Size structured population models with diffusion

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Outline

1. Course work
2. Introduction
3. Literature survey
4. Objectives
5. Mild solutions and various results
6. Conclusions
7. Future works

Course work

Sr. No.	Course code	Course name	Credits	Grade
1.	EE-509	Linear Dynamical Systems	3	O
2.	MA-780	Topics in Semigroup Theory	3	O
3.	MA-513	Ordinary Differential Equations	4	O
4.	MA-522	Partial Differential Equations	4	O
5.	ME-591	Special Topics in Elasticity	1	B

Introduction

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- Anjali Wadhwa. 7 Rare and Exotic Wildlife Species that can be found in India, THE BETTER INDIA.

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Many times single species can not be modeled as one population, but can be considered as structured population where individuals in the population are partitioned into classes or stages.

Structured population models help us to understand how individual characteristics such as age, size, location and movement affect these models and thus outcomes and consequences of the biological and epidemiological processes.

Literature survey

[1] F. Sharpe and A. Lotka. A problem in age-distribution. Philos. Mag., 21(124):435–438, 1911.

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[2] A. G. M'Kendrick. Applications of mathematics to medical problems. Proceedings of the Edinburgh Mathematical Society, 44:98–130, 1925.

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[3] Morton E. Gurtin and Richard C. MacCamy. Non-linear age-dependent population dynamics. Arch. Rational Mech. Anal., 54:281–300, 1974.

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This work gives a new direction to non linear age structured population models. They used nonlinear Volterra integral equations approach and established the existence, uniqueness and convergence to equillibrium of solutions to nonlinear version of Sharpe-Lotka-McKendrick model. They considered the model of following type

$$D\rho + \lambda\rho = 0$$

$$P(t) = \int_0^\infty \rho(a,t)\mathrm{da}$$

$$\rho(0,t) = \int_0^\infty \beta(a,P)\rho(a,t)\mathrm{da}, \quad \rho(0,a) = \phi(a)$$

$$\text{where} \quad D\rho(a,t) = lim_{h\to 0}\frac{\rho(a+h,t+h)-\rho(a,t)}{h}.$$

Cont...

$\rho(a, t)$ is the population at time t in the age interval $(a, a + da)$. $\beta(a)$ is the birth modulus i.e average number of offsprings produced by individual of age a, $\lambda(a)$ is the death modulus.

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Here, authors studied population models in which the parameters such as fecundity, mortality and interaction coefficient are assumed to be age dependent. Conditions for the existence, stability and global attractivity of state and periodic solutions are derived.

Cont...

[5] O. Diekmann, H. Heijmans and H. Thieme. On the stability of the cell size distribution, J. Math. Biol., 19:227–248, 1984.

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A model for the growth of a size structured population reproducing by fission into two identical daughters is formulated and analyzed. The studied model is of the form

$$\frac{\partial \eta}{\partial t}(t,x)+\frac{\partial}{\partial x}(g(x)\eta(t,x)) = -\mu(x)\eta(t,x)-b(x)\eta(t,x)+4b(2x)\eta(t,2x).$$

The functions μ,b and g are the rates at which cells of size x die, divide and grow respectively.

Cont...

[6] Angel Calsina and Joan Saldana. A model of physiologically structured population dynamics with a ‘ nonlinear individual growth rate. J. Math. Biol., 33(4):335–364, 1995.

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In this article, a size structured population model with a nonlinear growth rate depending on the individual's size and on the total population is studied. Existence and uniqueness of solutions for the model equations is shown, and also existence of a (compact) global attractor for the trajectories of the dynamical system defined by the solutions of the model is shown.

Cont...

[7] G. F. Webb. Population models structured by age, size, and spatial position. In Structured population models in biology and epidemiology, volume 1936 of Lecture Notes in Math., pages 1–49. Springer, Berlin, 2008.

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In this work population models incorporating age, size and spatial structure are analyzed. The method uses the theory of semigroups of linear and nonlinear operators.

[8] Syed Abbas, Malay Banerjee, and Norbert Hungerbuhler. Existence, uniqueness and stability analysis of allelopathic stimulatory phytoplankton model. J. Math. Anal. Appl., 367(1):249–259, 2010.

