Lab - 8

Rushi Rajpara (201801410)[*] and Ayan Khokhar (201801057)[†]
Dhirubhai Ambani Institute of Information & Communication Technology,
Gandhinagar, Gujarat 382007, India
CS-302, Modeling and Simulation

In this lab we will try to model and simulate 1D and 2D random walks and some of their variants and will try to analyze them.

1D Random Walk

Distribution followed by Random Walk

Lets assume that p is probability that random walker will move to the right side of x-axis (i.e. x co-ordinate will increase by l) and $1-p$ be probability of walker moving to the left side (i.e. x co-ordinate will decrease by l).

Now, lets try to find probability of walker being at position k after n steps. Lets assume walker took a left steps and b right steps, so we have,

$$la + lb = ln$$

$$-la + lb = k$$

From above equations, we get,

$$b = \frac{ln + k}{2l}$$

$$a = \frac{ln - k}{2l}$$

Probability of this event is,

$$p = \begin{cases} \binom{n}{a} * p^b * (1-p)^a & (ln-k)\%(2l) = 0, \\ & a \geq 0, \\ & b \geq 0 \\ 0 & otherwise \end{cases}$$

Thus, position of random walker after some number of steps have a binomial distribution. This can be approximated as a normal distribution, here np steps are expected in right direction and nq steps are expected in left direction, so mean is,

$$\mu = nl(2p-1) \tag{1}$$

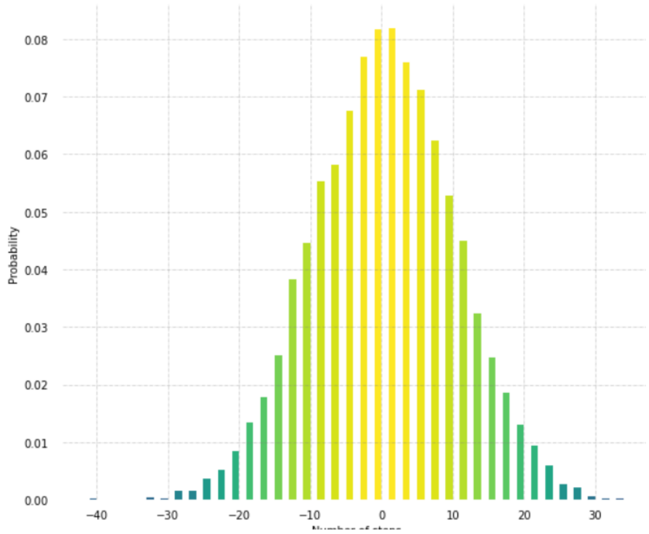

FIG. 1: Probability distribution of position of random walker after 100 steps with p = 0.5. Number of trails = 10000.

As decision made by random walker at any time instance is independent of other time instances, we have,

$$\sigma^2 = \sum_{i=1}^{n} (\sigma_i^2)$$

$$\sigma^2 = 4nl^2 p(1-p) \tag{2}$$

So, normal distribution, which approximates binomial distribution of random walk, has a mean of μ and variance of σ^2. In order to get probability of random walker being at position k after n steps we can use following formula,

$$p = \begin{cases} P(k - l < X \leq k + l) & (ln-k)mod(2l) = 0 \\ 0 & otherwise \end{cases}$$

Here, P is probability function of normal curve with mean μ and variance σ^2, given by equations (1) and (2). Figure (1) shows probability distribution of position of random walker.

[*]Electronic address: 201801410@daiict.ac.in
[†]Electronic address: 201801057@daiict.ac.in

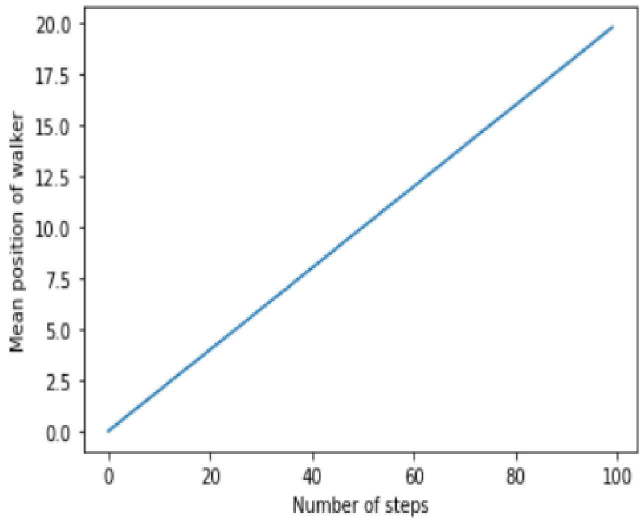

FIG. 2: Mean position of Walker v/s number of steps. Number of trials = 100000, p = 0.6, l = 1.

