2-3.8 Partial Ordering

Definition 2-3.16 A binary relation R in a set P is called a *partial order relation* or a *partial ordering* in P iff R is reflexive, antisymmetric, and transitive.

It is conventional to denote a partial ordering by the symbol $\leq$. This symbol does not necessarily mean "less than or equal to" as is used for real numbers. Since the relation of partial ordering is reflexive, we shall henceforth call it a relation on a set, say P. If $\leq$ is a partial ordering on P, then the ordered pair $\langle P, \leq \rangle$ is called a *partially ordered set* or a *poset*.

Definition 2-3.17 Let $\langle P, \leq \rangle$ be a partially ordered set. If for every $x, y \in P$ we have either $x \leq y \lor y \leq x$, then $\leq$ is called a *simple ordering* or *linear ordering* on P, and $\langle P, \leq \rangle$ is called a *totally ordered* or *simply ordered set* or a *chain*.

Note that it is not necessary to have $x \leq y$ or $y \leq x$ for every x and y in a partially ordered set P. In fact, x may not be related to y, in which case we say that x and y are *incomparable*.

If R is a partial ordering on P, then it is easy to see that the converse of R, namely $\tilde{R}$, is also a partial ordering on P. If R is denoted by $\leq$, then $\tilde{R}$ is denoted by $\geq$. This means that if $\langle P, \leq \rangle$ is a partially ordered set, then $\langle P, \geq \rangle$ is also a partially ordered set. $\langle P, \geq \rangle$ is called the dual of $\langle P, \leq \rangle$.

We now define another relationship which is associated with every partial ordering $\leq$ on P and which is denoted by $<$. This relation $<$ is defined, for every $x, y \in P$, as

$$x < y \Leftrightarrow x \leq y \land x \neq y$$

Similarly, corresponding to the converse partial ordering $\geq$, there is a relation $>$ such that

$$x > y \Leftrightarrow x \geq y \land x \neq y$$

Note that the relations $<$ and $>$ are antisymmetric and transitive. In addition, these relations are irreflexive. We now give some partial order relations which are frequently used.

1 Less Than or Equal to, Greater Than or Equal to: Let R be the set of real numbers. The relation "*less than or equal to*," or $\leq$, is a partial ordering on R. The converse of this relation, "*greater than or equal to*," or $\geq$, is also a partial ordering on R. Associated relations are "*less than*," or $<$, and "*greater than*," or $>$, respectively.

2 Inclusion: Let $\rho(A) = 2^A = X$ be the power set of A, that is, X is the set of subsets of A. The relation of inclusion ($\subseteq$) on X is a partial ordering. Associated with the relation $\subseteq$ is a relation called proper inclusion ($\subset$) which is irreflexive, antisymmetric, and transitive.

As a special case, we let $A = \{a, b, c\}$. Then

$$X = \rho(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$$

It is easy to write the elements of the relation $\subseteq$. Note that $\{a\}$ and $\{b, c\}$, $\{a, b\}$ and $\{a, c\}$, etc., are incomparable.

3 Divides and Integral Multiple: If a and b are positive integers, then we say "a divides b," written $a \mid b$, iff there is an integer c such that $ac = b$. Alternatively, we say that "b is an *integral multiple of* a." The relation "*divides*" is a partial order relation. Let X be the set of positive integers. The relations "divides" and "integral multiple of" are partial orderings on X, and each is the converse of the other.

As a special case, let $X = \{2, 3, 6, 8\}$ and let $\leq$ be the relation "divides" on X. Then

$$\leq\ = \{\langle 2, 2\rangle, \langle 3, 3\rangle, \langle 6, 6\rangle, \langle 8, 8\rangle, \langle 2, 8\rangle, \langle 2, 6\rangle, \langle 3, 6\rangle\}$$

The relation "integral multiple of," written as $\geq$, is given by

$$\geq\ = \{\langle 2, 2\rangle, \langle 3, 3\rangle, \langle 6, 6\rangle, \langle 8, 8\rangle, \langle 8, 2\rangle, \langle 6, 2\rangle, \langle 6, 3\rangle\}$$

4 Lexicographic Ordering: A useful example of simple or total ordering is the lexicographic ordering. We shall define it for certain ordered pairs first and then generalize it.

