Module: 2

Probability Distribution

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Introduction:-

In previous chapter we have learnt prob. distribution for univariate & bivariate r.v. We can express any pmf of discrete r.v in functional form. The prob. distribution which possess some common features can be applied to variety of real life situation.

Mean / Expected value of r.v X

$\Rightarrow$ $E(x) = \sum_{x} x\, p(x)$ if x is discrete r-V

$= \int_{-\infty}^{\infty} x f(x) dx$ if x is continuous r V

variance :- which gives variability in the distribution.

$$\sigma^2 = \sum (x-\mu)^2 p(x) = E(x^2) - \mu^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

S.D $\rightarrow$ it is +ve square root of variance

$\Rightarrow$ $\sigma = +\sqrt{\sigma^2}$

1. Discrete Uniform distribution :-

Discrete Uniform distribution is the simplest of all discrete prob. distribution.

Def^n:-

A discrete r.v X taking values $1,2 \cdots n$ is said to follow a discrete uniform distribution if its pmf is given by

$$P(x) = P[X=x] = \frac{1}{n} \qquad x = 1,2 \cdots n$$
$$0 \qquad \text{otherwise.}$$

Where n is called as the parameter of the Uniform distribution.

In other words a r.v X is said to follow uniform distribution if probability remains the same for any value of X ie $1/n$ for all $x = 1,2 \cdots n$.

(II) mean & variance :-

$$E(X) = \sum_{x=1}^{n} x\,p(x) = \sum_{x=1}^{n} x \frac{1}{n}$$

$$= \frac{1}{n}(1+2+3+\cdots+n) = \frac{n(n+1)}{n\,2} = \frac{n+1}{2}$$

&

$$E(X^2) = \sum_{x=1}^{n} x^2 P(x) = \sum_{x=1}^{n} x^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n}(1^2+2^2+\cdots+n^2) = \frac{n(n+1)(2n+1)}{n \cdot 6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

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Now,

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n^2-1}{12}$$

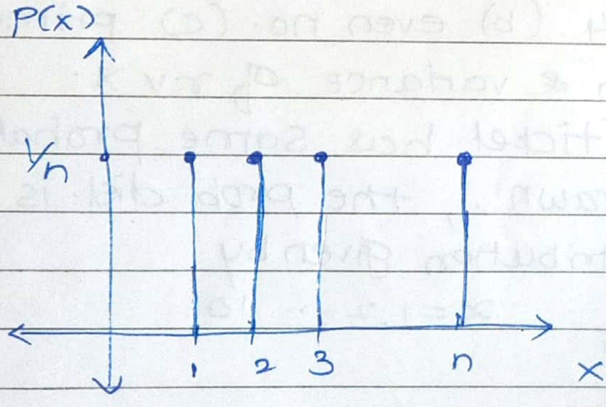
Remark :—

i) This distribution treats all the values of variable uniformly hence we call it as uniform distribution

(ii) The uniform distribution can be given in tabular form as

X	1	2	3	...	n
P(X)	1/n	1/n	1/n	...	1/n

& graphical representation is as follows.



Examples:

① Let X denote the no. on the face of an unbiased die obtained in a throw of single die. The pmf of X is $P(x) = \frac{1}{6}$ $x = 1, 2, 3, 4, 5, 6$

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2] suppose there are n Lottery tickets in a bag let x denote lottery ticket getting the first prize is uniform having pmf is

$$P(x) = \frac{1}{n} \quad x = 1, 2 \ldots n$$

3] The birthday of any person may occur on either sunday, monday ... saturday with same probability hence pmf is

$$P(x) = \frac{1}{7} \quad x = 1, 2, 3, 4, 5, 6, 7$$

where $x = 1$ interpreted as sunday etc.

4] If a ticket is drawn from a box containing 10 tickets numbered 1 to 10 inclusive. Find the probability that the no. x drawn is
a) less than 4 (b) even no. (c) prime no.
d] Find mean & variance of r v x.

→ since each ticket has same probability for being drawn, the prob. dist. is discrete uniform distribution given by

$$P(x) = \frac{1}{10} \quad x = 1, 2 \ldots 10.$$

i) $P(x < 4) = P(1) + P(2) + P(3) = \frac{3}{10}$

(ii) $P(\text{even}) = P(2) + P(4) + P(6) + P(8) + P(10) = \frac{5}{10} \frac{1}{2}$

(iii) $P(\text{prime}) = P(2) + P(3) + P(5) + P(7) = \frac{4}{10} = 0.4$

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iv) mean $= \frac{n+1}{2} = \frac{11}{2} = 5.5$

Variance $= \frac{n^2-1}{12} = \frac{100-1}{12} = \frac{99}{12} = 8.25$

5) If a card is selected at random from an ordinary pack of 52 cards. Find the prob. that
HW a) card is spade (ii) card is a face card ie jack, king or queen (iii) card is spade face card

Ans: (i) 13/52 (ii) 12/52 (iii) 3/52

6] Find the prob. that a card drawn at random from 50 cards numbered 1 to 50 is
HW a) prime (ii) ends in digit 2 (iii) divisible by 5

→ Ans (i) 3/10 (ii) 1/10 (iii) 1/5.

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Continuous Uniform distribution

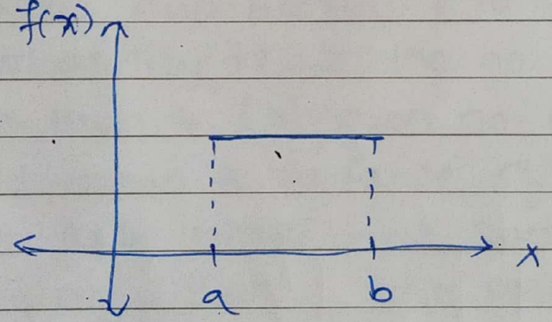
A continuous r.v is said to have continuous uniform distribution over an interval $[a, b]$ if its Pdf is given

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Remark:

I) We use notation $X \sim U(a, b)$ to describe X has uniform distribution over interval (a, b) where a & b are parameters of distribution

II) The graph of continuous uniform distribution is



Hence this distribution is also known as rectangular distribution.

