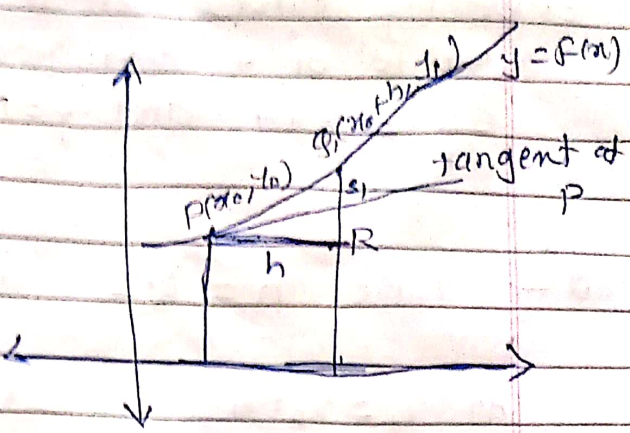
Numerical solutions to ordinary differential equations

* Euler's Method

consider the differential equation



$\frac{dy}{dx} = f(x,y)$ with

initial condition

$x = x_0$ and $y = y_0$

Let solution of the differential eqn be $y = f(x)$ (curve). P be any point $P(x_0, y_0)$ now we have to find the ordinate at point $Q(x_0+h, y_1)$ Let tangent at P makes an angle θ with x-axis. therefore

$$\frac{dy}{dx} = \tan\theta$$

$$\tan\theta = \frac{S_1R}{PR}$$

$$\therefore S_1R = PR \cdot \tan\theta$$

take approximately $S_1R \cong Q_1R$.

$$\therefore Q_1R \cong PR \cdot \tan\theta$$

$$y_1 - y_0 = h \cdot \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

lly we get

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1})$$

Ex (1) Using Euler's method find an approximate value of y corresponding to $x = 1$ given $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$

Soln:- Take $h = 0.2$ i.e. $h = \frac{x - x_0}{5} = \frac{1-0}{5} = 0$

At $x_1 = x_0 + h$ $x_0 = 0$ $y_0 = 1$

$= 0 + 0.2 = 0.2$

$y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.2(0+1)$

$=$ ~~[illegible]~~ $= 1 + 0.2$

$= 1.2$

$y_2 = y_1 + h \cdot f(x_1, y_1)$

Now $f(x_1, y_1) = x_1 + y_1 = 0.2 + 1.2 = 1.4$

$\therefore$ at $x_2 = 0.4$ $y_2 = y_1 + h \cdot f(x_1, y_1)$

$= 1.2 + 0.2(1.4) = 1.48$

$\therefore f(x_2, y_2) = x_2 + y_2 = 0.4 + 1.48 = 1.88$

At $x_3 = 0.6$ $y_3 = y_2 + h \cdot f(x_2, y_2)$

$= 1.48 + 0.2(1.88) = 1.856$

Now $f(x_3, y_3) = x_3 + y_3 = 0.6 + 1.856$

$= 2.456$

At $x_4 = 0.8$ $y_4 = y_3 + h \cdot f(x_3, y_3)$

$= 1.856 + 0.2(2.456)$

$= 2.3472$

$\therefore f(x_4, y_4) = x_4 + y_4 \quad 0.8 + 2.3472 = 3.1472$

$\therefore$ At $x_5 = 1 \quad y_5 = y_4 + h \cdot f(x_4, y_4)$

$= 2.3472 + 0.2(3.1472)$

$= 2.37664$

Ex: Use Euler's method to find an approximate value of y correct to 4 decimal places for $x = 0.1$ given $\frac{dy}{dx} = x - y^2$ and $x = 0, y = 1$

