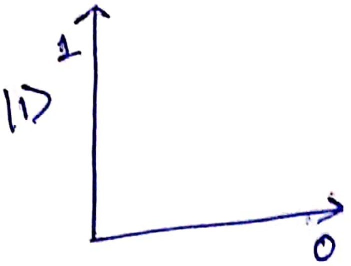
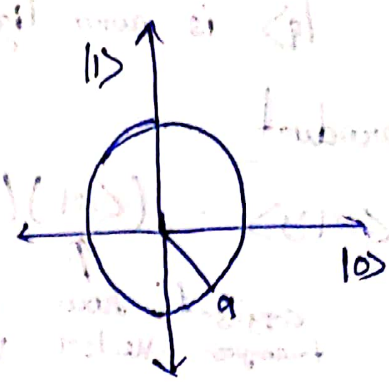
Quantum Algorithms

Classical bits : 0, 1

Quantum bits (Qbits) : Linear combination of 0, 1.





$$|q\rangle = \alpha|0\rangle + \beta|1\rangle$$

We use the notations.

$|a\rangle$ = column vector (Ket)

$\langle a|$ = row vector. (Bra)

In a 1 q-bit system, $|0\rangle$ & $|1\rangle$ are basis states.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In a 2 q-bit system, basis states are.

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$$|q\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \varepsilon|11\rangle$$

Define.

$$|ab\rangle = |a\rangle \otimes |b\rangle$$

$$|00\rangle = |0\rangle \otimes |0\rangle$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In a 1 qbit system.

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle \quad \alpha, \beta \in \mathbb{C}.$$

Here $|q\rangle$ is normalized.

<u>Inner product</u>

$$\langle x|y\rangle = (\langle x|)(|y\rangle)$$

conjugate transpose. row vector col vector.

$|q\rangle$ is normalized if.

- $\langle q|q\rangle = 1$

$$\Rightarrow \langle q| = \alpha\langle 0| + \beta\langle 1| \qquad \alpha, \beta \in \mathbb{C}$$

Now $\langle q|q\rangle = (\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$$= |\alpha|^2\langle 0|0\rangle + \alpha\beta\langle 0|1\rangle + \beta\alpha\langle 1|0\rangle + |\beta|^2\langle 1|1\rangle$$

Here the min terms $\langle 0|1\rangle$ and $\langle 1|0\rangle$ vanishes.

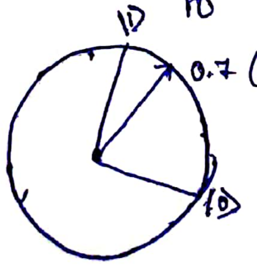
$$\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0. \quad \cancel{\text{The Hence } \langle q|q\rangle = 1}$$

Hence $\langle q|q\rangle = 1$

$$\Rightarrow |\alpha|^2 + |\beta|^2 = 1$$

Here $\underbrace{\langle 0|0\rangle \text{ and } \langle 1|1\rangle = 1}$

This is the probability of belonging to one of the states.



$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ hence $\alpha^2 = \frac{1}{2}, \beta^2 = \frac{1}{2}$

Bell States.

Two qubit system. They are defined as.

$$|\beta_{00}\rangle \equiv \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$

$$|\beta_{01}\rangle \equiv \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle]$$

$$|\beta_{10}\rangle \equiv \frac{1}{\sqrt{2}}[|00\rangle - |11\rangle]$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}[\cancel{|10\rangle}\ |01\rangle - |10\rangle]$$

These are called EPR pairs. Einstein, ~~poto~~ podoloski, Rosen. These are called entangled states, i.e. the states ~~can~~ that cannot be written as a tensor product of two 1 qubit states.

$$|ab\rangle = |a\rangle \otimes |b\rangle$$

Postulates of Quantum Mechanics (4 postulates)

1) Associated to any isolated ~~system~~ physical system is a complex vector space with inner product known as the state space of the system. The system is completely described by its state vector, which is a unit vector.

2) ~~Fuot~~ The evolution of a closed quantum system is described by a unitary transformation. That is the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at t_2 by a unitary operator U.

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i.e $|\psi'\rangle = U|\psi\rangle$

example

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$U|\psi\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a|1\rangle + b|0\rangle$$

Unitary matrices are the gates of quantum computers.

<u>Postulate 3)</u>

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur

If the state of the system is $|\psi\rangle$ immediatly before the measurement, then the probability that m occurs is given by $p(m) = \langle\psi| M^*_m M_m |\psi\rangle$

How : After measurement

$$M|\psi\rangle$$

To calculate that take inner product.

$$\langle\psi| M^*_m M_m |\psi\rangle = |\alpha_m|$$

They M_m satisisfies the completeness ~~eee~~ eqn.

$$|\psi_{x}\rangle = \frac{M_m|\psi\rangle}{\langle\psi M_m M \psi\rangle} \qquad \cancel{\sum_m \langle\psi|M_m^* M_m|\psi\rangle} = I$$

$$\cancel{1 = \sum_m p(m)} = \quad \sum_m (M_m|\psi\rangle)(M_m|\psi\rangle) = I$$

$$\cancel{1 = \sum_m p(m) = \sum_m \langle\psi|}$$

Example.

