• LU 분할

- 가우스 소거법.

$2u+v+w=5 \cdots ①$
$4u-6v \quad = -2 \cdots ②$
$-2u+7v+2w=9 \cdots ③$

$2u+v+w=5$
$-8v-2w=-12 \leftarrow ② - 2\times①$
$8v+3w=14 \leftarrow ③ + ①$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ 0 & 8 & 3 \end{bmatrix}$$

$$E_{31}E_{21}A \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$2u+v+w=5$
$-8v-2w=-12$
$w = 2 \cdots ③+②$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{E_{32}}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

◎ Elementary Matrix in G.E.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$② - ① \times l_{21} \Rightarrow ②'$
$\Rightarrow ② = ②' + l_{21} \times ①$

$E_{21}A \rightarrow \overset{②}{②'}$

$E_{21}^{-1}A' \Rightarrow \underset{②}{A}$

$2u+v+w=5 \cdots ①$
$4u-6v \quad = -2 \cdots ②$
$\Downarrow$
$-8v-2w=-12 \cdots ②'$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}E_{21}^{-1} = I$$

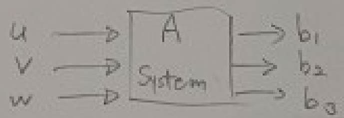
$E_{32}E_{31}E_{21}A = U$
$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}U$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{\text{Lower triangular Mat}}U = LU$$

1.5 Triangular Factors

$Ax = b$
$A = LU$
$\Rightarrow$ LU factorization
(decomposition)



$Ax = b$

$\underbrace{L^{-1}A}_{U}x = L^{-1}b = c$

$\underline{Ux = c}$

$A = LU \qquad Lc = b$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$\text{ⓔx } L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\text{ⓔx } A \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

$\Downarrow ② - 3\times①$

$$U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$\det(A) = 2$

$$\text{ⓔx } A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$= L = L\underbrace{I}_{\rightarrow U}$

$$U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & d_2 & & & \\ & & d_3 & & \\ & 0 & & \ddots & \\ & & & & d_n \end{bmatrix} = DU$$

$$= \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdots & \frac{u_{1n}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} & \cdots & \frac{u_{2n}}{d_2} \\ & & \ddots & & \vdots \\ 0 & & & & 1 \end{bmatrix}$$

Diagonal Matrix

$A = LU$

$\Rightarrow LDU$

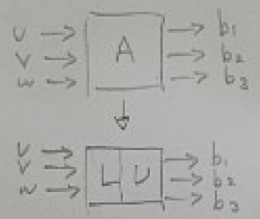
$$D^n = \begin{bmatrix} d_1^n & & & 0 \\ & d_2^n & & \\ & & \ddots & \\ 0 & & & d_n^n \end{bmatrix}$$

① LU factorization is unique

일대일 대응 ⇒ unique.

일대일 대응이 아니면, $x \to y$는 정의할 수 있으나

역과정인 $y \to x$를 정의할 수는 없다.



② Raw Exchange (Pivoting)

⇒ Permutation. P

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 \\ 2 & 1 & 1 \\ -2 & 7 & 2 \end{bmatrix}$$

↳ P_{21} (2, 1 행을 바꾼다)

∘ Permutation Matrix

⇒ has the same rows with I

⇒ There is a single "1" in every row and column.

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{32} P_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\neq$

$$P_{21} P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$A \Rightarrow LU$

$PA = LU$

$P^{-1} = P^T$

$P_{21} \to P_{21}^{-1}$

$A = P^T LU$

• 벡터공간과 힐버트공간

Chapter 1 Review

$$\left.\begin{aligned} 2u+v+w&=5 \\ 4u-6v&=-2 \\ -2u+7v+2w&=9 \end{aligned}\right\}$$ 선형 연립방정식.

