Normal Distribution:

Definition: A continuous random variable x is said to follow normal distribution with parameter m (called mean) & σ^2 (called variance). if its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$$-\infty < x < \infty$$

$$-\infty < m < \infty \qquad \sigma^2 > 0$$

Remark

① A continuous random variable x following normal distribution with mean m and Standard deviation σ is referred to as $X \sim N(m, \sigma)$

② If X is normal variate with parameter m, σ then $Z = \frac{X-m}{\sigma}$ is also a normal variate with mean $= 0$ & standard deviation $= 1$

it is called standard Normal Variate

* Importance of Normal Distribution

(i) The variable such as height weight intelligence etc. follow normal distribution.

(ii) Many other distributions occuring in practice such as Binomial, Poisson etc can be approximated by normal distribution.

(iii) Normal distribution has wide application in statistical quality control.

(iv). It is also useful in psychological & educational research.

I] Mean and variance of the Normal distribution.

Mean = m

$var(x) = \sigma^2$

For standard Normal variate put

$$z = \frac{x - m}{\sigma}$$

Now

$$P(m \le x \le x_1) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_m^{x_1} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

When $x = m$ $z = 0$ when

$x = x_1$, $z = \frac{x_1 - m}{\sigma} = z_1$ say

$$\therefore P(m \le x \le x_1) = P(0 \le z \le z_1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{1}{2}z^2} dz$$

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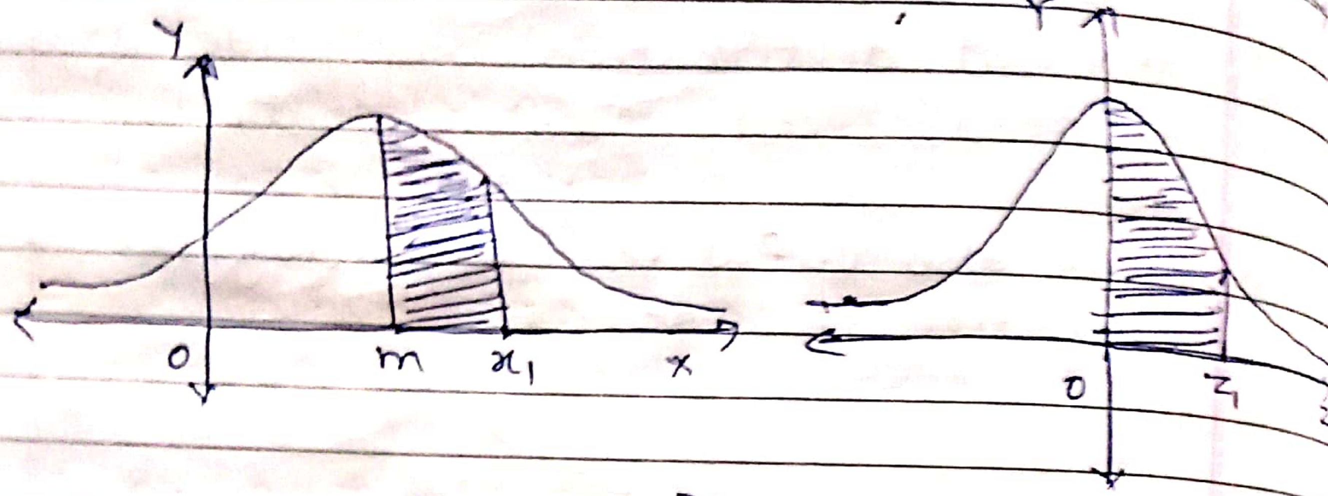
Thus the area under the normal curve from m to x_1 is equal to the area under the standard normal curve from 0 to z_1



The integral $\frac{1}{\sqrt{2\pi}}\int_0^{z_1} e^{-\frac{1}{2}z^2}\,dz$ is denoted by $\int_0^{z_1}\phi(z)\,dz$ and is known as normal probability integral.

