# Lab - 3

Rushi Rajpara (201801410)* and Ayan Khokhar (201801057)†
*Dhirubhai Ambani Institute of Information & Communication Technology,*
*Gandhinagar, Gujarat 382007, India*
*CS-302, Modeling and Simulation*

In this lab, we numerically and analytically analyze models of population growth. Then, we will analyze effect of harvesting on population growth with help of this models.

## Introduction

Population model is a mathematical model mainly used for studying population dynamics. Such population models provide a better understanding of how the numbers change over a long interval of time. Simple Logistic equations can be used to model population growth. However, a modified logistic equation which takes in consideration the fact that a species can not survive if its population becomes below a certain level, may be more appropriate. This logistics equations can be modified in order to study effect of constant rate harvesting and constant effort harvesting on population growth.

## Model

The population growth can be modelled using a differential equation as follows:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{1}$$

Where, dP/dt is the instantaneous rate of change of population, r (1/time) is the population's growth rate, and K is the carrying capacity of the system. Carrying capacity is defined as the maximum number of organisms an environment can support. Such differential equations show logistic behaviour and have a sigmoid shaped curve.

From the stability analysis of logistic equations, it is seen that $P = 0$ is an unstable root and $P = K$ is a stable root hence if we start with any population $P > 0$ then we will end up at carrying capacity.

From the Figure(1), we can see that if the initial population is less than half the carrying capacity $(K/2)$ then the curve shows exponential behaviour at the start, till the population reaches $(K/2)$ and after that it converges to the carrying capacity. Whereas if the initial population is more than half the carrying capacity then, the curve converges to the carrying capacity.

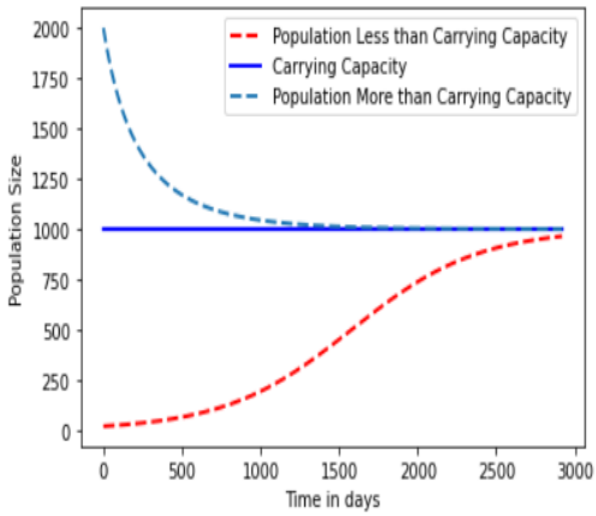


Figure 1: Carrying Capacity(K) = 1000, Initial Population $P_0 = 20$, Total Time = 8 years, Population Growth Rate(r) = 0.0009/year

If the system has a population more than the carrying capacity then the nearby environment tries to stabilize the population towards the carrying capacity.

## Population Growth with Harvesting

Depending on the rate at which harvesting is done harvesting can be of two types:

- Constant Rate Harvesting
- Harvesting dependent on the instantaneous population

## Constant Rate Harvesting

The model for population growth with constant harvesting rate is as follows:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) - h \tag{2}$$

Where, h (population/time) is the constant harvesting rate. Here, there are three free parameters r, K, h.

*Electronic address: 201801410@daiict.ac.in
†Electronic address: 201801057@daiict.ac.in

We can modify the above model to reduce the number of free parameters as below:

$$\frac{dn}{d\tau} = n(1-n) - \tilde{h} \tag{3}$$

Where, $n = \frac{P}{K}, \tau = rt$ and $\tilde{h} = \frac{h}{rK}$ Moreover, the nature of the above equation is parabolic hence only one stable fixed point can exist.

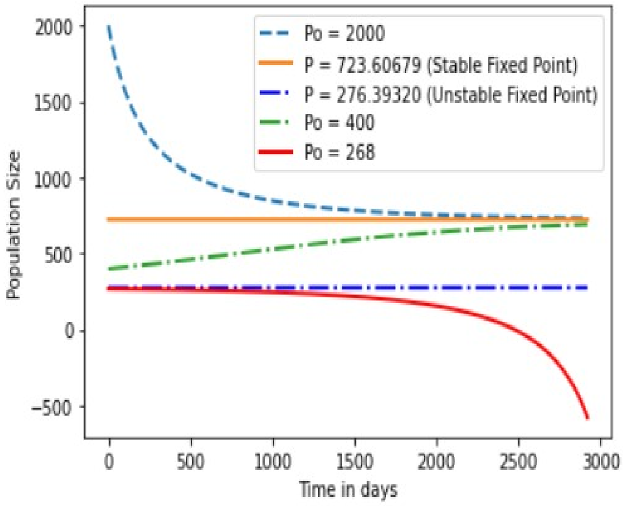


Figure 2: Plot of Population vs Time with constant harvesting rate h = 200, r = 1

From the Figure(2), we can see that as the harvesting rate h increases, the graph shifts downward, showing that the population growth decreases.

