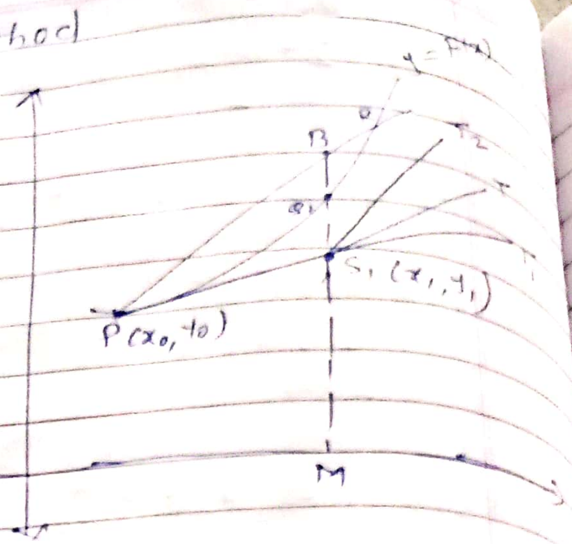
Euler's Modified Method



Consider the differential equation $\frac{dy}{dx} = f(x, y)$

P be the point on the solution curve $y = F(x)$ and let P be (x_0, y_0) and Q_1 be the point (x_0+h, y_1) we want to find the approximate value of y.

As we have seen in earlier Eulers method we find the y co-ordinate of S_1 where S_1 is the point of intersection of of the tangent PT_1 at P and the ordinate Q_1 this is approximately taken as y-coordinate of Q_1. Let's denote it by y_1 i.e.

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

Let S_1T_2 be the line through $S_1(x_1, y_1)$ and having slope $f(x, y)$. Let S_1T be the line through $S_1(x_1, y_1)$ and having slope equal to the average of the slopes of S_1T_1 & S_1T_2 i.e.

$$= \frac{f(x_0, y_0) + f(x_1, y_1)}{2}$$

We draw a line PR through $P(x_0, y_0)$ and parallel to S_1T, This line is used to find the approximate of y co-ordi[nate] of Q_1 this ordinate intersect the lin[e] PR in B. Then y co-ordinate of B is take[n] as the approximate value of y-cordinate Q_1

Now the equation of the PBR is

$$y - y_0 = (x - x_0)\left\{\frac{f(x_0, y_0) + f(x_1, y_1)}{2}\right\}$$

Now $BM = y_1$ & $x_1 - x_0 = h$

$$y_1^{(1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1)\}$$

where $y_1 = y_0 + h f(x_0, y_0)$

This is the first approximation of y_1 using this approximate value of $y_1^{(1)}$ in place of y_1 we get the second approximation.

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

We continue this process till we do not find any difference between two successive approximation.

Ex. ① Use Euler's modified method to find the value of y satisfying the equation $\frac{dy}{dx} = \log(x+y)$, $y(1) = 2$ for $x = 1.2$ and $x = 1.4$ correct to three decimal places by taking $h = 0.2$

Sol^n) We have $\frac{dy}{dx} = \log(x+y)$

(a) $f(x, y) = \log(x+y)$ $x_0 = 1,\ y_0 = 2$ & $h = 0.$

$\therefore y_1 = y_0 + h \cdot f(x_0, y_0)$

$f(x_0, y_0) = \log(x_0 + y_0) = \log(1+2) = 1.0986$

$\therefore$

$y_1 = 2 + (0.2)(1.0986) = 2.2197$

Now 1st approximation

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

But $f(x_0, y_0) + f(x_1, y_1) = \log(1+2) + \log(1.2 + 2.$
$= 1.0986 + 1.2295 = 2.328$

$$\therefore y_1^{(1)} = 2 + \frac{0.2}{2}(2.3281) = 2.23328$$

2nd approximation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 2 + \frac{0.2}{2}(2.332) = 2.2332$$

3rd approximation.

