7 - Lighting & Shading”,
- Just replace render() function

```

def render():
    global gCamAng, gCamHeight

    glClear(GL_COLOR_BUFFER_BIT|GL_DEPTH_BUFFER_BIT)

    glEnable(GL_DEPTH_TEST)

    glMatrixMode(GL_PROJECTION)
    glLoadIdentity()
    gluPerspective(45, 1, 1,10)

    glMatrixMode(GL_MODELVIEW)
    glLoadIdentity()

    gluLookAt(5*np.sin(gCamAng),gCamHeight,5*np.cos(gCamAng), 0,0,0, 0,1,0)

    # draw global frame
    drawFrame()

    glEnable(GL_LIGHTING)
    glEnable(GL_LIGHT0)
    glEnable(GL_RESCALE_NORMAL)

    # set light properties
    lightPos = (4.,5.,6.,1.)
    glLightfv(GL_LIGHT0, GL_POSITION, lightPos)

    ambientLightColor = (.1,.1,.1,1.)
    diffuseLightColor = (1.,1.,1.,1.)
    specularLightColor = (1.,1.,1.,1.)
    glLightfv(GL_LIGHT0, GL_AMBIENT, ambientLightColor)
    glLightfv(GL_LIGHT0, GL_DIFFUSE, diffuseLightColor)
    glLightfv(GL_LIGHT0, GL_SPECULAR, specularLightColor)

```

```

# ZYX Euler angles
t = glfw.get_time()
xang = t
yang = np.radians(30)
zang = np.radians(30)
M = np.identity(4)
Rx = np.array([[1,0,0],
               [0, np.cos(xang), -np.sin(xang)],
               [0, np.sin(xang), np.cos(xang)]]])
Ry = np.array([[np.cos(yang), 0, np.sin(yang)],
               [0,1,0],
               [-np.sin(yang), 0, np.cos(yang)]]])
Rz = np.array([[np.cos(zang), -np.sin(zang), 0],
               [np.sin(zang), np.cos(zang), 0],
               [0,0,1]])
M[:3,:3] = Rz @ Ry @ Rx
glMultMatrixf(M.T)

# # The same ZYX Euler angles with OpenGL functions
# glRotate(30, 0,0,1)
# glRotate(30, 0,1,0)
# glRotate(np.degrees(xang), 1,0,0)

glScalef(.25,.25,.25)

# draw cubes
glMaterialfv(GL_FRONT, GL_AMBIENT_AND_DIFFUSE, (.5,.5,.5,1.))
drawCube_glDrawArray()

glTranslatef(2.5,0,0)
glMaterialfv(GL_FRONT, GL_AMBIENT_AND_DIFFUSE, (1.,0.,0.,1.))
drawCube_glDrawArray()

glTranslatef(-2.5,2.5,0)
glMaterialfv(GL_FRONT, GL_AMBIENT_AND_DIFFUSE, (0.,1.,0.,1.))
drawCube_glDrawArray()

glTranslatef(0,-2.5,2.5)
glMaterialfv(GL_FRONT, GL_AMBIENT_AND_DIFFUSE, (0.,0.,1.,1.))
drawCube_glDrawArray()
glDisable(GL_LIGHTING)

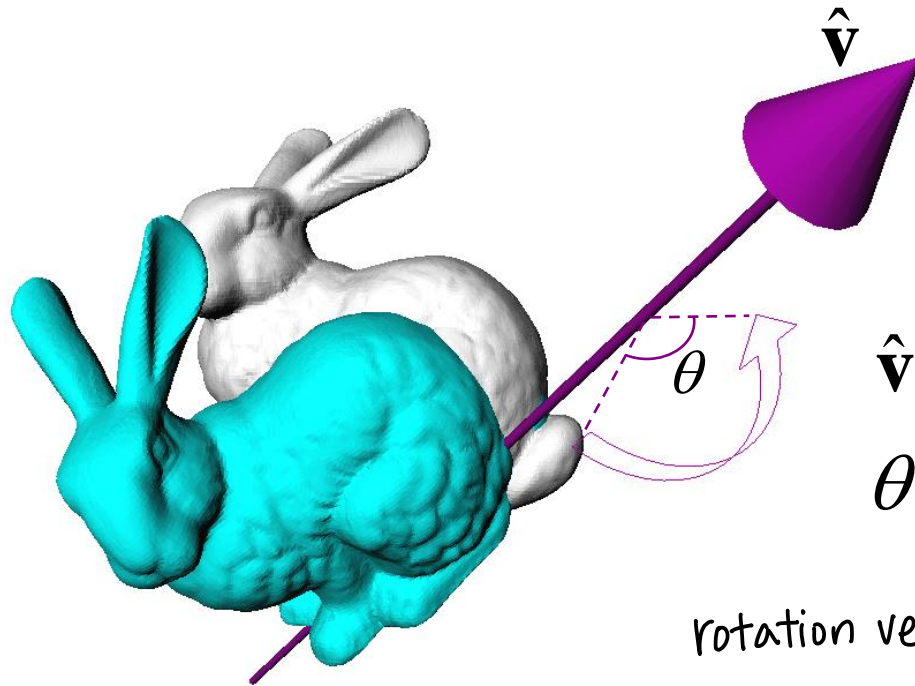
```

→ 최종적으로 local frame 구축에 대해 회전 시점에 따라

Quiz #1

- Go to <https://www.slido.com/>
- Join #cg-ys
- Click “Polls”
- Submit your answer in the following format:
 - **Student ID: Your answer**
 - e.g. **2017123456: 4)**
- Note that you must submit all quiz answers in the above format to be checked for “attendance”.

Rotation Vector (Axis-Angle)



$\hat{\mathbf{v}}$: rotation axis (unit vector)

