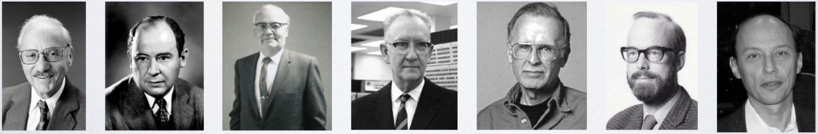
Matrix Computations Continued

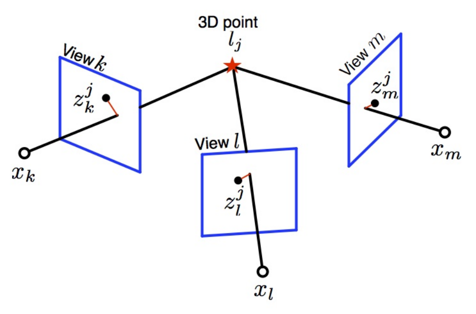
QR Method: One of the top 10 algorithms!

1946: The Metropolis Algorithm
1947: Simplex Method
1950: Krylov Subspace Method
1951: The Decompositional Approach to Matrix Computations
1957: The Fortran Optimizing Compiler
1959: QR Algorithm
1962: Quicksort
1965: Fast Fourier Transform
1977: Integer Relation Detection
1987: Fast Multipole Method



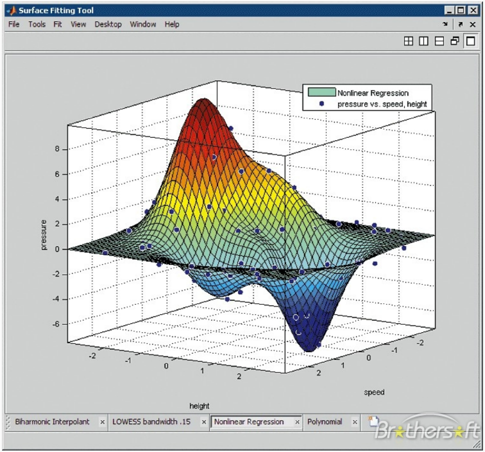
Dantzig von Neumann Hestenes Householder Backus Hoare Greengard

Motivation for QR Method: Computer Vision



- Identify 3D image from a set of 2D images taken from fixed camera positions
- Formulate the problem as nonlinear least squares, and solve it using QR method
- (Image source: Google images)

Motivation for QR Method



- Given a set of data points, fit a surface through it
- Formulate the problem as least squares, use QR to solve

Projections that preserve length

Given $x_1 \neq 0$, we have seen a projection P such that:

$$P \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Does all projections preserve lengths?
- Which projections preserve lengths?
- What do we mean by length?

Orthogonal Matrices

Inner product in $\mathbb{R}^n$: Given two vectors $x, y \in \mathbb{R}^n$, define inner product $\langle x, y \rangle$ as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = y^T x = x^T y$$

Properties of inner product:

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$
- $\langle x, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \alpha_1 \langle x, y_1 \rangle + \alpha_2 \langle x, y_2 \rangle$
- $\langle x, x \rangle \geq 0$, equality iff $x = 0$

Inner product

Euclidean Norm denoted by $\|x\|_2$ is defined as

$$\|x\|_2 = (\langle x, y\rangle)^{1/2}$$

- Cauchy Schwarz Inequality:

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$$

- Angle between vectors: Let θ be the angle between two vectors x and y, then

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2}$$

Orthogonal vectors

- Two vectors are called orthogonal if $\langle x, y \rangle = 0$
- When two vectors are orthogonal we say that the angle between them is $\pi/2$
- A matrix $Q \in \mathbb{R}^{n \times n}$ is called orthogonal if $QQ^T = I$
- Quiz: If Q_1 and Q_2 are orthogonal, then
  - Is Q_1^{-1} orthogonal?
  - Is $Q_1 Q_2$ orthogonal?
  - What is $\det(Q)$?
  - Is $\langle Qx, Qy \rangle = \langle x, y \rangle$? That is, does Q preserve angle?
  - Is $\|Qx\|_2 = \|x\|_2$? That is, does Q preserve length?
  - If $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all x, y, then Q is orthogonal?

Orthogonal transformations

- Want to find a matrix $Q \in \mathbb{R}^{2\times 2}$ that rotates a given vector $[x, y]^T$ by θ degrees
- Let

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

See Class Notes for More...

Rotators

- Rotators can be used to create zeros in a matrix. That is,

$$Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

See Class Notes for More ...

Using rotators to simplify a matrix

Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

See Class Notes for More...

Use QR to solve a linear system

Solve

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

See Class Notes for More...

Rotator picture

The (i, j) plane rotator:

$$Q = \begin{array}{cc} \begin{bmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& c &&&& -s &&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&& s &&&& c &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{bmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} \\ \begin{array}{ccccccccccc} &&& \uparrow &&&& \uparrow &&& \\ &&& i &&&& j &&& \end{array} & \end{array}$$

Rotators for high dimensional vectors

Let Q be a (ij) plane rotator. Show that the transformations $x \to Qx$ and $x \to Q^T x$ alter only the ith and jth entries of x and that the effect on these entries is the same as that of the 2 by 2 rotators

$$\hat{Q} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \quad \hat{Q}^T = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

Choose the rotators:

$$c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}, \quad s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$$