From stability analysis of fixed points, it is easy to see that for the above model the roots are:

$$Roots = \frac{K(1 \pm \sqrt{1-4\tilde{h}})}{2}$$

The nature of the roots can be controlled using the parameter $\tilde{h}$ . If $4\tilde{h} > 1$ then the roots will be imaginary else the roots will be real. So the curve will not converge for any value of $P_0$, $r$ and $k$ if $\tilde{h}$ greater than $\frac{1}{4}$.

The threshold value for $h$ to converge for a particular value of $P_0$, $r$ and $k$ is $\frac{(-r^2P_o^2+r^2kP_o)}{rk}$. If we set $h$ greater than this value, the population will gradually decrease and the population can become extinct.

**Constant effort harvesting:**

In this case harvesting rate is dependent on the instantaneous population. We can modify the population model with harvesting rate dependent on the instantaneous population as follows:

$$\frac{dP}{dt} = rP(1-\frac{P}{K}) - \epsilon P$$

Where $\epsilon P$ is the harvesting rate proportional to the instantaneous population.

We apply a similar modification $n = \frac{P}{K}, \tau = rt$ and $\tilde{\epsilon} = \frac{\epsilon}{r}$

$$\frac{dn}{d\tau} = n(1-n) - n\tilde{\epsilon}$$

The roots of the equation will be:

$$Roots = 0, K(1-\tilde{\epsilon})$$

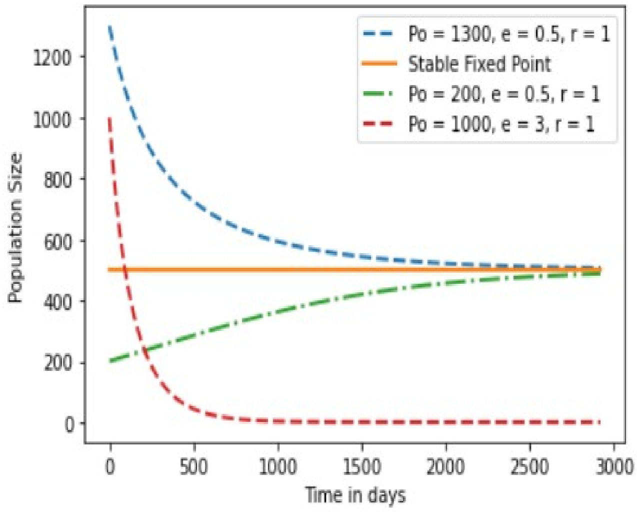


Figure 3: Plot of Population vs Time with constant effort harvesting. For the case $P_0 = 1000$, $\epsilon = 3$ and r = 1 the stable fixed point will be 0.

Where, $\tilde{\epsilon} = \epsilon/r$, in this case, using which the convergence of the curve can be controlled. If $\tilde{\epsilon} > 1$ then root will be negative and hence in that case 0 will be the stable fixed point and $K(1-\tilde{\epsilon})$ will be an unstable fixed point.

If $\tilde{\epsilon} < 1$, then the root $K(1-\tilde{\epsilon})$ will be positive and hence the curve tends to converge on that root and 0 will be the unstable fixed point.

If $\tilde{\epsilon} = 0$ then the only root of the equation will be 0 and hence that root will show bifurcations behaviour in that case. So if we start with a positive initial population then it will converge to 0 whereas if we start with a negative population then it will diverge to $-\infty$.

**Threshold Logistic Model**

Threshold Logistic Model is an improvement over the simple logistic model. It is observed that a species doesn't survive if its population falls below a certain threshold($T$). Simple logistic equation can be modified as shown in equation (4) to reflect this observation.

$$\frac{dP}{dt} = -rP(1-\frac{P}{K})(1-\frac{P}{T}) \tag{4}$$

We can divide both side of equation (...) with r to make it dimensionless,

$$\frac{dP}{d(rt)} = \frac{dP}{d\tau} = -P(1-\frac{P}{K})(1-\frac{P}{T}) \tag{5}$$

From the equation we can see that rate of change of $P$ will be negative for $P < T$ and thus population decreases. Due to this, P will remain lesser then $T$ and will decrease until it becomes zero (Here, we assume that $K > T$). On the other hand, if $K > P > T$, then $P$ will increase as $\frac{dP}{dt} > 0$ and will remain greater than T and thus it will keep increasing till it reaches $K$ where $\frac{dP}{dt} = 0$.