Remarks

1. Since standard normal curve is symmetrical about the y-axis it is enough to find the areas to the right. The areas to the left of y-axis at equal distances will be equal

2. The total area under the curve is unity. Hence, because of symmetry the area under S.N.V. to the right of the y-axis is 0.5

③ $P(x_1 \le x \le x_2) = P(z_1 \le z < z_2)$
$= \text{area between } z = z_1 \text{ \& } z = z_2$
under standard normal curve

* Properties of the Normal Distribution.

① The Normal curve is bell-shaped & symmetrical about the maximum ordinate at $x = m$, the mean. In other words the curve is divided into equal parts by this ordinate

(ii) mean = median = mode = m for the Normal distribution.

(iii) The curve never intersects the x-axis at any finite point. The x-axis touches it at infinity

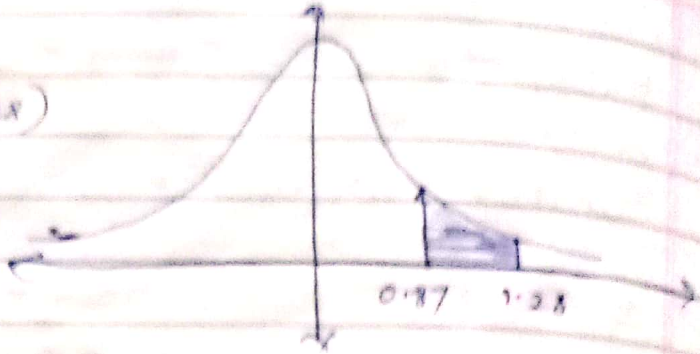
Examples

① Find the probability that a random variable having the standard normal distribution will take a value between 0.87 and 1.28

(Given Area from $z=0$ to $z = 0.87$ is 0.3078
Area bet $z=0$ to $z = 1.28$ is 0.3997

→ soln

Sol^n: $P(0.87 < z < 1.28)$



= Area between
$z = 0.87$ & $z = 1.28$

= (Area from $z=0$ to $z = 1.28$)
− (Area from $z=0$ to $z = 0.87$)

$= 0.3997 - 0.3078$
$= 0.0919$

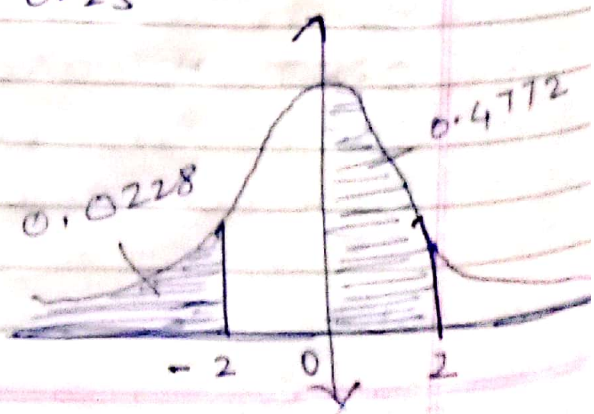
Ex. ② Sacks of sugar packed by an automatic loader have an average weight of hundred kilograms with standard deviation of two hundred fifty grams. Assuming a normal distribution find ~~odds~~ the chance of getting a sack weighing less than 99.5 kg
(Given for S.N.V. z area from $z=0$ to $z=2$ is 0.4772

Sol^n. S.N.V $z = \frac{x-m}{\sigma} = \frac{x-100}{0.25}$

when $x = 99.5$

$z = \frac{99.5 - 100}{0.250} = \frac{-0.5}{0.25}$

$= -2$



$P(X < 99.5) = P(z < -2)$

$= \text{Area to the left of } (z = -2)$

$= 0.5 - 0.4772$

$= 0.0228$

Ex ③ Weight's of 4000 students are found to be normally distributed with mean 50 kgs and standard deviation 5kgs. Find the number of students with weights

