Lab - 5

Rushi Rajpara (201801410)[*] and Ayan Khokhar (201801057)[†]
Dhirubhai Ambani Institute of Information & Communication Technology,
Gandhinagar, Gujarat 382007, India
CS-302, Modeling and Simulation

In this lab, we will analyse models of epidemiological, namely the SIR model and the SARS model. We will also look at the effects of various interventions on an epidemic.

SIR Model

Introduction

SIR model is a compartmental model which simplifies the study of epidemiology and is used to mathematically model various infectious diseases. The population flows from one compartment to others between S(Susceptible), I(Infected) and R(Recovered).

Mathematical Model

The following differential equations define the flow between the compartments of the SIR model.

$$\frac{dS}{dt} = -\beta SI \tag{1}$$

$$\frac{dI}{dt} = \beta SI - \alpha I \tag{2}$$

$$\frac{dR}{dt} = \alpha I \tag{3}$$

Where α and β are recovery rate and infection spread rate respectively. $\frac{dS}{dt}$, $\frac{dI}{dt}$ and $\frac{dR}{dt}$ represents the rate of flow of population in Susceptibles, Infected and Recovered Compartments respectively.

Role of S_0

We can see from the figure(2) that as the initial number of susceptible changes, the peak of the infected people increases as there are more available people for the infection to spread. Thus as the initial number of susceptibles (S_0) increases then the intensity and the duration of the epidemic increases.

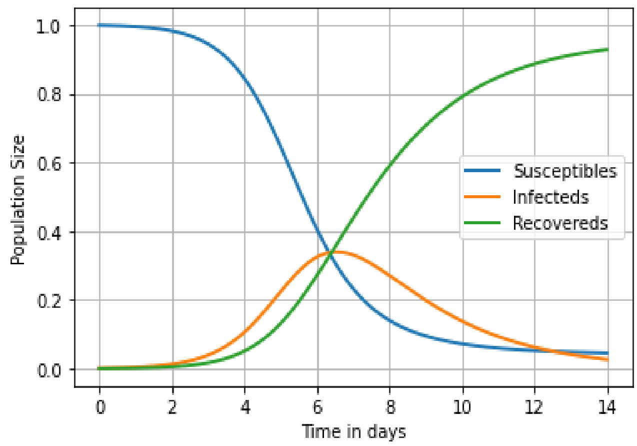

FIG. 1: Population Fraction vs Time (in days) with parameters $\alpha = 0.5$ and $\beta = 0.00218$

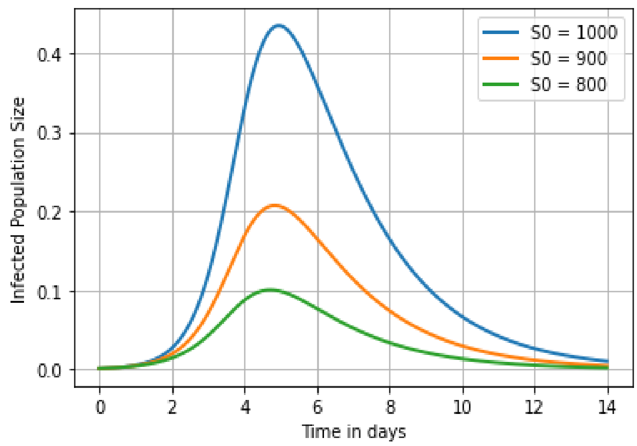

FIG. 2: Total Number of Infected Population vs S_0

Role Of R_0

We will examine role of R_0in size of epidemic and time at which maximum number of infected people will be present. By dividing equation (2) by equation (1) and normalising, we get,

$$\frac{di}{ds} = \frac{R_0 si - i}{-R_0 si}$$

Integrating above equation with starting values of s_0 for

*Electronic address: 201801410@daiict.ac.in
†Electronic address: 201801057@daiict.ac.in

s and i_0 for i, we get,

$$i + s - \frac{1}{R_0} ln(s) = i_0 + s_0 - \frac{1}{R_0} ln(s_0) \quad (4)$$

Also, from equation(2) we can see that i_{max} occurs when $s = \frac{s}{R_0}$. Substituting this value in equation (4) we get,

$$i_{max} = 1 - \frac{1}{R_0} + \frac{1}{R_0} ln(\frac{R_0}{s_0}) \quad (5)$$

Getting time at which number of infected is maximum (t_{max}) is not so easy to get and requires some numerical approximations. We used various regression techniques to get a curve which fits our data well enough.

