Elementary Row Operations

Here: e_i^T: ith row of identity matrix

1. Row switching: $R_i \leftrightarrow R_j$

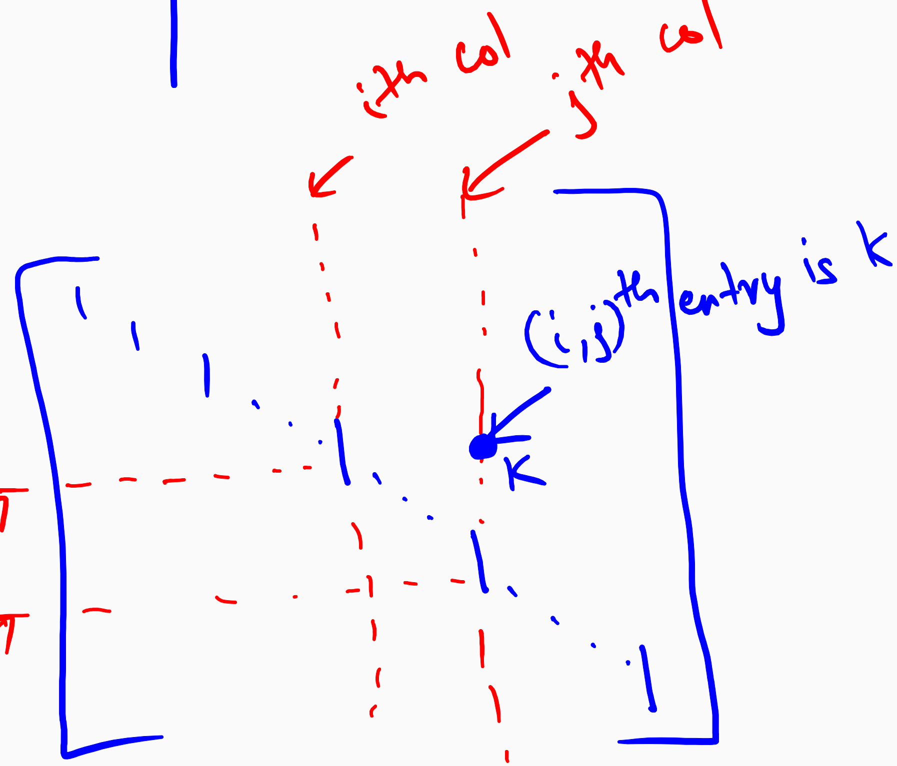
$$\begin{bmatrix} - & e_1^T & - \\ - & e_2^T & - \\ & \vdots & \\ - & e_i^T & - \\ & \vdots & \\ - & e_j^T & - \\ & \vdots & \\ - & e_n^T & - \end{bmatrix}$$

Interchange Row i & j of the identity matrix

2. Row multiplication: $R_i \leftarrow kR_i$

$$\begin{bmatrix} - & e_1^T & - \\ & \vdots & \\ - & k e_k^T & - \\ & \vdots & \\ - & e_n^T & - \end{bmatrix}$$

3. Row addition: $R_i \leftarrow R_i + kR_j$



Gauss Transforms

What is τ?

$$\begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

More generally, which matrix to multiply on the left to create zeros below v_k?

$$\begin{bmatrix} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Gauss Transforms

Suppose $v \in \mathbb{R}^n$ with $v_k \neq 0$. If

$$\tau^T = [0, \ldots, 0, \tau_{k+1}, \ldots, \tau_n], \quad \tau_i = \frac{v_i}{v_k}, \quad i = k+1 : n,$$

Define: $M_k = I_n - \tau e_k^T$, then

$$M_k v = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 & & 0 \\ 0 & & -\tau_{k+1} & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\tau_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Upper Triangularizing a Matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

1. Make zeros below the diagonal of 1st column:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix} \implies M_1 A = \begin{bmatrix} 1 & 4 & 7 \\ \boxed{0} & -3 & -6 \\ \boxed{0} & -6 & -11 \end{bmatrix}$$

$\tau^{(1)}$ became zero

2. Make zeros below the diagonal of the 2nd column of above matrix

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-2} & 1 \end{bmatrix} \implies M_2 M_1 A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & \boxed{0} & 1 \end{bmatrix}$$

$\tau^{(2)}$ became zero

Remarks on upper triangularization

1. At the start of the *kth* loop we have a matrix

$$A^{(k-1)} = M_{k-1} \cdots M_1 A$$

   that is upper triangular in columns 1 through $k-1$
2. The multipliers in the kth Gauss transform M_k are based on $A^{(k-1)}(k+1:n, k)$ and $a_{kk}^{(k-1)}$ must be non-zero in order to proceed

Solving simultaneous linear systems: Matrix view

$$\begin{bmatrix} 3 & 5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Idea: Keep making zeros below the main diagonal. Then matrix becomes upper triangular, which can be solved using backward sweep.

Exercise

Existence of LU factorization

If no zero pivots are encountered, then Gauss transforms $M_1, \ldots, M_{n-1}$ are generated such that

$$M_{n-1} \cdots M_1 A = U,$$

Diff. is only in sign

is upper triangular.

If $M_k = I_n - \tau^{(k)} e_k^T$, then $M_k^{-1} = I_n + \tau^{(k)} e_k^T$, and

$$A = LU,$$

where

$$L = M_1^{-1} \cdots M_{n-1}^{-1}$$

Check $M_k M_k^{-1}$

$= (I_n - \tau^k e_k^T)(I_n + \tau^k e_k^T)$

$= I_n - \cancel{\tau^k e_k^T} + \cancel{\tau^k e_k^T} - \tau^k e_k^T \tau^k e_k^T$

$= I_n - \tau^k e_k^T \tau^k e_k^T$

This is zero!

