벡터의 선형독립과 기저벡터

$Ax=b$

$\boxed{}\,[\,]=\boxed{b}$

$b=0$. $Ax=0 \Rightarrow$ Null space $N(A)$

$$\begin{bmatrix}1&3&3&2\\2&6&9&7\\-1&-3&3&4\end{bmatrix}\begin{bmatrix}u\\v\\w\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix}1&3&3&2\\0&0&3&3\\0&0&0&0\end{bmatrix}$$ Echelon form U

$Ux=0$

$\Downarrow$

$$\begin{bmatrix}1&3&0&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$ Row Reduced form R.

$Rx=0$

$u=-3v+z$

$w=-z$

pivot variable $\Rightarrow u, w$

free variable $\Rightarrow v, z$

$$\begin{bmatrix}u\\v\\w\\z\end{bmatrix}=v\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$$

special solution

$Ax=b\ (\neq 0)$

$$\begin{bmatrix}1&3&3&2\\2&6&9&7\\-1&3&3&4\end{bmatrix}\begin{bmatrix}u\\v\\w\\z\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$\Downarrow$

$$\left[\begin{array}{cccc|c}1&3&3&2&b_1\\0&0&3&3&b_2-2b_1\\0&0&6&6&b_3+b_1\end{array}\right]$$

$\Downarrow$

$$\left[\begin{array}{cccc|c}1&3&3&2&b_1\\0&0&3&3&b_2-2b_1\\0&0&0&0&b_3-2b_2+5b_1\end{array}\right]$$

$b_3-2b_2+5b_1=0$

$Ax=b \quad b\in C(A)$

$\begin{bmatrix}1\\2\\-1\end{bmatrix} \rightarrow 5b_1-2b_2+b_3=0$.

$b=\begin{bmatrix}1\\5\\5\end{bmatrix}$

$$\left[\begin{array}{cccc|c}1&3&3&2&1\\0&0&3&3&3\\0&0&0&0&0\end{array}\right]$$

$\Downarrow$

$$\left[\begin{array}{cccc|c}1&3&0&-1&-2\\0&0&1&1&1\\0&0&0&0&0\end{array}\right]$$

$u+3v-z=-2$

$w+z=1$

$u=-3v+z-2$

$w=-z+1$

$$\begin{bmatrix}u\\v\\w\\z\end{bmatrix}=\underbrace{v\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}+z\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}}_{X_n}+\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}$$

$X = X_n + X_p$.

$Ax=A(X_n+X_p)$

$=\underbrace{AX_n}_{=0}+AX_p=b$

Finding the solution $Ax=b$.

1) $[A \,\vdots\, b] \xrightarrow{G.E} [R \,\vdots\, d]$

2) separate (pivot variables / free variables)

3) find the special solutions for null space. from R.

$$\begin{bmatrix}1&0&*&0&3&*&0\\0&1&*&0&-2&*&0\\0&0&0&1&1&*&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{bmatrix}$$

$x_1 \sim x_7$

pivot : x_1, x_2, x_4, x_7

free : x_3, x_5, x_6

$$x_3\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}+x_5\begin{bmatrix}-3\\2\\0\\1\\0\end{bmatrix}+x_6\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}$$

(4) Find part

$$\begin{bmatrix} d_1 \\ d_2 \\ 0 \\ d_3 \\ 0 \\ 0 \\ d_4 \end{bmatrix}$$

2.3 Linear Independence

Basis (vectors), Dimension

→ linear independent

$c_1V_1 + c_2V_2 + \cdots + c_nV_n = 0$

→ only $c_1 = c_2 = \cdots = c_n = 0$

ex)

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$ → column vector

◎ If G.E of A generates m non-zero rows → m independent column vectors in A

• Rank of A

= number of Independent column vectors

= number of Independent row vectors

= number of pivots in G.E

= Dim of C(A)

ex)

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix}$ ⇒ x,y 평면 (2차원)

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

◎ Spanning

all linear combinations of vectors $\{v_1, v_2 \cdots v_n\}$ construct a vector space

- $\{v_1, v_2 \cdots v_n\}$ span vector space

ex)

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix}$ → xy평면 $c_1 = c_2 = 2$ $\begin{bmatrix}2\\2\\0\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix}$ → xy평면 $c_1 = 0$, $c_2 = 2$ $\begin{bmatrix}2\\2\\0\end{bmatrix}$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix}$ → x·y 평면