$$t(S_0, I_0, R_0) = 24.655 * (R_0)^{-1.0592} * (\frac{S_0}{800 * I_0})^{0.1} \quad (6)$$

Above equation, gives almost correct output of t_{max}, when $I_0 << S_0$ and R_0 is not very small (very close to 1) and is not unrealistically high (in order of 10^4).

From equation (5), we can see that value of I_{max} increases with increase in R_0. Figure(3) represents this fact.

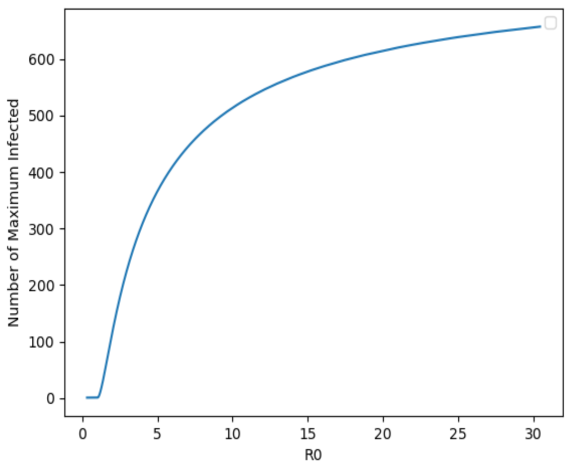

FIG. 3: I_{max}vs R_0 for SIR model. Values used : $\alpha = 0.5$, $S_0 = 762$ and $I_0 = 1$

From equation (6), we can observe that t_{max} decreases as R_0 increases. Figure(4) compares our solution to actual answer.

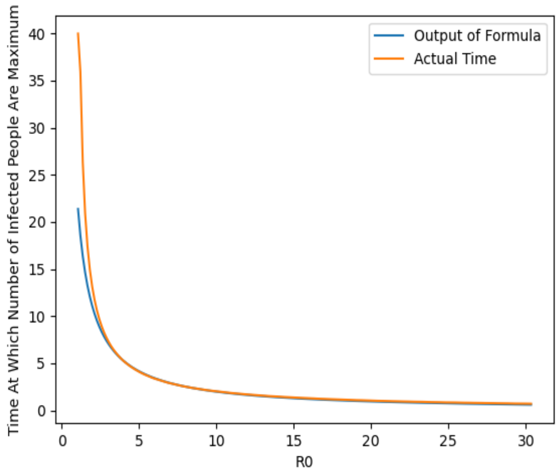

FIG. 4: t_{max}vs R_0.Values used : $\alpha = 0.5$, $S_0 = 762$ and $I_0 = 1$

Role of Vaccination

The modified equations to accomodate the role of vaccination are as follows:

$$\frac{dS}{dt} = -\beta SI - \gamma S$$

$$\frac{dI}{dt} = \beta SI - \alpha I$$

$$\frac{dR}{dt} = \alpha I$$

$$\frac{dV}{dt} = \gamma S$$

Where V is the vaccination compartment and the γ is the vaccination fraction. $\frac{dV}{dt}$ is the rate at which the population flows into the vaccination compartment.

Figure(7) clearly shows that, as the vaccine percentage increases and more the effective it becomes the peak value of the infected population decreases.

Due to the flow of population from the susceptibles to vaccinated the maxima of the infected decreases and thus vaccination prevents infection spread among the population.

