- $e^{(a1+a2+\cdots+an)x}$

  Coefficient of x^k is $\frac{(a1+a2+\cdots+an)^k}{k!}$
- Linear Recurrence:

  GF: A(x).

  Recurrence: $A(i) = C_0A_{i-1}+C_1A_{i-2}+\ldots+C_{k-1}A_{i-k}$

  Let $Q(x) = 1-C_0x-C_2x^2-\ldots-C_{k-1}x^{k-1}$. We can show each k -order recurrence can be described by A(x) = P(x) / Q(x) such that

  - ✓ Q(x) is of degree k, and contains the coefficients of the recurrence. Additionally, the constant term is 1 (or Q(0) = 1).
  - ✓ P(x) is of degree < k.

  The generating functions P(x) / Q(x) and P(x)R(x) / Q(x)R(x) generate the same sequence. If we let R(x) = Q(-x) then all odd terms of the denominator will be vanished.

  Another way to look at it:

  Let $M(x) = x^k-C_0x^{k-1}-C_1x^{k-2}-\ldots-C_{k-1}$

  Let $m_lx^l + m_{l-1}x^{l-1} + m_{l-2}x_{l-2} + \ldots + m_0$ be any polynomial divisible by M(x). Then:

  $m_lA_l+m_{l-1}A_{l-1}+m_{l-2}A_{l-2}+\cdots+m_0=0$

  $S(x) = s_{k-1}x^{k-1}+s_{k-2}x^{k-2}+\cdots+s_1x+s_0 = x^N \% M(x)$.

  Then $A_N=A_{k-1}S_{k-1}+A_{k-2}S_{k-2}+\ldots+A_1S_1+A_0S_0$

  A LR can be shown by the help of many different recurrences.

  If $A(n) = \sum_{i=0}^{k-1} A\big(n-b(i)\big) * c(i)$ then,

  $x^N \% M(x) = \sum_{i=0}^{k-1} x^{n-b(i)} * c(i) \% M(x)$

- The generating function for the Catalan numbers is

$$\frac{1-\sqrt{1-4x}}{2x}$$

- $(1+ax)^n = \sum \binom{n}{k} a^k x^k$
- $\frac{1}{1-ax} = \sum a^i x^i$
- $\frac{1}{(1-ax)^n} = \sum \binom{i+n-1}{n-1} a^i x^i$
- $\left(\sum_{i=0}^{inf} S_i x^i\right)^2 = \sum_{i=0}^{inf} S_i S_{n-i} x^n$
- Fibonacci Numbers:

$$f[0]=0, f[1]=1, f[n]=f[n-1]+f[n-2]$$

$$\xrightarrow{yields} G(x) = xG(x) + x^2G(x) + 0 + x$$

- Catalan numbers:

$$C_{n+1} = C_0C_n + C_1C_{n-1} + \cdots + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}.$$

$\xrightarrow{yields} G(x) = xG(x) + 1$, [+1 is beacause C[0] = 1]

- $f[0] = K, f[n] = A * f[n-1] + B * \sum_{i=1}^{n-1} f[i] * f[n-i]$

$$\xrightarrow{yields} G(x) = K + AxG(x) + BxG(x)^2$$

$f[0] = 1$

$$f[s] = \sum_{w \in \{c_1, \ldots, c_n\}} \sum_i f[i] \times f[s - w - i]$$

Let F(z) be the generating function of f. That is, $F(z) = \sum_{k \geq 0} f[k] z^k$

And then let $C(z) = \sum_{k=1}^{n} z^{c_k}$

So we have:

$$F(z) = C(z)F(z)^2 + 1$$

" $+1$" is for $f[0] = 1$.

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$$(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

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- Find $\sum \frac{N!}{x_1! x_2! \ldots x_k!}$ over all $x_1 + x_2 + \ldots + x_k = N$ where x_i is odd.

  This is equivalent to the coefficient of

  x^N in $\left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)^k$ The generator function of the above polynomial is $\left(\frac{e^x - e^{-x}}{2}\right)^k$

  Thus the solution is the coefficient of x^N of the above eqn, which on expanding binomially we get

  $\sum_{r=0}^{k} \frac{(-1)^r \binom{k}{r} e^{(k-2r)x}}{2^k}$ Coefficient of x^N in $e^b x = \frac{b^N}{N!}$; thus coefficient of x^N in the above eqn will be, multiplied by $N!$, is $\sum_{r=0}^{k} \frac{(-1)^r \binom{k}{r} (k-2r)^n}{2^k}$

  Polynomial for even x is $\frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$ Whose generator is $\left(\frac{e^x + e^{-x}}{2}\right)$

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + k_2 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

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- $f[n][k] = f[n-1][k] + f[n][k-1] + (n-1) * f[n-1][k-1]$, $f[0][0] = 1$, $f[0][n] = f[n][0] = 0$, $f[n][1] = n * (n+1)/2$

$$F(x) = \prod_{i=1}^{n} (x + i), \quad f[n][k] = \sum_{i=1}^{k} \binom{k-1}{i-1} F(x)[n-i]$$