$c_1 = c_2 = 2$, $c_3 = 0$

$c_1 = 0$ $c_2 = 0$, $c_3 = 2$

$c_1 = c_2 = c_3 = 1$

⋮

◎ Basis (vectors)

⇒ number of minimum linearly independent vectors to span the vector space

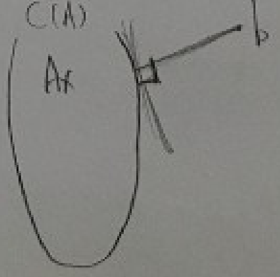
→ linear combination is unique from basis

◦ Basis is not unique for a vector space

ex) $\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}$

$\boxed{A}\boxed{x} = \boxed{b}$



1차 연립방정식 풀이와 직교벡터 구하기

1) Square System ($A_{m \times n}$, $m=n$)

$$A x = b$$

$\Rightarrow$ G.E $\rightarrow$ unique solution

$$X = A^{-1} b$$

2) under constrained System ($m < n$)

$$A x = b$$

$\Rightarrow$ 일반적으로 무수히 많은 해

$\Rightarrow X = X_n + X_p$

$$N(A) = \{ X_n \mid AX_n = 0 \}$$

$\Rightarrow$ how to find the solutions?

$\rightarrow$ G.E $\Rightarrow$ Reduced Row Echelon Form

$\rightarrow$ special sol $\rightarrow$ pivot

particular sol $\rightarrow$ free

3) Overconstrained System ($m > n$)

$$A x = b$$

$\rightarrow$ 일반적으로 $\rightarrow$ no solution $\Rightarrow \min \| Ax - b \|^2$

$\rightarrow$ Least square

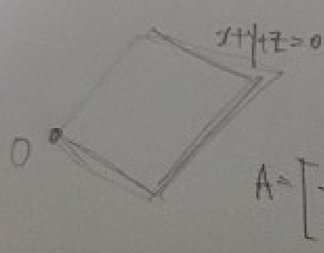
$\Rightarrow$ projection

$$A^T(A\hat{x} - b) = 0$$

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$p = Pb = A(A^TA)^{-1}A^Tb$$

• Projection



basis (vectors)

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{not unique}$$

orthonormal basis (vectors)

$$\begin{cases} v_1, v_2 \cdots v_n & \|v_i\| = 1 \\ v_i^T v_j = 0 \end{cases}$$

$$X = \sum_{i=1}^{n} c_i v_i \rightarrow \text{unique for a basis}$$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} X \end{bmatrix}$$

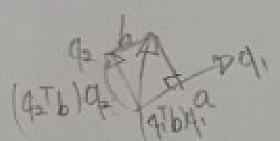
$$c_i = v_i^T X = \frac{v_i^T X}{v_i^T v_i}$$

· If given independant vectors

$a_1, a_2, a_3 \cdots\cdots$

$\rightarrow$ find the orthonormal basis vectors

$\Rightarrow$ Gram-Schmidt orthogonalization



1) $a \rightarrow \frac{a}{\|a\|} = q_1$

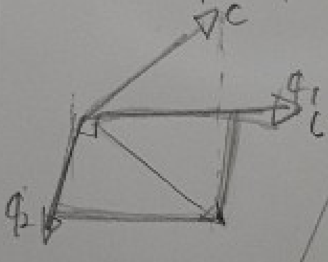
2) project b onto q_1

$$\frac{q_1^Tb}{q_1^Tq_1} q_1$$

$$b - (q_1^Tb)q_1 \perp q_1$$

$$\frac{b - (q_1^Tb)q_1}{\| b - (q_1^Tb)q_1 \|} = q_2$$

$$b = (q_1^Tb)q_1 + (q_2^Tb)q_2$$



$$c - \left((q_1^Tc)q_1 + (q_2^Tc)q_2 \right)$$

$\perp q_1$

q_2

normalization

q_3

$$c = (q_1^Tc)q_1 + (q_2^Tc)q_2 + (q_3^Tc)q_3$$

$a_1, a_2, \dots$

1) $q_1 = \frac{a}{\|a\|}$

2) $a_j - \sum_{i=1}^{j-1} (q_i^T a_j) q_i = A_j$

3) $\frac{A_j}{\|A_j\|} = q_j$

$$a_j = \sum_{i=1}^{j} (q_i^T a_j) q_i$$

$$X = \sum_{i=1}^{n} (q_i^T X) q_i$$

Projection → Least square.