Take $h = 0.02$

Sol^n:- Here h is given $h = 0.02$

or $h = \frac{0.1 - 0}{5} = 0.02$

$x_0 = 0, \; y_0 = 1$

$f(x, y) = x - y^2 \qquad f(x_0, y_0) = x_0 - y_0^2 = 0 - 1^2 = -1$

$\therefore$ At $x_1 = \overset{x_0+h}{0.02} \qquad y_1 = y_0 + h\,f(x_0, y_0)$

$= 1 + 0.02(-1) = 0.98$

Now $f(x_1, y_1) = x_1 - y_1^2 = 0.02 - (0.98)^2 = -0.9404$

At $x_2 = x_1 + h$

$= 0.02 + 0.02 \qquad y_2 = y_1 + h \cdot f(x_1, y_1)$

$= 0.04 \qquad = 0.98 + 0.02(-0.9404)$

$= 0.9612$

Now $f(x_2, y_2) = x_2 - y_2^2 = 0.04 - (0.9612)^2$

$= -0.8839$

At $x_3 = 0.06$ $\quad y_3 = y_2 + h\, f(x_2, y_2)$

$= 0.9612 + (0.02)(-0.8839)$

$= 0.9435$

$f(x_3, y_3) = x_3 - y_3^2 = 0.06 - (0.9435)^2$

$= -0.8302$

At $x_4 = 0.08$, $y_4 = y_3 + h\, f(x_3, y_3)$

$= 0.9435 + (0.02)(-0.8302)$

$= 0.9269$

$f(x_4, y_4) = x_4 - y_4^2 = 0.08 - (0.9269)^2 = -0.7791$

At $x_5 = 0.10$ $\quad y_5 = y_4 + h \cdot f(x_4, y_4)$

$= 0.9269 + 0.02(-0.7791)$

$= 0.9113$

This is approximate value of y at $x = 0.1$

Ex (3) Using Euler's method find the approximate value of y where $\frac{dy}{dx} = x + \sqrt{y}$ $\quad y(2) = 4$

taking $h = 0.2$ at $x = 3$

Sol^n Let $h = 0.2$, $x_0 = 2$, $y_0 = 4$

$f(x_0, y_0) = x_0 + \sqrt{y_0} = 2 + \sqrt{4}$

$= 4$

At $x_1 = x_0 + h$ $\qquad$ $y_1 = y_0 + h \cdot f(x_0, y_0)$

$= 2 + 0.2$

$= 2.2$ $\qquad$ $= 4 + 0.2 \times 4 = 4 + 0.8 = 4.8$

$f(x_1, y_1) = x_1 + \sqrt{y_1} = 2.2 + \sqrt{4.8} = 4.3908$

At $x_2 = x_1 + h$ $\qquad$ $y_2 = y_1 + h \cdot f(x_1, y_1)$

$= 2.2 + 0.2$ $\qquad$ $= 4.8 + 0.2 \times 4.3908$

$= 2.4$ $\qquad$ $= 5.6781$

$f(x_2, y_2) = x_2 + \sqrt{y_2} = 2.4 + \sqrt{5.6781}$

$= 4.7828$

At $x_3 = x_2 + h$

$= 2.4 + 0.2$ $\qquad$ $y_3 = y_2 + h \cdot f(x_2, y_2)$

$= 2.6$ $\qquad$ $= 5.6781 + 0.2 \times 4.7828$

$= 5.6781 + 0.9565 = 6.63466$

$f(x_3, y_3) = x_3 + \sqrt{y_3} = 2.6 + \sqrt{6.63466}$

$= 5.1757$

At $x_4 = 2.8$

$y_4 = y_3 + h \cdot f(x_3, y_3)$

$= 6.63466 + 0.2 \times 5.1757$

$= 7.6698$

$f(x_4, y_4) = x_4 + \sqrt{y_4} = 2.8 + \sqrt{7.6698} = 5.56944$

At $x_5 = 3$ $\qquad$ $y_5 = y_4 + h \cdot f(x_4, y_4)$

$= 7.6698 + 0.2 \times 5.56944$

$= 8.7836$

$\therefore$ Approximate value at $x = 3$ is 8.7836