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle$$

Define $M_0 = |0\rangle\langle 0| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

$M_1 = |1\rangle\langle 1| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Measure the outcome 1.

$$M_1|q\rangle = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\left(\alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}\right) = \beta|1\rangle$$

$$p(1) = \langle q|M_1^* M_1|q\rangle$$

$$= (\beta^*\langle 1|)(\beta|1\rangle) = |\beta|^2$$

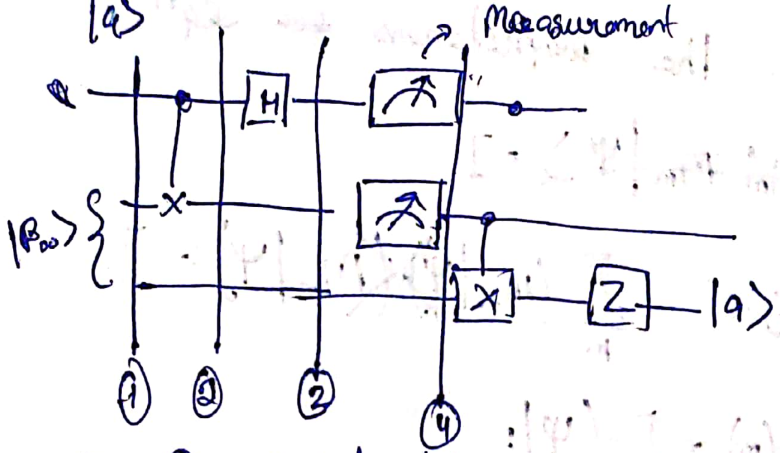
4) Multi-qubit System.

The state space of a composite physical system is the ~~the~~ tensor product of the state space of the component physical system.

$$|0\rangle \otimes |1\rangle$$
$$|0\rangle \otimes |0\rangle.$$

Quantum Teleportation.

Transmit an ~~the~~ unknown qubit instantaneously over any distance.



Step ① ~~Initial state~~ Initial state

$|B_{00}\rangle$ is an entangled pair.

first qbit → Alice.

second qbit → Bob.

q First bit of B second bit of B

① $|q\rangle \otimes |B_{00}\rangle$

$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \alpha \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

② ~~Apply~~ Alice applies CNOT to her two qubit system.

$$|x,y,z\rangle \rightarrow |x, x\oplus y, z\rangle$$

After applying CNOT

$$\frac{1}{2}(\underbrace{\alpha|000\rangle}_{a} + \overbrace{\beta\alpha|011\rangle}^{b} + \overbrace{\beta|110\rangle}^{c} + \underbrace{\beta|101\rangle}_{d})$$

③ Hadamard gate.

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$(+)$ $(-)$

a) $|\underline{0}00\rangle \rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$ Hadamard only on first qbit.

$|\underline{0}11\rangle \rightarrow \frac{1}{\sqrt{2}}(|011\rangle + |111\rangle)$

$|\underline{1}10\rangle \rightarrow |010\rangle - |110\rangle$

$|\underline{1}01\rangle \rightarrow |001\rangle - |101\rangle$

Hence $\frac{1}{2}\{\alpha|\underline{00}0\rangle + \alpha|\underline{10}0\rangle + \alpha|\underline{01}1\rangle + \alpha|\underline{11}1\rangle$

normalization constraints

$+\beta|\underline{01}0\rangle - \beta|\underline{11}0\rangle + \beta|\underline{00}1\rangle - \beta|\underline{10}1\rangle\}$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

First two bits take common (Alice bits).

$= \{|00\rangle \frac{\alpha|0\rangle+\beta|1\rangle}{2} + |01\rangle \frac{\alpha|1\rangle+\beta|0\rangle}{2}$

$+ |10\rangle \frac{\alpha|0\rangle-\beta|1\rangle}{2} + |11\rangle \frac{\alpha|1\rangle-\beta|0\rangle}{2}\}$

Measurement operation on $|00\rangle$ will collapse the other parts.