θ : scalar angle

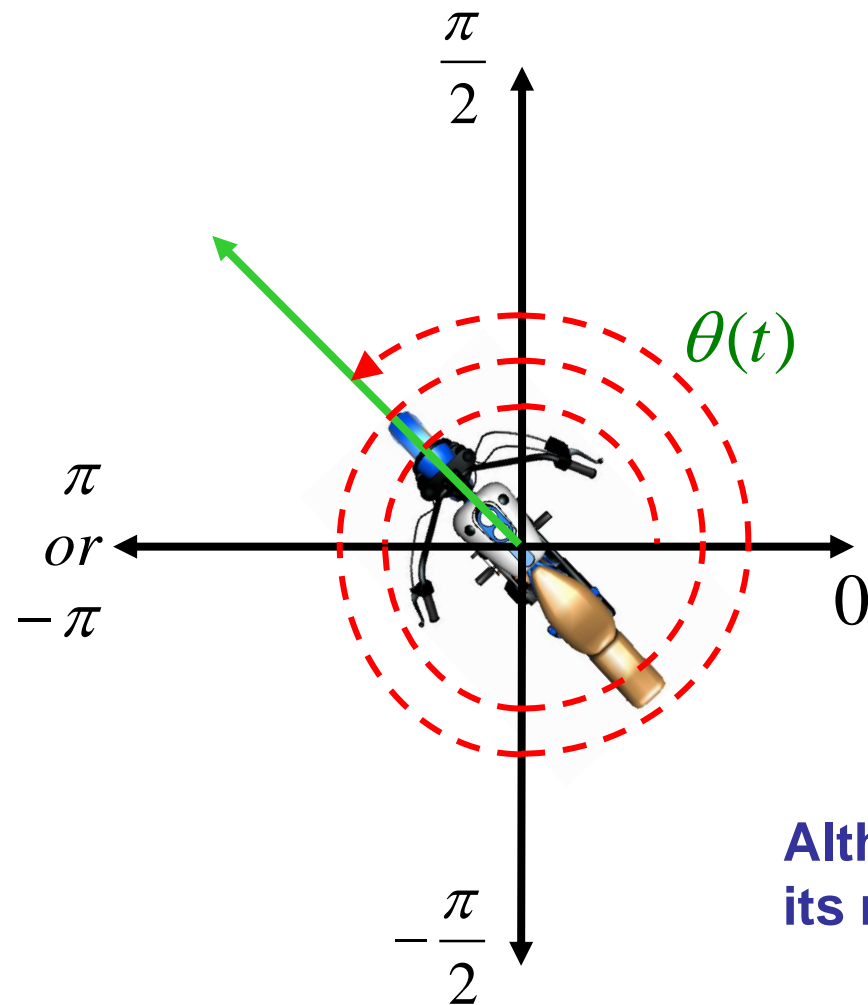
rotation vector의 길이가
 \Rightarrow rotation angle

- Rotation vector $\mathbf{v} = \theta \hat{\mathbf{v}} = (x, y, z)$
- Axis-Angle $(\theta, \hat{\mathbf{v}})$

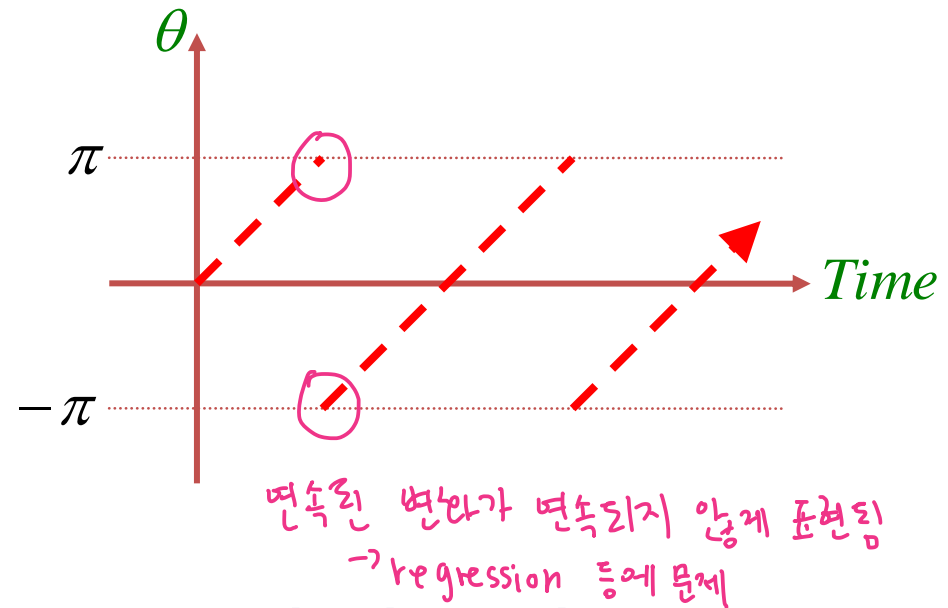
Discontinuity (or Many-to-one Correspondences) Problem

- Euler angles and rotation vector use 3 parameters.
- But, expressing 3D orientation using 3 parameters has problems:
- Euler angles
 - Discontinuity (or many-to-one correspondences)
 - **Gimbal lock**
- Rotation Vector (Axis-Angle)
 - Discontinuity (or many-to-one correspondences)

