11. 고유값과 고유벡터 및 대각화

Chap 5. Eigenvalue , Eigenvector

$$\underset{n\times n}{A}x = \lambda x$$

x → eigenvector

λx → scalar multiplication

λ → eigenvalue

$$(A-\lambda I)x = 0$$

↑ non-zero

⇒ $\det(A-\lambda I) = 0$ → singular. 역행렬 존재 X

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & a_{22}-\lambda & \\ \vdots & \vdots & \ddots \end{vmatrix} = 0$$

$$= \lambda^n \cdots (\quad)\lambda^{n-1} \cdots$$

ex) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda)+10 = 0$$

$$\lambda = 2, -1$$

$\lambda=2$. $\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → null space

$(A-\lambda I)x = 0$

x : eigenvector → $N(A-\lambda I)$

→ how to find null space (eigenvector)

Row Echelon form

$2x_1 - 5x_2 = 0$ $\quad \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\lambda = -1$.

$\begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 5x_1 = 5x_2$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

⇩

$\begin{bmatrix} 5 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ & 0 \end{bmatrix}$

ex)

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & -\frac{1}{2}-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\tfrac{3}{4}-\lambda)(-\tfrac{1}{2}-\lambda) = 0$$

triangular

$$\det A = \prod_{i=1}^{n} \text{pivot}_i = \prod_{i=1}^{n} d_i$$

$$= \prod_{i=1}^{n} \lambda_i$$

$$\text{Trace of } A = \sum_{i=1}^{n} a_{ii} = (a_{11}+a_{22}+\cdots+a_{nn})$$

$$= \sum_{i=1}^{n} \lambda_i$$

$$\begin{vmatrix} a_{11}-\lambda & & \\ & a_{22}-\lambda & \\ & & \ddots \end{vmatrix} = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = 0$$

$$= \lambda^n + \lambda^{n-1}(\quad) \cdots$$

5.2. Diagonalization of matrix.

1) $A = LU = LDU$

2) $A = QR = \begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix}\begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & \cdots \\ \vdots & (q_2^T a_2) & \\ & & \ddots \end{bmatrix}$

3) $A = S\Lambda S^{-1}$

$$S = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$Ae_i = \lambda_i e_i$$

$$\begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_n e_n \end{bmatrix}$$

$$A\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

$$AS = S\Lambda$$

$$A = S\Lambda S^{-1}$$

ⓔⓧ $A=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} \rightarrow \begin{vmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}-\lambda\end{vmatrix}=0$

$\lambda=1, 0$

$\lambda=1$

$\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$e_1=\begin{bmatrix}1\\ 1\end{bmatrix}$

$\lambda=0$

$\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$e_2=\begin{bmatrix}1\\ -1\end{bmatrix}$

$S=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \quad \Lambda=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$A=S\Lambda S^{-1}$

ⓔⓧ $K=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

$\begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix}=\lambda^2+1=0$

$\lambda=\pm i$

$\lambda=i$

$\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}1\\ -i\end{bmatrix}$

$\lambda=-i$

$\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \begin{bmatrix}1\\ i\end{bmatrix}$

$S=\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix} \quad \Lambda=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$

Remark 1>

If $\lambda_1 \lambda_2 \cdots \lambda_n$ are different,
then $e_1, e_2 \cdots e_n$ are linearly independent

pf) assume that e_2 is dependent on e_1

$e_2 = ce_1 \Rightarrow \lambda_2 e_2 = c\lambda_2 e_1$

$Ae_2 = cAe_1 \Rightarrow \lambda_2 e_2 = c\lambda_1 e_1$

$c\lambda_2 e_1 = c\lambda_1 e_1$

$c(\lambda_2-\lambda_1)e_1=0$

$\lambda_2 \neq \lambda_1, e_1 \neq 0$

$c=0$

$e_1\ e_2 \rightarrow$ independent.

$e_n = c_1e_1 + c_2e_2 \cdots + c_{n-1}e_{n-1}$

$\times A \quad Ae_n = c_1Ae_1 + c_2Ae_2 + \cdots + c_{n-1}Ae_{n-1}$

$\lambda_n e_n = c_1\lambda_1 e_1 + c_2\lambda_2 e_2 + \cdots + c_{n-1}\lambda_{n-1}e_{n-1}$

$\lambda_n e_n = c_1\lambda_n e_1 + c_2\lambda_n e_2 + \cdots + c_{n-1}\lambda_n e_{n-1}$

$c_1(\lambda_n-\lambda_1)e_1 + c_2(\lambda_n-\lambda_2)e_2 \cdots + c_{n-1}(\lambda_n-\lambda_{n-1})e_{n-1}$
$=0$

Remark 2>

S is not unique

since $ke \rightarrow$ eigenvector.