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In this article, two species competitive delay plankton allelopathy stimulatory model system studied.

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[9] Nobuyuki Kato. Abstract linear partial differential equations related to size-structured population models with diffusion. J. Math. Anal. Appl., 436(2):890–910, 2016.

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In this article, author studied size structured model with diffusion. Existence and uniqueness of mild solutions is shown. Author also studied the dual problem and after introducing weak solutions, shows the uniqueness of weak solutions.

Objectives

Our objective is to study a size structured population model with time varying diffusion rate.

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Cont...

Let $\Omega_T = (0, T) \times \Omega$, $\Omega_s = (0, s_f) \times \Omega$, $\Omega_{Ts} = (0, s_f) \times (0, T) \times \Omega$ and $\Sigma_{Ts} = (0, s_f) \times (0, T) \times \partial\Omega$.

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$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial s}(g(s,t)p) = k(t)\Delta p - \mu(s,t,x)p + f(s,t,x) \quad \text{in} \quad \Omega_{Ts}$$

$$g(0,t)p(0,t,x) = C(t,x) + \int_0^{s_f} \beta(s,t,x)p(s,t,x)ds \quad \text{in} \quad \Omega_T$$

$$\frac{\partial p}{\partial \nu}(s,t,x) = 0 \quad \text{in} \quad \Sigma_{Ts}$$

$$p(s,0,x) = p_0(s,x) \quad \text{in} \quad \Omega_s.$$

Cont...

Let $\mathcal{A}(t)$ be realization of $-k(t)\Delta$ in $L^q(\Omega)$, q in $(1,\infty)$. For each $t \in [0, T]$,

$$D(\mathcal{A}(t)) = \left\{ v \in W^{2,q}(\Omega) \mid \frac{\partial v}{\partial \nu} = 0 \quad \text{a.e} \quad \text{on} \quad \partial\Omega \right\}$$

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For $\phi \in L^q(\Omega)$, define the bounded linear operator $\mathcal{B}(s,t)$ by

$$[\mathcal{B}(s,t)\phi](x) = \beta(s,t,x)\phi(x), \quad [C(t)](x) = C(t,x).$$

Cont...

So, in operator form our model reduces to

$$\frac{\partial p}{\partial t}(s,t)+\frac{\partial}{\partial s}(g(s,t)p(s,t)) = [-\mathcal{A}(t)-\mu(s,t)]p(s,t)+f(s,t), \quad (s,t)\in \overline{S}_{T_f}$$

$$g(0,t)p(0,t) = C(t) + \int_0^{s_f} \mathcal{B}(s,t)p(s,t)ds, \quad t \in [0,T]$$

$$p(s,0) = p_0(s), \quad s \in [0,s_f].$$

Abstract formulation

Let Z be a Banach space with norm $\|\cdot\|$ and $s_f \in (0,\infty)$, $T \in (0,\infty)$ are fixed numbers, let $S_{T_f} = (0, s_f) \times (0, T)$ and $\overline{S}_{T_f} = [0, s_f] \times [0, T]$ be its closure.

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$$\begin{aligned}\frac{\partial p}{\partial t} + \frac{\partial}{\partial s}(g(s,t)p) = [-\mathcal{A}(t) - \mu(s,t)]p + f(s,t), \quad (s,t) \in S_{T_f} \\ g(0,t)p(0,t) = C(t) + \int_0^{s_f} \mathcal{B}(s,t)p(s,t)\mathrm{ds}, \quad t \in (0,T) \\ p(s,0) = p_0(s), \quad s \in (0,s_f).\end{aligned} \tag{1}$$

Assumptions

(K1) $g\colon \overline{S}_{T_f} \mapsto [0,\infty)$ is continuously differentiable w.r.t both s,t. Also $g(s,t)$ is positive on S_{T_f}.

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(K3) $\mathcal{B} \in L^\infty(S_{T_f}; \mathcal{L}(Z))$, where $\mathcal{L}(Z)$ is the space of all bounded linear operators in Z.