Let R be the set of real numbers and let $P = R \times R$. The relation $\geq$ on R is assumed to be the usual relation of "greater than or equal to." For any two ordered pairs $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ in P, we define the total ordering relation S as follows:

$$\langle x_1, y_1\rangle\, S\, \langle x_2, y_2\rangle \Leftrightarrow (x_1 > x_2) \vee ((x_1 = x_2) \wedge (y_1 \geq y_2))$$

It is clear that if $\langle x_1, y_1\rangle\, \not S\, \langle x_2, y_2\rangle$, then we must have $\langle x_2, y_2\rangle\, S\, \langle x_1, y_1\rangle$, so that S is a total ordering on P. The partial ordering S is called the *lexicographic ordering*. The significance of the terminology will become clear after we generalize the above ordering relation. The following are some of the ordered pairs of P which are S-related:

$$\langle 2, 2\rangle\, S\, \langle 2, 1\rangle$$
$$\langle 3, 1\rangle\, S\, \langle 1, 5\rangle$$
$$\langle 2, 2\rangle\, S\, \langle 2, 2\rangle$$
$$\langle 3, 2\rangle\, S\, \langle 1, 1\rangle$$

We now generalize this concept. For this purpose, let R be a total ordering relation on a set X and let

$$P = X \cup X^2 \cup X^3 \cup \cdots \cup X^n = \bigcup_i X^i \qquad (n = 1, 2, 3, \ldots)$$

This equation means that the set P consists of strings of elements of X of length less than or equal to n. We may assume some fixed value of n. A string of length p may be considered as an ordered p-tuple. We now define a total ordering S on P called lexicographic ordering. For this purpose, let $\langle u_1, u_2, \ldots, u_p\rangle$ and $\langle v_1, v_2, \ldots, v_q\rangle$, with $p \leq q$, be any two elements of P. Note that before starting, to compare two strings to determine the ordering in P, the strings are interchanged if necessary so that $p \leq q$. Now

$$\langle u_1, u_2, \ldots, u_p\rangle\, S\, \langle v_1, v_2, \ldots, v_q\rangle$$

if any one of the following holds:

1. $\langle u_1, u_2, \ldots, u_p\rangle = \langle v_1, v_2, \ldots, v_p\rangle$
2. $u_1 \neq v_1$ and $u_1 \; R \; v_1$ in X
3. $u_i = v_i$, $i = 1, 2, \ldots, k$ $(k < p)$, and $u_{k+1} \neq v_{k+1}$ and $u_{k+1} \; R \; v_{k+1}$ in X

If none of these conditions is satisfied, then

$$\langle v_1, v_2, \ldots, v_q\rangle \; S \; \langle u_1, u_2, \ldots, u_p\rangle$$

As a special case of lexicographic ordering, let $X = \{a, b, c, \ldots, z\}$ and let R be a simple ordering on X denoted by $\leq$ where $a \leq b \leq c \leq \cdots \leq z$ and $P = X \cup X^2 \cup X^3$. Thus, P consists of all "words" or strings of 3 or fewer than 3 letters from X. Let S denote the lexicographic ordering on P described earlier. We will have

me S met	by condition 1
bet S met	by condition 2
beg S bet	by condition 3
get S go	by the last rule

since "go" and "get" are compared and the conditions 1, 2, and 3 are not satisfied.

The order in which the words in an English dictionary appear is a familiar example of lexicographic ordering. Instead of using S to denote the lexicographic ordering, it is customary to use names such as "lexically less than or equal to" or "lexically greater than."

We shall now describe how the lexicographic ordering is used in sorting character data on a computer. For this purpose, let X denote the set of characters available on a particular computer. It is necessary first to define a simple ordering on the elements of X (frequently called the collating sequence). One method is to compare the numeric values of the coded representation of each character in the computer by using the relation "less than or equal to." This ordering may vary from one computer to another. An example of a code which has such an ordering is the Extended Binary Coded Decimal Interchange Code (EBCDIC). In any case, we have a totally ordered set $\langle X, \leq\rangle$, and character strings are formed from the elements of X. Since blanks are also permitted to appear in such strings, a blank is treated as a character, i.e., an element of X. It is convenient to assume that a blank is less than all other elements of X. Not only do blanks appear inside a string, but sometimes it will be convenient to add blanks at the end of a string. It will be assumed that such additions do not alter the relative ordering of a string.

Now we consider how two given strings of equal length are compared for the purpose of ordering them lexicographically. If one string is shorter than the other, we simply assume that it is padded at the right end (because we shall assume the scanning is done from left to right) with the number of blanks sufficient that both strings to be compared are of equal length. In some cases, it may be necessary to distinguish between a given string and the one to which some blanks are added. This distinction can be made by comparing the strings for lexical equality and then comparing them according to their lengths.