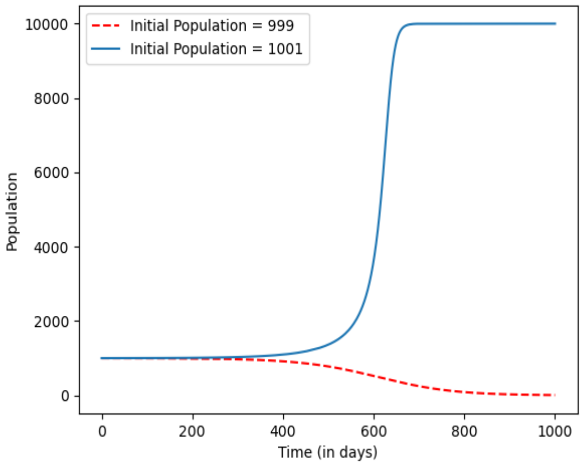


Figure 4: Plot of Population v/s Time. Here, $r = 0.0125$ $da^{-1}$, $T = 1000$ and $K = 10000$.

**Constant Rate Harvesting**

We will assume that rate of harvesting is constant and is equal to $h$. Thus, we can modify equation(4) as below,

$$\frac{dP}{dt} = -rP(1-\frac{P}{K})(1-\frac{P}{T}) - h \tag{6}$$

Here, for different value of h, roots of $\frac{dP}{dt} = 0$ will change and therefore carrying capacity and value minimum threshold population needed for survival of a species will also change. Impact of h on those values are shown in figure (5). We can clearly see in the figure (5) that initially we will have three fixed points, out of which two are stable and one is unstable. Here, we can interpret that dashed line is for threshold value and solid line which is above dashed line is for carrying capacity.

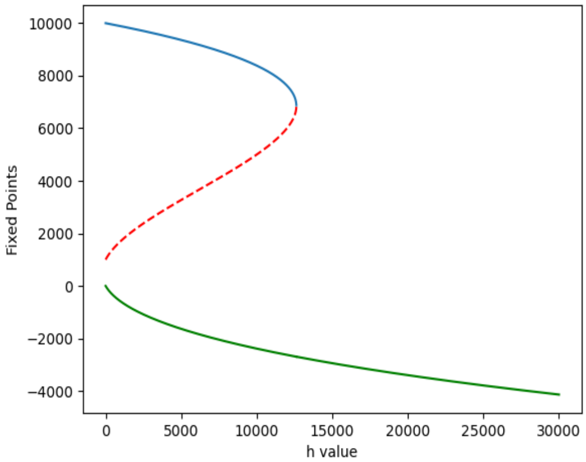


Figure 5: Positions of fixed point v/s value of h. Here, $r = 1$ $da^{-1}$, $K = 10000$ and $T = 1000$. Dashed line represents position of unstable fixed points and solid line represents position of stable fixed points.

Thus, as we increase h, threshold will increase and carrying capacity will decrease. Also, after a certain value of h, fixed points representing carrying capacity and threshold are going to get disappeared, i.e, after that h value we can have any value of initial population but species will not survive. And for a particular value of h, there will be two roots so, if initial population is equal to value of the positive root, then species will survive and their population will remain constant else species will go extinct. Note that if initial population is less than threshold then population will get saturated at less than zero value (if $h > 0$), it is because of the fact that rate of harvesting is not dependent on population (this is a limitation of assuming that harvesting happens at constant rate). Figure (6) provides insight on how curves changes if we change value of h.

If we want exact values of new threshold and carrying capacity then we will have to find solution of the equation $\frac{dP}{dt} = 0$ (it will be a cubic equation). We should carefully select the value of h such that new value of threshold is below initial population so that species will not go extinct (in figure (6), threshold for $h = 970$ was above the initial population and for $h = 980$, it was below the initial population).'

**Constant Effort Harvesting**

In the case of constant effort harvesting, we will assume that rate of harvesting at any time is proportional to population. So equation(4) can be modified as,

$$\frac{dP}{dt} = -rP(1-\frac{P}{K})(1-\frac{P}{T}) - \varepsilon P \tag{7}$$

$\varepsilon$ is fraction of the population harvested at any time t.