Problem of Representing 2D Orientation by Angle θ : Discontinuity

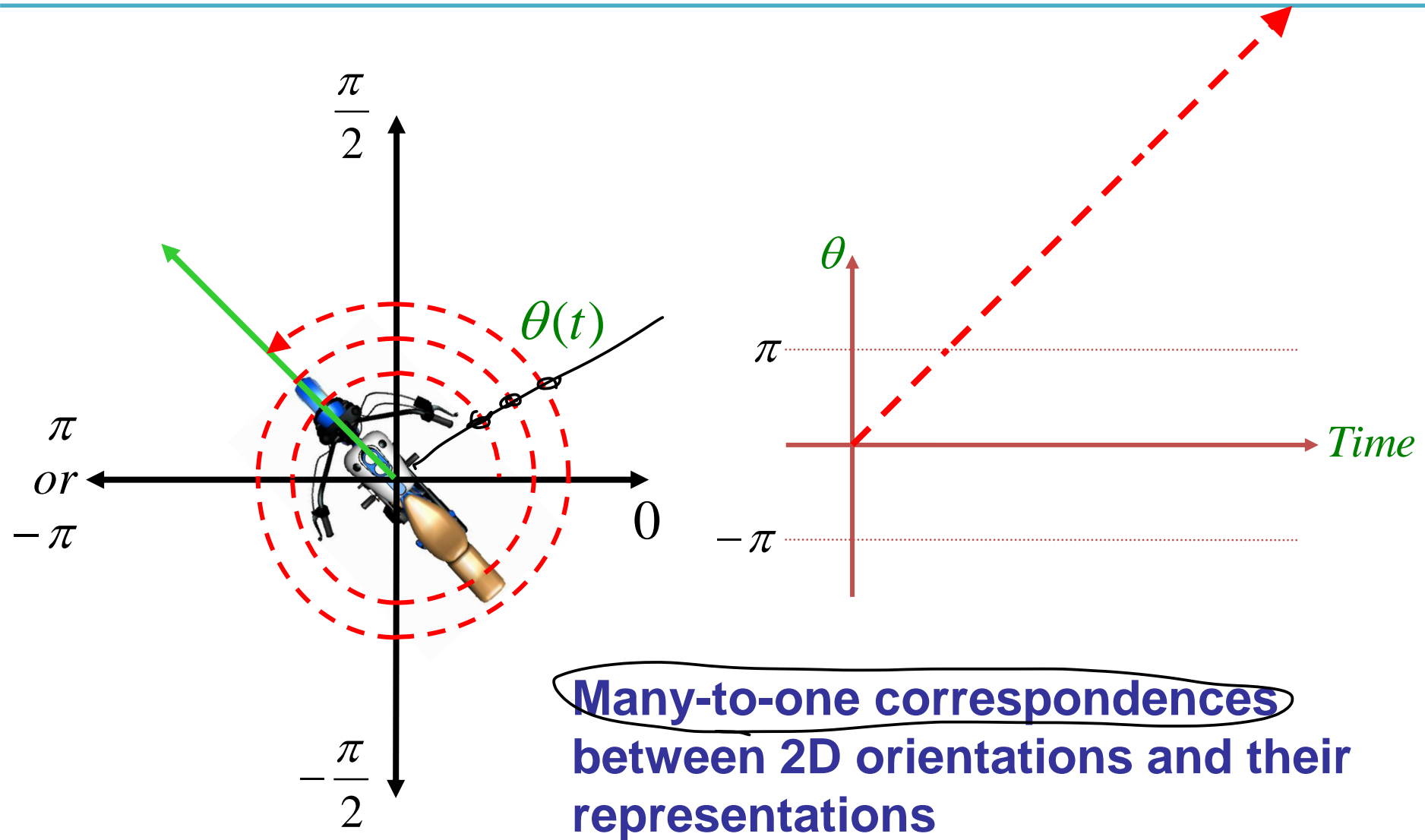


orientation을
angle θ 하나로 표현해버려!
↪



**Although the motion is continuous,
its representation could be discontinuous**

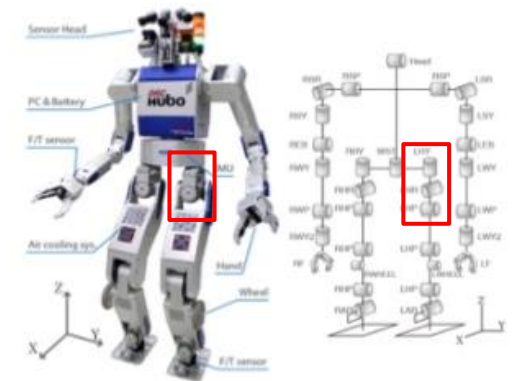
Problem of Representing 2D Orientation by Angle θ : Many-to-one Correspondence



Discontinuity (or Many-to-one Correspondences) Problem

- To avoid these problems, we need more parameters than DOFs
 - Rotation matrices $(3 \times 3) \rightarrow$ 숫자 9개 씩
 - Unit quaternions \rightarrow 숫자 4개 씩
- But Euler angles is still meaningful because
 - It's the most common way to implement actuated 3 DOF rotational joints in real world.

3자유도 회전
모터 3개 붙여서
힘 조인트
← 모질거
사용



Rotation Matrices

- Rotation in 3D space can be represented as 3x3 matrix:

**Rotation matrix
about x, y, z axis**

Rotation about x axis:

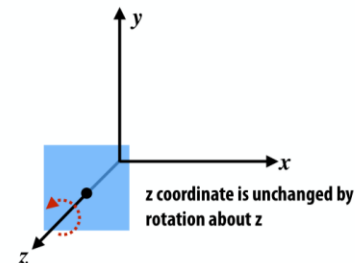
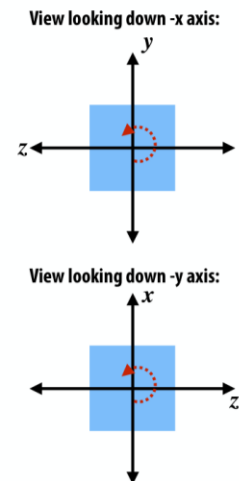
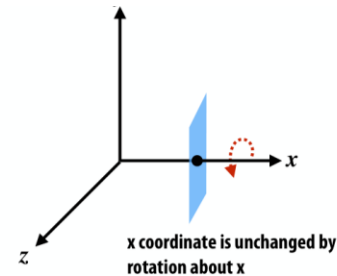
$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

Rotation about y axis:

$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

Rotation about z axis:

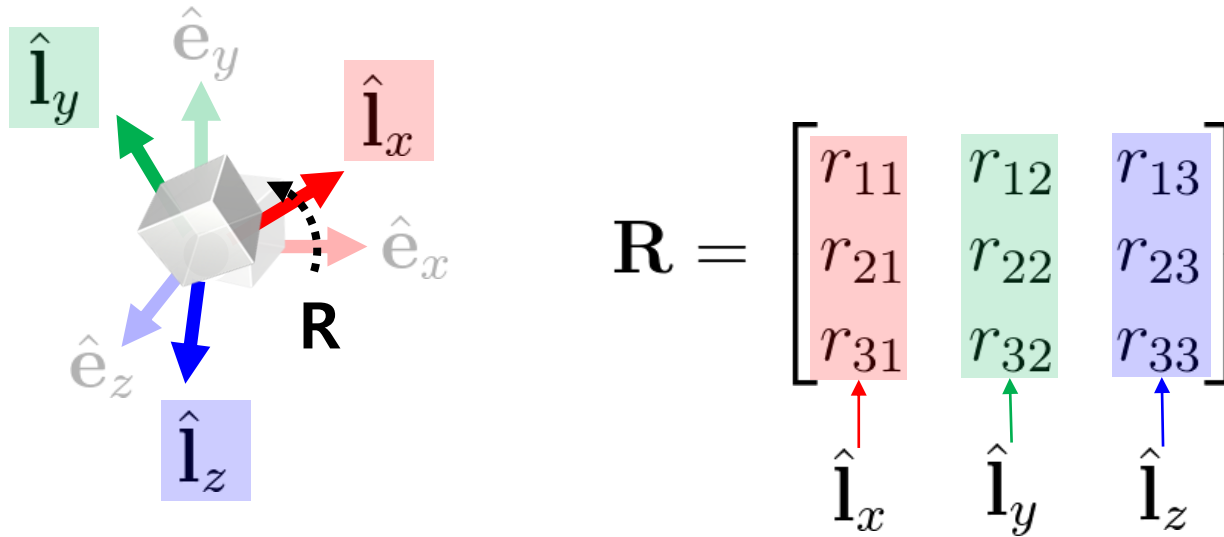
$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



**Rotation matrix from
ZXZ Euler angles**

$$\mathbf{R} = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Meaning of Rotation Matrix