$e_1=\begin{bmatrix}1\\ 1\end{bmatrix}, \ e_2=\begin{bmatrix}1\\ -1\end{bmatrix} \quad \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

$\downarrow$

$\begin{bmatrix}-1\\ 1\end{bmatrix} \quad \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

$A=S\Lambda S^{-1}$

Remark 3>

The order of eigenvalues is same with that of eigenvectors

Remark 4>

Not all matrices have n linearly independent eigenvectors

$\rightarrow A=S\Lambda S^{-1}$

is not always established.

- Powers

$A \rightarrow \lambda, e$

$A^k \rightarrow \lambda^k, e$

$Ae = \lambda e$

$A^2e = \lambda Ae = \lambda^2 e$

$A^k = (S\Lambda S^{-1})(S\Lambda S^{-1}) \cdots$

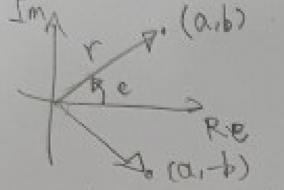
$= S\Lambda^k S^{-1}$

$\hookrightarrow \begin{bmatrix}\lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k\end{bmatrix}$

14. 복소행렬과 에르미트 행렬

복소. complex Matrix

· Complex number $z = \underbrace{a}_{\text{real part}} + j\underbrace{b}_{\text{imaginary part}}$



$j^2 = -1$

$r = |z| = \text{magnitude} = \sqrt{a^2+b^2}$

$\theta = \arg\{z\} = \tan^{-1}\left(\frac{b}{a}\right)$

$a = r\cos\theta$

$b = r\sin\theta$

$z = a+jb = \underbrace{r(\cos\theta + j\sin\theta)}_{\text{Euler form}}$

$= \underbrace{re^{j\theta}}_{\text{polar form}}$

$|z|^2 = a^2+b^2 = (a+jb)(a-jb) = zz^*$

· Complex conjugate

$z^* = a - jb$: symmetic on real-axis

$= r \cdot e^{-j\theta}$

$|z|^2 = z \cdot z^* = re^{j\theta} \cdot re^{-j\theta} = r^2$

· Complex vector

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $x_k = a_k + jb_k$

$\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$

$= x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n$

$= (x^T)^* x$ → conjugate Transpose (Hermite)

· inner product of complex vector

→ real ~~number~~ vector. $x^T y = y^T x$

→ complex vector $(x^T)^* y$ $x^H y \neq y^H x$

$(1+j)^*(1+2j)$

$\neq (1+2j)^*(1+j)$

$\begin{bmatrix} 2+j & 3j \\ 4+j & 5 \\ -1 & 0 \end{bmatrix} \xrightarrow{H} \begin{bmatrix} 2-j & 4-j & -1 \\ -3j & 5 & 0 \end{bmatrix}$

1) orthogonal $x^H y = 0$ $(\neq y^H x)$

if $x^H y = y^H x$ → real

$(x^H y)^H$

$(z_1 z_2)^* = z_1^* z_2^*$

2) $\|x\|^2 = x^H x$

3) $(AB)^H = B^H A^H$

◎ Hermitian Matrix

· for real matrix A

$A^T = A$ → symmetric

· for complex matrix A

$A^H = A$ → Hermition

$a_{ij} = a_{ji}^*$ $(i \neq j)$

a_{ii} → real $(i = j)$

· Properties of Hermition or Symmetric Matrix

(1) $x^H A x$ (quadratic form) is real

$(x^H A x)^H = x^H A^H x = x^H A x$ ⇒ real

$[x\ y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 = 1$

$[x\ y]\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + by^2 = 1$

$[x\ y]\begin{bmatrix} a\cos\theta & -b\sin\theta \\ b\sin\theta & a\cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$R = A^H A$ $x^H R x = x^H A^H A x = (Ax)^H (Ax)$

$= \|Ax\|^2 > 0$

2) every eigenvalue is real

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ & \cdots & & a_{nn}-\lambda \end{vmatrix}$$

$$= (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

$$= \lambda_n + \alpha_1 \alpha^{n-1} \cdots\cdots + \alpha_n^{n}$$

$$Ax = \lambda x$$

$$x^H A x = \lambda x^H x = \lambda \|x\|^2$$

$$\lambda = \frac{x^H A x}{\|x\|^2} \to \lambda = \frac{x^H R x}{\|x\|^2} > 0$$

3) eigenvectors are orthogonal

$$Ax_1 = \lambda_1 x_1 \quad Ax_2 = \lambda_2 x_2 \quad \lambda_1 \neq \lambda_2$$

$$\underbrace{(Ax_1)^H x_2}_{\downarrow} \quad x_1^H A^H x_2 = x_1^H A x_2 = \lambda_2 x_1^H x_2^{2}$$

$$= (x_1 x_1)^H x_2 = \lambda_1^{*} x_1^H x_2$$

$$= \lambda_1 x_1^H x_2 = \lambda_2 x_1^H x_2$$

$$\underbrace{(\lambda_1-\lambda_2)}_{\neq 0} \underbrace{x_1^H x_2}_{\overset{\parallel}{0}} = 0$$

