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Distributions.

## Binomial Distribution :-

consider an experiment which results in either success or failure. Let it be repeated n times, the probability p of success remaining constant every time and let $q = 1-p$, the probability of failure

then probability of r success and hence $(n-r)$ failures in a trial in a particular order say as given by multiplication theorem on Prob. is

$$= ppp \ldots r \text{ times} \times qq \ldots (n-r) \text{ times}$$

$$= p^r q^{n-r}$$

but r success can occur in $nC_r$ ways and the probability of each of these ways is the $p^r q^{n-r}$

Hence. $P(X=r) = nC_r \, p^r q^{n-r}$

**Definition** :- A random variable is said to follow Binominal distribution if probability of x is given by

$$P(X=r) = nC_r \, p^r q^{n-r}, \quad r = 0, 1, 2, 3 \ldots$$

The two constants n and p are called the parameters of the distribution

* When do we get binomial distribution

→ We get a binomial distribution when the following conditions are satisfied

① A trial is repeated n times where n is finite number

② ~~Each~~ Each trial results only in two ways success or failure

③ These possibilities are mutually exclusive exhaustive but not necessarily equally likely

④ If p and q are the probabilities of success and failure then $p+q=1$

⑤ The events are independent i.e. the probability p of success in each trial remains constant in all trials.

* Uses : we can use binomial distribution when these conditions are satisfied. Thus in problems involving (i) the tossing of a coin heads or tails
ii) The result of an examination sucess or failure
iii) The result of election success or failure

* Properties of the binomial distribution

(i) If x denotes binomial variate, the mean of the distribution is given by

$$\boxed{\bar{x} = n \cdot p}$$

ii) The variance of the distribution is npq

$$\boxed{V(x) = npq}$$

iii) If the experiment, each of n trials, is repeated N times then the expectation of r successes i.e. the expected frequency of r successes in N experiments is $N \cdot {}^nC_r \, p^r q^{n-r}$

$$\text{Freq}^n = N \cdot P(X=r) = N \cdot {}^nC_r \, p^r q^{n-r}$$

Remark : For a binomial variate $\frac{V(x)}{\bar{x}} = \frac{npq}{np}$

and since $q<1$ variance of x cannot be greater than the mean

Ex. ① Find the mean of the probability distribution of the number of heads obtained in three flips of a balanced coin

Sol$^n$:- We have $p = \frac{1}{2}$ $n=3$

$$\text{Mean} = n \cdot p = 3 \times \frac{1}{2} = 1.5$$

Ex. ② If x is binomial variate mean is 2 & variance is 4/3, find the ~~probability~~ constants of the distribution.

$$\bar{x} = n \cdot p \qquad V(x) = npq$$

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$$\frac{npq}{np} = \frac{4/3}{2} \Rightarrow q = \frac{2}{3}$$

$$p = 1 - q = \frac{1}{3}$$

$$np = 2 \quad \therefore \quad n \cdot \frac{1}{3} = 2 \Rightarrow n = 6$$

Ex. ③ Prove that for all Binomial distributions with the same parameter $n$, the variance is maximum when $p = 1/2$

Sol$^n$:- We have the binomial distribution

$$P(x) = {}^nC_r\, p^r q^{n-r}$$

Variance is given by $V = npq$

where $q = 1 - p$

$$\therefore V = np(1-p)$$

$$= np - np^2$$

For maxima $\frac{dv}{dp} = 0$ & $\frac{d^2v}{dp^2} = -ve$

Now $\frac{dv}{dp} = n - 2np$ & $\frac{d^2v}{dp^2} = -2n$ always $-ve$

$\therefore \frac{dv}{dp} = 0$ gives $n - 2np = 0 = n(1-2p) = 0$

but $n \neq 0$ $\quad 1 - 2p = 0 \quad \therefore p = \frac{1}{2}$

Hence, the variance is maximum when $p = \frac{1}{2}$

Ex. (3) Find the binomial distribution if the mean is 4 and variance is 3

Sol$^n$. We have mean $= n \cdot p = 4$ & variance $= npq = 4$

$\therefore \frac{np}{npq} = \frac{4}{3}$ $\therefore \frac{1}{q} = \frac{4}{3}$ $\therefore q = 3/4$