$\boxed{A}\boxed{x}=\boxed{b} \xrightarrow[A^{-1}]{GE} x \rightarrow$ unique! / no solution

미지수의 개수 = 식의 개수

Chapter 2

미지수의 개수 > 식의 개수

$\boxed{A}\boxed{x}=\boxed{b}$

⇒ 무한히 많은 해 / 해가 없다

$2u+v+w=5$

$4u-6v=-2$

※ 미지수의 개수 < 식의 개수

Chapter 2. Vector Space

2.1. Vector Space Subspace

• space ⇒ set (덧셈 : 스칼라 곱에 닫혀있음)

• for any vectors $x, y \in R^n$
  for any scalar $c \in R$

$x, y \in V$

$$\left.\begin{aligned} x+y &\in V \\ cx &\in V \end{aligned}\right\} c_1x+c_2y \in V$$

→ Vector Space

1) $x+y=y+x$

2) $x+(y+z)=(x+y)+z$

3) Zero vector가 존재. $x+0=0+x=x$

4) 각각의 벡터 x에 대해,
$x+(-x)=(-x)+x=0 \Rightarrow -x$: 역벡터 unique

5) $1 \cdot x = x$

6) $c(x+y)=cx+cy$

7) $(c_1+c_2)x=c_1x+c_2x$

ex) $m \times n$ matrix $\in R^{mn}$

$2 \times 2 \in R^4$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\alpha_1\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}+\alpha_2\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

ex) 최대 2차 다항함수

$ax^2+bx+c \in R^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

ex) $f(x)=ae^x$

Tayler Series

$$f(x)=f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}+\cdots$$

ex) $f(x)=e^x$

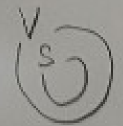
$$\Rightarrow 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \quad \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2!} \\ \frac{1}{3!} \\ \vdots \end{bmatrix} \in R^\infty$$

$\alpha_1e^x+\alpha_2e^x=(\alpha_1+\alpha_2)e^x$ Hilbert Space

◦ Subspace

subset of the whole Vector Space.

Vector Space의 조건들을 모두 만족해야 한다.



$s_1, s_2 \in S \subset V$

$c_1s_1+c_2s_2 \in S$

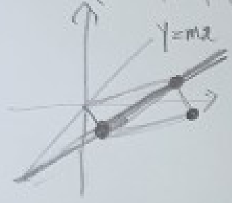
Ex) Lower Triangular Mat in R^{n^2}
(Upper)

$L_{n\times n}$ $c_1L_1+c_2L_2 \in L$

$L_{n\times n} \subset A_{n\times n} \in R^{n^2}$

Ex) $y=mx$ $(m\neq 0)$ $(x,y)\in R^2$

$S=\{(x,y) \mid y=mx,\ m\neq 0\} \in R^2$



① Column Space of A (C(A))

⇒ set of all linear Combinations from column vectors in A

$$A=\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \Rightarrow \left\{\sum_{i=1}^{n} c_i a_i\right\}$$

$Ax=b$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=b$$

$= x_1a_1+x_2a_2+\cdots+x_na_n=b$

$2x_1+3x_2=3$
$-x_1+4x_2=1$

$$x_1\begin{bmatrix} 2 \\ -1 \end{bmatrix}+x_2\begin{bmatrix} 3 \\ 4 \end{bmatrix}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

if $b\in C(A)$
then there is at least one solution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Ex)

$$u\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}+v\begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

if $b_1, b_2 \in C(A)$

$A\underline{x_1}=b_1$
$A\underline{x_2}=b_2$

$b_1+b_2=b$

$Ax_1+Ax_2=A\underbrace{(x_1+x_2)}_{x}=b$

$b=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ $Ax=b$

$Ax_1=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$Ax_2=\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

$Ax_3=\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

$Ax_1=b_1$
$cb_1=b$
$A(cx_1)=cb_1=b$

$$b=2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x=2x_1+3x_2+4x_3$

• 열벡터공간과 허집합

Vector Space (V).

set of vector in R^n

1) closed under addition

$v_1 \in V,\ v_2 \in V$

$v_1 + v_2 \in V$

2) closed under scalar multiplication

$v \in V,\ c \in R$

$cv \in V$

3) V include zero vector (0)

$\hookrightarrow$ origin

$Ax = b \qquad A_{n \times n}$

$x = A^{-1} b$

if A^{-1} exists.

always $b \in C(A)$

$\Downarrow$

$C(A)$: whole space

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ whole space is constructed by linear combinations $\rightarrow$ span

Null Space of A (N(A))

$\Rightarrow$ set of vectors such that $Ax = 0$ 인 x들 모든것들.