Q orthogonal matrix

$$Q = [q_1\ q_2 \cdots q_n] \quad q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$Q^T Q = I \quad Q^{-1} = Q^T$$

A: symmetric, $\dfrac{\lambda_1\ \lambda_2 \cdots \lambda_n}{x_1\ x_2 \cdots x_n}$

$$\|x_n\|^2 = 1$$

$$A = S \Lambda S^{-1} = Q \Lambda Q^{-1}$$

$$= Q \Lambda Q^T = [x_1\ x_2 \cdots x_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix} \begin{bmatrix} \cdots x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix}$$

$$= [\lambda_1 x_1\ \lambda_2 x_2 \cdots \lambda_n x_n] \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x^T \end{bmatrix}$$

$$A \to \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T \cdots + x_n y_n y_n^T$$

spetial theorem

U. unitary matrix

↳ complex matrix

$$U^H U = U U^H = I$$

$$U^H = U^{-1}$$

① angle, length preserved

$$(Qx)^T (Qx) = x^T Q^T Q x = x^T y$$

$$\|Qx\|^2 = (Qx)^T (Qx) = x^T Q^T Q x = \|x\|^2$$

$$(Ux)^H (Uy) = x^H y$$

$$\|Ux\|^2 = \|x\|^2$$

15. 특이값 분해

Complex Matrix / vector

$(x^*)^T = x^H$ (Hermitian)

$\|x\|^2 = \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} x_i^* x_i$

$x^H y \neq y^H x$

$=$ iff $x^H y$ is real

Hermitian Matrix for Complex

Symmetric Matrix for Real

$A^H = A \; / \; A^T = A$

$\begin{cases} A^H A - R \\ A^T A - R \end{cases}$

1) $x^H A x = \lambda x^H x = \lambda \|x\|^2$

$\lambda = \frac{x^H A x}{\|x\|^2} \rightarrow$ real

$A^H A x = \lambda x$

$x^H A^H A x = \lambda x^H x$

$\lambda = \frac{\|Ax\|^2}{\|x^2\|} \geq 0$

2) eigenvectors are orthonormal

$A = S \Lambda S^{-1}$

$= Q \Lambda Q^{-1} = Q \Lambda Q^T$

$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \dots + \lambda_n x_n x_n^T$

spectral theorem

∘ Unitary Matrix

$Q^{-1} = Q^T = Q^T Q = I$

$U^T U = U U^T = I$

$\|Qx\|^2 = (Qx)^T (Qx) = x^T Q^T Q x = \|x\|^2$

$\|Ux\|^2 = (Ux)^T (Ux) = \|x\|^2$

$(Qx)^T (Qy) = x^T y$

$(Ux)^H (Uy) = x^H y$

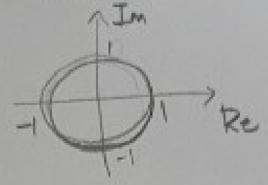
∘ eigenvalues $|\lambda_i| = 1$

$Ux = \lambda x$

$\|Ux\| = \|\lambda x\| = \|\lambda\| \|x\|$

$\|x\| = |\lambda| \|x\|$

$\|\lambda\| = 1$



∘ eigenvectors are orthonormal

$Ux_1 = \lambda_1 x_1 , \quad Ux_2 = \lambda_2 x_2$

$x_1^H x_2 = (Ux_1)^H (Ux_2)$

$= (\lambda_1 x_1)^H (\lambda_2 x_2)$

$= \lambda_1^H \lambda_2 x_1^H x_2$

$\underbrace{(1 - \lambda_1^* \lambda_2)}_{\neq 0.} \underbrace{x_1^H x_2}_{= 0} = 0$

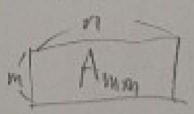
얘 0이 되는

$\lambda_1 = \lambda_2$인 경우.

∘ Singular Value Decomposition (SVD)

$A = S \Lambda S^{-1}$ for $n \times n$ square Matrix.