- A rotation matrix defines
 - **Orientation** of new rotated frame or,
 - **Rotation** from a global frame to be that rotated frame

Mathematical Properties of Rotation Matrix

- A square matrix \mathbf{R} is a rotation matrix if and only if

$$\boxed{1. \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}} \&\& \boxed{2. \det(\mathbf{R}) = 1}$$

- For details, see 9 - *reference-rotmat-properties.pdf*
- A rotation matrix is an **orthogonal matrix** with **determinant 1**
 - Sometimes it is called *special orthogonal matrix*
 - A set of rotation matrices of size 3 forms a *special orthogonal group*, $SO(3)$

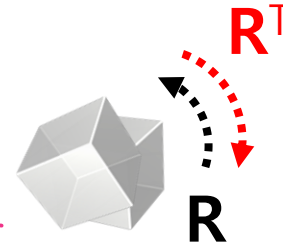
↳ 3차원 공간상의 rotation matrix

Geometric Properties of Rotation Matrix

- \mathbf{R}^T is an inverse rotation of \mathbf{R}

– Because, $\mathbf{R}\mathbf{R}^T = \mathbf{I} \iff \mathbf{R}^{-1} = \mathbf{R}^T$

반대방향 회전 : matrix에 transpose 취함



- $\mathbf{R}_1\mathbf{R}_2$ is a rotation matrix as well (composite rotation)

– proof) $(\mathbf{R}_1\mathbf{R}_2)^T(\mathbf{R}_1\mathbf{R}_2) = \mathbf{R}_2^T\mathbf{R}_1^T\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2^T\mathbf{R}_2 = \mathbf{I}$

and $\det(\mathbf{R}_1\mathbf{R}_2) = \det(\mathbf{R}_1) \cdot \det(\mathbf{R}_2) = 1$

- The length of vector \mathbf{v} is not changed after applying a rotation matrix \mathbf{R}

– proof) $\|\mathbf{R}\mathbf{v}\|^2 = (\mathbf{R}\mathbf{v})^T(\mathbf{R}\mathbf{v}) = \mathbf{v}^T\mathbf{R}^T\mathbf{R}\mathbf{v} = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$

$$\begin{array}{c} \boxed{\mathbf{v}^T} \\ \downarrow \\ \boxed{\mathbf{v}} \end{array} = \mathbf{v} \cdot \mathbf{v}$$

[Practice] Properties of Rotation Matrix

- Start with the previous practice code
- Just replace `render()` function

```

def render():
    global gCamAng, gCamHeight
    glClear(GL_COLOR_BUFFER_BIT|GL_DEPTH_BUFFER_BIT)
    glEnable(GL_DEPTH_TEST)

    glMatrixMode(GL_PROJECTION)
    glLoadIdentity()
    gluPerspective(45, 1, 1, 10)

    glMatrixMode(GL_MODELVIEW)
    glLoadIdentity()

    gluLookAt(5*np.sin(gCamAng), gCamHeight, 5*np.cos(gCamAng),
0, 0, 0, 0, 1, 0)
    drawFrame() # draw global frame

    glEnable(GL_LIGHTING)
    glEnable(GL_LIGHT0)
    glEnable(GL_RESCALE_NORMAL) # rescale normal

    glLightfv(GL_LIGHT0, GL_POSITION, (1., 2., 3., 1.))
    glLightfv(GL_LIGHT0, GL_AMBIENT, (.1, .1, .1, 1.))
    glLightfv(GL_LIGHT0, GL_DIFFUSE, (1., 1., 1., 1.))
    glLightfv(GL_LIGHT0, GL_SPECULAR, (1., 1., 1., 1.))

    # ZYX Euler angles
    t = glfw.get_time()
    xang = t
    yang = np.radians(30)
    zang = np.radians(30)
    M = np.identity(4)
    Rx = np.array([[1, 0, 0],
                    [0, np.cos(xang), -np.sin(xang)],
                    [0, np.sin(xang), np.cos(xang)]]])
    Ry = np.array([[np.cos(yang), 0, np.sin(yang)],
                    [0, 1, 0],
                    [-np.sin(yang), 0, np.cos(yang)]]])
    Rz = np.array([[np.cos(zang), -np.sin(zang), 0],
                    [np.sin(zang), np.cos(zang), 0],
                    [0, 0, 1]])

```

```

R = Rz @ Ry @ Rx
# # check inverse rotation
# R = Rz @ Ry @ Rx.T x축 회전만 반대방향

# # check R @ R.T
# print(R @ R.T) → identity 3x3 matrix

# # check determinant
# print(np.linalg.det(R)) → 1
# determinant=1 이 계산됨

M[:3, :3] = R
glMultMatrixf(M.T)

glScalef(.25, .25, .25)

# draw cubes
glMaterialfv(GL_FRONT,
GL_AMBIENT_AND_DIFFUSE, (.5, .5, .5, 1.))
drawCube_glDrawArray()

glTranslatef(2.5, 0, 0)
glMaterialfv(GL_FRONT,
GL_AMBIENT_AND_DIFFUSE, (1., 0., 0., 1.))
drawCube_glDrawArray()

glTranslatef(-2.5, 2.5, 0)
glMaterialfv(GL_FRONT,
GL_AMBIENT_AND_DIFFUSE, (0., 1., 0., 1.))
drawCube_glDrawArray()

glTranslatef(0, -2.5, 2.5)
glMaterialfv(GL_FRONT,
GL_AMBIENT_AND_DIFFUSE, (0., 0., 1., 1.))
drawCube_glDrawArray()

glDisable(GL_LIGHTING)

```

Rotation Matrix for Rotation about an Arbitrary Axis

- Euler's Rotation Theorem tells us that an arbitrary 3D rotation equals to one rotation around an axis.
- Then, how to compute the rotation matrix for given axis vector $u=(u_x, u_y, u_z)$ by angle θ ?
- A naive, inefficient method:
 - Step 1: rotate the axis u so that it is aligned with the Z-axis
 - Step 2: rotate about the Z-axis by the angle θ
 - Step 3: rotate the Z-axis back to the original axis
 - For details, see 9 - *reference-naive-rotvec2rotmat.pdf*