$N(A) = \{x \mid Ax = 0\}$

(1) closed under addition

for $Ax_1 = 0,\ Ax_2 = 0$

$x_1 + x_2 \in N(A)$

$A(x_1 + x_2) = Ax_1 + Ax_2 = 0$

(2) closed under scalar mul.

for $Ax = 0$ for any c

$cx \in N(A)$

$\Rightarrow A(cx) = 0$

$cAx = 0$

$cx \in N(A)$

ex

$$\begin{cases} u + w = 0 & -① \\ 5u + 4v + 9w = 0 & -② \\ 2u + 4v + 6w = 0 & -③ \end{cases}$$

$② - 5 \times ① \rightarrow 4v + 4w = 0$

$③ - 2 \times ① \rightarrow 4v + 4w = 0$

$u + w = 0$

$v + w = 0$

$w = c.$

$u = v = -c$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Infinitely many solution

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = 0$

$$c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = (c_1 + c_2) \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \in N(A)$$

2.2. Solving $Ax = 0$ and $Ax = b$

$\boxed{A_{m \times n}}\ \boxed{x} = \boxed{b}$

$m < n$. 식의 개수 < 미지수의 개수.

해가 아예 없거나 무수히 많다.

∘ Echelon form U

ex)

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$C(A) \subset R^m$

$N(A) \subset R^n$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Downarrow$ all pivots $\rightarrow 1$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ Row Reduced form R

$Ax = 0 \longrightarrow Ux = 0 \Rightarrow Rx = 0$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} u + 3v - z = 0 \\ w + z = 0 \end{cases}$$

pivot variables : u, w

free variables : v, z

$u = -3v + z$

$w = -z$

$$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} -3v+z \\ v \\ -z \\ z \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\in N(A)$

$$\begin{cases} x + y + z = 0 \\ x - z = 0 \end{cases}$$

$2x + 2y = 0$

$\boxed{x + y = 0}$

$x = -y$

$$c \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$Ax = 0$

$N(A)$

$$= \left\{ \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \middle| \; c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\hookrightarrow Dim(N(A)) = 2$

• Dimension of Vector Space

$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}$ in R^4

$Dim(N(A))$ = number of independent special solution vectors

· 벡터의 직교성과 직선투영

Chap 3 Orthogonality

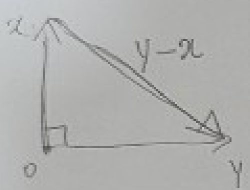
3.1 orthogonal vectors and subspace

벡터가 수직이다 ⇒ independent basis

⇒ linear combination을 계산하기 쉽다.

· length of vectors $\|X\|$

$\Rightarrow \|X\|^2 = X^T X = \sum_{i=1}^{n} x_i^2$



$$\|y-x\|^2 = \|x\|^2 + \|y\|^2$$

$$(y-x)^T(y-x) = x^T x + y^T y$$

$$\Rightarrow \cancel{y^T y} - y^T x - x^T y + \cancel{x^T x} = \cancel{x^T x} + \cancel{y^T y}$$

$x^T y = y^T x \qquad x^T y = 0$

$\vec{x} \cdot \vec{y}$

· for vector inner product $x^T y$

1) $x^T y = 0 \Rightarrow$ 각도 $= 90°$

2) $x^T y < 0 \Rightarrow$ 각도 $> 90°$

3) $x^T y > 0 \Rightarrow$ 각도 $< 90°$

◎ If non-zero vectors $v_1, v_2 \cdots v_n$ 가 모두 수직일 때, $(v_i^T v_j = 0)\ (i \neq j)$

$\|v_i\| \neq 0$

그러면 vector들은 linearly independent

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$

일 때, linearly independent인 경우는 해가 오직 $c_1 = c_2 = \cdots = c_n = 0$ 일 때이다.

$$v_i^T (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = 0$$

$c_i \|v_i\|^2 = 0$ for all i.