(III) Distribution func (df) or cdf :-

If $X \sim U(a, b)$ then pdf is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & o.w \end{cases}$$

For $x \geq b$,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{a} f(t)\,dt}_{0} + \int_{a}^{b} f(t)\,dt + \underbrace{\int_{b}^{x} f(t)\,dt}_{0}$$

$$= \frac{1}{b-a}\left[t\right]_a^b = 1$$

We have

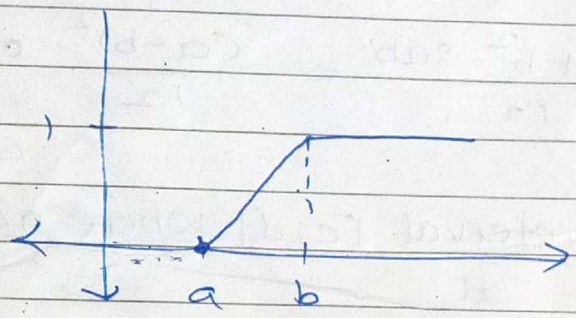
$$F(x) = P[X \leq x]$$

$$= \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$$= \frac{1}{b-a}(x-a)$$

$$= \frac{x-a}{b-a} \qquad a \leq x \leq b$$

The graph of df is as



(iv) Mean & variance:-

If $X \sim U(a,b)$ then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{a}^{b} \frac{x}{b-a}\,dx$$

$$\cancel{= \frac{1}{b-a}(b-a) \neq x}$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}(b^2-a^2) = \frac{b+a}{2}$$

$$E(X^2) = \int_{a}^{b} x^2 f(x)\,dx = \int_{a}^{b} \frac{x^2}{b-a}\,dx = \frac{1}{(b-a)3}(b^3-a^3)$$

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$$= \frac{b^2+ab+a^2}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4a^2+4b\cancel{a}a+4b^2 - 3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12} \text{ or } \frac{(b-a)^2}{12}$$

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If x is uniformly distributed in $-2 \leq x \leq 2$. find (i) $P(x<1)$ (ii) $P(|x-1| \geq 1/2)$

→ Here $f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{2-(-2)} & -2 \leq x \leq 2 \\ 0 & \text{o.w} \end{cases}$

1) $P(x<1) = \int_{-2}^{1} f(x)\,dx$

$= \int_{-2}^{1} \frac{1}{4}\,dx = \frac{1}{4}[x]_{-2}^{1} = \frac{3}{4}$

(ii) $P(|x-1| \geq 1/2) = P((x-1) \geq 1/2 \text{ or } -(x-1) \geq 1/2)$

$= P(x \geq 3/2 \text{ or } -x \geq -1/2)$

$= P(x \geq 3/2 \text{ or } x \leq 1/2)$

$= P(3/2 \leq x \text{ or } x \leq 1/2)$

$= P(3/2 \leq x \leq 2) + P(-2 \leq x \leq 1/2)$

$= \int_{3/2}^{2} f(x)\,dx + \int_{-2}^{1/2} f(x)\,dx$

$= \int_{3/2}^{2} 1/4\,dx + \int_{-2}^{1/2} 1/4\,dx$

$= \frac{1}{4}(2-3/2) + \frac{1}{4}(1/2-(-2))$

$= \frac{1}{8} + \frac{5}{8} = 6/8 = 3/4$

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(2) If x is uniformly distributed in $[-\alpha, \alpha]$ with $\alpha > 0$ then determine α such that $P(x>1) = 1/3$

→

As,

$$P(x>1) = \frac{1}{3}$$

$$\int_1^{\alpha} f(x)\,dx = \frac{1}{3}$$

$$\int_1^{\alpha} \frac{1}{2\alpha}\,dx = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2\alpha}(\alpha-1) = \frac{1}{3} \Rightarrow \frac{\alpha-1}{2\alpha} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2\alpha} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2\alpha} = \frac{1}{6} \Rightarrow \boxed{\alpha=3}$$

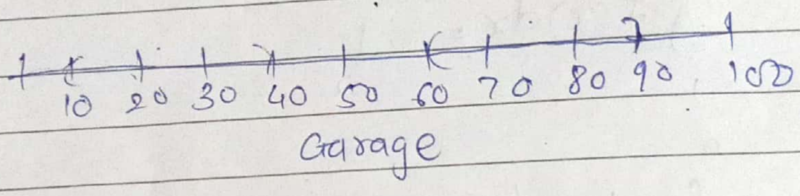
(3) A bus travels betn two cities A & B which are 100 miles apart. If the bus has a breakdown the distance x of point of breakdown from city ~~B~~ A has uniform distribution U[0, 100]

I) There are services garages in the city A, city B & midway betn. cities A & B. If a breakdown occurs, a tow truck is sent from garage closest to the point of breakdown. What is prob. that the tow truck has to travel more than 10 miles to reach bus.

⑤ Would it be more efficient if the three garages were placed at 25, 50, 75 miles from city A? Explain.

→

a) If the bus breakdown in the intervals [10, 40] miles or [60, 90] miles then bus have to be towed for more than 10 miles.



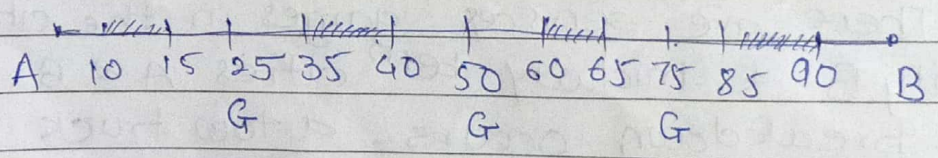
P(bus has to be towed more than 10 miles)

$$= P(10 < x < 40) + P(60 < x < 90)$$

$$= \int_{10}^{40} \frac{1}{100}\,dx + \int_{60}^{90} \frac{1}{100}\,dx$$

$$= \frac{1}{100}(40-10) + \frac{(90-60)}{100} = \frac{60}{100} = \frac{3}{5} = 0.6$$

b) Suppose 3 garages are placed at 25, 50, 75 miles from city A.



In this case, the bus is to be towed for more than 10 miles if the bus breaksdown in any one of four intervals (10, 15), (35, 40), (60, 65), (85, 90) miles

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$\Rightarrow P(10<x<15) + P(35<x<40) + P(60<x<65) + P(85<x<90)$

$= \frac{1}{100}\cdot 5 + \frac{1}{100}\cdot 5 + \frac{1}{100}\cdot 5 + \frac{1}{100}\cdot 5$

$= \frac{20}{100} = 0.2$

since prob. is small (0.2) compared to (0.6) in case a, b is more effective.

The averge daily procurement (acquirement) of fresh milk by a milk producer's union is 40,000 lit & mini is 25,000 lit per day. Assuming a uniform dist. find the max^m milk procurement in a day & what % of days of the procurement will exceed 35,000 lit?

→

Let x — daily procurement of milk.

$x \sim U(a, b)$

$a = 25,000$ & mean of is $40,000$.

we have

$E(x) = \frac{a+b}{2}$

$\Rightarrow 40000 = \frac{25000+b}{2}$

$\Rightarrow b = 55000$

∴ max daily procurement would be 55,000 Lit

To find % of days of procurement will exceed 35000 lit.

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We have

$X \sim U(25,000, 55,000)$, pdf is

$$f(x) = \begin{cases} \frac{1}{30,000} & 25,000 < x < 55,000 \\ 0 & o.w \end{cases}$$

$$P(x > 35,000) = \int_{35,000}^{\infty} f(x)\,dx$$

$$= \int_{35,000}^{55,000} \frac{1}{30,000}\,dx$$

$$= \frac{2}{3} = 0.67$$

Thus

67% of the days the procurement of milk is beyond 35000 lit

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Normal Distribution / Gaussian

Normal distribution is the prob. distribution of continuous r.v X known as normal random variable or normal variate. It is given by

$$N(\bar{x}, \sigma) = f(x) = Y(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$

Here $\bar{x}$ = mean

σ = S.D be two parameter of cont. prob distribution.