\Rightarrow 비효율적

Rotation Matrix for Rotation about an Arbitrary Axis

- More efficient solution: Rodrigues' rotation formula
- Rotation about a normalized axis vector
 $\mathbf{u}=(u_x, u_y, u_z)$ by angle θ : $= \text{normal vector}$

$$R = \begin{bmatrix} \cos \theta + u_x^2 (1 - \cos \theta) & u_x u_y (1 - \cos \theta) - u_z \sin \theta & u_x u_z (1 - \cos \theta) + u_y \sin \theta \\ u_y u_x (1 - \cos \theta) + u_z \sin \theta & \cos \theta + u_y^2 (1 - \cos \theta) & u_y u_z (1 - \cos \theta) - u_x \sin \theta \\ u_z u_x (1 - \cos \theta) - u_y \sin \theta & u_z u_y (1 - \cos \theta) + u_x \sin \theta & \cos \theta + u_z^2 (1 - \cos \theta) \end{bmatrix}$$

(You do not have to memorize this)

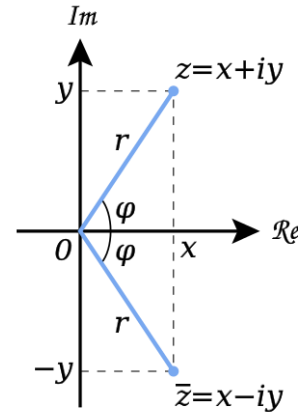
Quiz #2

- Go to <https://www.slido.com/>
- Join #cg-ys
- Click “Polls”
- Submit your answer in the following format:
 - **Student ID: Your answer**
 - e.g. **2017123456: 4)**
- Note that you must submit all quiz answers in the above format to be checked for “attendance”.

Quaternions

- Complex numbers can be used to represent 2D rotations

$$z = x + iy \quad \text{where} \quad i^2 = -1$$



- Basic idea: Quaternion is its extension to 3D space

$$q = w + ix + jy + kz \quad \text{where}$$

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1 \\ ij &= k, \quad jk = i, \quad ki = j \\ ji &= -k, \quad kj = -i, \quad ik = -j \end{aligned}$$

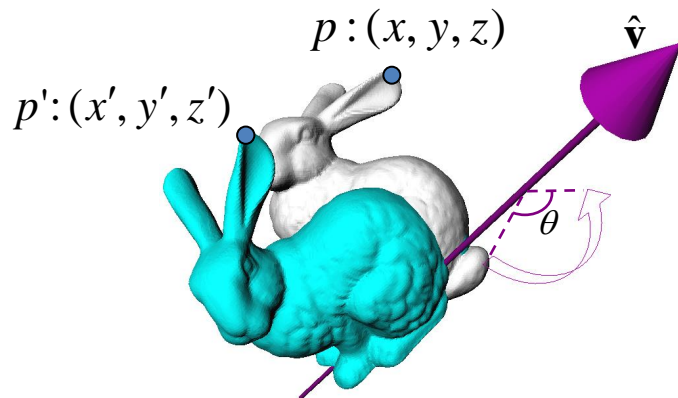
Unit Quaternions

- Unit quaternions represent 3D rotations

$$\begin{aligned}\mathbf{q} &= w + ix + jy + kz \\ &= (w, x, y, z) \\ &= (w, \mathbf{v})\end{aligned}$$

$$w^2 + x^2 + y^2 + z^2 = 1$$

- Rotation about axis $\hat{\mathbf{v}}$ by angle θ



$$\mathbf{q} = \left(\cos \frac{\theta}{2}, \hat{\mathbf{v}} \sin \frac{\theta}{2} \right)$$

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1} \quad \text{where} \quad \mathbf{p} = (0, x, y, z)$$

For details, see 9 - *reference-quaternions.pdf*

Which Representation to Use?

- 3D orientation & rotation representation
 - Euler angles
 - Rotation Vector (Axis-Angle)
 - Rotation matrices
 - Unit quaternions
- Which one to use?
- General recommendation: **rotation matrices or unit quaternions.**
 - Reason: Euler angles and rotation vector have gimbal lock or discontinuity (many-to-one correspondences) problems.
- But you may need other representations depending on the context.
 - Euler angles are useful for hardware implementation of ball joints.

Which Representation to Use?

- Rotation matrices and unit quaternions do not have discontinuity or gimbal lock problems
 - Because they use more parameters (rotation matrix: 9, unit quaternion: 4) than DOFs of 3D orientation/rotation (3)
- Rotation matrices vs. Unit quaternions ?

Rotation Matrix vs. Unit Quaternion

- Equivalent in many aspects
 - Redundant
 - No singularity
 - Can be converted from & to axis-angle representation
- Why quaternions ?
 - Fewer parameters
 - Simpler algebra
 - Easy to fix numerical error
- Why rotation matrices ?
 - One-to-one correspondence (quaternion: 2-to-1 correspondence)
 - Handle rotation and translation in a uniform way
 - Eg) 4x4 homogeneous matrices

Conversion Between Representations

- **Rotation vector \rightarrow Rotation matrix**
 - Rodrigues' rotation formula, ...
- **Rotation matrix \rightarrow Rotation vector**
 - We'll see one of the methods soon.
- **Euler angles \rightarrow Rotation matrix**
 - Building canonical rotation matrices (\mathbf{R}_x , \mathbf{R}_y , \mathbf{R}_z) and composing them
- **Rotation matrix \rightarrow Euler angles**
 - Not covered in this class (but you can easily google it)
- **Unit quaternion \leftrightarrow Rotation matrix**
 - Not covered in this class (but you can easily google it)

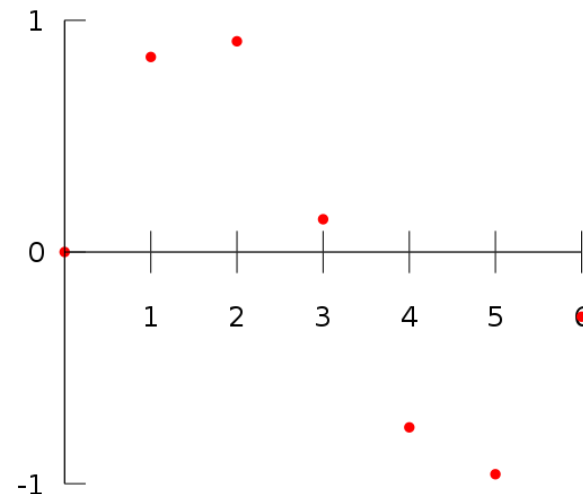
3D Orientation Interpolation

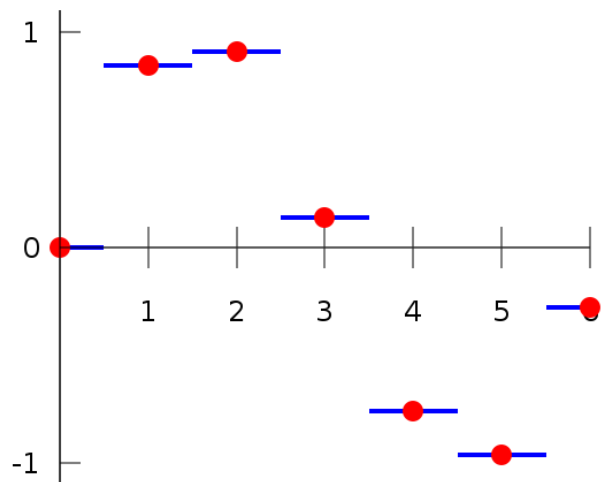
Interpolation

- A method of constructing new data points within the range of a discrete set of known data points.
- In other words, guessing unknown function $f(x)$ from known data points $(x_i, f(x_i))$.