(x_i, y_i) ⟹ line fitting

$y = ax+b$

$$\begin{aligned} y_1 &= ax_1 + b \\ y_2 &= ax_2 + b \\ &\vdots \\ y_n &= ax_n + b \end{aligned} \qquad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$Ax = b$

$$A^T A = \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{bmatrix}$$

$$\|Ax - b\|^2 = \sum_{i=1}^{n} (ax_i + b - y_i)^2 = f(a,b)$$

$$\left.\begin{aligned} \frac{\partial f}{\partial a} &= 0 \\ \frac{\partial f}{\partial b} &= 0 \end{aligned}\right] a, b$$

$$\hat{x} = (A^T A)^{-1} A b$$

$y = ax^2 + bx + c$

$y_i = ax_i^2 + bx_i + c$

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\|Ax - b\|^2 = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2 = f(a,b,c)$$

$$\left.\begin{aligned} \frac{\partial f}{\partial a} &= 0 \\ \frac{\partial f}{\partial b} &= 0 \\ \frac{\partial f}{\partial c} &= 0 \end{aligned}\right) a, b, c$$

Random Sample Consensus

RANSAC

일반최소자승법과 QR 분해

◎ Generalized Least Square

$Ax = b \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow x_i \rightarrow w_i$

$\rightarrow$ weight : 차등

$wAx = wb$

$A^TAx = A^Tb$.

$A^TW^TWAx = A^TW^Tb$

$a_{11}x_1 + a_{12}x_2 + \cdots = b_1$
$a_{21}x_2 + a_{22}x_2 + \cdots = b_2$

$wAx = wb$

3.4. Orthogonal Basis

- orthogonal vectors $\rightarrow$ independent
  $\rightarrow$ basis vectors
- Let $q_1, q_2 \cdots q_n$ be orthonormal

$q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$Q = \begin{bmatrix} q_1, q_2 \cdots q_n \end{bmatrix}$

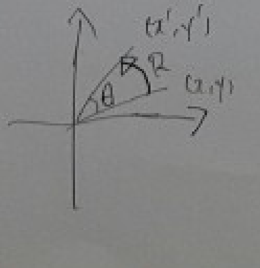
$Q^TQ \Rightarrow Q^T = Q^{-1}$ (Left Inverse)

$\begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \begin{bmatrix} q_1 \, q_2 \cdots q_n \end{bmatrix} = I$

◎ Q examples

1) Rotation matrix

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



2) Permutation Matrix

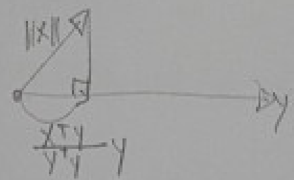
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

◎ Q Rotation preserves the length and angle

$\Rightarrow \|x\|^2 = x^Tx \qquad \|Qx\| = x^T\underbrace{Q^TQ}_{I}x$
$= x^Tx$

$\Rightarrow x^Ty. \quad (Qx)^T(Qy) = x^T\underbrace{Q^TQ}_{I}y = x^Ty$

- projection reduces the length



$\|Px\| \leq \|x\|$

$\|x^Ty\| \leq \|x\|\|y\|$

- for $q_1, q_2 \cdots q_n \in R^n$ (square system)

$x = \sum_{i=1}^{n} c_i q_i$

$x = \begin{bmatrix} q_1 q_2 \cdots q_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}^{-1} \begin{bmatrix} x \end{bmatrix}$

$A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$Q \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Q^T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix} x$$

$$c_1 = \frac{q_1^T x}{q_1^T q_1 = 1}$$

$$x = \sum_{j=1}^{n} c_j q_j = \sum_{i=1}^{n} (q_i^T x) q_i$$

$$\begin{aligned} q_i^T x &= q_i^T \Big(\sum_{j=1}^{n} c_j q_j\Big) \\ &= q_i^T (c_1 q_1 + \cdots + c_n q_n) \\ &= c_i \|q_i\|^2 = c_i \end{aligned}$$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_m \end{bmatrix} \qquad m \lesseqgtr n$$

$$Q = \begin{bmatrix} \ \end{bmatrix}$$

$$Qx = b$$

$$x = \underbrace{(Q^T Q)^{-1}}_{I} \underline{\underline{Q^T b}}$$

$$\boxed{Q^T}\ \boxed{Q} = \boxed{I}$$

$$\underset{(n \times m)}{\boxed{Q}}\ \underset{(m \times n)}{\boxed{Q^T}} = \underset{n \times n}{\begin{bmatrix} I & 0 \\ 0 \cdots & 0 \end{bmatrix}}$$

ex)