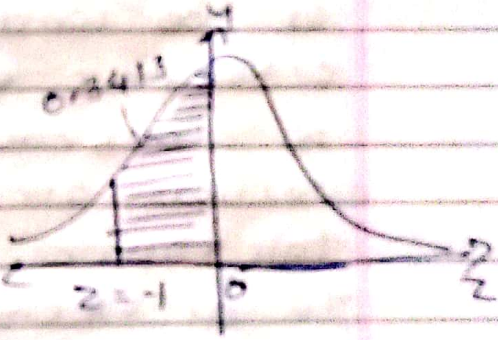
i) less than 45kgs

ii) between 45 & 60 kgs

(For a standard normal variate z area under the curve z=0 to z=1 is 0.3413 & that between z=0 to z=2 is 0.4772

Sol) We have $z = \dfrac{x-m}{\sigma}$

$= \dfrac{x-50}{5}$



(i) $x = 45,\ z = \dfrac{45-50}{5} = -1$

$P(x < 45) = P(z < -1)$

$= \text{Area left to } (z = -1)$

$= 0.5 - 0.3413 = 0.1587$

$\therefore$ No of students $= N \cdot P$

$= 4000 \times 0.1587$

$= 634.8 \doteq 635$

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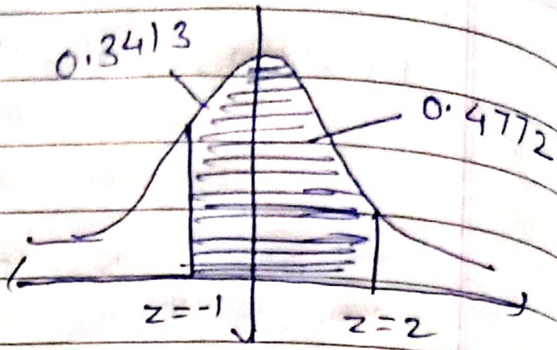
(ii)

When $x = 60$

$z = \frac{60 - 50}{5} = 2$



$P(45 < x < 60) = P(-1 < z < 2)$

= Area between

$z = -1$ & $z = 2$

$= 0.3413 + 0.4772$

$= 0.8185$

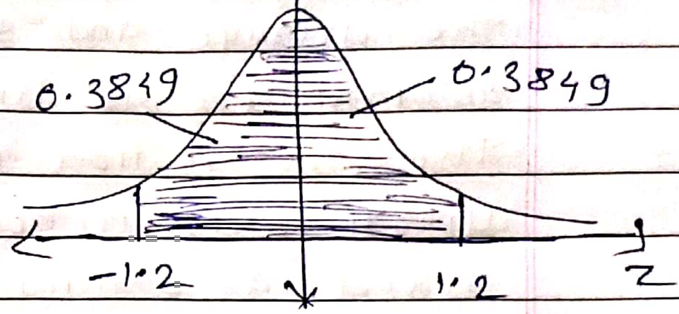
$\therefore$ No of Students $= N \cdot P = 4000 \times 0.8185$

$= 3274$

Ex. ④ The mean inside diameter of a sample of 200 washers produced by a machine is 0.502 cms and the standard deviation is 0.005 cms. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 0.496 to 0.508 cms. Determine the percentage of detective washers produced by the machine assuming the diameters are normaly distributed. (Area under the normal curve from $z=0$ to $z=1.2$ is 0.3849, from $z=0$ to $z=1.4$ is 0.4192 from $z=0$ to $z=1.0$ is 0.3413 from $z=0$ to $z=1.6$ is 0.4452