The primitive algorithm *SORT* which follows is based on lexicographic comparisons.

Algorithm *SORT* Given a vector *NAME* which contains m character strings, it is required to sort these strings into the proper lexicographic order. To assist in this process, the character string variable *TEMP*, the logical variable *FLAG*, and parameters i and j are used.

1 [Repeat for $m - 1$ passes] Repeat steps 2 to 4 inclusive for $i = 1, 2, \ldots, m - 1$ and then Exit.

2 [Scan to element $m - i$] Set $FLAG \leftarrow T$. Repeat step 3 for $j = 1, 2, \ldots, m - i$.

3 [Exchange required?] If $NAME[j] > NAME[j + 1]$ then set $TEMP \leftarrow NAME[j]$, $NAME[j] \leftarrow NAME[j + 1]$, $NAME[j + 1] \leftarrow TEMP$ and $FLAG \leftarrow F$.

4 [Pass without exchange?] If *FLAG* then Exit.

The purpose of this algorithm is to sort the m character sequences in *NAME* into an ascending lexicographic order; that is, $NAME[j]$ must not be lexically greater than $NAME\ [j + 1]$ for $j = 1, 2, \ldots, m - 1$. When i is equal to 1, all adjacent elements are compared and those which are out of order are exchanged. When this comparison is done for elements j and $j + 1$, with j ranging from 1 to $m - i$, in this specific case we are assured that element m will contain the lexically greatest string. When i is equal to 2, the pass will be completed with the second greatest string in position $m - 1$ of *NAME*. A maximum of $m - 1$ such passes are performed, and on the ith pass, $m - i$ comparisons must be made. Of course, on completion of a pass without any exchanges, all elements are in the proper order and the algorithm is complete, as indicated by *FLAG*.

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2-3.9 Partially Ordered Set: Representation and Associated Terminology

In a partially ordered set $\langle P, \leq \rangle$, an element $y \in P$ is said to *cover* an element $x \in P$ if $x < y$ and if there does not exist any element $z \in P$ such that $x \leq z$ and $z \leq y$; that is,

$$y \text{ covers } x \Leftrightarrow (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$$

Sometimes the term "immediate predecessor" is also used. Note that "cover" as used here should not be confused with the "cover" of a set defined in Sec. 2-3.4.

A partial ordering $\leq$ on a set P can be represented by means of a diagram known as a Hasse diagram or a partially ordered set diagram of $\langle P, \leq \rangle$. In such a diagram, each element is represented by a small circle or a dot. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, and a line is drawn between x and y if y covers x. If $x < y$ but y does not cover x, then x and y are not connected directly by a single line. However, they are connected through one or more elements of P. It is possible to obtain the set of ordered pairs in $\leq$ from

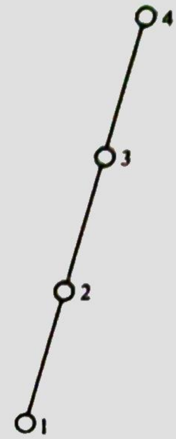

FIGURE 2-3.23 Hasse diagram.

such a diagram. Several examples of partially ordered sets and their Hasse diagrams follow.

For a totally ordered set $\langle P, \leq \rangle$, the Hasse diagram consists of circles, one below the other, as in Fig. 2-3.23. Thus a totally ordered set is called a chain. If we let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation "less than or equal to," then the Hasse diagram is as shown in Fig. 2-3.23.

Consider the set $P = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$ and the relation of inclusion $\subseteq$ on P. The Hasse diagram of $\langle P, \subseteq \rangle$ is similar to that given in Fig. 2-3.23 except that the nodes are relabeled.

The two relations defined above are not equal, but they have the same Hasse diagram. Such situations will be shown to occur frequently, and the reason for these occurrences is explained in Chap. 4 in the discussion of the order isomorphism of two partially ordered sets.

EXAMPLE 1 Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $\langle X, \leq \rangle$.

SOLUTION The Hasse diagram is given in Fig. 2-3.24. ////

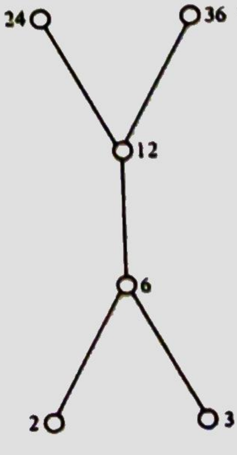

FIGURE 2-3.24 Hasse diagram of divides relation.