Vaccination with partial effect

Assuming vaccination is only partially effective the individuals can become susceptible at rate μ. Then, the modified equations for such model will be:

$$\frac{dS}{dt} = -\beta SI - \gamma S + \mu V$$

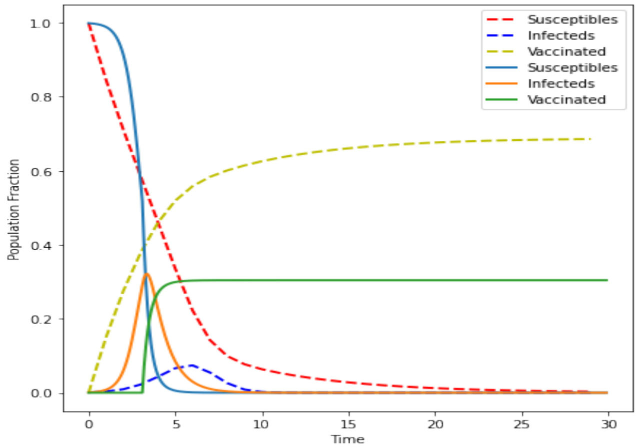

FIG. 5: Population Fraction vs Time $\beta = 0.00218$ $\alpha = 0.5$ and dotted is with vaccination rate = 0.15 and Immunization begins immediately and solid lines is with vaccination rate = 0.15 and Immunization begins after 3 days

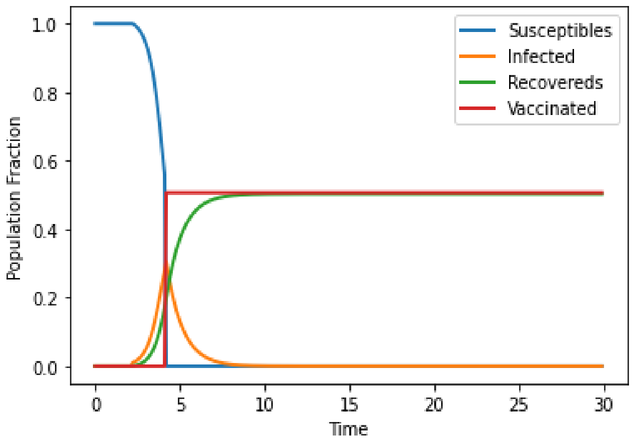

FIG. 6: Population Fraction vs Time (days) $\beta = 0.00218$ $\alpha = 0.5$ where children are vaccinated before and Immunization begins after 4 days

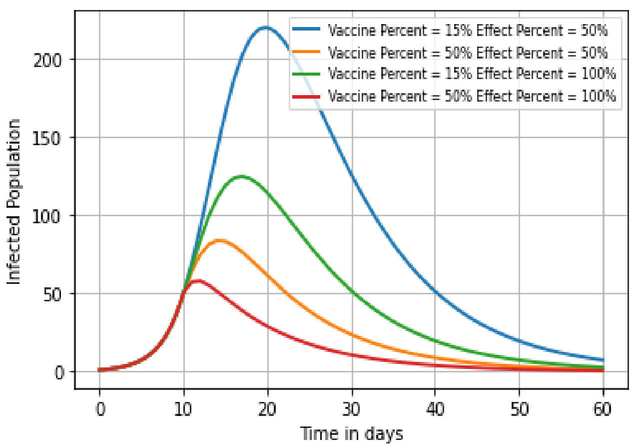

FIG. 7: Infected Population vs Time (days) with varying vaccine amount and effect percent of the vaccine

$$\frac{dI}{dt} = \beta SI - \alpha I$$

$$\frac{dR}{dt} = \alpha I$$

$$\frac{dV}{dt} = \gamma S - \mu V$$

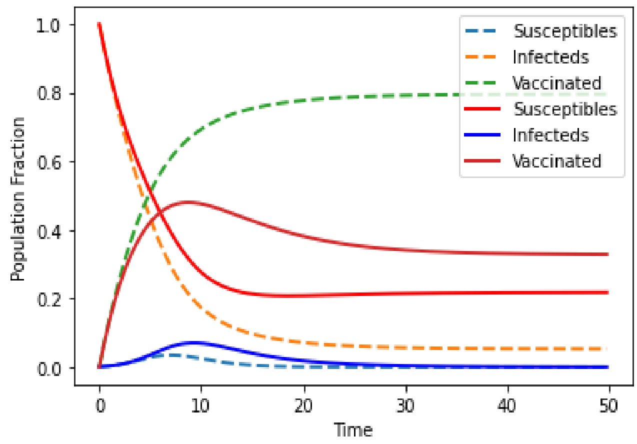

FIG. 8: Population Fraction vs Time (days) where dotted is with $\mu = 0.01$ and solid lines are with $\mu = 0.1$

From figure(8), we can see that as the value of μ increases then the population flow increases from the vaccinated compartment to the susceptible compartment as the effect of the vaccine decreases. Thus instead of the susceptibles approaching to zero value they tend to become stable after sometime. Vaccinated population reaches to a maxima and then decreases and finally stabilizes.