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(K4) $\mu \in L^1(S_{T_f}; Z), f \in L^1(S_{T_f}; Z), C \in L^1(0, T; Z)$ and $p_0 \in L^1(0, s_f; Z)$.

Cont...

We can extend the function $g(s,t)$ on $\mathbb{R} \times [0, T]$ by defining:

$$g(s,t) = \begin{cases} g(0,t) & \text{if} \quad (s,t) \in (-\infty, 0) \times [0, T] \\ g(s_f, t) & \text{if} \quad (s,t) \in (s_f, \infty) \times [0, T]. \end{cases}$$

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We can extend the function $g(s,t)$ on $\mathbb{R} \times [0, T]$ by defining:

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Because of assumption (K1), characteristic curve is given by the unique solution of

$$\frac{d}{dt} s(t) = g(s(t), t), \quad s(t_0) = s_0 \quad \text{where} \quad s_0 \in \mathbb{R}. \tag{2}$$

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Let us define
$s(t) = \phi(t; t_0, s_0)$, $z_0(t) = \phi(t; 0, 0)$, and $z_1(t) = \phi(t; T, s_f)$.

Cont...

Now, we consider the following four cases:

Case 1: For the case (K1)-(a), we have $z_0(t) > 0$ for $t > 0$ and $z_1(t) < s_f$ for $t < T$. Now, for $(s,t) \in \overline{S}_{T_f}$ satisfying $s \leq z_0(t)$ there exists a unique $\tau_0 \in [0, T]$ such that $\phi(t; \tau_0, 0) = s$.

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$$\tau_0(t,s) = \begin{cases} \tau_0, & \text{if} \quad s \leq z_0(t) \\ 0, & \text{if} \quad s > z_0(t). \end{cases} \tag{3}$$

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Similarly for $(s,t) \in \overline{S}_{T_f}$ satisfying $z_1(t) \leq s$, there exists a unique $\tau_1 \in [0, T]$ such that $\phi(t; \tau_1, s_f) = s$ and the final time is defined by

$$\tau_1(t,s) = \begin{cases} \tau_1, & \text{if} \quad s \geq z_1(t) \\ T, & \text{if} \quad s < z_1(t). \end{cases} \tag{4}$$

Cont...

Case 2: For the case (K1)-(b), we have $z_0(t) > 0$ for $t > 0$ and $z_1(t) \equiv s_f$, initial time will be defined in similar manner as in (3) and here does not exists $\tau_1 \in [0, T]$ such that $\phi(t; \tau_1, s_f) = s$, unless $s = s_f$. Here, final time $\tau_1(t, s)$ for $(s, t) \in \overline{S}_{T_f}$ is defined by $\tau_1(t, s) = T$ for $s < s_f$.

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Case 3: For the case (K1)-(c), we have $z_0(t) \equiv 0$ and $z_1(t) < s_f$. So, final time will be defined in similar manner as in (4) and initial time $\tau_0(t, s)$ for $(s, t) \in \overline{S}_{T_f}$ is defined as $\tau_0(t, s) = 0$ for $s > 0$.

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Case 4: For the case (K1)-(d), we have $z_0(t) \equiv 0$ and $z_1(t) \equiv s_f$. Initial time and final time is defined in similar way as defined in **Case 2** and **Case 3** respectively.

Main results

Suppose $p(s,t)$ satisfies (1) in strict sense and let

$$u(\sigma; t, s) = \exp\left(\int_{\tau_0}^{\sigma} \tilde{\mu}(s(\eta),\eta))d\eta\right) p(s(\sigma),\sigma),$$

$$\text{where} \quad \tilde{\mu}(s(\eta),\eta)) = \partial_s g(s(\eta),\eta) + \mu(s(\eta),\eta).$$

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It can be further written as

$$u(\sigma; t, s) = \exp\left(\int_{\tau_0}^{\sigma} \tilde{\mu}(\phi(\eta; t, s),\eta))d\eta\right) p(s(\sigma),\sigma), \text{ where } \phi(\eta; t, s) = s(\eta).$$

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Differentiating $u(\sigma;t,s)$ with respect to σ, we get

$$\frac{d}{d\sigma}(u(\sigma;t,s)) = -\mathcal{A}(\sigma)u(\sigma;t,s) + \exp\left(\int_{\tau_0}^{\sigma} \tilde{\mu}(s(\eta),\eta)d\eta\right) f(s(\sigma),\sigma) \tag{5}$$

Before moving further, let us state some established results.