$|00\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

$M_{|00\rangle} = |00\rangle\langle 00| = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}[1\,0\,0\,0]$

Measurement operator

$= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$M_{|00\rangle}|00\rangle = |00\rangle$

on all other states

eg $|01\rangle$ $M_{|00\rangle}|01\rangle = 0$

① If $M_{|00\rangle}$ is used, Bob applies $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② If $M_{|01\rangle}$ is used Bob applies $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

③ If $M_{|10\rangle}$ is used " " $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

④ If $M_{|00\rangle}$ is used " " $ZX = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Deutsch Algorithm

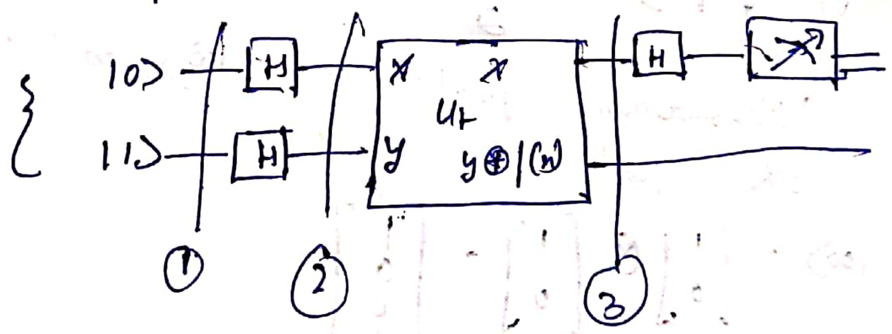
Problem. Find whether or not a Boolean fn $f(x)$ is constant.

Classically 2 steps are needed to find whether f is constant or not.

Assume a unitary operator U_f that achieves

$$U_f|xy\rangle = |x, y \oplus f(x)\rangle$$

ancilla qbits {



① $|\psi_1\rangle = |0\rangle \otimes |1\rangle = |01\rangle$

② Apply H to both q bits.

$$|\psi_2\rangle = H_1H_2|\psi_1\rangle = H_1|0\rangle \otimes H_2|1\rangle$$

$$= \frac{1}{2}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{4}\left(|00\rangle - |01\rangle + |10\rangle - |11\rangle\right)$$

(3) $|\psi_3\rangle = U_f|\psi_2\rangle = \frac{1}{4}\Big(|0, 0\oplus f(0)\rangle - |0, 1\oplus f(0)\rangle$

$+ |1, 0\oplus f(1)\rangle - |1, 1\oplus f(1)\rangle\Big)$

(4) Apply H. $|\psi_4\rangle = H|\psi_3\rangle$

$= \frac{1}{4}\frac{1}{\sqrt{2}}\Big(|+, 0\oplus f(0)\rangle - |+, 1\oplus f(0)\rangle$

$+ |-, 0\oplus f(1)\rangle - |-, 1\oplus f(1)\rangle\Big)$

$= \frac{1}{4\sqrt{2}}\Big(|0, 0+f(0)\rangle + |1, 0\oplus f(0)\rangle - |0, 1\oplus f(0)\rangle$

$- |1, 1\oplus f(0)\rangle - |1, 1\oplus f(0)\rangle + |0, 0+f(1)\rangle$

$- |1, 0\oplus f(1)\rangle - |0, 1\oplus f(1)\rangle + |1, 1\oplus f(1)\rangle$

$= \frac{1}{4\sqrt{2}}\Big(|0\rangle\big(|0\oplus f(0)\rangle + \cancel{|0\rangle \oplus f} \; |0\oplus f(1)\rangle$

$- |1\oplus f(1)\rangle - |1\oplus f(1)\rangle\big)$

$+ |1\rangle\big(|0\oplus f(0)\rangle - |0\oplus f(1)\rangle -$

$|1\oplus f(0)\rangle + |1\oplus f(1)\rangle\big)\Big)$

Two cases.

a) $a = f(0) = f(1)$

b) $a = f(0) \neq f(1) = \bar{a}$

$\cancel{|\psi_4\rangle\, f(0) = f(1)}$ $\quad |\psi_4\rangle \; f(0) = f(1) =$

$\frac{1}{4\sqrt{2}}\Big\{|0\rangle\big(|a\rangle + |a\rangle - |\bar{a}\rangle - |\bar{a}\rangle\big)$

$+ |1\rangle\big(\cancel{|a\rangle} - \cancel{|a\rangle} - \cancel{|\bar{a}\rangle} + \cancel{|\bar{a}\rangle}\big)$

$= \frac{1}{a\sqrt{2}} \; |0\rangle\big(|a\rangle - |\bar{a}\rangle\big)$

$$|\psi_4\rangle_{f(0)\neq f(1)} = \frac{|1\rangle(|a\rangle - |\bar{a}\rangle)}{2\sqrt{2}}$$

We note that.

$$f(0) \oplus f(1) = 0 \quad \text{if } f(0) = f(1)$$

$$f(0) \oplus f(1) = 1 \quad \text{if } f(0) \neq f(1)$$

Using this property.

$$|a\rangle - |\bar{a}\rangle = \pm(|0\rangle - |1\rangle)$$

$$|\psi_4\rangle \equiv |f(0) \oplus f(1)\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$