Role of Lock Down

Lock down can serve for many purpose during an epidemic outbreak. We will focus on how lock down can be used to limit maximum number of infected people present at any time and to decrease number of deceased people. Lock down of different type serves different purpose. We will look into 2 types of lock down : single lock down and intermittent lock down.

Single Lock Down

Single lock down can be used to decrease number of deceased and number of total infected during epidemic. We will assume that a certain percentage of recovered will be deceased. So in order to decrease number of deceased, we should try to increase number of susceptible at the end of epidemic. Also, we will assume that as soon as lock down starts value of β becomes $\rho\beta$ where $0 <= \rho <= 1$. Here, ρ is measure of strictness of lock down, lower the ρ, stricter the lock down.

Lets try to make susceptible at the end of epidemic to be kS_0. Lets say that we start lock down at time $t = t_s$ and at that time $S = aS_0$, $I = bI_0$ and $R = (1 - a - b)R_0$.Also, lets assume t_e be the end time of lock down and $S_e = kS_0$, $I_e \approx 0$ and $R_e \approx (1 - k)S_0$. Now , by dividing equation (1) and (3) we get,

$$\frac{dS}{dR} = \frac{-\rho\beta S}{\alpha N}$$

Integrating above equation and putting appropriate values we get following relation,

$$ln(\frac{k}{a}) <= \frac{-\rho\beta}{\alpha}(a + b - k) \tag{7}$$

If we start lock down when values of a and b is such that equation(4) hold true and end lock down when $I \approx 0$, then number of susceptible at the end of epidemic will be greater than kS_0. Figure(9) shows results obtained by using equation (7) for $\rho = 0.15$, $\beta = 1.66116$, $\alpha = 0.5$, $S_0 = 762$, $I_0 = 1$ and $k = 0.25$.

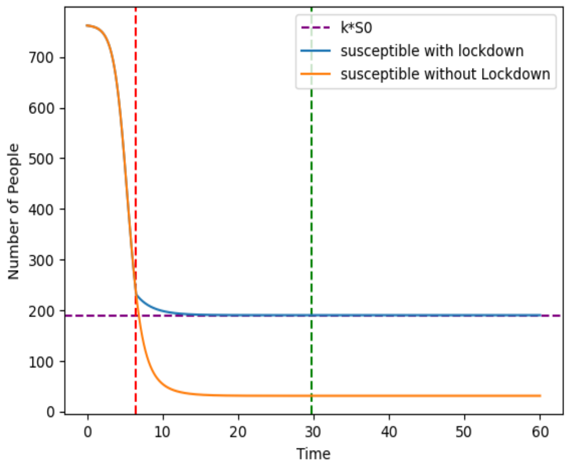

FIG. 9: Number of susceptible with and without lock down v/s time(in days). Red dotted line represents start of lockdown and green dotted line represents its end.

Intermittent Lock Down

All countries have limited medical resources and for that reason they don't want number of infected people to go beyond a particular limit. Intermittent lock down can be used to keep total number of infected below a certain level at any time. Lets say we want to keep number of infected people to be lesser than I_0. In order to do so lock down must be initiated when I is just a little lesser than I_0 and we will open lock down when I is below a certain limit (I_l). Figure(10) represents result obtained by implementing this idea. We will again assume that β changes to $\rho\beta$ during lock down ($0 <= \rho <= 1$).