This theoretical distribution is most imp., simple & useful & is corner stone of modern statistics because a) discrete prob distributions such as Binomial, Poission, Hypergeometric can be approximated by N.D.

b) Sampling distribution 't', 'F', χ^2 ~~can to~~ tend to be normal for large samples

c) It is applicable in statistical quality control.

Properties :

(1) The graph of normal ~~curve~~ distribution $y = f(x)$ in the XY plane is known as normal curve (NC). NC is (a) symmetric about Y axis.

(ii) it is bell shaped

(iii) The mean, mode, median coincide & NC is unimodal (has max^m only one point).

(iv) NC has inflection pt. at $\bar{x} \pm \sigma$

v) NC is asymptotic to both +ve & -ve x axis.

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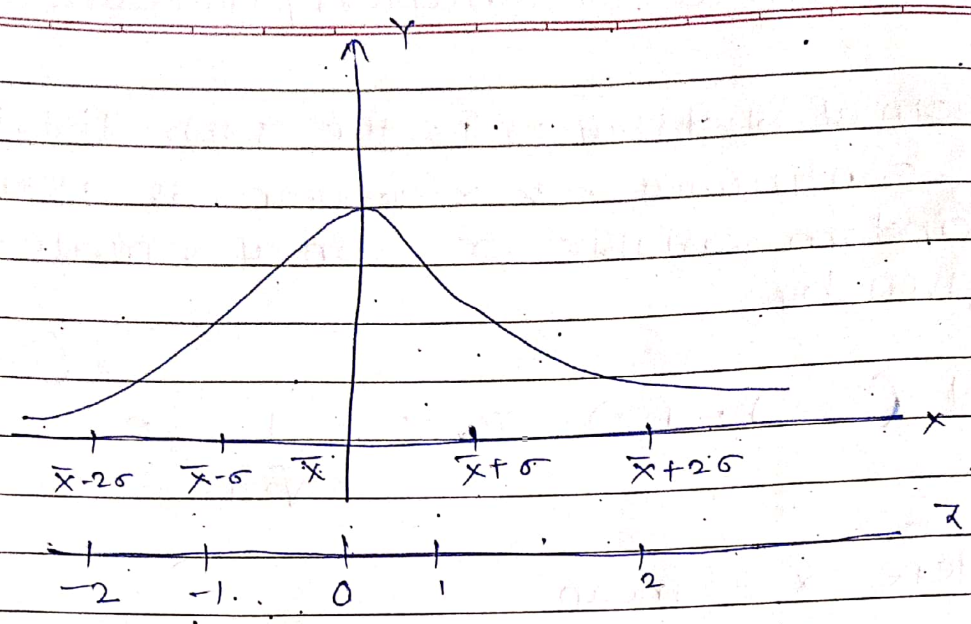

2. Area under normal curve is 1.

3. Probability that cont. r.v X lies betn x_1 & x_2 is given by

$$p(x_1 < x < x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2}$$

This depends on two parameter $\bar{x}$ & σ. we get different normal curves for different values of $\bar{x}$ & σ & it is impracticable task to plot all such normal curve. Instead, by introducing $z = \frac{x-\bar{x}}{\sigma}$

R.H.S of integral becomes independent of two parameter $\bar{x}$ & σ. Here z is known std. variate.

4. change of scale from x axis to z axis

$$p(x_1 < x < x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2} dx$$

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$$P(z_1 \le z \le z_2) = \int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma\, dz$$

$$= \int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad \longrightarrow (*)$$

Where

$$z_1 = \frac{x_1 - \bar{x}}{\sigma}, \quad z_2 = \frac{x_2 - \bar{x}}{\sigma}$$

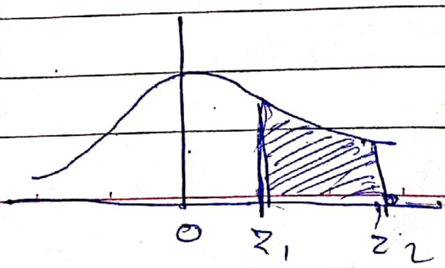
ie Normal distribution $N(\bar{x}, \sigma)$ transformed by std. variate z is

$$N(0,1) = \gamma(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$ with mean 0 & S.D 1.

is known as std. Normal distribution & its normal curve is known as std. normal curve.

The probability integral (*) is tabulated for various values of z varying from 0 to 3.9 is known as normal table. Thus entries in the normal table gives area under normal curve betn ordinates $z=0$ to z (shaded in fig).

ie $P(x_1 \le x \le x_2) = P(z_1 \le z \le z_2)$
= Area under normal curve from (0 to z_2) − (Area under normal curve from 0 to z_1)



(1) Given a r.v x having normal distribution with $\mu = 16.2$ & $\sigma^2 = 1.5625$. Find the prob. that it will take a value

i) greater than 16.8

ii) less than 14.9 (iii) betn 13.6 & 18.8

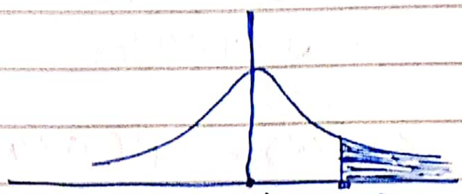
iv) betn 16.5 & 17.4 (v) Greater than

→ Given $m = \mu = 16.2$, $\sigma = 1.25$

(i) $P(x > 16.8) = P\left(\frac{x-m}{\sigma} > \frac{16.8-m}{\sigma}\right)$

$= P\left(z > \frac{16.8-16.2}{1.25}\right)$

$= P(z > 0.48)$



0 0.48

$= 0.5 - P(0 < z < 0.48)$.

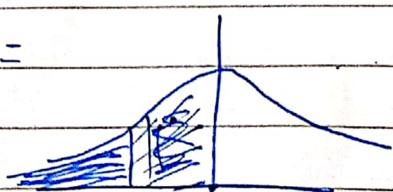
$= 0.5 - 0.1844$ (From table)

$= 0.3156$.

(ii) $P(x < 14.9) = P\left(\frac{x-\mu}{\sigma} < \frac{14.9-\mu}{\sigma}\right)$

$= P(z < -1.04)$

$=$



1.04 0

$= 0.5 - P(0 < z < 1.04)$

$= 0.5 - 0.3508 = 0.1492$

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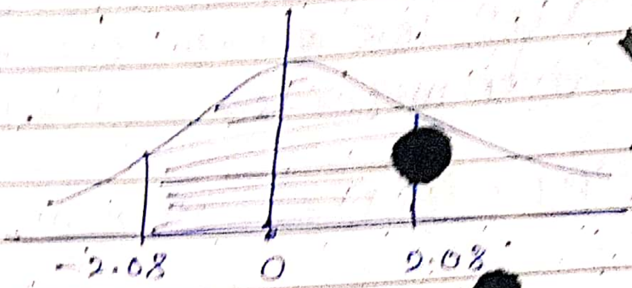
iii) $P(12.6 < x < 12.8) = P\left(\frac{12.6-m}{\sigma} < \frac{x-m}{\sigma} < \frac{12.2-m}{\sigma}\right)$