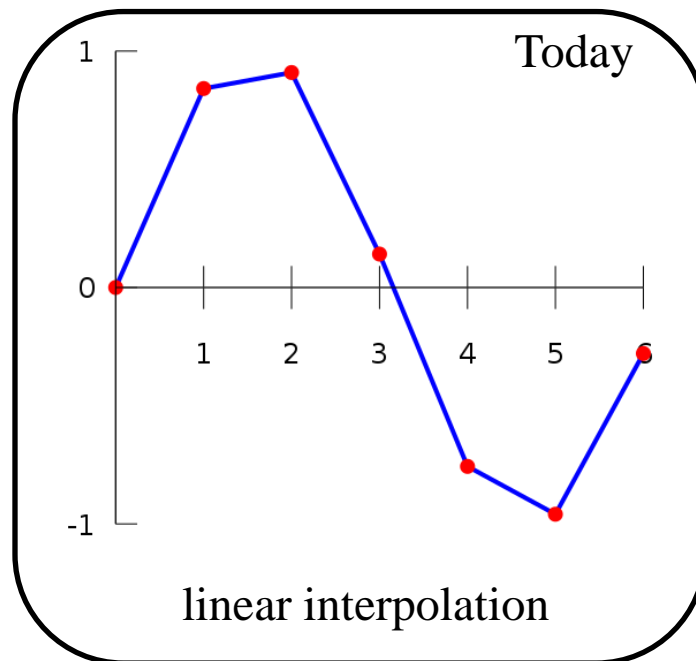
Ex) Known data points

x	f(x)
0	0
1	0.8415
2	0.9093
3	0.1411
4	-0.7568
5	-0.9589
6	-0.2794

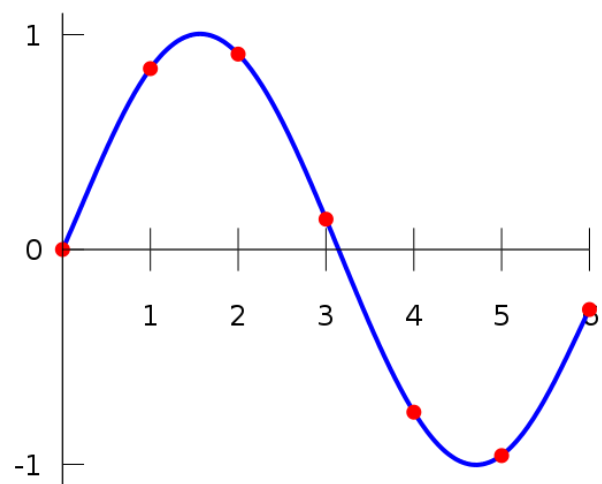




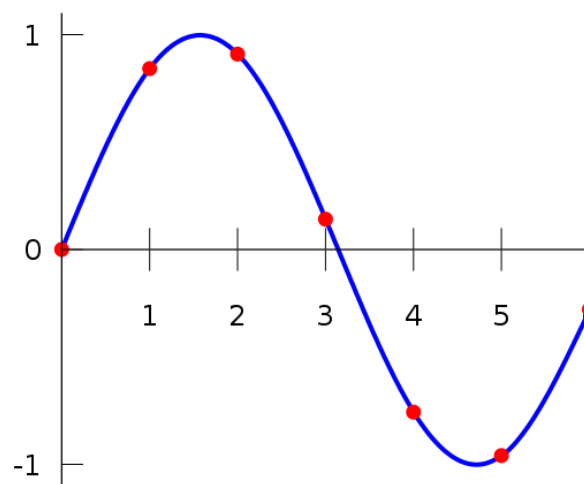
nearest-neighbor interpolation



linear interpolation

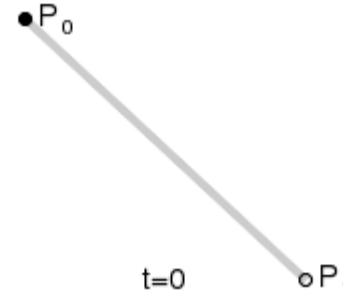
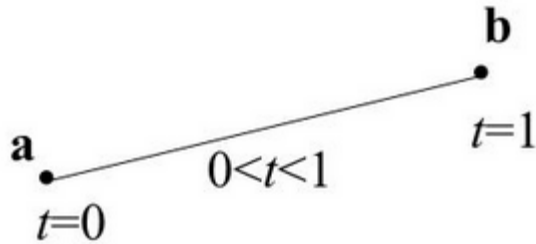


polynomial interpolation



spline interpolation

Linear Interpolation



https://upload.wikimedia.org/wikipedia/commons/0/00/B%C3%A9zier_1_big.gif

$$\text{lerp}(\mathbf{a}, \mathbf{b}, t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

```
float lerp(float v0, float v1, float t)
{
    return (1 - t) * v0 + t * v1;
}
```

- A straight line between two points
- This is fine for translations

Linear Interpolation for 3D Orientations?

- Recall: 3D orientation & rotation representation
 - Euler angles
 - Rotation vector
 - Rotation matrices
 - Unit quaternions
- How to linearly interpolate **two orientations** in these representations?

Interpolating Each Element of Rotation Matrix?

- Let's try to interpolate \mathbf{R}_0 (identity) and \mathbf{R}_1 (rotation by 90° about x-axis)

$$\text{lerp}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, 0.5\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

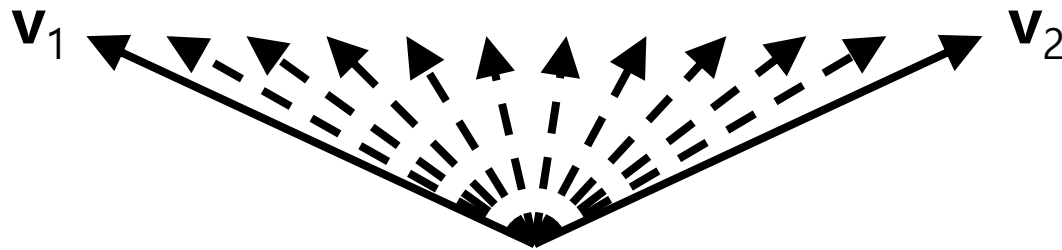
is not a rotation matrix!
does not make sense at all!

$$\mathbf{R} \cdot \mathbf{R}^T \neq \mathbf{I}$$

- Similarly, interpolating each number (w, x, y, z) in unit quaternions does not make sense.