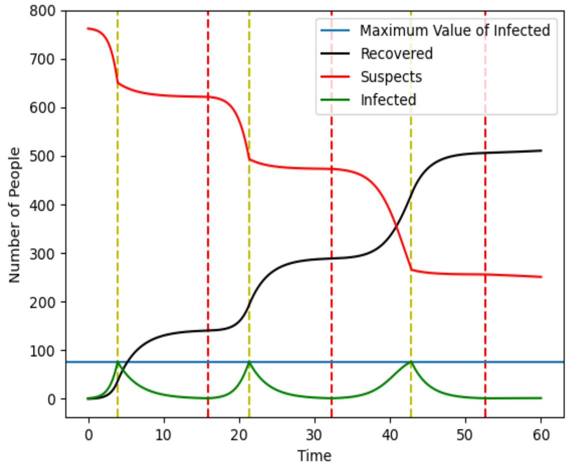

FIG. 10: Number of People v/s Time. Values used are $I_0 = 0.1S_0$ and $\rho = 0.1$. Yellow and red dotted line represents starting and ending of lock down respectively.

We observed that higher I_0 will lead to less number of lock downs and lesser interval of lock downs. Higher I_l will lead to reaching the $I \approx 0$ quickly but number of infected people overall will be more. Also, larger the ρ, longer will be the lock down but number of infected people will decrease overall. Note that, when $I = I_0$, if $\frac{\rho\beta}{\alpha} > 1$ then we will not be able to limit maximum number of infected to I_0.

Effects of Time Varying Beta

In previous discussions, we assumed that beta value becomes $\rho\beta$ as soon as lock down starts and gets back to β as soon as lock down is over. But in real life scenario, this is not usually the case. In real life scenario, β decreases slowly to the value of $\rho\beta$ when lock down is initiated and ones lock down is over, value of beta return to β slowly once lock down in over. We can model this change with following equations,

For $0 < t <= st$,

$$\beta(t) = \beta_0$$

For $st < t <= et$,

$$\beta(t) = \beta_0 - (1 - \rho)\beta_0(1 - e^{-\gamma(t-st)})$$

For $t > et$,

$$\beta(t) = \frac{\beta_0}{1 + \frac{\beta_0 - \beta(et)}{\rho\beta(et)} e^{-\gamma(t-et)}}$$

Here, st is starting time of lock down and et is ending time of lock down. Lock down should be done such that $st <= I_{max} <= et$. In case of time varying β, we will lock down for longer time if we want to produce same result as that of ideal case (instantaneous change in β at start and end of lock down). Here, we observed that if γ then peak of infected (size of epidemic) will be lesser as compared to size of epidemic when γ is higher for same time frame.

Also, if value of γ is higher then we will need to increase our time frame of lock down in order to get more susceptible (lesser deaths) at the end of epidemic ($I \approx 0$), this happens because if we open lock down too early then there will be considerable increase in number infected people after lock down (second wave). Figure(11) represents above situation.

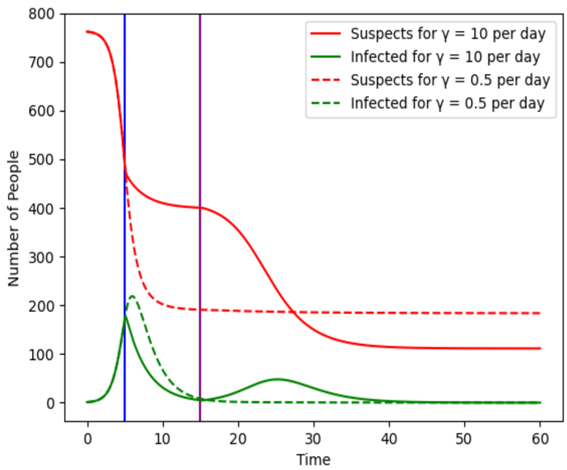

FIG. 11: Number of People v/s Time(in days). Start and end of lock down is represented by blue and purple line respectively.

SARS Model

Introduction

This model was developed by Marc Lipsitch and other collaborators for modeling outbreak of SARS disease. This model is an extension of the standard SEIR model. It has more compartments than SEIR as it distinguishes quarantined and non-quarantined people and also recovered and deceased people. Figure(12) shoes flow from various compartments of SARS model.