$= P(-2.08 < z < 2.08)$



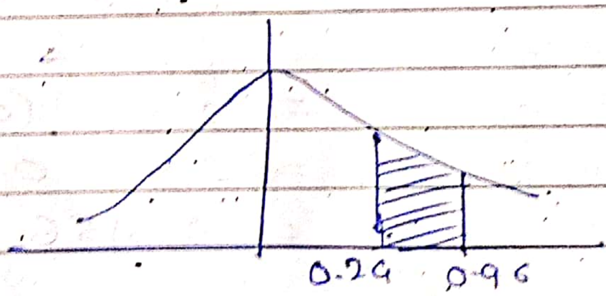
$= P(-2.08 < z < 0) + P(0 < z < 2.08)$

$= 2 \cdot P(0 < z < 2.08)$

$= 2(0.4812)$

$= 0.9624$

iv) $P(16.5 < x < 17.6) = P(0.24 < z < 0.96)$



$= P(0 < z < 0.96) - P(0 < z < 0.24)$

0.3315
0.0948

$= 0.3315 - 0.0948$

$= 0.2366$

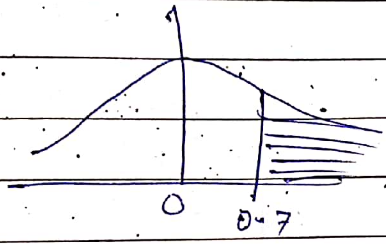
(v)

② The mean height of 1000 students at a certain college is 165 cms & SD 10 cms. Assuming Normal distribution find the no. of students whose height is
i) greater than 172 cms (ii) betⁿ 159 & 178 cm

→ (For SNV, Area from $z = 0$ to $z = 0.7$ is 0.2580
Area from $z = 0$ to $z = 0.6$ is 0.2258
$z = 0$ to $z = 1.3$ is 0.4032)

→ Here $\mu = m = 165$, $\sigma = 10$
Here x — height of student.

$P(x > 172) = P(z > 0.7)$



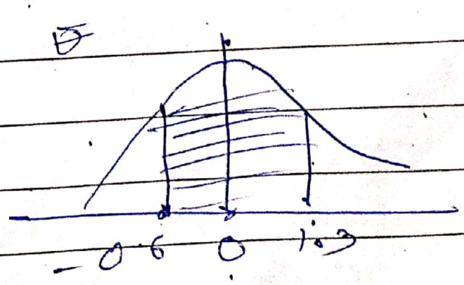
$= 0.5 - P(0 < z < 0.7)$

$= 0.5 - 0.2580$

$= 0.2420$

∴ No. of students whose height is greater than 172 is $0.2420 \times 1000 = 241.96 \Rightarrow 242$.

(ii) $P(159 < x < 178) = P(-0.6 < z < 1.3)$



$= P(-0.6 < z < 0) + P(0 < z < 1.3)$

$= 0.2258 + 0.4032 = 0.6290$

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No. of students $= 1000 \times P[159 < X < 178)$

$= 628.95 \approx 629.$

* Common For all problems :→

* For SNV,

Area from $z=0$ to $z=2$ is 0.4772

—"— $z=0$ to $z=1$ is 0.3413

—"— $z=0$ to $z=0.525$ is 0.2

—"— $z=0$ to $z=1.28$ is 0.4

—"— $z=0$ to $z=0.25$ is 0.1

—"— $z=0$ to $z=0.52$ is 0.2

—"— $z=0$ to $z=1.645$ is 0.45

—"— $z=0$ to $z=0.5$ is 0.19

—"— $z=0$ to $z=1.4$ is 0.42

(You can verify all these values from normal table.]

(1) Students of class give an aptitude test. Their marks are normally distributed with mean 60, SD 5. What is % of students scored more than 60 marks.

→

Given $m = 60$

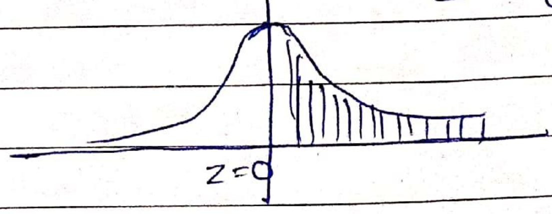
$\sigma = 5$.

Let x be marks.

$$P(x > 60) = P\left(\frac{x-m}{\sigma} > \frac{60-m}{\sigma}\right)$$

$$= P(z > 0)$$

$$= 0.5$$



ie % of students = 50%.

(2) Sacks of sugar packed by an automatic loader have an average weight of 100 kg with S.D 250gm. Assuming a normal distribution, Find a chance of getting a sack weighing less than 99.5 kg.

→ Let x be weight.

$m = 100kg$

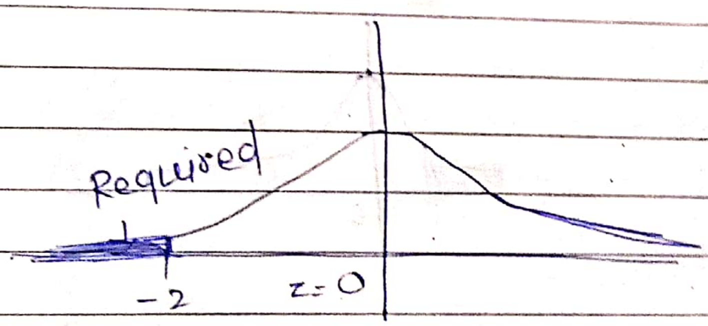
$\sigma = 250\,g = \frac{250}{1000} = 0.25kg$

Required prob =

$P(x < 99.5) = P\left(\frac{x-m}{\sigma} < \frac{99.5-m}{\sigma}\right)$

$= P\left(z < \frac{99.5-100}{0.25}\right)$

$= P(z < -2)$



$= 0.5 - P(-2 < z < 0)$

$= 0.5 - P(0 < z < 2)$

(due symmetric curve)

$= 0.5 - 0.4772$

$P(x < 99.5) = 0.0228$

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(3) Weights of 4000 students are found to be normally distributed with mean 50kgs & S.D 5kgs Find the no. of students with weights

1) less than 45kg

11) betn 45 & 60kg.