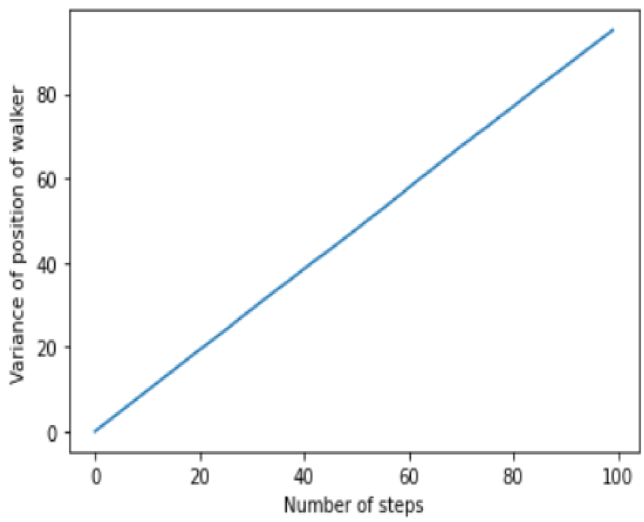

FIG. 3: Variance of walker v/s number of steps. Number of trials = 100000, p = 0.6, l = 1.

Mean of Random Walk

Mean of position of random walker at any time is given by equation (1). Figure (2) shows how mean position of walker changes with number of steps.

Variance of Random Walk

Variance of position of random walker at any time is given by equation (2). It is linearly dependent on number of steps. Figure (3) shows show variance of position of random walker changes with number of steps.

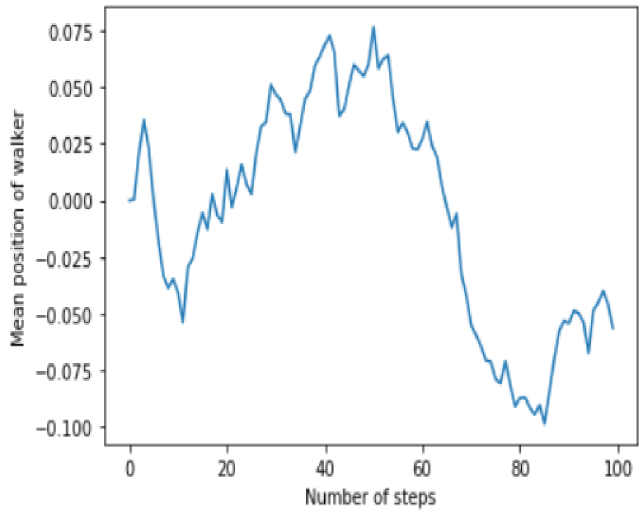

FIG. 4: Mean position of Walker v/s number of steps. Number of trials = 100000, p = 0.5, l = 1.

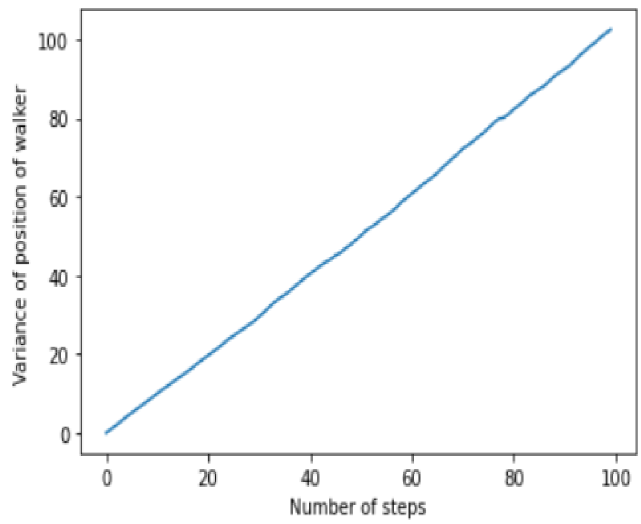

FIG. 5: Variance of walker v/s number of steps. Number of trials = 100000, p = 0.5, l = 1.

Unbiased Random Walk

Unbiased random walk is just a special case of biased random walks, where p = 0.5. Figure (4) and Figure (5) shows how mean and variance of position of random walker changes with number of steps. Note, that mean of unbiased random walk is same for any number of steps taken (i.e. any n value) and also variance depends only on number of steps taken since starting (if l is fixed), so we can say that unbiased random walk is stationary process. However, as mean keeps changing in biased random walk (i.e., $p \neq 0.5$), so it is not a stationary process.