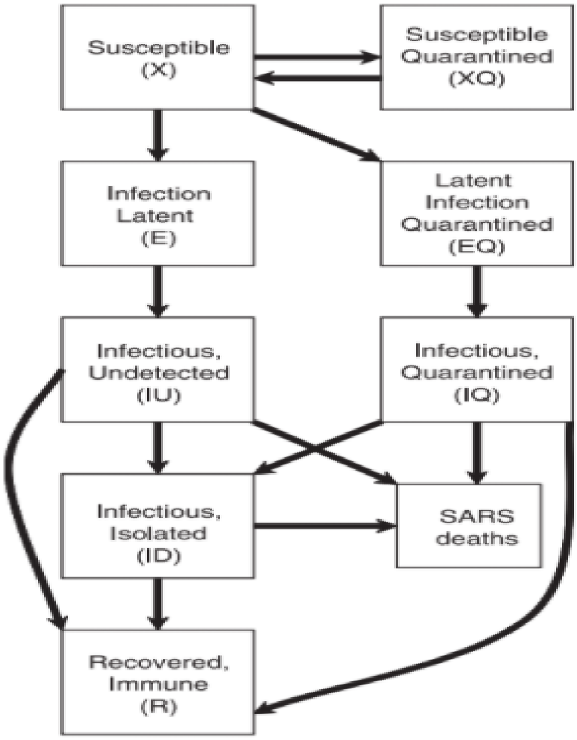

FIG. 12: Flow Diagram Of SARS Model

Mathematical Model

$$\frac{dS}{dt} = uS_Q - \frac{k(1-b)qSI_U}{N} - \frac{kb(1-q)SI_U}{N} - \frac{kbqSI_U}{N}$$

$$\frac{dS_Q}{dt} = \frac{k(1-b)qSI_U}{N} - uS_Q$$

$$\frac{dE}{dt} = \frac{kb(1-q)SI_U}{N} - pE$$

$$\frac{dI_U}{dt} = pE - (v+m+w)I_U$$

$$\frac{dE_Q}{dt} = \frac{kbqSI_U}{N} - pE_Q$$

$$\frac{dI_Q}{dt} = pE_Q - (v+m+w)I_Q$$

$$\frac{dI_D}{dt} = w(I_U + I_Q) - (v + m)I_D$$

$$\frac{dR}{dt} = v(I_Q + I_U + I_D)$$

$$\frac{dD}{dt} = m(I_Q + I_U + I_D)$$

where,

b: is the probability that the contact between S and I_U results in transmission of SARS.

k: mean number of contacts per day someone from infectious undetected (IU) has.

m: is the per capita death rate.

N: is initial number of people in population.

p: is fraction per day of exposed people who become infectious.

q: is fraction per day of individuals in susceptible (S) who have had exposure to SARS that go into quarantine.

u: is fraction per day of those in susceptible quarantined (SQ) who are allowed to leave quarantine.

v: is the per capita recovery rate.

w: fraction per day of those in infectious undetected (IU) who are detected and isolated and thus transferred to category infectious isolated (ID).

Reproduction Number

From equation of $\frac{dI_U}{dt}$ we can say that, average time a person remains infected is $\frac{1}{v+m+w}$. Also, a person who is infected and undiagnosed (IU category) will make on average k contacts per unit time with susceptibles. So overall contacts made by a person from IU category while he/she is still infected is $\frac{k}{v+m+w}$. Now, out of those contacts only b portion of contacts will result in decease transmission, so portion of susceptible people who will get infected will be $\frac{bk}{v+m+w}$. However, out of this people, q fraction of people will go to quarantine, so effective increase in infected undiagnosed is $\frac{bk(1-q)}{v+m+w}$. So, reproduction number is,

$$R_0 = \frac{bk(1-q)}{v+m+w}$$

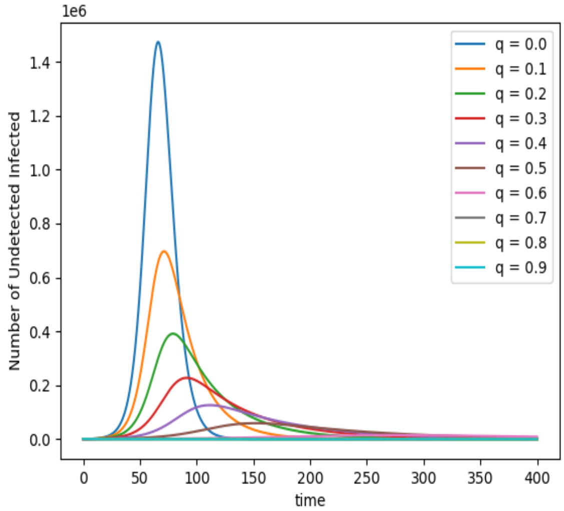

FIG. 13: Undetected Infected v/s Time(in days) for varying q.