EXAMPLE 2 Let A be a given finite set and $\rho(A)$ its power set. Let $\subseteq$ be the inclusion relation on the elements of $\rho(A)$. Draw Hasse diagrams of $\langle \rho(A), \subseteq \rangle$ for (a) $A = \{a\}$; (b) $A = \{a, b\}$; (c) $A = \{a, b, c\}$; (d) $A = \{a, b, c, d\}$.

SOLUTION The required Hasse diagrams are given in Fig. 2-3.25*a* to *d*.

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The following points may be noted about Hasse diagrams in general. For a given partially ordered set, a Hasse diagram is not unique, as can be seen from Fig. 2-3.25*b*. From a Hasse diagram of $\langle P, \leq \rangle$, the Hasse diagram of $\langle P, \geq \rangle$, which is the dual of $\langle P, \leq \rangle$, can be obtained by rotating the diagram through 180° so that the points at the top become the points at the bottom. Some Hasse diagrams have a unique point which is above all the other points, and similarly some Hasse diagrams have a unique point which is below all other points. Such was the case for all the Hasse diagrams given in Example 2, while the Hasse diagram given in Example 1 does not possess this property. The Hasse diagrams become more complicated when the number of elements in the partially ordered set is large.

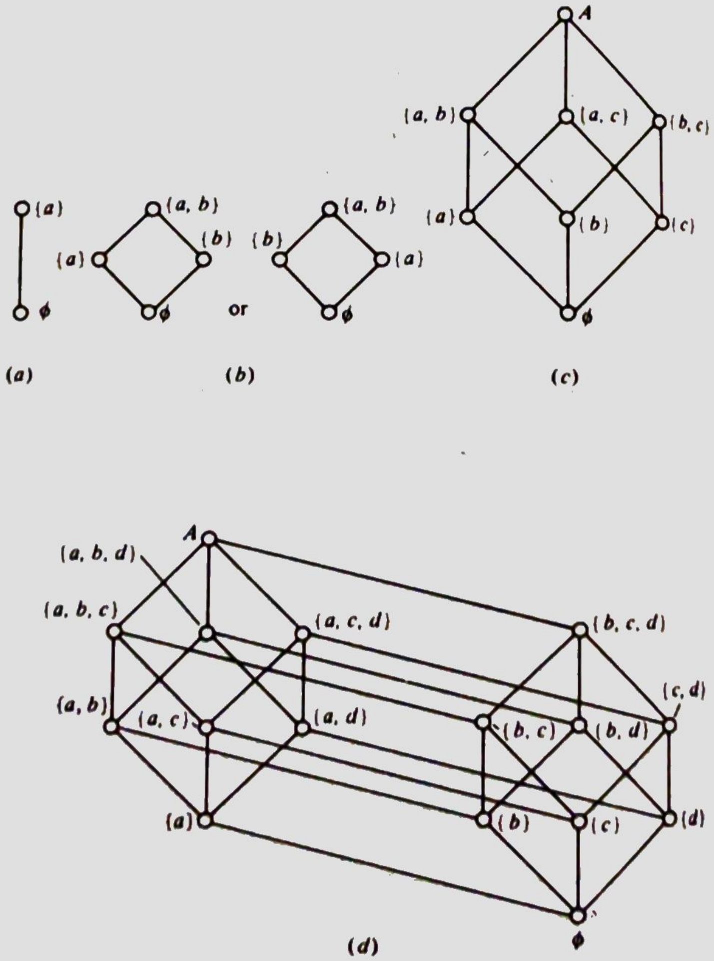

FIGURE 2-3.25 Hasse diagrams of $\langle \rho(A), \subseteq \rangle$.

EXAMPLE 3 Let A be the set of factors of a particular positive integer m and let $\leq$ be the relation divides, i.e.,

$$\leq \; = \{\langle x, y\rangle \mid x \in A \wedge y \in A \wedge (x \text{ divides } y)\}$$

Draw Hasse diagrams for (*a*) $m = 2$; (*b*) $m = 6$; (*c*) $m = 30$; (*d*) $m = 210$; (*e*) $m = 12$; and (*f*) $m = 45$.