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Q^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

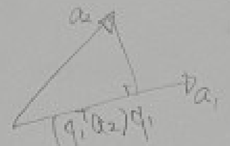
$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q : Left inves for rectangular system

◎ Gram-Schmitt Orthogonalization

· given linearly independent vectors $\{a_1, a_2, \cdots a_n\}$

to find the orthonormal basis vectors



1) $a_1 \to \frac{a_1}{\|a_1\|} \to q_1$

2) project a_2 onto q_1

$$a_2 - (q_1^T a_2) q_1 \perp q_1$$

$$\frac{a_2 - (q_1^T a_2) q_1}{\|a_2 - (q_1^T a_2) q_1\|} \to q_2$$

3) project a_3 onto q_1, q_2

$$a_3 - \big((q_1^T a_3) q_1 + (q_2^T a_3) q_2\big) \to q_3.$$

$$\Rightarrow a_j - \sum_{i=1}^{j-1} (q_i^T a_j) q_i \to \text{normalize} \Downarrow q_j$$

$$a_j = \sum_{i=1}^{j} (q_i^T a_j) q_i$$

ex)

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad a_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \to q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$q_3 : a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = q_3$$

- $A = QR$ factorization
- from G-E $\Rightarrow A = LU$

$$\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A} = \begin{bmatrix} (q_1^T a_1) q_1 & (q_1^T a_2) q_1 \\ & (q_2^T a_2) q_2 \end{bmatrix}$$

$$= \begin{bmatrix} (q_1^T a_1) q_1 & (q_1^T a_2) q_1 + (q_2^T a_2) q_2 & \cdots & (q_1^T a_n) q_1 + \cdots + (q_n^T a_n) q_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_1 a_2 \cdots a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$$\overset{AB}{\begin{bmatrix} a_1 a_2 \cdots a_n \end{bmatrix} \begin{bmatrix} b_1 b_2 \cdots b_n \end{bmatrix}} = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix}$$

$$\underset{Q}{\begin{bmatrix} q_1 q_2 \cdots q_n \end{bmatrix}} \underset{R}{\begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & \cdots & (q_1^T a_n) \\ 0 & (q_2^T a_2) & & \vdots \\ \vdots & 0 & & \vdots \\ 0 & 0 & & (q_n^T a_n) \end{bmatrix}}$$

$$Ax = b$$

$$x = (A^T A)^{-1} A^T b$$

$$= (R^T \underbrace{Q^T Q}_{I} R)^{-1} R^T Q^T b$$

$$= R^{-1} R^{-T} R^T Q^T b$$

- Function Space (Hilbert) and Fourier Series

Vector space (R^n)	Function Space (R^∞)
vectors $v_1, v_2 \cdots$	function $x_1(t), x_2(t) \cdots$
independent	independent (basis function)
$\rightarrow$ linear combination $X = \sum_{i=1}^{n} c_i q_i$	$x(t) = \sum_{i=1}^{\infty} a_i b_i(t)$
$\rightarrow$ inner product $\rightarrow$ orthogonal	$\rightarrow$ inner product $\rightarrow$ orthogonal

$1, t, t^2, t^3 \cdots$

$\rightarrow \begin{cases} \cos nt \\ \sin nt \end{cases} \quad 0 \le t < 2\pi$

고유값과 고유벡터 및 대각화

Chapter 5 Eigenvalue, Eigenvector

$\|Ka - x\|$

$Ax = \lambda x$ → eigenvector

→ scalar multiplication

→ eigenvalue

$(A - \lambda I)x = 0$

non-zero

$\det(A - \lambda I) = 0$ → singular

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & \\ \vdots & & \ddots & \\ \vdots & & & \end{vmatrix} = 0$$

$= \lambda^n - (\quad)\lambda^{n-1} \cdots$

ex)

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10 = 0$$

$\lambda = 2, -1$

$\lambda = 2$.

$$\begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ → null space : eigenvectors

$(A - \lambda I)x = 0$

x : eigenvector → $N(A - \lambda I)$

→ how to find null space (eigenvector)

→ reduced row echelon form

$2x_1 - 5x_2 = 0$ $\quad \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{e_1}$

$\lambda = -1$

$$\begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ $\quad 5x_1 = 5x_2$. $\quad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{e_2}$

ex)

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & -\frac{1}{2}-\lambda \end{vmatrix}$$