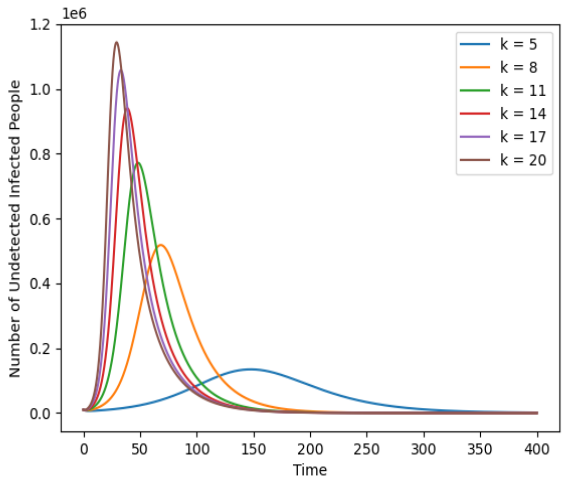

FIG. 14: Undetected Infected v/s Time(in days) for varying k.

Observations

- Parameter q is directly proportional to the number of people gets quarantined. So, higher the value of q, more number of people will be quarantined and thus overall number of infected people (total of all three classes of infected) will decrease. Due to this number of susceptible at the end of epidemic will also be larger and number of deaths will be lower. However, maxima of number of people who are quarantined infected(IQ) don't simply decrease with increase in q as opposed to maxima of infected detected(ID) and infected undetected(IU). Maxima of IQ first increases, as more are getting quarantined, Iq will get more share from total number of infected people. However, after certain limit, maximum of IQ starts decreasing with increase in q, as number of total infected people will decrease. With increase in q, time to reach I_{max}(maximum number of total infected over all time) will also decrease. Figure(13) shows how size of epidemic increases due to decrease in q.

- Parameter k describes mean number of contacts per day for someone who is infected undetected (IU). So if k is increased, then more number of contacts will be made by people who are in IU, and so more infection will get spread. Thus, increase in k increases size of epidemic (maximum of IU, IQ, ID all increases with increase in k). Also, note that it also decreases time it takes to reach to I_{max}. As value of k increases, number of susceptible at the end of epidemic will be less and thus there will be more deaths due to epidemic. Figure(14) shows how size of epidemic increases due to increase in k.

- 1/(u + w + v) represents average time a person remains infected. If this average increases, then size of epidemic will increase and time to reach I_{max} will decrease.

- Parameter b represents probability that a contact between person in infectious_undetected (IU) and someone in susceptible (S) results in transmission of SARS. So lesser the probability, lesser will be size of epidemic.

Effect of Late Start of Quarantine

In realistic scenario, quarantine and isolation measures are not instituted since start of epidemic. It generally takes time to notice that a virus is spreading and time is also consumed by other tasks that need to be performed before instituting isolation measures. We should see how a late start of quarantine can affect pressure on medical facilities and number of people who will be in quarantine.

As we know that lesser value of q means there will be higher number of infected people and undiagnosed people will be there and so size of epidemic will be larger. Thus, if we start our quarantine late, then there will already be high number of people who have got the disease and it will put a lot of pressure on health care system to handle them. So, it is better to start quarantine while still there are only a manageable number of infected people. Figure(15) shows effect of late quarantine on health care system.

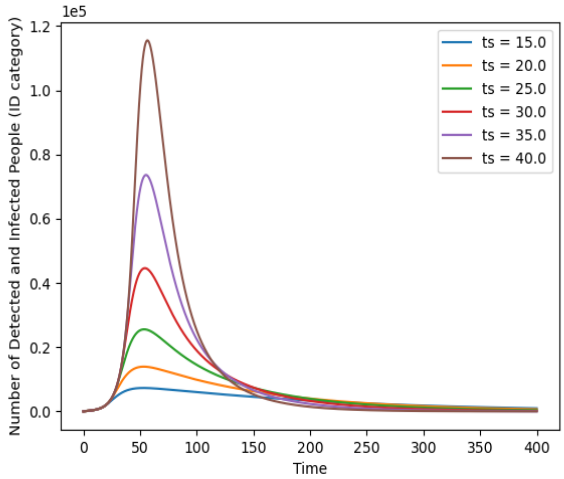

FIG. 15: Number of People in ID Category v/s Time(in days) for different start time(ts) of quarantine.