Sol^n We have

$$z = \frac{x-m}{\sigma}$$



$m = 0.502, \ \sigma = 0.005$

$$z = \frac{x - 0.502}{0.005}$$

$x = 0.496 \quad z = \dfrac{0.496 - 0.502}{0.005} = -1.2$

when $x = 0.508, \ z = \dfrac{0.508 - 0.502}{0.005} = 1.2$

$P(0.496 < x < 0.508) = P(-1.2 < z < 1.2)$

$= \text{Area from } (z = -1.2 \text{ to } z = 1.2)$

$= 2(0.3849)$

$= 0.7698$

$\therefore$ percentage of washers between tolerance limits

$= 76.98 = 77\%$

$\therefore$ percentage of defective washers

$= 100 - 77 = 23\%$

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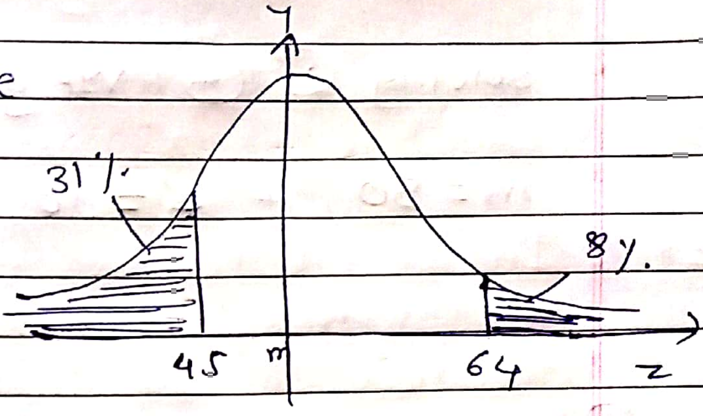
EX ⑤ In a normal distribution 31% items are under 45 & 8% are over 64. Find it's mean and standard deviation
(Given: For a normal distribution the area between $z=0$ to $z=0.5$ is 0.19 and that between $z=0$ to $z=1.4$ is 0.42)

→

Let m and σ be the mean and the standard deviation of the distribution



Since, 31% items are below 45

$50-31=19\%$ items are between 45 & m.

Since 8% items are above 64

$50-8=42\%$ items are between m & 64

Now, we are given that the area from $z=0$ to $z=0.5$ is 0.19 but the area from $z=0$ to $z=-0.5$ is also 0.19 but area in the normal distribution is equal to the probability.

$$\therefore P(45<x<m) = P(-0.5<z<0) = 0.19$$

but $z = \dfrac{x-m}{\sigma}$

when $x=45$ $z=-0.5$

$$\therefore -0.5 = \frac{45-m}{\sigma} \quad \therefore -0.5\sigma = 45-m \quad —①$$

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similarly $P(45 < X < 64) = P(0 < Z < 1.4) = 0.42$

When $X = 64$, $Z = 1.4$

$$1.4 = \frac{64 - m}{\sigma}$$

$$1.4\sigma = 64 - m \quad \text{——} \; (2)$$

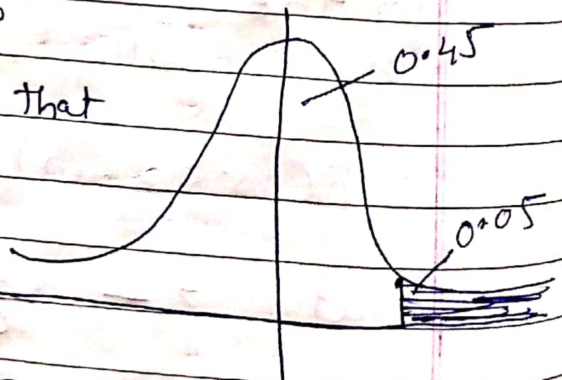
solving (1) & (2) we get

$m = 50$, & $\sigma = 10$

Ex (6) Monthly salary X in a big organization is normally distributed with mean Rs. 3000 and standard deviation Rs 250 what should be the minimum salary of worker in this organization so that the probability that he belongs to top 5% workers (Given Area from $z=0$ to $z=1.64$ is 0.45)

→ We have $z = \frac{X - m}{\sigma} = \frac{X - 3000}{250}$

We want to find z_1 such that



$$P(z > z_1) = \frac{5}{100} = 0.05$$

Since $0.5 - 0.05 = 0.45$ and and corresponding to 0.45 the entry in the area table is 1.64 $\therefore z_1 = 1.64$