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(2)})\right]$$

$$= 2 + \frac{0.2}{2}(2.3321) = 2.2332$$

Thus $y_1^{(2)} = y_1^{(3)}$

Hence the correct value of y at $x = 1.2$ is

$$y(1.2) = 2.2332$$

(ii) To find the value of y at $x = 1.4$ we take $x_0 = 1.2$ and $y_0 = 2.2332$ as obtained above

Now $y_1 = y_0 + h f(x_0, y_0)$

But $f(x_0, y_0) = \log(x_0 + y_0)$

$$= \log(1.2 + 2.2332) = 1.2335$$

$$y_1 = 2.2332 + (0.2) \times (1.2335) = 2.4799$$

Now 1st approximation

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[\log[(1.2) + 2.2332] + \log(1.4 + 2.4799)\right]$$

$$= 2.2332 + (0.1) \times (2.5893)$$

$$= 2.4921$$

2nd approximation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x, y_1^{(1)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[\log(1.2 + 2.2332) + \log(1.4 + 2.4921)\right]$$

$$= 2.2332 + (0.1)(2.5924) = 2.4924$$

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3^{rd} approximation is

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[\log(1.2 + 2.2332) + \log(1.4 + 2.4924)\right]$$

$$= 2.2332 + (0.1)(2.5925)$$

$$= 2.4924$$

∵ 2^{nd} & 3^{rd} approximation gives the same value. Therefore

$$y(1.4) = 2.4924$$

Ex. Use Euler's modified method to find the value of upto 4 places of decimals satisfying the equation $\frac{dy}{dx} = 1 - y$ $y(0) = 0$ for $x = 0.2$ by taking $h = 0.1$

→ (a) We have $\frac{dy}{dx} = 1 - y$ $x_0 = 0$, $y_0 = 0$

$h = 0.1$ $x_1 = x_0 + h = 0 + 0.1 = 0.1$

∴ $y_1 = y_0 + h \cdot f(x_0, y_0)$

$y_1 = 0 + 0.1(1 - 0)$

$y_1 = 0.1$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

$$y_1^{(1)} = 0 + \frac{0.1}{2}\left[(1-0) + (1-0.1)\right]$$

$$y_1^{(1)} = 0 + 0.05\left[1 + 0.9\right]$$
$$= 0.05 \times 1.9$$
$$y_1^{(1)} = 0.095$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 0 + 0.05\left[(1-0) + (1-0.095)\right]$$

$$= 0.05(1 + 0.905)$$
$$= 0.05 \times 1.905$$
$$= 0.09525$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 0 + 0.05\left[(1-0) + (1-0.09525)\right]$$

$$= 0.05\left[1 + 0.90475\right]$$
$$= 0.09523$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[(1-0) + (1-0.09523)\right]$$
$$y_1^{(4)} = 0 + 0.05\left[1 + 0.90477\right]$$

$$y_1^{(4)} = 0.05 \times 1.90477$$
$$= 0.09523$$

$$y_1^{(3)} = y_1^{(4)}$$

$$\therefore \quad y(0.1) = 0.09523$$

(1) $x_0 = 0.1, \quad y_0 = 0.09523$

$x_1 = x_0 + h = 0.1 + 0.1 = 0.2$

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$= 0.09523 + 0.1\,(1 - 0.09523)$

$= 0.09523 + 0.1 \times (0.90477)$

$= 0.09523 + 0.090477$

$= 0.185707$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$

$= 0.09523 + 0.05\left[(1 - 0.09523) + (1 - 0.1857\right.$

$= 0.09523 + 0.05\,(0.90477 + 0.8143)$

$= 0.09523 + 0.05\,(1.71907)$

$= 0.09523 + 0.08595$

$= 0.18118$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 0.09523 + 0.05\,(0.90477 + 0.81882)$

$= 0.09523 + 0.05\,(1.72359)$

$= 0.09523 + 0.086179$

$= 0.1814095$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$

$= 0.09523 + 0.05\,(0.90477 + 0.81859)$

$= 0.09523 + 0.05 \times 1.72336$

$= 0.09523 + 0.086168$

$= 0.181398$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 0.09523 + 0.05\left[0.90477 + 0.8186\right]$$

$$= 0.09523 + 0.05 \times 1.72337$$

$$= 0.09523 + 0.08616$$

$$= 0.181398$$

$$y_1^{(3)} = y_1^{(4)}$$

$$\therefore y(0.2) = 0.1813$$

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