1D Random Walk With 3-Options

In this variant of 1D random walk, random walker has 3 options at each step, either to move forward, backward or to stay at same position. Lets try to find out probability of walker being at position k after n steps. Lets say number of steps in which walker moves backward, forward and stand still is l, r and s respectively. Without losing generality we will assume that length of steps will be 1. So, we have,

$$l + r + s = n$$

$$r - l = k$$

In order to find probability of above mentioned event happening we will have to use summation over values of s,

$$P(n,k) = \sum_{s=0}^{n-k} \begin{cases} \frac{n!}{l!r!s!} * p^r * q^l * (1-p-q)^s & (n-s-k)\%2 = 0 \\ 0 & otherwise \end{cases}$$

In above equation, it is assumed that probability of taking a step forward, backward and staying still, is p,q and $1 - p - q$ respectively. Also, note that r and l in above equation are $\frac{n-s+k}{2}$ and $\frac{n-s-k}{2}$ for a particular s.

Mean of the distribution is,

$$\mu = n(p - q)$$

Variance of the distribution is,

$$\sigma^2 = n(p + q - p^2 + 2pq - q^2)$$

Figure (6) shows our simulation's result.

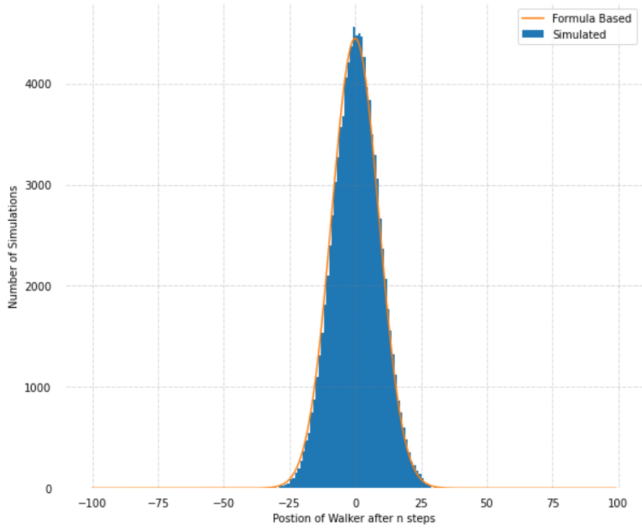

FIG. 6: Number of Simulations v/s Position of Random Walker After n Steps. n = 100, p = 0.4, q = 0.4

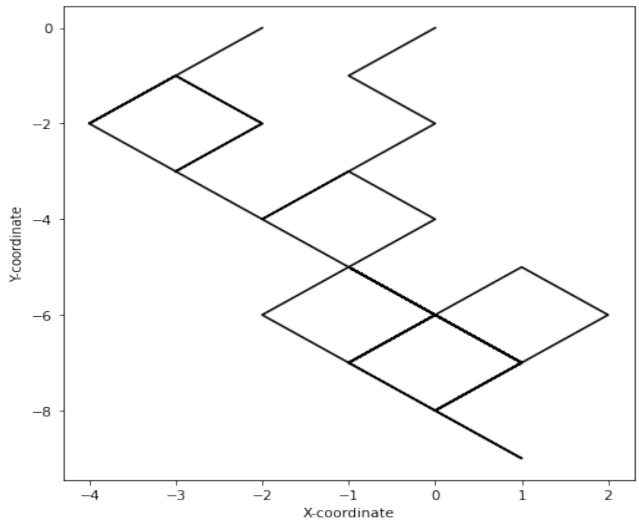

FIG. 7: Position of the Random Walker on the coordinate axis

2D Random Walk

In unbiased 2D random walk, the random walker can move in any of the four directions North-East, North-West, South-East, South-West with equal probabilities.

Figure 7 shows one such simulation of a 2D random walk. Thus in each step the random walker travels a euclidean distance of $\sqrt{2}$ units.

Let p, q, r and s be the probabilities of going in NE, NW, SE and SW directions respectively then the mean of the position in x and y directions are given by following equations:

$$\mu_X = n(p + q - r - s)$$

$$\mu_Y = n(p + r - q - s)$$

Thus for an unbiased 2D random walk the mean position of random walker will be $(0, 0)$ as all p, q, r and s values will have the same probability 1/4.