→ $N = 4000$

$m = 50$ Here x be weight

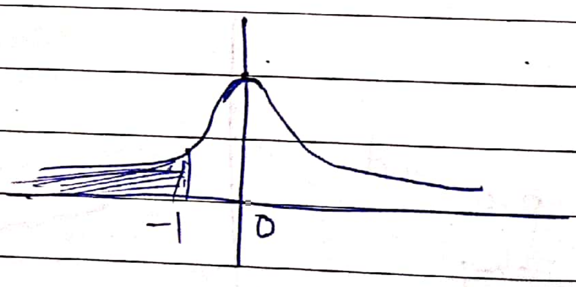
$\sigma = 5$

(1)

$$P(x < 45) = P\left(\frac{x-m}{\sigma} < \frac{45-m}{\sigma}\right)$$

$$= P\left(z < \frac{45-50}{5}\right)$$

$$= P(z < -1)$$



$$= 0.5 - P(-1 < z < 0)$$

$$= 0.5 - P(0 < z < 1) \quad (\because \text{symmetry})$$

$$= 0.5 - 0.3413$$

$$= 0.1587$$

No. of students weighing less than 45 kg $= 4000 \times 0.1587$

$= 634$

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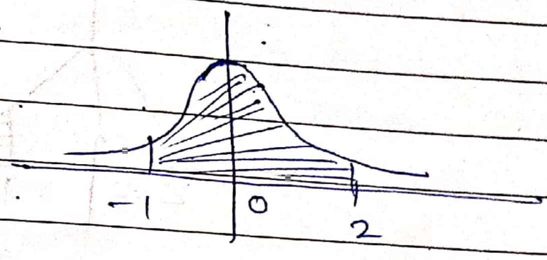
ii) $P(45 < x < 60) = P\left(\frac{45-m}{\sigma} < \frac{x-m}{\sigma} < \frac{60-m}{\sigma}\right)$

$= P\left(\frac{45-50}{5} < z < \frac{60-50}{5}\right)$

$= P(-1 < z < 2)$



$= P(-1 < z < 0) + P(0 < z < 2)$

$= P(0 < z < 1) + P(0 < z < 2)$

$= 0.3413 + 0.4772$

$= 0.8185$

$\therefore$ No. of students having weight betn 45 & 60 kg $= 4000 \times 0.8185$

$= 3274.$

④ In an examination given by 500 Candidates the average & S.D of marks obtained are 40 & 10 resp. Assuming the distribution of marks to be normal. Find approximately

i) how many will pass if 50 is fixed as minimum

ii) What should be minimum if 350 candidates are to pass?

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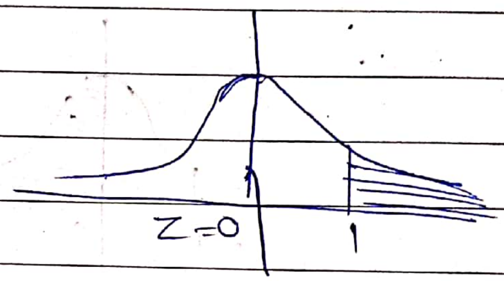
$N = 500$

$m = 40$, $\sigma = 10$.

Let x be marks.

$P(x \geq 50) = P\left(\frac{x-m}{\sigma} \geq \frac{50-m}{\sigma}\right)$

$= P(z \geq 1)$



$= 0.5 - P(0 < z < 1)$

$= 0.5 - 0.3413$

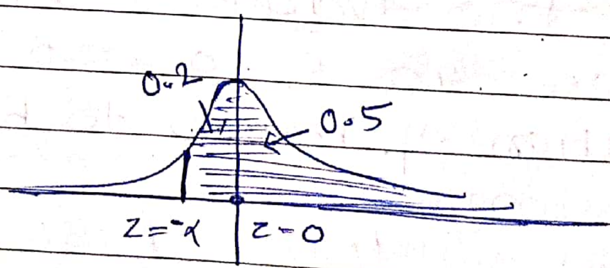
$= 0.1587$

No. of students passing $= 500 \times 0.1587$

$= 79$

(II) $P(\text{passing}) = \frac{350}{500} = 0.7$

Let x be minimum marks.



Area from $z = -\alpha$ to $z = \infty$ is 0.7

we know that

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Area from $z=0$ to $z=\infty$ is 0.5
& Area from $z=-\alpha$ to $z=0$ is 0.2

From given values

Area from $z=0$ to $z=0.525$ is 0.2

By symmetricity

$\therefore \quad \alpha = -0.525$

we have

$$z = \frac{x-m}{\sigma}$$

$$\Rightarrow -0.525 = \frac{x-40}{10}$$

$$\Rightarrow x = 34.75.$$

(5) Determine the min. marks of student in order to get A grade if top 10% students are awarded A grade in an exam where mean of marks is 72 & S.D is 9

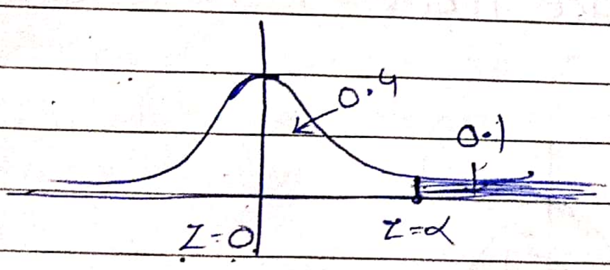
$\rightarrow$

$m = 72$

$\sigma = 9$.

Let x be min. marks.

Given that 10% awarded with A grade



Given

Area from $z=\alpha$ to $z=\infty$ is 0.1

& Area from $z=0$ to $z=\alpha$ is 0.4

By given data

Area from $z=0$ to $z=1.28$ is 0.4

By compairing
$z = 1.28$
we have

$$z = \frac{x-m}{\sigma}$$

$$\Rightarrow 1.28 = \frac{x-72}{9}$$

$$\Rightarrow x = 83.52 \simeq 84$$

ie student must get 84 marks to get A grade.

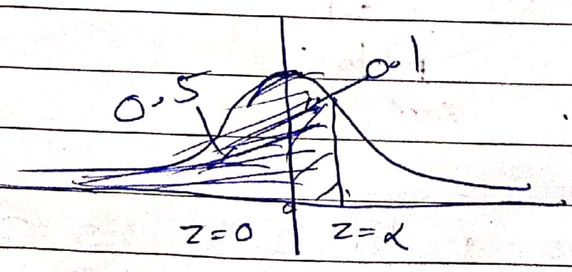
⑥ When the mean of marks was 50% & SD 5% then 60% students failed in an exam. Determine the grace marks to be awarded in order to show that 70% of students passed. Assume marks are normally distributed.

→

$m = 50\% = 0.5$

$\sigma = 5\% = 0.05$

Let x be marks obtained in exam. Before grace marks were awarded, 60% failed. ie



Shaded area = 0.6
ie Area from $z=0$ to $z=\alpha$ is 0.1

By given data

Area from $z=0$ to $z=0.25$ is 0.1

$\therefore z = \alpha = 0.25$

Now,

$$z = \frac{x-m}{\sigma}$$

$$0.25 = \frac{x - 0.5}{0.05}$$

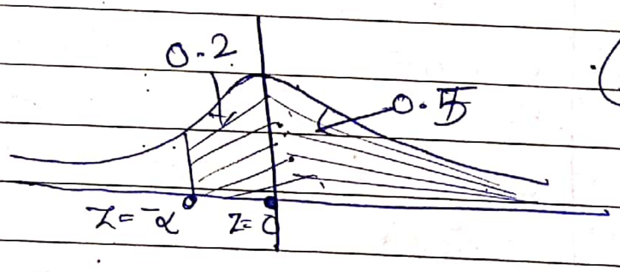
$\Rightarrow x = 0.5125$

min pass marks before grace = 51.25%.