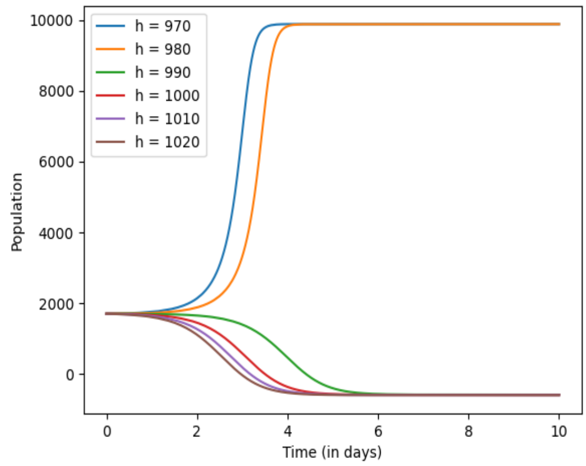


Figure 6: Plot of Population v/s Time for different values of h. Here, $r = 1da^{-1}$, $K = 10000$, Initial Population = 1700 and $T = 1000$.

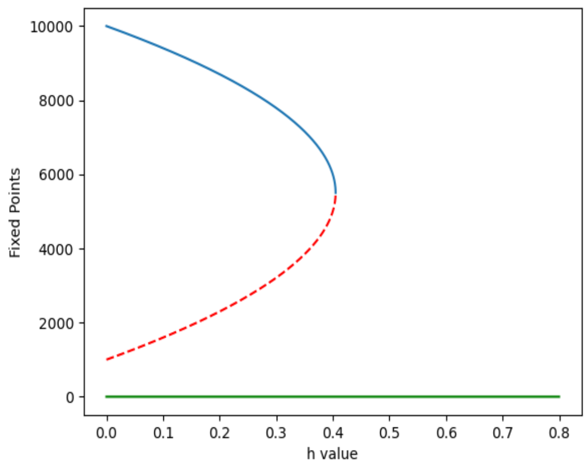


Figure 7: Positions of fixed point v/s value of h. Here, $r = 0.2\ da^{-1}$, $K = 10000$ and $T = 1000$. Dashed line represents position of unstable fixed points and solid line represents position of stable fixed points.

We can see in figure(7) that, same as in the case of constant harvesting, here too threshold's value and value of carrying capacity changes as we increase value of $\varepsilon$. In this case too, threshold value will increase and carrying capacity will decrease although not in same way as in constant harvesting. We should notice that in this case one stable fixed point is always located at 0 and after a certain value of h only that fixed point remains. Also, at a particular h value, two roots will remain, out of which one root is 0. In that case, if initial population is equal to the root which has a positive value then species will survive.

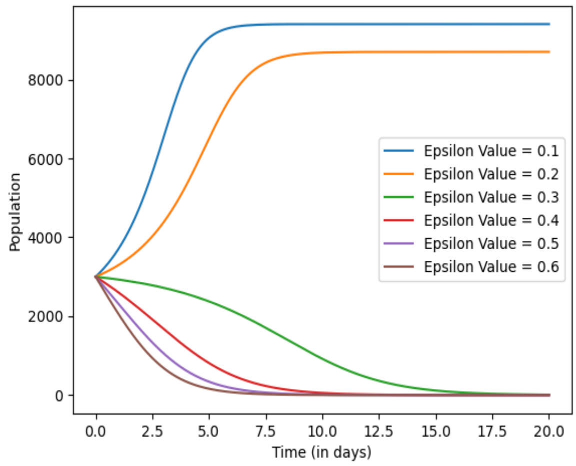


Figure 8: Plot of Population v/s Time for different values of h. Here, $r = 0.2da^{-1}$, $K = 10000$, Initial Population = 3000 and $T = 1000$.

We can find numerically the maximum value of $\varepsilon$ such that species will not go extinct. In order to do that we will find roots of equation $\frac{dP}{dt} = 0$. Roots of this equation are,

$$P = 0 \tag{8}$$

$$P = \frac{rK + rT \pm \sqrt{(rK - rT)^2 - 4r\varepsilon KT}}{2r} \tag{9}$$

Here, we should make sure that second smallest root should be less than initial population. By doing that comparison we found equation(10) that must hold true for survival of species.

$$\varepsilon < \frac{r(2P_0 - 2T)(2K - 2P_0)}{4KT} \tag{10}$$

## Conclusion

Population Model is an excellent tool to analyze and predict future population growth. We started with a basic logistic model for population growth, to make the model more realistic we added an important feature 'Death Due to Isolation'. For both models we discussed harvesting analysis, fixed point stability analysis, the nature of curves, and derived the conditions for which the population stabilizes to a maximum permissible capacity or can become extinct.