$\therefore \; 1.64 = \dfrac{X - 3000}{250}$

$X = 3000 + 250 \times 1.64 = 3410$

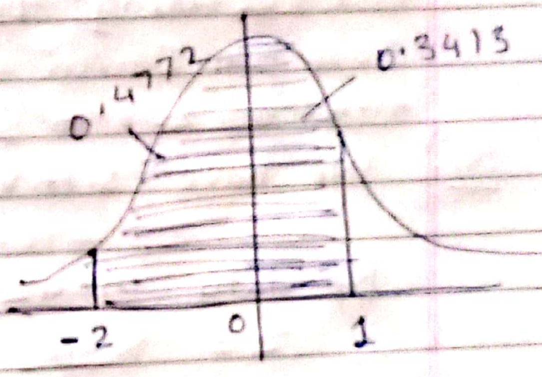
Ex ⑦ The marks obtained by 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be

(i) between 60 & 75

(ii) more than 75

Sol^n^ We have S.N.V Z

$Z = \dfrac{X - m}{\sigma} = \dfrac{X - 70}{5}$



(i) When $X = 60$, $Z = \dfrac{60-70}{5}$

$= -2$

When $X = 75$ $\quad Z = \dfrac{75-70}{5} = 1$

$P(60 \le X \le 75) = P(-2 < Z < 1)$

= Area from $Z = 0$ to $Z = 2$ + Area from $Z = 0$ to $Z = 1$

$= 0.4772 + 0.3413 = 0.8185$

$\therefore$ No. of Students getting marks bet^n^ 60 & 75

$= N \cdot P = 1000 \times 0.8185 = 818$

$P(x \geq 75) = P(z \geq 1)$

$= $ Area to the right of $z=1$

$= 0.5 - ($ Area betn $z=0$ to $z=1)$

$= 0.5 - 0.3413$

$= 0.1587$

$\therefore$ No. of students getting more than 75 m

$= N \cdot P = 1000 \times 0.1587 = 159$

Ex. (8) In an intelligence test administered to 1000 students, the average was 42 and standard deviation was 24. Find the number of students i) exceeding the score 50 & ii) between 30 & 54. [Given Area from z=0 to z=0.33 = 0.1293, Area from z=0 to z=0.5 = 0.1915]

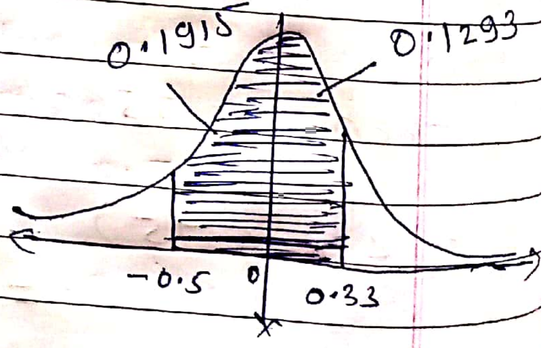
Soln: We have S.N.V. $z = \frac{x-m}{\sigma}$

By data $m = 42$, & $\sigma = 24$

$\therefore z = \frac{x-42}{24}$

i) When $x = 50$

$z = \frac{50-42}{24} = \frac{1}{3} = 0.33$



$P(x \geq 50) = P(z \geq 0.33)$

$= $ Area to right of 0.33

$= 0.5 - ($area betn $z=0$ to $z=0.33)$

$= 0.5 - 0.1293$

$= 0.3707$

ii) When $X = 30$ & $X = 54$ we get

$$z = \frac{30-42}{24} = -0.5 \ \&$$

$$z = \frac{54-42}{24} = 0.5$$

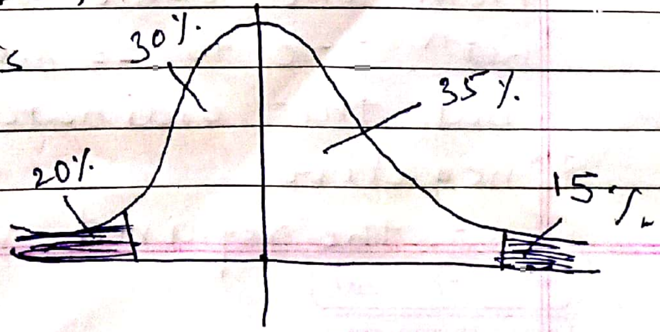
$P(30 \le z \le 54)$ = Area bet^n $z = -0.5$ to 0.5
= 2 (Area bet^n $z = 0$ to $z = 0.5$)
= 2 (0.1915)
= 0.3830