$c_i = 0$

→ orthogonal vectors → basis vectors

$X = \sum_{i=1}^{n} c_i v_i \qquad v_i^T X = c_i \|v_i\|^2$

orthonormal $\|v_i\| = 1$

linearly independent vectors

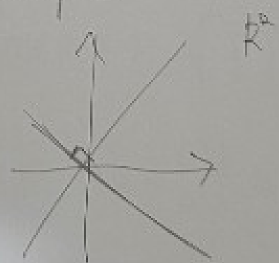
$a_1, a_2 \cdots a_n$

$X = \sum_{i=1}^{n} c_i a_i$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} X \end{bmatrix}$$

◎ orthogonal Subspaces

⇒ Every vector in one subspace is orthogonal to every vector in the other subspace



◎ Row Space ⊥ Null Space

$C(A^T)$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_m^T \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\sum_{k=1}^{m} c_k a_k \perp x = 0$$

◎ Column Space ⊥ Left null Space

$$\boxed{A^T}\boxed{y} = \boxed{0}$$

Dim of V.S = number of indep vectors to span the V.S = rank of A (r)

$$\underbrace{Dim(C(A))}_{r} + \underbrace{Dim(N(A^T))}_{m-r} = m$$

$$\underbrace{Dim(C(A^T))}_{r} + \underbrace{Dim(N(A))}_{n-r} = n$$

◎ Orthogonal Complement Subspace

$v \in V \quad w \in W$ in R^n

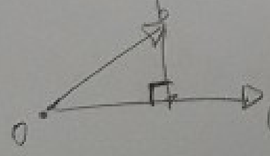
$v \perp w \rightarrow$ orthogonal Subspace

$Dim(V) + Dim(W) = n$

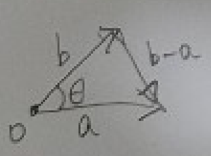
$1+1=2$

orthogonal subspace ≠ orthogonal Com. Subs

3.2 Cosines and Projection onto line



project b onto a



$$\|b-a\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\|\cos\theta$$

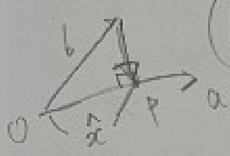
제2 코사인법칙

$$(b-a)^T(b-a) = a^Ta + b^Tb - 2\|a\|\|b\|\cos\theta$$

$$a^Tb = \|a\|\|b\|\cos\theta$$

$\|b\|\cos\theta \ \vec{a}$

$\|c\|\cos(\frac{\pi}{2}-\theta)\ \vec{c}$



$(b - \hat{x}a) \perp a$.

$$a^T(b - \hat{x}a) = 0$$

$$\hat{x} = \frac{a^Tb}{a^Ta}$$

$$p = \hat{x}a$$

ex

$a = \begin{bmatrix}1\\1\\1\end{bmatrix} \quad b = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{6}{3} = 2$$

$$\hat{x}a = p = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$p = \hat{x}a = \frac{a^Tb}{a^Ta}a = \frac{aa^T}{a^Ta}b$$

$P = \frac{aa^T}{a^Ta}$ projection matrix.