If we start quarantine late then size of epidemic will have reached a significant size and in order to put an end to epidemic, more number of people will need to be quarantined. Figure(16) shows how start time of quarantine can affect total number of people in quarantine.

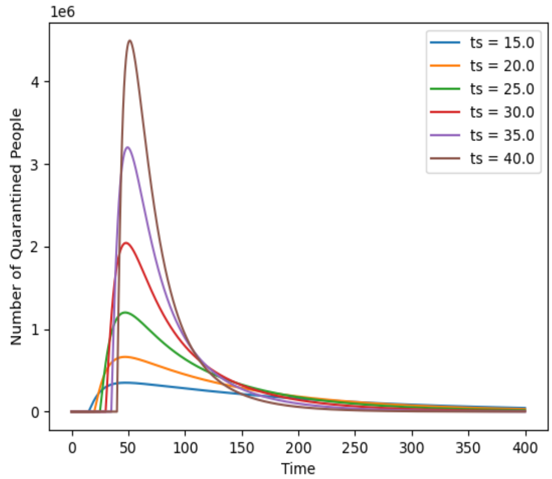

FIG. 16: Total Number of Infected v/s Time(in days) for different start time of quarantine.

Thus, we can say that in order to put a lesser pressure on health system and people, quarantine and isolation measures should be started as soon as possible.

Interventions Effects

The scale of interventions needed to control an epidemic depends on the number of infectious people present at that time and on the use of control measures such as isolation and quarantine. Isolation and quarantine will be less effective if the number of cases increases over time. Thus such control measures if implemented in the early stage of the epidemic helps to prevent more stricter measures when the epidemic spreads. The number of days spent by in the quarantine typically depends on the effectiveness of the quarantine and other control measures. Above a threshold, Quarantining a larger fraction of the infected population helps to decrease the number of overall person days spent in the quarantine. This threshold and the overall number of people needed to quarantine can be lowered further using isolation which can be used as a control measure to stop the further transmission.

During the interventions, R (reproduction number) changes to R_{int} and D (average time for an infected person to say infected) changes to D_{int}, which are related by following equation,

$$R_{int} = R(1-q)\frac{D_{int}}{D}$$

Figure(17) shows contour representing R_{int} as function of proportional decrease in D due to interventions and q.

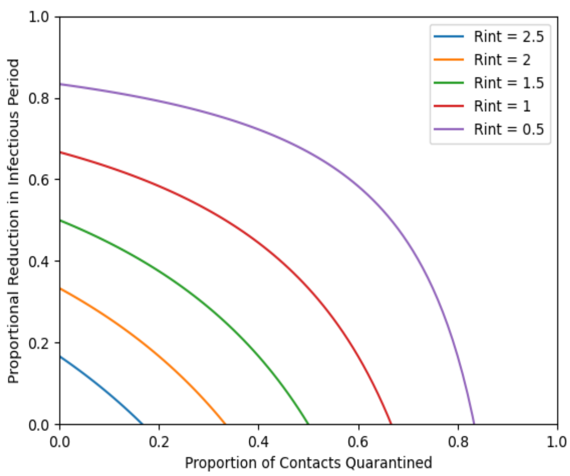

FIG. 17: Total Number of Infected v/s Time for different start time of quarantine.

Conclusion

We looked at how various parameters can affect epidemic in both SIR and SARS model. We also studied some intervention techniques to bring down the size of an epidemic. From our discussion, we can surely say that in order to prevent pressure on health care system and to keep number of deceased people to the minimum, measures like lockdown, social distancing and vaccination should be initiated as soon as possible to keep value of reproduction number to be less then 1. However, choosing an efficient strategy to deal with an epidemic will depend on availability of resources, people's behaviour and various other factors.