SOLUTION The required Hasse diagrams for (*a*) to (*d*) are the same as given in Fig. 2-3.25*a* to *d*. Hasse diagrams of (*e*) and (*f*) are given in Fig. 2-3.26. ////

In Examples 2 and 3 we saw that the Hasse diagrams (*a*) to (*d*) are identical. However, Hasse diagrams (*e*) and (*f*) of Example 3 cannot be given by the Hasse diagram of any power set of a set, because a power set has 2^n elements, while in (*e*) and (*f*) we only have 6 elements in each of the partially ordered sets. Of course, in all the cases given in Example 3 we again have a single element at the top and a single element at the bottom because if p is any divisor of m, we have $1 \leq p \leq m$.

Hasse diagrams can also be drawn for any relation which is antisymmetric and transitive but not necessarily reflexive. Examples of such relations are proper inclusion and any relation $<$ associated with the partial ordering relation $\leq$. Any family tree or organization chart of the military or of any establishment is a Hasse diagram in this sense. We shall, however, assume that a Hasse diagram represents a partial ordering unless otherwise stated. Some Hasse diagrams are given in Fig. 2-3.27.

We shall now introduce terminology for partially ordered sets which will be found useful in Chap. 4. To this end, let $\langle P, \leq\rangle$ denote a partially ordered set.

If there exists an element $y \in P$ such that $y \leq x$ for all $x \in P$, then y is called the *least member* in P relative to the partial ordering $\leq$. Similarly, if there exists an element $y \in P$ such that $x \leq y$ for all $x \in P$, then y is called the *greatest*

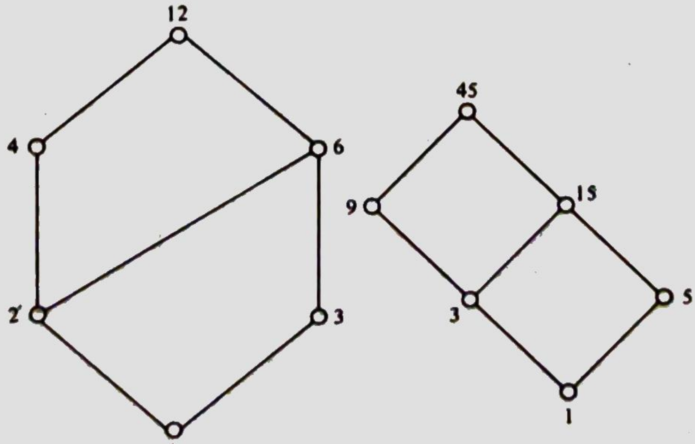

FIGURE 2-3.26

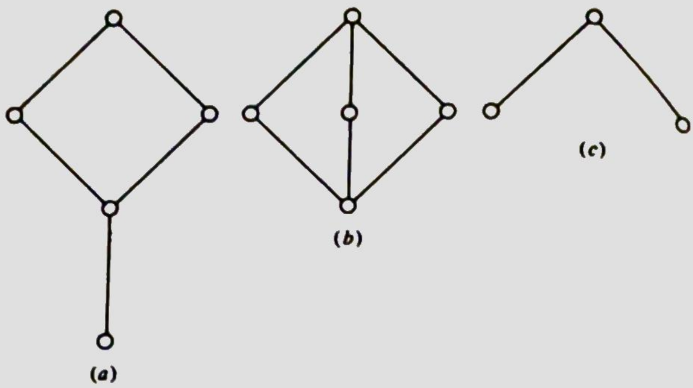

FIGURE 2-3.27

member in P relative to $\leq$. From the definition it is clear that the least member, if it exists, is unique; so also is the greatest member. It may happen that the least or the greatest member does not exist. The least member is usually denoted by 0 and the greatest by 1.

If the Hasse diagram of a partially ordered set is available, then it is easy to see whether the least or the greatest member exists. From Fig. 2-3.23 it is clear that the least member is 1 and the greatest is 4. In Example 1 there is no least or greatest member, while in Example 2 the least member is $\varnothing$ and the greatest member is A in all cases. In every simple ordering or chain, the least and the greatest members always exist. The Hasse diagram of Fig. 2-3.27c shows that the greatest member exists but there is no least member.

An element $y \in P$ is called a *minimal member* of P relative to a partial ordering $\leq$ if for no $x \in P$ is $x < y$. A minimal member need not be unique. All those members which appear at the lowest level of a Hasse diagram of a partially ordered set are minimal members. Similarly, an element $y \in P$ is called a *maximal member* of P relative to a partial ordering $\leq$ if for no $x \in P$ is $y < x$. In the Hasse diagram of Fig. 2-3.27c, there are two minimal members and one maximal member. Distinct minimal members are incomparable, and distinct maximal members are also incomparable.