$Pb = p$

ex $a = \begin{bmatrix}1\\1\\1\end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$a = \begin{bmatrix}2\\2\\2\end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{1}{12}\begin{bmatrix}4&4&4\\4&4&4\\4&4&4\end{bmatrix}$$

$$P\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$

$$P^T = \frac{aa^T}{a^Ta} = \frac{aa^T}{a^Ta} = P, \quad P^2 = \frac{aa^Taa^T}{(a^Ta)(a^Ta)} = P$$

• 선형대수학과 최소제곱법

• Basis (vectors) to span the V.S

$v_i \in V \quad w_i \in W$ — basis vectors

$X = \sum_{i=1}^{m} c_i v_i \quad Y = \sum_{i=1}^{n} b_i w_i$

for any v_i, and for any w_j

$v_i^T w_j = w_j^T v_i = 0$

for all i, j

$X^T Y = \left(\sum_{i=1}^{m} c_i v_i^T\right)\left(\sum_{j=1}^{n} b_j w_j\right) = 0$

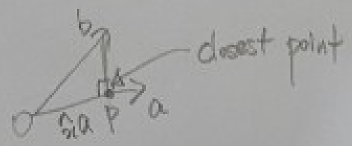
$x + 2y - 3z = 0$

$\begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

• orthogonal vectors

→ linearly independent vectors

◎ Projection onto line



$b - \hat{x}a \perp a$

$a^T(b - \hat{x}a) = 0$

$a^T a \hat{x} = a^T b \Rightarrow \hat{x} = \frac{a^T b}{a^T a}$

$p = \hat{x}a = \frac{a^T b}{a^T a} a = \frac{a a^T}{a^T a} b = Pb$

3-3. Projections and Least square

$\boxed{A}\boxed{x} = \boxed{b}$

$\min \|Ax - b\|^2$

$Ax \Rightarrow$ column vector linear combination

$\hookrightarrow \in (C(A))$

$b - A\hat{x} \perp A$

$b - A\hat{x} \perp a^T{}_i$ → column vectors in A.

$$\left.\begin{aligned} a_1^T(b - A\hat{x}) &= 0 \\ a_2^T(b - A\hat{x}) &= 0 \\ &\vdots \\ a_n^T(b - A\hat{x}) &= 0 \end{aligned}\right) A^T(b - A\hat{x}) = 0$$

$A^T A \hat{x} = A^T b$

$\hat{x} = (A^T A)^{-1} A^T b$. best estimate

$P = A\hat{x}$

$= A(A^T A)^{-1} A^T b$

$= Pb$

normal form

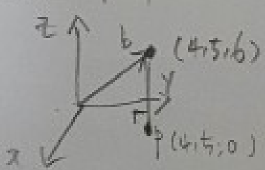
$A^T A \hat{x} = A^T b$

2) $P^2 = P$

1) If $b \in C(A)$

$P(b) = b$.

ex) $x-y$ plane $b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$



$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$

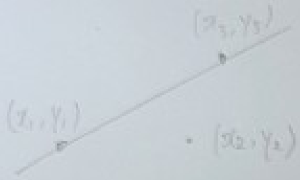
$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}$

$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 13 & -5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $Pb = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

$x+2y-3z=0$

$$A=\begin{bmatrix}1 & 3\\ 1 & 0\\ 1 & 1\end{bmatrix}$$

◎ Least square for line fitting



$y=ax+b$

$$\begin{aligned} y_1&=ax_1+b\\ y_2&=ax_2+b\\ y_3&=ax_3+b\\ &\vdots\\ y_n&=ax_n+b \end{aligned}$$

$$\begin{bmatrix}x_1 & 1\\ x_2 & 1\\ \vdots & \vdots\\ x_n & 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} \Rightarrow Ax-b=\begin{bmatrix}ax_1+b-y_1\\ ax_2+b-y_2\\ \vdots\\ ax_n+b-y_n\end{bmatrix}$$

$Ax=b$

$$\|Ax-b\|^2=\sum_{i=1}^{n}(ax_i+b-y_i)^2$$

$$f(x)=(x-a_1)^2+(x-a_2)^2+\dots+(x-a_n)^2$$

$$\min f(x) \Rightarrow x=\frac{\sum_{i=1}^{n} a_i}{n}$$