Cont...

Let Z be a Banach space and $z \in Z$ and consider the following IVP:

$$\begin{cases} \frac{du(t)}{dt} + \mathcal{A}(t)u(t) = 0 & 0 \leq r < t \leq T \\ u(r) = z \end{cases} \tag{6}$$

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(A_1) The domain $D(\mathcal{A}(t)) = D$ of $\mathcal{A}(t)$, $0 \leq t \leq T$ is dense in Z and independent of t.

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(A_2) For $t \in [0, T]$, the resolvent $R(\lambda \colon \mathcal{A}(t))$ of $A(t)$ exist for all λ with $\text{Re}\lambda \leq 0$ and there is a constant M such that

$$\|R(\lambda \colon \mathcal{A}(t))\| \leq \frac{M}{|\lambda| + 1} \quad \text{for} \quad \text{Re}\lambda \leq 0, t \in [0, T].$$

Let Z be a Banach space and $z \in Z$ and consider the following IVP:

$$\begin{cases} \frac{du(t)}{dt} + \mathcal{A}(t)u(t) = 0 & 0 \leq r < t \leq T \\ u(r) = z \end{cases} \tag{6}$$

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(A_3) There exist constants L and $0 < \alpha \leq 1$ such that

$$\|(\mathcal{A}(t) - \mathcal{A}(r))\mathcal{A}(\tau)^{-1}\| \leq L|t - r|^{\alpha} \quad \text{for} \quad r, t, \tau \in [0, T].$$

† **Theorem** Let Z be a Banach space and suppose that the assumptions $(A_1)-(A_3)$ holds, then there is a unique evolution system corresponding to (6) satisfying:

(E_1) $\|U(t,r)\| \leq C \quad$ for $0 \leq r \leq t \leq T$.

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(E_2) For $0 \leq r < t \leq T, \quad U(t,r)\colon Z \mapsto D$ and $t \mapsto U(t,r)$ is strongly differentiable in Z. The derivative $\frac{\partial U(t,r)}{\partial t} \in B(Z)$ and it is strongly continuous on $0 \leq r < t \leq T$. Moreover,

$$\frac{\partial U(t,r)}{\partial t} + \mathcal{A}(t)U(t,r) = 0 \quad \text{for} \quad 0 \leq r < t \leq T,$$

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† A. Pazy. Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

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(E_3) For $v \in D$ and $t \in (0, T]$, $U(t,r)v$ is differentiable with respect to r on $0 \leq r \leq t \leq T$ and

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Suppose that the assumptions $(A_1) - (A_3)$ holds, then the existence of evolution system is guaranteed and for $\eta \in [\tau_0, \tau_1]$, we have
$u(\eta; t, s) =$
$U(\eta, \tau_0)u(\tau_0; t, s) + \int_{\tau_0}^{\eta} U(\eta, \sigma)\text{exp}\left(\int_{\tau_0}^{\sigma} \tilde{\mu}(s(\eta), \eta)d\eta\right) f(s(\sigma), \sigma)d\sigma.$

$$\text{Since} \quad u(\eta; t, s) = \exp\left(\int_{\tau_0}^{\eta} \tilde{\mu}(s(\sigma), \sigma)d\sigma\right) p(s(\eta), \eta),$$

which implies

$$u(t; t, s) = \exp\left(\int_{\tau_0}^{t} \tilde{\mu}(s(\sigma), \sigma)d\sigma\right) p(s, t),$$

Cont...

which further implies that

$p(s,t) =$

$Q(\tau_0)\left[U(t,\tau_0)p(0,\tau_0) + \int_{\tau_0}^{t} U(t,\sigma)\exp\left(\int_{\tau_0}^{\sigma}\tilde{\mu}(s(\eta),\eta)d\eta\right) f(s(\sigma),\sigma)d\sigma\right],$