Note, here:

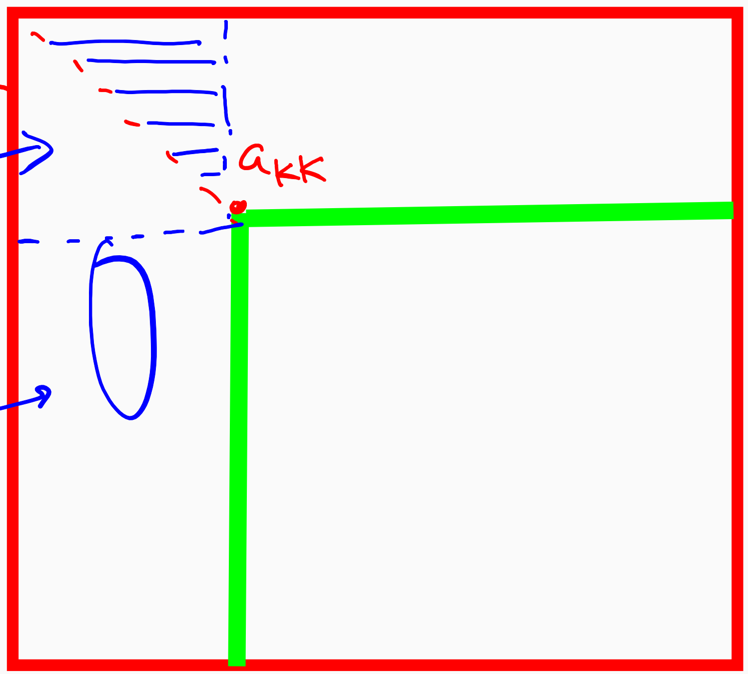
$\tau^k = [0, 0, \ldots, 0, \tau^k_{k+1}, \tau^k_{k+2}, \ldots, \tau^k_n]^T$

$e_k^T = [\underbrace{0, 0, \ldots, 0,}_{k\text{ terms}} 1, 0, \ldots, 0]$

LU Factorization

If $A \in \mathbb{R}^{n\times n}$ and $\det(A(1:k,1:k)) \neq 0$ for $k = 1:n-1$, then there exists a unit lower trianguar $L \in \mathbb{R}^{n\times n}$ and an upper triangular $U \in \mathbb{R}^{n\times n}$ such that $A = LU$. If this is the case and A is nonsingular, then the factorization is unique and $\det(A) = u_{11}u_{22}\cdots u_{nn}$.

LU can't proceed after k^{th} step if a_{kk} is zero, but this is not possible because $\det(A(1:k,1:k)) \neq 0$
$\Rightarrow \prod_{i=1}^{k} a_{ii} \neq 0$, in particular, $a_{kk} \neq 0$.



$A(1:k,1:k)$ is upper triangular
$\Rightarrow \det(A(1:k,1:k)) \neq 0$
$\Rightarrow \prod_i \lambda_i$
λ_i: eig. val. of $A(1:k,1:k)$

$\prod \lambda_i = \prod_{i=1}^{k} a_{ii}$, where a_{ii} are diagonal entries of $A(1:k,1:k)$

$\Uparrow$

Since, $A(1:k,1:k)$ is upper triag. $\Rightarrow$ entries on diag. are eig. vals

Simplify L

$$L = M_1^{-1} \cdots M_{n-1}^{-1}$$

Construction of L is not complicated:

$$\begin{aligned} L &= M_1^{-1} \cdots M_{n-1}^{-1} \\ &= (I_n - \tau^{(1)} e_1^T)^{-1} \cdots (I_n - \tau^{(n-1)} e_{n-1}^T)^{-1} \\ &= (I_n + \tau^{(1)} e_1^T) \cdots (I_n + \tau^{(n-1)} e_{n-1}^T) \end{aligned}$$

L looks complicated
but it is not.

Here $\tau^k = [0, \cdots, 0, \tau^{k+1}, \cdots, \tau^n]^T$.

Have a look at "mix" terms:

$$\tau^{(i)} e_i^T \tau^{(j)} e_j^T$$

Simplify L

Does these "mix" terms:

$$\tau^{(i)} e_i^T \tau^{(j)} e_j^T$$

survive?

$e_i^T = [0, 0, \dots, \underset{\uparrow\, i^{th} \text{ term}}{1}, 0, \dots, 0]$, $\tau^{(j)} = [0, 0, \dots, 0, \tau^j_{j+1}, \tau^j_{j+2} \dots \tau^j_n]^T$

For $j \geq i$, $e_i^T \tau^{(j)} = 0$

Hence, L now becomes after removing the mixed terms, we get:

$$\underbrace{L = I_n + \sum_{k=1}^{n-1} \tau^{(k)} e_k^T}_{\uparrow \text{ This means}}, \quad L(k+1:n, k) = \tau^{(k)}(k+1:n)$$

$$\Rightarrow \begin{bmatrix} L_{k+1,k} \\ L_{k+2,k} \\ \vdots \\ L_{n,k} \end{bmatrix} = \begin{bmatrix} \tau^k(k+1) \\ \tau^k(k+2) \\ \vdots \\ \tau^k(n) \end{bmatrix}$$