$A = QR$

Fact 1

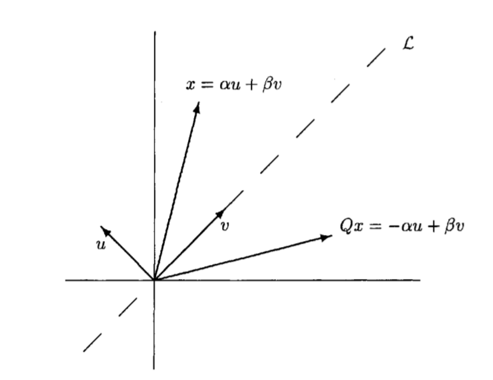
Let $A \in \mathbb{R}^{n \times n}$. Then there exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

Proof in class.

Reflectors

Start with case $n = 2$ as done for rotators. Let $\mathcal{L}$ be a line in $\mathbb{R}^2$ that passes through origin. The operator that reflects each line in $\mathbb{R}^2$ through the line $\mathcal{L}$ is a linear transformation: hence, can be represented by a matrix. We wish to find that matrix.

Reflector: picture



- Let v be a vector lying on $\mathcal{L}$
- Let u be a vector orthogonal to v
- Any $x \in \mathbb{R}^2 : x = \alpha u + \beta v$

Reflection through a line

We look for a matrix Q such that

$$Q(x) = Q(\alpha u + \beta v) = -\alpha u + \beta v, \quad \forall \alpha, \beta$$

In other words, we require

$$Qu = -u, \quad Qv = v.$$

W.L.O.G, assume $\|u\|_2 = 1$. Consider $P = uu^T$.

- Check Pu, Pv. Does it satify our requirement?
- If not, then how to modify P to get desired Q?
- How about $Q = I - 2P$?

Example

Reflect the vector $[1, 2]^T$ about the line $\alpha[1, 1]^T, \alpha \in \mathbb{R}$.

Properties of Projectors

Fact 2

Let $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, and define $P \in \mathbb{R}^{n \times n}$ by $P = uu^T$. Then

1. $Pu = u$
2. $Pv = 0$ if $\langle u, v \rangle = 0$
3. $P^2 = P$
4. $P^T = P$

- Projector: A matrix that satisfies: $P^2 = P$
- Orthoprojector: A projector that is also symmetric
- $P = uu^T$ is a rank-1 orthoprojector

Properties of Reflectors

Fact 3

Let $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, and define $Q \in \mathbb{R}^{n \times n}$ by $Q = I - 2uu^T$. Then

1. $Qu = -u$
2. $Qv = v$ if $\langle u, v \rangle = 0$
3. $Q = Q^T$
4. $Q^T = Q^{-1}$
5. $Q^{-1} = Q$

- Matrices Q are called reflectors or Householder transformations, after A.S. Householder.

Properties of Reflectors

Fact 4

Let $u \in \mathbb{R}^n, u \neq 0$. Define $\gamma = 2/\|u\|_2^2$ and $Q = I - \gamma u u^T$.
Then Q is a reflector satisfying

1. $Qu = -u$
2. $Qv = v$ if $\langle u, v \rangle = 0$

Proof on chalk board.

Properties of Reflectors

Fact 5

Let $x, y \in \mathbb{R}^n$ with $x \neq y$ but $\|x\|_2 = \|y\|_2$. Then there is a unique reflector Q such that $Qx = y$.

Proof on chalkboard.

Choose a reflector to create zeros

Fact 6

Let $x \in \mathbb{R}^n$ be any non-zero vector. Then there exists a reflector Q such that

$$Q \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$