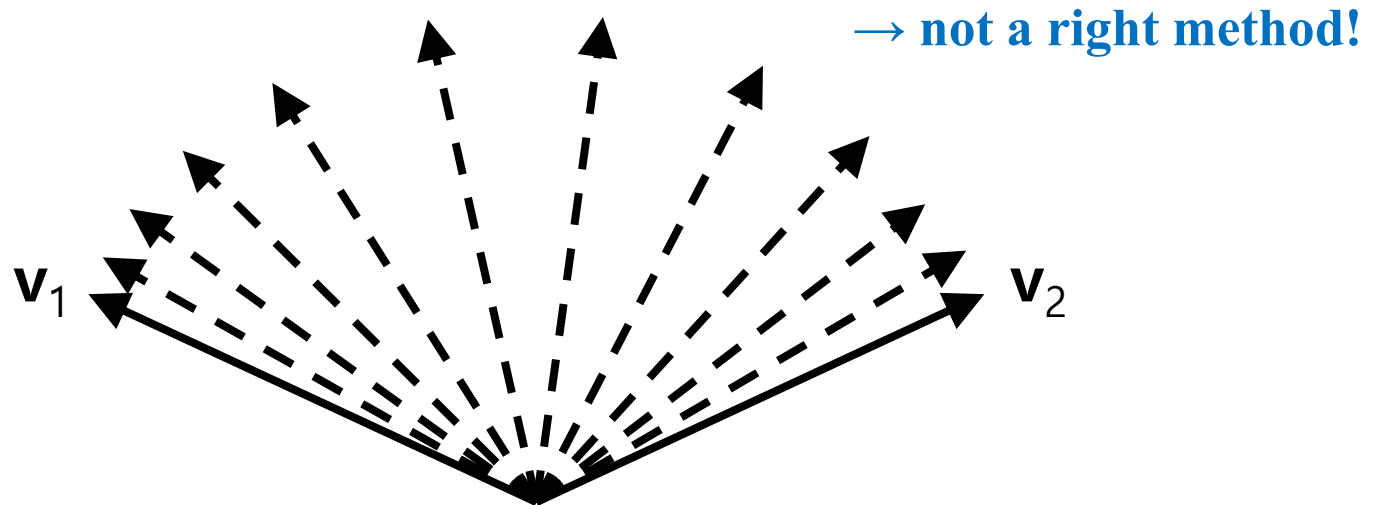
Interpolating Rotation Vector?

- Let's say we have two rotation vectors \mathbf{v}_1 & \mathbf{v}_2 of the same length
- Linear interpolation of \mathbf{v}_1 & \mathbf{v}_2 produces even spacing



Interpolating Rotation Vector?

- Let's say we have two rotation vectors \mathbf{v}_1 & \mathbf{v}_2 of the same length
- Linear interpolation of \mathbf{v}_1 & \mathbf{v}_2 produces even spacing
- But it's not evenly spaced in terms of orientation!

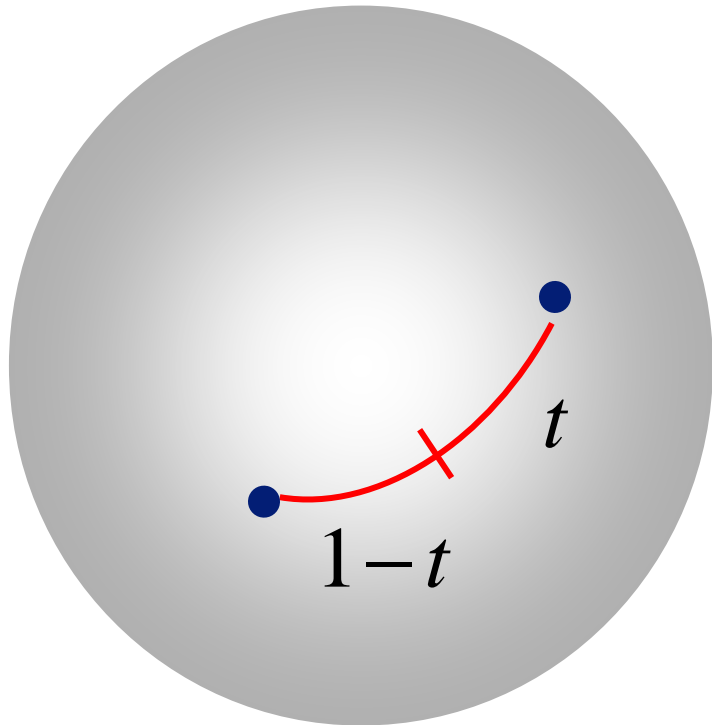


Interpolating Euler Angles?

- Interpolating two tuples of Euler angles does not make correct result
 - + angular velocity is not constant
 - + still suffer from gimbal lock: jerky movement occurs near gimbal lock configuration

Slerp

- The right answer: **Slerp** [Shoemake 1985]
 - Spherical linear interpolation
 - Linear interpolation of two orientations



“t” refers power, not transpose

$$\begin{aligned}\text{slerp}(\mathbf{R}_1, \mathbf{R}_2, t) &= \mathbf{R}_1 (\mathbf{R}_1^T \mathbf{R}_2)^{\overset{\uparrow}{t}} \\ &= \mathbf{R}_1 \exp(t \cdot \log(\mathbf{R}_1^T \mathbf{R}_2))\end{aligned}$$

Slerp

$$\text{slerp}(\mathbf{R}_1, \mathbf{R}_2, t) = \mathbf{R}_1 (\mathbf{R}_1^T \mathbf{R}_2)^t$$

$$= \mathbf{R}_1 \exp(t \cdot \log(\mathbf{R}_1^T \mathbf{R}_2))$$

difference 인 rotate vector가
scaling 될

↓ exp

rotation matrix

↓

\mathbf{R}_1 에 더할

($t=0.1$ 이면, \mathbf{R}_1 에 더 가까운
 $t=0.9$ 이면, \mathbf{R}_2 에 더 가까운
orientation이 나옴