Lets say mean distance of walker from origin is R_n^2 after n steps. Lets assume that x_0, x_1, x_2,...,x_n be change in x co-ordinate that occurred in n steps. Similarly, let y_0, y_1, y_2,...,y_n be change in y co-ordinate that occurred in n steps. So, we can write,

$$R_n^2 = (x_0 + x_1 + ...x_n)^2 + (y_0 + y_1 + ... + y_n)^2$$

$$R_n^2 = (x_0^2 + x_1^2 + ... + x_n^2 + y_0^2 + y_1^2 + ... + y_n^2) + (x_0x_1 + x_0x_2$$

$$+ .... + y_0y_1 + y_0y_2 + ...)$$

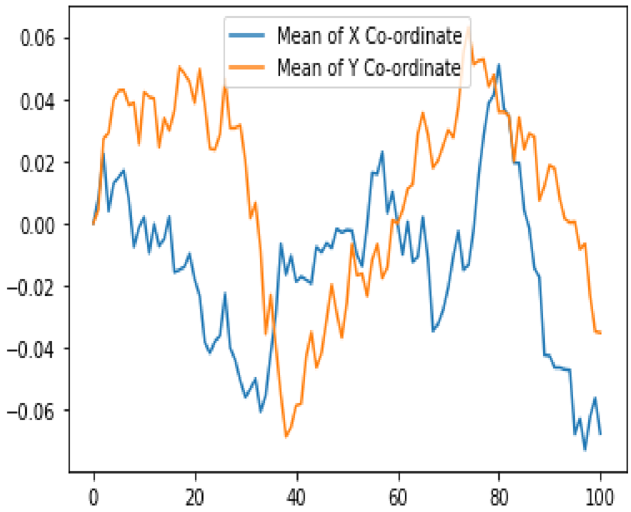

FIG. 8: Mean Position of the Random Walker

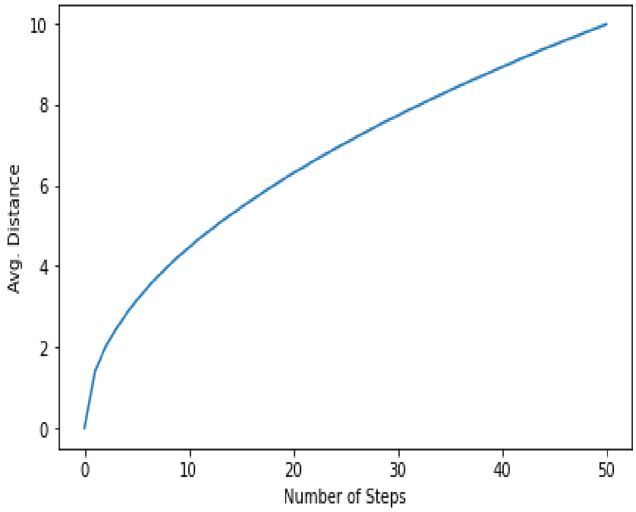

FIG. 9: RMS Distance vs Number of steps where number of simulations = 500000

As $x_i x_j$ and $y_i y_j$ can be -1 and 1 with equal probability, those terms will get canceled. And also $x_i^2 + y_i^2 = 2$, So,

$$R_n^2 = 2n$$

$$R_n = \sqrt{2n}$$

Thus, root mean square of distance after n steps is $\sqrt{2n}$. Figure 8 shows the plot of RMS distance vs the number of steps.

2D Random Walk with 8 permissible directions

In this variant of 2 dimensional random walk, the random walker is permitted to move in all the 8 permissible directions namely north, east, west, south, north-east, north-west, south-east, south-west with different probabilities.

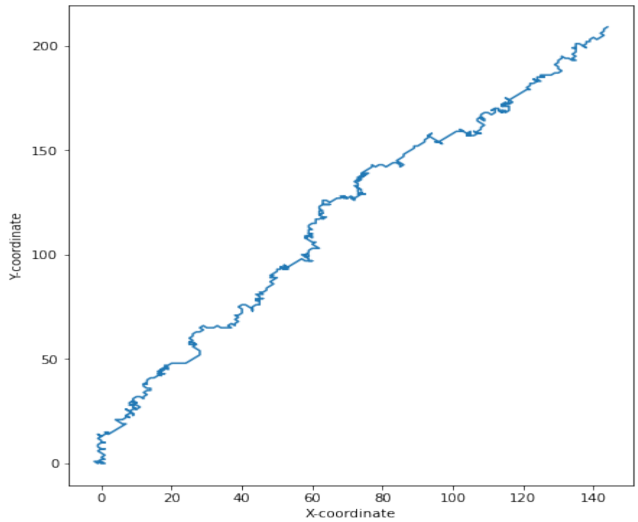

FIG. 10: Position of a Random Walker in which all the 8 possible directions are permitted with different probabilities

Figure 10 shows one such example where probabilities of moving in N = 0.19, NE = 0.24, E = 0.17, SE = 0.1, S = 0.02, SW = 0.03, W = 0.1, NW = 0.15. Thus in such a case for large number of steps the random walker tends to move in the direction which has the higher probability.

Conclusion

In this report, we discussed about the 1D and 2D random walks and their different variants such as biased, unbiased and random walk with three states. We also looked at how mean and variance changes with number of steps for different type of random walks. We also found probability distributions of different random walks after certain number of steps.