$P = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$np = 4 \Rightarrow n \cdot \frac{1}{4} = 4 \Rightarrow n = 16$

$P(X = r) = {}^nC_r \, p^r q^{n-r}$

$= {}^{16}C_r \left(\frac{1}{4}\right)^r \cdot \left(\frac{3}{4}\right)^{16-r}$

Ex (5) It has been claimed that in 60% of all Solar heat installations, the utility bill is reduced by at least one third. Accordingly what are the probabilities that the utility bill will be reduced at least one third in (i) Four of five installations (ii) atleast four of five installations.

→ For a binomial distribution $P(X = r) = {}^nC_r \, p^r q^{n-r}$

We have $n = 5$ $r = 4$ $p = 0.6$, $q = 0.4$

(i) $P(X = 4) = {}^5C_4 (0.6)^4 (0.4)^1 = 0.259$

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(ii) P(atleast 4 of 5 installations)

$= P(x \geq 4) = P(x=4) + P(x=5)$

$= 5C_4 (0.6)^4 (0.4)^1 + 5C_5 (0.6)^5 (0.4)^0$

$= 0.259 + 0.078$

$= 0.337$

Ex ⑥ The probability that a bomb dropped from a plane will strike the target is 1/5. If six such a bombs are dropped find the probability that

i) exactly two bombs hit the target

ii) at least two will hit the target

→ We have $p = \frac{1}{5}$, $q = \frac{4}{5}$, $n = 6$

By binomial distribution

$P(X=r) = nC_r \, p^r \, q^{n-r} = 6C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{6-r}$

$P(X=2) = 6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$

∴ ~~B(6,2)~~

$= 0.24576$

~~P(At least two)~~ $= 1 - [P(x=1) + P(x=0)]$

$= 1 - [6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5 + 6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^6$

$= \cancel{0.3932}$

$= \cancel{0.6068}$

$= 1 -$

$$P(\text{at least two}) = 1 - P(X=1)$$
$$= 1 - 6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5$$
$$= 1 - 0.3932$$
$$= 0.6068$$

Ex. (7) The probability that at any moment one telephone line out of 10 will be busy is 0.2

(i) What is the probability that 5 lines are busy
ii) Find the expected number of busy lines and also find the probability of this number
iii) What is the probability that all lines are busy?

Sol$^n$:- We have $P = 0.2$, $q = 0.8$, $n = 10$
By Binomial distribution.

$$P(X=r) = nC_r \, p^r \, q^{n-r}$$

$$\therefore P(X=r) = 10C_r (0.2)^r (0.8)^{10-r}$$

(i) $P(X=5) = 10C_5 (0.2)^5 (0.8)^{0.5}$

(ii) Expected number of busy lines $= \text{mean} = n \cdot p$
$$= 10 \times \frac{2}{10} = 2$$

$$P(X=2) = 10C_2 (0.2)^2 (0.8)^{0.8}$$
$$=$$

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(iii) Probability of all lines busy

$P(X=10) = 10C_{10}\ (0.2)^{10}\ (0.8)^0$

$= (0.2)^{10}$

Ex. ⑧ out of 1000 families of 3 children each, how many would you expect to have 2 boys and 1 girl

$P(\text{getting boy}) = p = \frac{1}{2}$

$P(\text{getting girl}) = q = \frac{1}{2}$

$n = 3, \quad r = 2$

$P(2 \text{ boys and } 1 \text{ girl}) = 3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) = \frac{3}{8}$

∴ Expected number of families.