Now,

After grace marks, 70% passed

ie



(on left side z +ve is negative)

shaded area = 0.7

ie Area from $z = -\alpha$ to $z=0$ is 0.2

By data

Area from $z=0$ to $z=0.525$ is 0.2

By symmetricity

Area from $z=-0.525$ to $z=0$ is 0.2

$\therefore z = -0.525$

As $z = \frac{x-m}{\sigma} \Rightarrow -0.525 = \frac{x-0.5}{0.05}$

$\Rightarrow x = 0.4740$

$\therefore$ min pass marks After grace = 47.40%.

Grace marks = 51.25 − 47.40 = 3.85%

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(7) The income of 10,000 persons is normally distributed with mean Rs 520 & SD Rs 60. Find

i) lowest income of richest 500

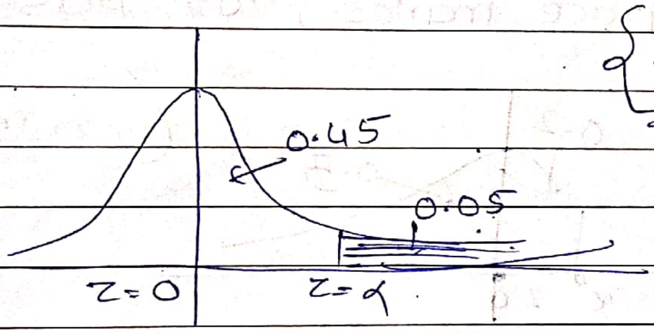
ii) highest income of poorest 500

→

$m = 520, \sigma = 60$

i) Let x be income.

If we consider the richest 500 persons the prob. that a person selected at random is $\frac{500}{10,000} = 0.05$.



richest people take at extreme right & poorest people take at left side) just for convenience.

we have shaded area = 0.05

Area from $z = 0$ to $z = \infty$ is 0.5

Area from $z = 0$ to $z = \alpha$ is 0.45

—"— $z = \alpha$ to $z = \infty$ is 0.05

By given data

Area from $z=0$ to $z = 1.645$ is 0.45

$\therefore z = \alpha = 1.645$

we have

$z = \frac{x-m}{\sigma}$

$\Rightarrow 1.645 = \frac{x-520}{60} \rightarrow x = 618.7$

The lowest income of richest 500 is Rs 619

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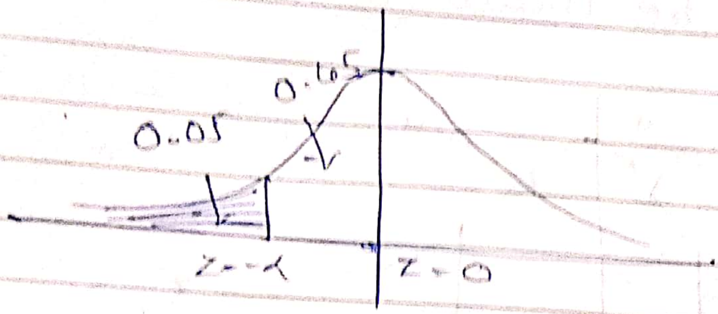
(ii) If we consider poorest 500 person. Then probability that a person selected at random is $\frac{500}{10000} = 0.05$



Here

Area from $z = -\infty$ to $z = -\alpha$ is 0.05
" " $z = -\alpha$ to $z = 0$ is 0.45

By given data

Area from $z = 0$ to $z = 1.645$ is 0.45

By symmmetricity

$z = -1.645$

we have

$$z = \frac{x - m}{\sigma}$$

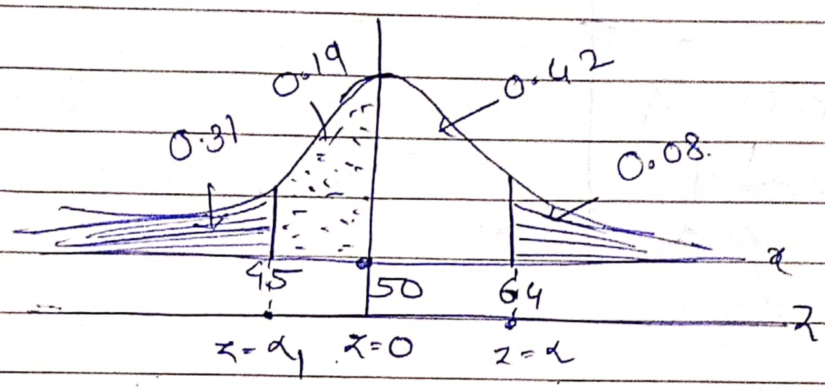
$$-1.645 = \frac{x - 520}{60}$$

$\Rightarrow$ $x = 421$

i.e. The highest income of poorest 500 person is Rs 421.

(8) In a normal distribution 31% items are under 45 & 8% are over 64. Find its mean & S.D.

→ Let m be mean & σ be S.D.



Area from $z = \alpha_1$ to $z = 0$ is 0.19

By given data

Area from $z = 0$ to $z = 0.5$ is 0.19

ie. $z = -0.5$ (corresponding x is 45)

we have

$$z = \frac{x - m}{\sigma}$$

$$\therefore -0.5 = \frac{45 - m}{\sigma}$$

$$\Rightarrow -0.5\sigma = 45 - m$$

$$\Rightarrow -0.5\sigma + m = 45 \quad \text{— (I)}$$

Now,

Area from $z = 0$ to $z = \alpha$ is 0.42

By data

Area from $z = 0$ to $z = 1.4$ is 0.42

$\therefore$ $\underline{z = 1.4}$ (corresponding to $x = 64$)

$$z = \frac{x-m}{\sigma}$$

$$1.4 = \frac{64-m}{\sigma}$$

$$\Rightarrow 1.4\sigma = 64-m$$

$$\Rightarrow m + 1.4\sigma = 64 \quad \text{— (II)}$$

on solving (I) & (II)

$$-0.5\sigma + m = 45$$
$$1.4\sigma + m = 64$$
$$- \quad - \quad -$$
$$-1.9\sigma = -19$$

$$\boxed{\sigma = 10}$$

put $\sigma = 10$ in (I) or (II)

$$m = 64 - 1.4\sigma$$
$$= 64 - 14$$
$$\boxed{m = 50}$$

$\therefore$ mean $= 50$, S.D $= \sigma = 10$.

H/W

In a sample of 1000 students the mean & S.D of marks obtained by students in a certain test are 14 & 2.5 Assuming distribution to be normal find no. of students getting marks (I) betⁿ 12 & 15 (II) below 8 (III) above 18

→ (I) 443 (II) 8 (III) 55

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Exponential Distribution

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Exponential distribution is related to Poisson distribution. In some sense. The no. of events which occur in time interval $[0, t]$ has Poisson distribution. The E.D can be used to obtain distribution of time that elapses betⁿ such events. ie The E.D is often concerned with the amount of time until some specific event occurs.

eg: i) Failures of a system to work follow Poisson distribution. Then time betⁿ two failures ie the time for which the system is working has exponential distribution.

ii) The amount of time (beginning now) until earthquake occurs has E.D.