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For the case (K1)-(a) and (K1)-(b), $g(0,t) > 0$, so $p(0,t)$ is defined by

$$p(0,t) = \frac{1}{g(0,t)}[C(t) + \int_0^{s_f}\mathcal{B}(s,t)p(s,t)ds] \quad \text{for} \quad t \in (0,T) \tag{7}$$

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Also, for the case (K1)-(a) and (K1)-(b), we have

$$\tau_0(t,s) = \tau_0 \quad \text{for} \quad s \in (0, z_0(t)) \quad \text{and} \quad \tau_0(t,s) = 0 \quad \text{for} \quad s \in (z_0(t), s_f).$$

We define the concept of mild solutions as follows:
Mild solution: For the case K1(a) and K1(b), a function $p \in L^{\infty}(0, T; L^1(0, s_f; Z))$ given by

$$\begin{cases} Q(\tau_0)U(t,\tau_0)p(0,\tau_0) + \int_{\tau_0}^{t} U(t,\sigma)Q(\sigma)f(s(\sigma),\sigma)d\sigma & , s \in (0, z_0(t)) \\ Q(0)U(t,0)p_0 + \int_0^t U(t,\sigma)Q(\sigma)f(s(\sigma),\sigma)d\sigma & , s \in (z_0(t), s_f) \end{cases}$$

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is called the mild solution of (1) on $(0, T)$. For the other cases (K1)-(c) and (K1)-(d), the mild solution of (1) is defined as a function $p \in L^{\infty}(0, T; L^1(0, s_f; Z))$ given by

$$p(s,t) = Q(0)U(t,0)p_0 + \int_0^t U(t,\sigma)Q(\sigma)f(s(\sigma),\sigma)d\sigma, \quad \text{a.e.} \quad (s,t) \in S_{T_f}.$$

Cont...

Suppose $D_\phi p(s,t)$ is the derivative along the characteristic ϕ

$$D_\phi p(s,t) := \lim_{h\to 0} \frac{[p(\phi(t+h;t,s),t+h) - p(s,t)]}{h}.$$

Then, it is easy to see that

$$D_\phi p(\phi(\eta;t,s),\eta) = \frac{d}{d\eta} p(\phi(\eta;t,s),\eta).$$

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Theorem.1: For the **Case (K1)-a** and **Case (K1)-b**, any mild solution $p \in L^\infty(0,T;L^1(0,s_f;Z))$ of (1) is continuously differentiable along the characteristic and satisfies

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$$D_\phi p(s,t) = -\mathcal{A}(t)p(s,t) - \partial_s g(s,t)p(s,t) - \mu(s,t)p(s,t) + f(s,t) \quad \text{a.e} \quad (s,t$$

$$g(0,t)p(0,t) = C(t) + \int_0^{s_f} \mathcal{B}(s,t)p(s,t)ds, \quad \text{a.e} \quad t \in (0,T),$$

$$p(s,0) = p_0(s), \quad \text{a.e} \quad s \in (0,s_f),$$

Cont...

where $p(s,0)$ and $p(0,t)$ are the limits along the characteristic curve which are defined as follows:

$$p(s,0) = \lim_{\eta \to 0^+} p(\phi(\eta; 0, s), \eta) \quad \text{a.e} \quad s \in (0, s_f),$$

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Theorem.2 Suppose that the assumptions $(K_1) - (K_4)$ holds, then there exists a unique mild solution of (1) in $L^\infty(0, T; L^1(0, s_f; Z))$.

Note: For a particular case, we have also shown the existence of global attractor.

Conclusions

For a structured population model with time dependent diffusion rate, we proved the existence of mild solutions and their uniqueness, also for a particular case we have shown the existence of global attractor.

Future works

Our next plan is to study delay in size structured population model with or without diffusion. Because in most of the biological processes birth process start after some stage, which can be captured by delay in the given mathematical model. We will also consider the harvesting problem for given size structured population model, our aim is to prove the existence of optimal harvesting effort to maximize total harvest.


Thank you