$= N \cdot P$

$= 1000 \times \frac{3}{8} = 375$

Ex. ⑨ Seven coins are tossed and the number of heads obtained is noted. The experiment is repeated 128 times and the following distribution is obtained

No. of heads : 0, 1, 2, 3, 4, 5, 6, 7 Total

Frequency : 7, 6, 19, 35, 30, 23, 7, 1 128

Fit a binomial distribution if

i) the coins are unbiased
ii) if the nature of the coin is not known

Sol$^n$. To fit a distribution to given data means to find the constants of the distribution which will adequately describe the given situation

(i) When the coins are unbiased

$p = \frac{1}{2}, \quad q = \frac{1}{2} \quad n = 7$

$P(X = r) = 7C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{7-r}$

put $r = 0, 1, 2, 3 \dots 7$ we get

$P(0) = \frac{1}{2^7}$

$P(1) = \frac{7}{2^7}$

$P(2) = \frac{21}{2^7} \quad \dots\dots \quad P(7) = ?$ (Find)

Expected frequency $= N \cdot P \quad N = 128$

Multipling the above probabilities by 128 i.e. by $2^7$ we get the expected frequencies as

1, 7, 21, 35, 35, 21, 7, 1

(e.g. $N \cdot P(0) = 2^7 \times \frac{1}{2^7} = 1$)

ii) When the nature of the coin is not known

We have

$$\bar{X} = \frac{\Sigma f_i x_i}{N} = \frac{0\times7 + 1\times6 + 2\times19 + \cdots 7\times1}{128}$$

$$= \frac{433}{128} = 3.38$$

But $\bar{x} = n \cdot p$

$\therefore p = \frac{\bar{x}}{n} = \frac{3.38}{7} = 0.48$

$q = 1 - p = 0.52$

$P(X=r) = 7C_r (0.48)^r (0.52)^{7-r}$

putting $r = 0, 1, 2, 3 \cdots 7$ we get

$P(0) = 0.01$, $P(1) = 0.066$, $P(2) = 0.184 \cdots$

Multiply these probabilities by 128 we get expected frequencies as

1, 8, 23, 36, 33, 18, 6, 3

total should be $\underline{\underline{128}}$

Ex. (10) In a hurdle race a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$ what is the probability that he will knock down less than two hurdles?

$n = 10$, $q = \frac{5}{6}$, $p = 1 - q = 1 - \frac{5}{6}$

$$P(X<2) = P(X=0) + P(X=1)$$

$$P(X<2) = {}^{10}C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^9$$

$$P(X<2) = 0.1615 + 0.3230$$

$$P(X<2) = \underline{\underline{0.4845}}$$

Ex. (11) 10 % of the tools produced in a certain manufacturing process turn out to be defective.

(a) Find the probability that in a sample of 10 tools chosen at random exactly two will be defective

(b) Find the probability that out of 20 tools selected at random there are (i) exactly two defectives (ii) at least two defective

→ $P = 0.1$ $\quad q = 0.9$

(a) $n = 10$

$$P(X=2) = {}^{10}C_2 (0.1)^2 (0.9)^8$$

$$= 0.1937$$

(b) $n = 20$

$P = 0.1,\ q = 0.9$

(i) $P(X=2) = {}^{20}C_2 (0.1)^2 (0.9)^{18}$
$= 0.2852$

(ii) $P(\text{At least two defective}) = 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[{}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19}\right]$

$= 1 - \cancel{(0.0348\ldots)}\ 0.3916$
$= 0.6084$

Ex. (12) If hens of a certain breed lay eggs on 5 days a week on an average, find on how many days during season of 100 days, a poultry keeper with 5 hens of this breed will expect to receive at least 4 eggs

Sol$^n$: Probability of an hen laying an egg

$p = 5/7$

$q = 1 - p = \frac{2}{7}$ (not laying an egg)

$P(X=r) = {}^nC_r\, p^r q^{n-r}$ $\quad n=5,\ p=\frac{5}{7},\ q=\frac{2}{7}$

$P(x \geq 4) = P(X=4) + P(X=5)$

$= {}^5C_4 \left(\frac{5}{7}\right)^4 \left(\frac{2}{7}\right)^1 + {}^5C_5 \left(\frac{5}{7}\right)^5 \left(\frac{2}{7}\right)^0$

$= 0.5578$

Expectation $= N \cdot P = 100 \times 0.5578$

$= 55.78 = 56$