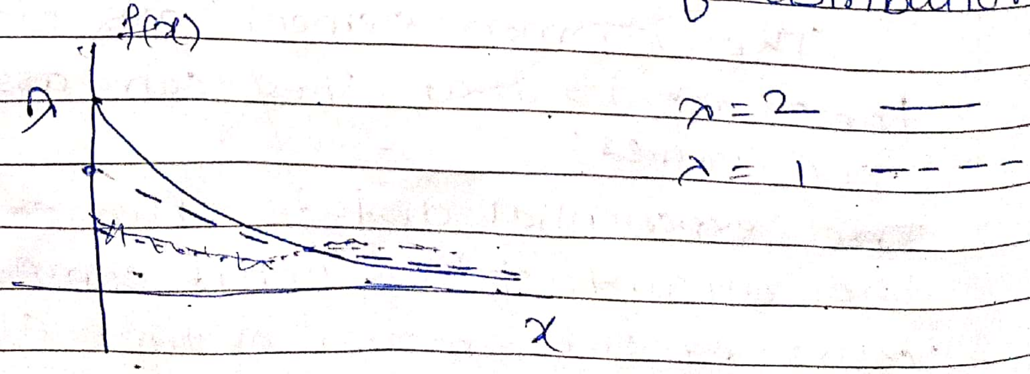
It plays an imp. role in reliability Theory & queuing theory.

Defⁿ:—

A continuous r.v X has following pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Where $\lambda > 0$ is called rate of distribution.



* $\Gamma n = (n-1)!$ if n is +ve int.
* $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$

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Real life situations ~~where~~ of application of E.D

E.D can be applied in the study of
i) life of electronic component
ii). time required for repairs.
iii) Time period betn two successive arrivals of customers at service station
(iv) Time requirement to find fault in machine.

mean & varian :→ Let X follow exponential distrib- with pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x} dx$$

$$= \lambda \cdot \frac{\Gamma 2}{\lambda^2} = \frac{1}{\lambda}.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \cdot \frac{\Gamma 3}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$\therefore \sigma = \frac{1}{\lambda}$

ie mean = S.D = $\frac{1}{\lambda}$

so one can see that as λ gets larger, the thing in process we are waiting for to happen tends to often more quickly hence λ is rate

Distribution fun^c or cdf:

Let x follow exponential distribution with pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

We have

$$F(x) = P[X \leq x]$$

$$= \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt$$

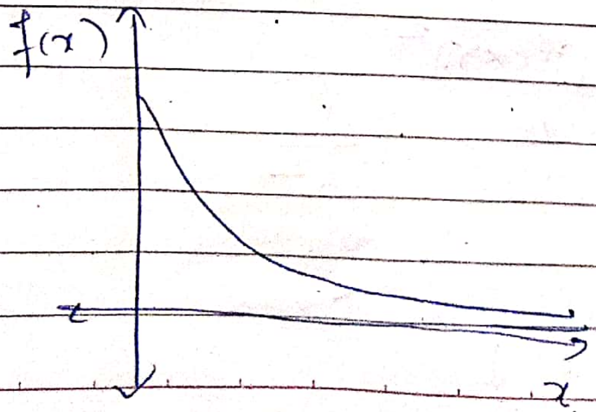
$$= \int_{0}^{x} \lambda e^{-\lambda t}\,dt = \lambda \left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^x$$

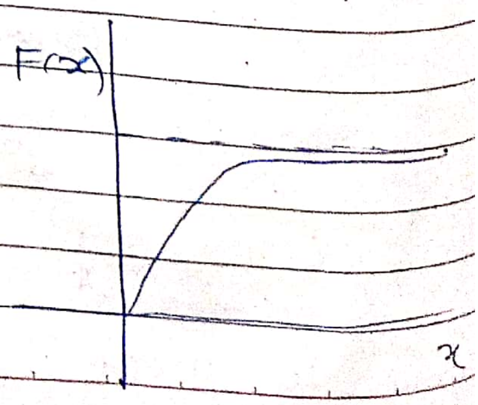
$$= -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x} \qquad x \geq 0$$

$$F(x) = 0 \qquad x < 0$$

$F(x)$ gives the prob that system will die before x units of time have passed.





survival fun^c:

It gives the probability that system survives more than x units of time

$$P[X > x] = 1 - P[X \le x]$$
$$= 1 - F(x) = \begin{cases} e^{-\lambda x} & x \ge 0 \\ 1 & x < 0 \end{cases}$$

Lack of memory property / memoryless / forgetfulness or markov property.

The exponential dist. characterised by property known as markov property. Among all distributions of non-ve continuous variables only E.D has no memory.

This property means that a given prob. distribution is independent of its history. Any time may be marked down as time zero. If prob dist. has a memoryless property, the possibility of something happening in the future has no relation to whether or not it has happened in the past. ie The history of fun^c is irrelevent of future. Every instant is like ~~a new~~ beginning of new random period which has same distribution regardless of how much time has already elapsed.

Ex:- i] Discrete case:

Tossing of fair coin is an example prob dist. that is memoryless. Every time you toss a coin you have 50%. chance of it coming up head. It doesn't matter whether or not last five times, you threw a coin it came up consistently tails, the prob of heads in next throw is always goingtozero.

(ii) For real life examples consider independent failure of computer hardware. When figuring the probability of a (new ind.) hardwar failure, it doesn't matter ~~whether~~ how frequently or when your hardware failed in the past. The prob. it will fail in five min. from now is independent of fact it hasn't failed for 3 months.

statement:

~~Let x follow exponential distribution~~

Let x be exponential rv with mean $1/\lambda$

then

$$P\left[x > a+b \,\middle|\, x > a\right] = P[x > b] \qquad a \geq 0, \; b \geq 0$$

→

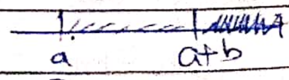
As x follows exponential distribution, its pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P[x > b] = 1 - P[x \leq b]$$
$$= 1 - F(b)$$
$$= 1 - e^{-\lambda b}$$

Now,

$$P[x > a+b \mid x > a] = \frac{P[(x > a+b) \cap (x > a)]}{P[x > a]}$$



$$= \frac{P[x > a+b]}{P(x > a)}$$

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$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}$$

$$= P[X > b]$$

From the point of view of waiting time until arrival of a customer, the memory less property means that it does not matter how long you have waited so far. If you have not observed a customer until time a, the distribution of waiting time (from time a) until next customer is same as when you started at time zero.

Ex:- If x is life of electronic component then x is not failed upto a time 'a' then the chance that it will not fail in the next time period of b units ie it will survive upto (a+b) units is same as the new component will not fail b units starting from zero which means the knowledge of first component has already worked for time a has no significance. Hence there is no improvement due to aging in life of component.

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(1) The failure time of a component X is assumed to have an exponential distribution with mean 100 hrs. Find the prob. that any particular component will

i) last atleast for 200 hrs

ii) last betn 250 & 300 hrs.