- **exp(): rotation vector to rotation matrix**

- **log(): rotation matrix to rotation vector**

- Implication

- $\mathbf{R}_1^T \mathbf{R}_2$: difference between orientation \mathbf{R}_1 and \mathbf{R}_2 ($\mathbf{R}_2(-)\mathbf{R}_1$)

- \mathbf{R}^t : scaling rotation (scaling rotation angle) $t=2: \mathbf{R} \cdot \mathbf{R}$ $t=0: \mathbf{R}^0 = \mathbf{I}$

- $\mathbf{R}_a \mathbf{R}_b$: add rotation \mathbf{R}_b to orientation \mathbf{R}_a ($\mathbf{R}_a(+)\mathbf{R}_b$)

difference \approx
scaling & rotation

결과 $\Rightarrow \mathbf{R}_1$ 과 \mathbf{R}_2 사이에 있는
어떤 orientation이
나온다

Exp & Log

- **Exp (exponential): rotation vector to rotation matrix**

- Given normalized rotation axis $\mathbf{u}=(u_x, u_y, u_z)$, rotation angle θ

$$R = \begin{bmatrix} \cos \theta + u_x^2 (1 - \cos \theta) & u_x u_y (1 - \cos \theta) - u_z \sin \theta & u_x u_z (1 - \cos \theta) + u_y \sin \theta \\ u_y u_x (1 - \cos \theta) + u_z \sin \theta & \cos \theta + u_y^2 (1 - \cos \theta) & u_y u_z (1 - \cos \theta) - u_x \sin \theta \\ u_z u_x (1 - \cos \theta) - u_y \sin \theta & u_z u_y (1 - \cos \theta) + u_x \sin \theta & \cos \theta + u_z^2 (1 - \cos \theta) \end{bmatrix}$$

(Rodrigues' rotation formula)

- **Log (logarithm): rotation matrix to rotation vector**

Given rotation matrix \mathbf{R} , compute axis \mathbf{v} and angle θ

$$\theta = \cos^{-1}((\underbrace{R_{11} + R_{22} + R_{33}}_{\text{trace}} - 1)/2)$$

$$v_1 = (R_{32} - R_{23})/(2 \sin \theta)$$

$$v_2 = (R_{13} - R_{31})/(2 \sin \theta)$$

$$v_3 = (R_{21} - R_{12})/(2 \sin \theta)$$

→ But this formula has a singularity at $\theta=k\pi$, where k is an integer.

Algorithm for Log

Algorithm for Computing the Logarithm of a Rotation Matrix

Objective: Given $R \in SO(3)$, find $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$, and $\theta \in [0, \pi]$ such that

$$R = e^{[\omega]\theta} = I + \sin \theta [\omega] + (1 - \cos \theta)[\omega]^2. \quad (3.62)$$

$$\Rightarrow \theta = \cos^{-1}((R_{11} + R_{22} + R_{33} - 1)/2) \text{ or } \cos^{-1}((1 - R_{33})/2) = \cos^{-1}1 = 0 = \theta$$

(i) If $\text{tr } R = 3$, then set $\omega = 0$, $\theta = 0$. (422X)

trace: diagonal element의 합

(ii) If $\text{tr } R = -1$, then set $\theta = \pi$, and ω to any of the three following vectors that is nonzero:

$$\cos^{-1}(-1) = \pi = \theta$$

$$\omega = \frac{1}{\sqrt{2(1 + r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \quad (3.63)$$

or

$$\omega = \frac{1}{\sqrt{2(1 + r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \quad (3.64)$$

or

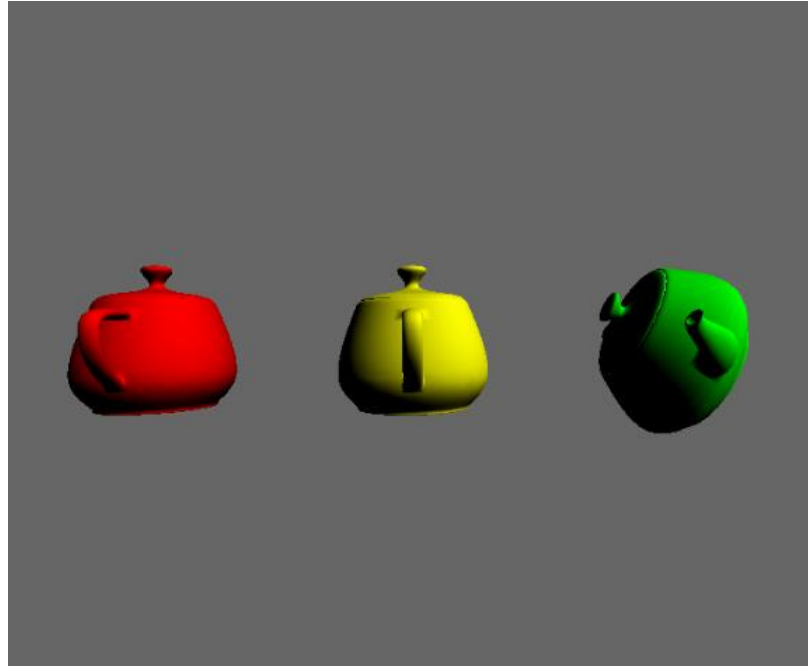
$$\omega = \frac{1}{\sqrt{2(1 + r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \quad (3.65)$$

(iii) Otherwise set $\theta = \cos^{-1} \left(\frac{\text{tr } R - 1}{2} \right) \in [0, \pi)$ and $[\omega] = \frac{1}{2 \sin \theta} (R - R^T)$.

See section 3.1.3 of INTRODUCTION TO ROBOTICS for detail:

http://robotics.snu.ac.kr/fcp/files/_pdf_files_publications/a_first_coruse_in_robot_mechanics.pdf

[Practice] Slerp Online Demo



<https://nccastaff.bournemouth.ac.uk/jmacey/WebGL/QuatSlerp/>

- Change “Start Rotation” & “End Rotation”
- Move “Interpolate” slider

Quiz #3

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 - e.g. **2017123456: 4)**
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Next Time

- Lab in this week:
 - Lab assignment 9
- Next lecture:
 - 10 - Kinematics & Animation
- Acknowledgement: Some materials come from the lecture slides of
 - Prof. Jehee Lee, SNU, http://mrl.snu.ac.kr/courses/CourseGraphics/index_2017spring.html
 - Prof. Taesoo Kwon, Hanyang Univ., <http://calab.hanyang.ac.kr/cgi-bin/cg.cgi>
 - Prof. Kayvon Fatahalian and Prof. Keenan Crane, CMU, <http://15462.courses.cs.cmu.edu/fall2015/>
 - Prof. Sung-Hee Lee, KAIST