It is not always necessary to draw the Hasse diagram of a partially ordered set in order to determine the least, greatest, maximal, and minimal members. However, their determination becomes simple when such a diagram is available.

We now extend these ideas to the subsets of a partially ordered set.

Definition 2-3.18 Let $\langle P, \leq \rangle$ be a partially ordered set and let $A \subseteq P$. Any element $x \in P$ is an *upper bound* for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is a *lower bound* for A if for all $a \in A$, $x \leq a$.

Let us consider the partially ordered set $\langle \rho(A), \subseteq \rangle$ in Example 2c. We choose a subset B of $\rho(A)$ given by $\{\{b, c\}, \{b\}, \{c\}\}$. Then $\{b, c\}$ and A are upper bounds for B, while $\varnothing$ is its lower bound. For the subset $C = \{\{a, c\}, \{c\}\}$, the upper bounds are $\{a, c\}$ and A while the lower bounds are $\{c\}$ and $\varnothing$. In Example 1, if $A = \{2, 3, 6\}$, then 6, 12, 24, and 36 are upper bounds for A, and there is no lower bound.

Note that upper and lower bounds of a subset are not necessarily unique. We therefore define the following terms.

Definition 2-3.19 Let $\langle P, \leq \rangle$ be a partially ordered set and let $A \subseteq P$. An element $x \in P$ is a *least upper bound*, or *supremum*, for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A. Similarly, the *greatest lower bound*, or *infimum*, for A is an element $x \in P$ such that x is a lower bound and $y \leq x$ for all lower bounds y.

A least upper bound, if it exists, is unique, and the same is true for a greatest lower bound. The least upper bound is abbreviated as "LUB" or "sup," and the greatest lower bound is abbreviated as "GLB" or "inf."

For a simply ordered set or a chain, every subset has a supremum and an infimum. Similarly, the partially ordered sets given in Examples 2 and 3 are such that every subset has a supremum and an infimum. This, however, is not generally the case, as can be seen from Example 1 in which the set $A = \{2, 3, 6\}$ has the LUB $A = 6$, while the GLB A does not exist. Similarly, for the subset $\{2, 3\}$, the supremum is again 6, but there is no infimum. For the subset $\{12, 6\}$, the supremum is 12 and the infimum is 6. The partially ordered sets which are such that every subset has a supremum and an infimum form an important subclass of partially ordered sets. Such sets are discussed in Chap. 4.

For a partially ordered set $\langle P, \leq \rangle$, we know that its dual $\langle P, \geq \rangle$ is also a partially ordered set. The least member of P relative to the ordering $\leq$ is the greatest member in P relative to the ordering $\geq$, and vice versa. Similarly, the maximal and minimal elements are interchanged. For any subset $A \subseteq P$, the GLB A in $\langle P, \leq \rangle$ is the same as the LUB A in $\langle P, \geq \rangle$.

We shall end this section by defining a property which has important applications in the use of the principle of transfinite induction.

✓**Definition 2-3.20** A partially ordered set is called *well-ordered* if every nonempty subset of it has a least member.

As a consequence of this definition, it follows that every well-ordered set is totally ordered, because for any subset, say $\{x, y\}$, we must have either x or y as its least member. Of course, every totally ordered set need not be well-ordered. A finite totally ordered set is also well-ordered.

A simple example of a well-ordered set is the set $\mathbf{I}_n = \{1, 2, \ldots, n\}$ or the set $\mathbf{I} = \{1, 2, \ldots\}$. Similarly the sets $\mathbf{I}_n \times \mathbf{I}_n$ or $\mathbf{I} \times \mathbf{I}$ are well-ordered under the natural ordering of "less than or equal." It is possible, however, to define a certain partial ordering on $\mathbf{I} \times \mathbf{I}$ such that it is no longer a well-ordered set.

EXERCISES 2-3.9

1 Draw the Hasse diagrams of the following sets under the partial ordering relation "divides," and indicate those which are totally ordered.

$\{2, 6, 24\}$ $\quad$ $\{3, 5, 15\}$ $\quad$ $\{1, 2, 3, 6, 12\}$ $\quad$ $\{2, 4, 8, 16\}$ $\quad$ $\{3, 9, 27, 54\}$

2 If R is a partial ordering relation on a set X and $A \subseteq X$, show that $R \cap (A \times A)$ is a partial ordering relation on A.