→

x - failure time of component

$\therefore$ mean $= \frac{1}{\lambda} = 100 \Rightarrow \lambda = \frac{1}{100}$

& hence pdf $f(x) = \begin{cases} \frac{1}{100} e^{-x/100} & x \geq 0 \\ 0 & x < 0 \end{cases}$

i) P(component will last atleast for 200 hr)

$= P(x > 200)$

$= e^{-200\lambda} = e^{-2} = 0.1353$

(ii) P(component will last betn 250 & 300 hrs)

$= P(250 < x < 300)$

$= \int_{250}^{300} f(x)\, dx = \int_{250}^{300} \frac{1}{100} e^{-x/100} dx$

$= \frac{1}{100} \left[\frac{e^{-x/100}}{-1/100} \right]_{250}^{300}$

$= -\left(e^{-3} - e^{-2.5} \right)$

$= 0.03229$

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(2) The lifetime of a microprocessor is exponentially dist. with mean 3000 hrs. Find the prob. that
(i) The microprocessor will fail within 300 hrs
(ii) The microprocess will func ~~max~~ for more than 6000 hrs.

→ Here X — lifetime of microprocessor

mean $= \frac{1}{\lambda} = 3000 \Rightarrow \lambda = \frac{1}{3000}$

& pdf $f(x) = \begin{cases} \frac{1}{3000} e^{-x/3000} & x \geq 0 \\ 0 & x < 0 \end{cases}$

i) P(microprocessor will fail within 300 hrs)

$= P(x < 300)$

$= 1 - P[x \geq 300]$

$= 1 - e^{-\lambda 300} = 1 - e^{-1/10} = 0.09516$

(ii) P[microprocessor will func more than 6000 hrs)

$= P(x > 6000)$

$= e^{-6000\lambda} = e^{-2} = 0.1353$

(3) The amount of time that a watch will run without having to be reset is r.v having an ED with mean 120 days. Find prob. that such a watch will
(i) have to be set is less than 24 days.
(ii) not have to be reset in atleast 180 days.

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$\rightarrow$ X - amount of time that a watch will run without having to be reset.

& mean $= \frac{1}{\lambda} = 120 \Rightarrow \lambda = \frac{1}{120}$.

& pdf

$$f(x) = \begin{cases} \frac{1}{120} e^{-x/120} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(i) $P(x<24) = \int_0^{24} f(x)\,dx$

$$= \int_0^{24} \frac{1}{120} e^{-x/120}\,dx$$

$$= \frac{1}{120}\left[\frac{e^{-x/120}}{-1/120}\right]_0^{24} = -\left(e^{-1/5} - 1\right)$$

$$= 1 - e^{-0.2}$$

$$= 0.1813$$

(ii) $P(x>180) = e^{-\lambda 180} = e^{-\frac{180}{120}} = e^{-1.5}$

$$= 0.2231$$

4) If X has exponentially distribution with mean λ. With $P[X \leq 2] = P[X>2]$. find var(X)

$\rightarrow$ X has exponentially dist with mean λ. so its pdf is

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda} \quad x \geq 0$$

We have X exponentially distributed with mean $1/\lambda$.

$f(x) = \lambda e^{-\lambda x}$ $x \geq 0$

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$P[X \le 2] = 1 - P[X > 2]$

But $P[X \le 2] = P[X > 2]$

$P[X \le 2] = 1 - P[X \le 2]$

$2 P[X \le 2] = 1$

$\Rightarrow P[X \le 2] = \frac{1}{2}$

$$\Rightarrow \int_0^2 f(x)\, dx = \frac{1}{2}$$

$$\Rightarrow \int_0^2 \frac{1}{\lambda} e^{-x/\lambda} dx = \frac{1}{2} \Rightarrow \frac{1}{\lambda}\left[\frac{e^{-x/\lambda}}{-1/\lambda}\right]_0^2 = \frac{1}{2}$$

$$= -\left[e^{-2/\lambda} - 1\right] = \frac{1}{2}$$

$$= 1 - e^{-2/\lambda} = \frac{1}{2}$$

$$\Rightarrow e^{-2/\lambda} = \frac{1}{2}$$

$$\Rightarrow \frac{-2}{\lambda} = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{2}{\lambda} = \ln(2)$$

$$\Rightarrow \lambda = \frac{\ln(2)}{2} \text{ - mean.}$$

$$\text{Variance} = \lambda^2 = \frac{(\ln 2)^2}{4}.$$

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(5) Let the milage (in thousands of miles) of a particular tyre to be r.v X having pdf

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20} & x > 0 \\ 0 & x \le 0 \end{cases}$$

find the prob. that one of these tyres will last (a) atmost 10,000 miles

(b) anywhere from 16000 to 24000 miles

(c) atleast 30,000 miles.

(d) Find the mean.

(e) Find the variance.

→

(a) $P(x < 10) = \int_0^{10} f(x)\,dx = \int_0^{10} \frac{1}{20} e^{-x/20}\,dx$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_0^{10}$$

$$= -\left[e^{-2} - 1\right] = 1 - e^{-1/2}$$

$$= 0.3934$$

(b) $P(16 \le x \le 24) = \int_{16}^{24} f(x)\,dx$

$$= \int_{16}^{24} \frac{1}{20} e^{-x/20}\,dx = \frac{1}{20}\left(\frac{e^{-x/20}}{-1/20}\right)_{16}^{24}$$

$$= -\left[e^{-6/5} - e^{-4/5}\right] = e^{-4/5} - e^{-6/5}$$

$$= 0.148$$

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c] $p(x>30) = e^{-30\lambda} = e^{-1.5} = 0.2231$

d] mean $= \frac{1}{\lambda} = \int_0^{\infty} x f(x)\, dx$

$$= \int_0^{\infty} x \cdot \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{20} \cdot \frac{\sqrt{2}}{(1/20)^2} = 20$$

e] variance $= E(X^2) - [E(X)]^2$ — (*)

$$E(X^2) = \int_0^{\infty} x^2 f(x)\, dx = \int_0^{\infty} x^2 \cdot \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{20} \frac{\sqrt{3}}{(1/20)^3} = \frac{(20^2)(2)}{\cancel{3}}$$

$$\sigma^2 = \frac{(2)(20)^2}{\cancel{4}} - (20)^2 = (20)^2 = 400$$

~~The length of time for one person to be served at cafeteria is a r.v X having exponentially distribution with mean of 4 min.~~

(6) suppose the life of automobile batteries is exponentially distributed with mean 1000 days

i) What is the prob. that such a battery will last more than 1200 days.

ii) What is prob. that battery will last more than 1200 days given that it has already served 1000 days.

→

X – life of automobile batteries.

$$\text{mean} = \frac{1}{\lambda} = 1000 \Rightarrow \lambda = \frac{1}{1000}$$

& pdf

$$f(x) = \begin{cases} \frac{1}{1200} e^{\frac{x}{1200}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(i) $P(x > 1200) = e^{-1200\lambda} = e^{-1.2} = 0.30119$

(ii) $P\left[x > 1200 \mid x > 1000\right] = P\left[x > 1000 + 200 \mid x > 1000\right]$

$= P\left[x > 200\right]$

(lack of memory property)

$= e^{-200\lambda}$

$= e^{-0.2} = 0.81873$