$= (1-\lambda)(\frac{3}{4}-\lambda)(-\frac{1}{2}-\lambda) = 0$

for triangular (or diagonal) matrix

→ eigenvalues = diagonal elements of A

$\det A = \prod_{i=1}^{n} \text{pivot}_i = \prod_{i=1}^{n} d_i = \prod_{i=1}^{n} \lambda_i$

trace of $A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$

$= \sum_{i=1}^{n} \lambda_i$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \vdots \\ \vdots & a_{22}-\lambda & \vdots \\ \vdots & \vdots & \vdots \end{vmatrix} = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = 0}{\lambda^n - \lambda^{n-1}(\lambda_1+\lambda_2+\cdots+\lambda_n) + \cdots}$$

$(a_{11}-\lambda)C_{11} + a_{12}C_{12} + \cdots$

5.2 Diagonalization of Matrix

1) $A = LU = LDU$

2) $A = QR = \begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & \cdot \\ \vdots & (q_2^T a_2) & \cdot \\ \vdots & & \vdots \\ \vdots & \vdots & \cdot \end{bmatrix}$

3) $A = S \Lambda S^{-1}$

$S = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ \vdots & & \ddots \\ & & & \lambda_n \end{bmatrix}$

$Ae_i = \lambda_i e_i$

$\begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} \lambda e_1 & \lambda e_2 & \cdots & \lambda e_n \end{bmatrix}$

$AB = A\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix}$

$A\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 & \cdots \\ 0 & \vdots \\ 0 & 0 \end{bmatrix}$

$AS = S\Lambda$

ex)

$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} = 0$

$\lambda = 1, 0$

$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A = S\Lambda S^{-1}$

ex)

$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\lambda = \pm i$

$\lambda = i$

$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda = -i$

$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ i \end{bmatrix}$

$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \quad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Remark 1 >

If $\lambda_1, \lambda_2 \cdots \lambda_n$ are different,
then $e_1, e_2 \cdots e_n$ are linearly independent

$\Rightarrow$ pf) assume that e_2 is dependent on e_1

$e_2 = ce_1$

$\times A \quad Ae_2 = cAe_1 \Rightarrow \lambda_2 e_2 = c\lambda_1 e_1$

$\times \lambda_2 \to \lambda_2 e_2 = c\lambda_2 e_1$

$c\lambda_2 e_1 = c\lambda_1 e_1$

$c(\lambda_2 - \lambda_1)e_1 = 0$

$\lambda_2 \neq \lambda_1 \quad e_1 \neq 0. \quad c = 0.$

$e_1, e_2 \to$ independent

$e_n = c_1 e_1 + c_2 e_2 + \cdots + c_{n-1} e_{n-1}$

$\times A \quad Ae_n = c_1 Ae_1 + c_2 Ae_2 + \cdots + c_{n-1} Ae_{n-1}$

$\lambda_n e_n = c_1 \lambda_1 e_1 + c_2 \lambda_2 e_2 + \cdots + c_{n-1} \lambda_{n-1} e_{n-1}$

$\times \lambda_n \quad \lambda_n e_n = c_1 \lambda_n e_1 + c_2 \lambda_n e_2 + \cdots + c_{n-1} \lambda_n e_{n-1}$

$c_1(\lambda_n - \lambda_1)e_1 + c_2(\lambda_n - \lambda_2)e_2 + \cdots + c_{n-1}(\lambda_n - \lambda_{n-1})e_{n-1}$
$= 0$

$\downarrow$

$A^T = A$

Remark 2 >

S is not unique.

since $ke \to$ eigenvector

$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$A = S \Lambda S^{-1}$

Remark 3 >

The order of eigenvalues is same with that of eigenvectors

$$\begin{bmatrix} e_2 & e_3 & e_1 \end{bmatrix} \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

Remark 4 >

Not all matrices have n linearly independent eigenvectors

$\to A = S \Lambda S^{-1}$

is not always established

· Powers

$A \to \lambda, e$

$A^k \to \lambda^k, e$

$$\times A \left(\begin{array}{l} Ae = \lambda e \\ A^2 e = \lambda A e = \lambda^2 e \end{array} \right.$$

$$A^k = (S\Lambda \underbrace{S^{-1})(S}_{I}\Lambda S^{-1}) \cdots (S \Lambda S^{-1})$$

$$= S \Lambda^k S^{-1}$$

$\downarrow$

$$\begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix}$$