$\underset{m \times n}{A} = \underset{m \times m}{U} \underset{m \times n}{\Sigma} \underset{n \times n}{V^T}$



$C(A) \in R^m \perp N(A^T) \in R^m$

$C(A^T) \in R^n \perp N(A) \in R^n$

1) first find the eigenvalues, eigenvectors of $A^T A$

$\Rightarrow A^T A \rightarrow$ symmetric

$x^T A^T A x = \lambda x^T x$

$\lambda = \frac{\|Ax\|^2}{\|x\|^2} \geq 0$

assume that there are r non-zero eigenvalues

$$\underbrace{\begin{matrix}\lambda_1, & \lambda_2 & \cdots & \lambda_r \\ \downarrow & \downarrow & & \downarrow \\ v_1 & v_2 & \cdots & v_r\end{matrix}}_{\text{orthonormal}} \quad \underbrace{\lambda_{r+1} \cdots \lambda_n}_{\overset{\parallel}{0}}$$

$\sqrt{\lambda_i} = \sigma_i \quad (i=1,2,3 \cdots r)$

↳ singular value

$$\Sigma_1 = \begin{bmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \; \sigma_r\end{bmatrix} \Rightarrow \underset{m\times n}{\Sigma} = \begin{bmatrix}\Sigma_1 & 0 \\ 0 & 0\end{bmatrix}$$

$V_1 = [v_1 v_2 \cdots v_r] \in R^n$

↳ $C(A^T)$ row space basis

for $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$

$A^TAx_j = \lambda_j x_j = 0 \quad (j=r+1, \cdots n)$

$A^TAx_j = 0 \qquad x_j \Rightarrow N(A^TA)$

$\qquad\qquad\qquad\quad = N(A)$

$V_2 = [v_{r+1} \; v_{r+2} \cdots\cdots v_n]$

↳ $N(A)$ basis

$V = [V_1 V_2]$

$V^TV = I$.

$$\underset{m\times n}{AV} = \underset{n\times m}{U} \; \underset{m\times n}{\Sigma} = [u_1 \, u_2 \cdots u_r \, u_{r+1} \cdots u_n]\begin{bmatrix}\lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots \lambda_r & & \\ & & & 0 & \\ & & & & \ddots 0\end{bmatrix}$$

$A[v_1 \; v_2 \cdots v_r \, v_{r+1} \cdots v_n]$

$= [Av_1 \;\; Av_2 \cdots Av_r \;\; 0 \cdots 0] = \underset{m\times n}{[\lambda_1 u_1 \;\; \lambda_2 u_2 \cdots \lambda_r u_r \, 0 \cdots 0]}$

$Av_i = \sigma_i u_i$

$u_i = \frac{1}{\sigma_i} Av_i \in R^m \qquad i=1,2,\cdots r$

$$u_i^T u_j = \frac{1}{\sigma_i}(Av_i)^T \frac{1}{\sigma_j} Av_j$$

$$= \frac{1}{\sigma_i \sigma_j} v_i^T A^T A v_j = \frac{1}{\sigma_i \sigma_j} v_i^T \lambda_j v_j$$

$$= \delta_{ij}$$

→ u_i ⇒ orthonormal → $C(A)$ basis

$U_1 = [u_1 \; u_2 \cdots\cdots u_r]$

- $u_{r+1}, u_{r+2} \cdots u_m \in R^m$

$\quad m-r$

⇒ Left null space $N(A^T)$

↳ orthonormal basis

$U_2 = [u_{r+1} \; u_{r+2} \cdots u_m]$

$$\underset{m\times m}{U} = [\underset{1\sim r}{U_1} \; \underset{r+1\sim m}{U_2}]$$

$A = U\Sigma V^T$

$$= [U_1 U_2]\begin{bmatrix}\Sigma_1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^T \\ V_2^T\end{bmatrix}$$

ⓔⓧ

$$A = \begin{bmatrix}1 & 1 \\ 1 & 1 \\ 0 & 0\end{bmatrix} \qquad A^TA = \begin{bmatrix}2 & 2 \\ 2 & 2\end{bmatrix}$$

$$\det\begin{bmatrix}2-\lambda & 2 \\ 2 & 2-\lambda\end{bmatrix} = 0$$

$\lambda_1 = 4, \quad \lambda_2 = 0$

$\lambda_1 = 4$

$$\begin{bmatrix}-2 & 2 \\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \quad x_1 = x_2. \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} = v_1$$

$\lambda = 0$

$$\begin{bmatrix}2 & 2 \\ 2 & 2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \quad x_1 + x_2 = 0 \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -1\end{bmatrix} = v_2$$

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}$$

$$u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{2}\begin{bmatrix}1 & 1 \\ 1 & 1 \\ 0 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ 1 \\ 0\end{bmatrix}$$

$$A^T = \begin{bmatrix}1 & 1 & 0 \\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = 0$$

$x_1 + x_2 = 0$

$$u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 \\ -1 \\ 0\end{bmatrix}$$

$$u_3 = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}$$

$$A = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 \\ 0 & 0 \\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$$