Number of students getting more than 50 marks
= N.P = 1000 × 0.3707
= 371

Number of students getting marks between 30 & 54 = N.P
= 1000 × 0.383 = 383

Ex 9 In a competitive examination the top 15% of the students appeared will get grade A while the bottom 20% will be declared fail. If the grades are normally distributed with mean % of marks 75 & S.D. 10, determine the lowest % of marks to receive grade A and the lowest % of marks that passes

[Given: Area $z = 0$ to $z = 1.04$ is 0.35
$z = 0$ to $z = 0.84$ is 0.3]



This is reverse problem

Solⁿ: We have $z = \frac{x-m}{\sigma} = \frac{x-75}{10}$

i) Grade A is given for 15% we have to find the value of z to the right of which the area is 0.15

But the area to the right of z = 0 is 0.5

∴ Area from (z=0 to z= this value)

$= 0.5 - 0.15 = 0.35$

from the table we find that the area betⁿ z = 0 to z = 1.04 is 0.35

∴ The required value of z = 1.04

but $z = \frac{x-75}{10}$

∴ $1.04 = \frac{x-75}{10}$

∴ $x = 75 + 10.4 = 85.4$

ii) Lowest 20% students are declared fail

We have to find the value of z to the left of which the area is 0.20. But the area to the left of z = 0 is 0.5

∴ Area from (z=0 to z= this value)

$= 0.5 - 0.2 = 0.3$

From the table we find that the area betⁿ z=0 to z= 0.84 is 0.3

but this ordinate is on left & hence negative

∴ the required value of z = − 0.84

but $z = \frac{x-75}{10}$

$$-0.84 = \frac{x-75}{10}$$

$\therefore x = 75 - 8.4 = 66.6$

Ex (10) Find the mean and the standard deviation of a normal distribution of marks in an examination where 58% of the candidates obtained marks below 75, 4% got above 80 and the rest between 75 and 80 (For S.N.V the area under the curve between $z = \pm 0.2$ is 0.16 & between $z = \pm 1.8$ is 0.92

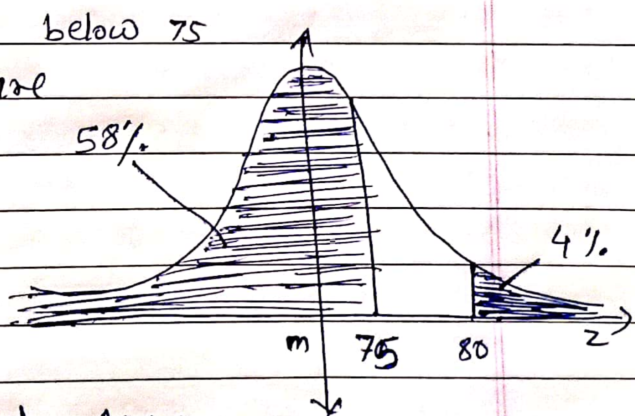
Soln :-

Let m and σ be the mean & Standard deviation of the variate

Since 58% students are below 75

58 - 50 = 8% Students are between 75 & m

since 4% students are above 80, 50-4 = 46%. Students are between m & 80



We are given that Area betn $z = \pm 0.2$ is 0.16 & that betn $z = \pm 1.8$ is 0.92

Hence the area between $z=0$ & $z = 0.2$ is $\frac{0.16}{2} = 0.8$ and that betn $z=0$ to $z=1.8$

is $\frac{0.92}{2} = 0.46$

In other words area 0.08 (8%), $z = 0.2$
For area 0.46 (46%), $z = 1.8$

$$\therefore \frac{75-m}{\sigma} = 0.2 \quad \& \quad \frac{80-m}{\sigma} = 1.8$$

$$75 - m = 0.2\sigma \quad \& \quad 80 - m = 1.8\sigma$$

Solving these two eqns we get

$\sigma = 3.125$

$m = 74.4$