4 Show that $A \subseteq B \Leftrightarrow A \cup B = B$.
5 If $S = \{a, b, c\}$, find nonempty disjoint sets A_1 and A_2 such that $A_1 \cup A_2 = S$. Find other solutions to this problem.
6 Prove the equalities in Eqs. (4) and (5).
7 Given $A = \{2, 3, 4\}$, $B = \{1, 2\}$, and $C = \{4, 5, 6\}$, find $A + B$, $B + C$, $A + B + C$, and $(A + B) + (B + C)$.

2-1.5 Venn Diagrams

Introduction of the universal set permits the use of a pictorial device to study the connection between the subsets of a universal set and their intersection, union, difference, and other operations. The diagrams used are called Venn diagrams. A *Venn diagram* is a schematic representation of a set by a set of points. The universal set E is represented by a set of points in a rectangle (or any other figure), and a subset, say A, of E is represented by the interior of a circle or some other simple closed curve inside the rectangle. In Fig. 2-1.1 the shaded areas represent the sets indicated below each figure. The Venn diagram for $A \subseteq B$ and $A \cap B = \varnothing$ are also given. From some of the Venn diagrams it is

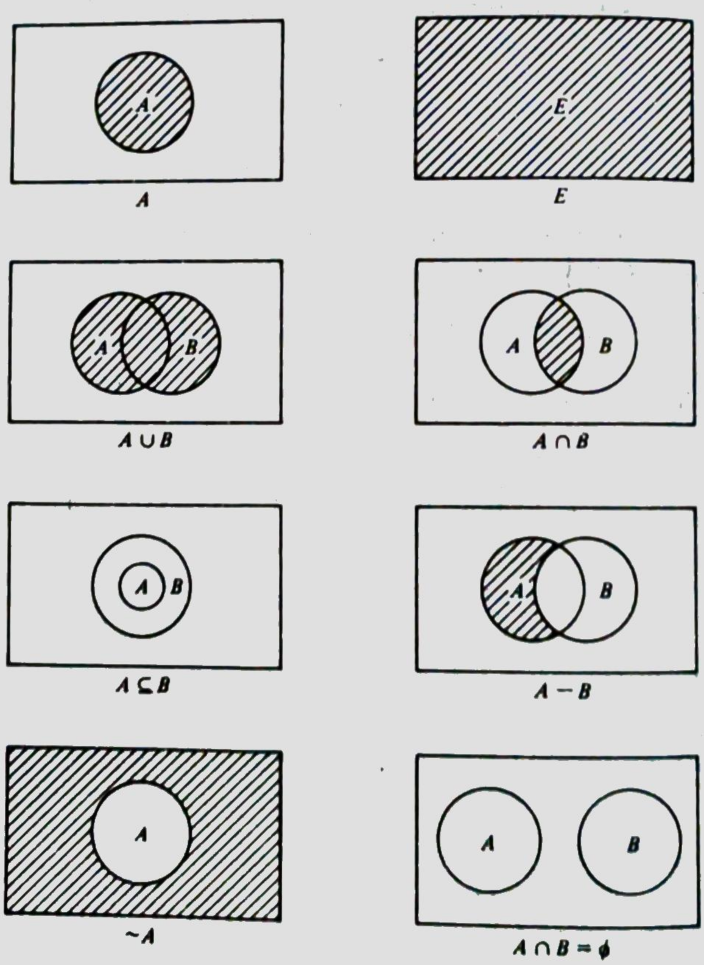


FIGURE 2-1.1 Venn diagrams.

easy to see that the following hold:

$$A \cup B = B \cup A \qquad A \cap B = B \cap A \qquad \sim(\sim A) = A$$

Furthermore, if $A \subseteq B$, then

$$A - B = \varnothing \qquad A \cap B = A \qquad \text{and} \qquad A \cup B = B$$

It should be emphasized that the above relations between the subsets are only suggested by the Venn diagram. Venn diagrams do not provide proofs that such relations are true in general for all subsets of E. We shall demonstrate this point by a particular example.

Consider the Venn diagrams given in Fig. 2-1.2. From the first two Venn diagrams it appears that

$$A \cup B = (A \cap \sim B) \cup (B \cap \sim A) \cup (A \cap B) \tag{1}$$

From the third Venn diagram it appears that

$$A \cup B = (A \cap \sim B) \cup (B \cap \sim A)$$

This equality is not true in general, although it happens to be true for the two disjoint sets A and B.

A formal proof of Eq. (1) will now be outlined. For any x,

$$x \in A \cup B \Leftrightarrow x \in \{x \mid x \in A \vee x \in B\}$$

$$\begin{aligned}
&x \in (A \cap \sim B) \cup (B \cap \sim A) \cup (A \cap B)\\
&\quad \Leftrightarrow x \in \{x \mid x \in (A \cap \sim B) \vee x \in (B \cap \sim A) \vee x \in (A \cap B)\}\\
&\quad \Leftrightarrow x \in (A \cap \sim B) \vee x \in (B \cap \sim A) \vee x \in (A \cap B)\\
&\quad \Leftrightarrow (x \in A \wedge x \in \sim B) \vee (x \in B \wedge x \in \sim A) \vee (x \in A \wedge x \in B)\\
&\quad \Leftrightarrow (x \in A \wedge (x \in \sim B \vee x \in B)) \vee (x \in B \wedge x \in \sim A)\\
&\quad \Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in \sim A)\\
&\quad \Leftrightarrow (x \in A \vee x \in B)\\
&\quad \Leftrightarrow x \in \{x \mid x \in A \vee x \in B\}\\
&\quad \Leftrightarrow x \in A \cup B
\end{aligned}$$

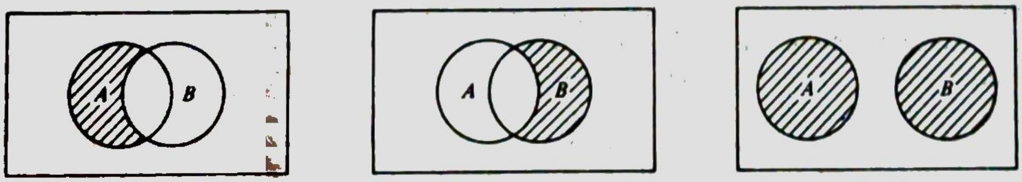
Consider the Venn diagrams in Fig. 2-1.3. From the third and fifth Venn diagrams it appears that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{2}$$

Similarly, one can show that for any three sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{3}$$

Equations (2) and (3) are known as the *distributive laws of union and intersection.*



$A - B = A \cap \sim B$ $\qquad$ $B - A = B \cap \sim A$ $\qquad$ $A \cup B$

FIGURE 2-1.2

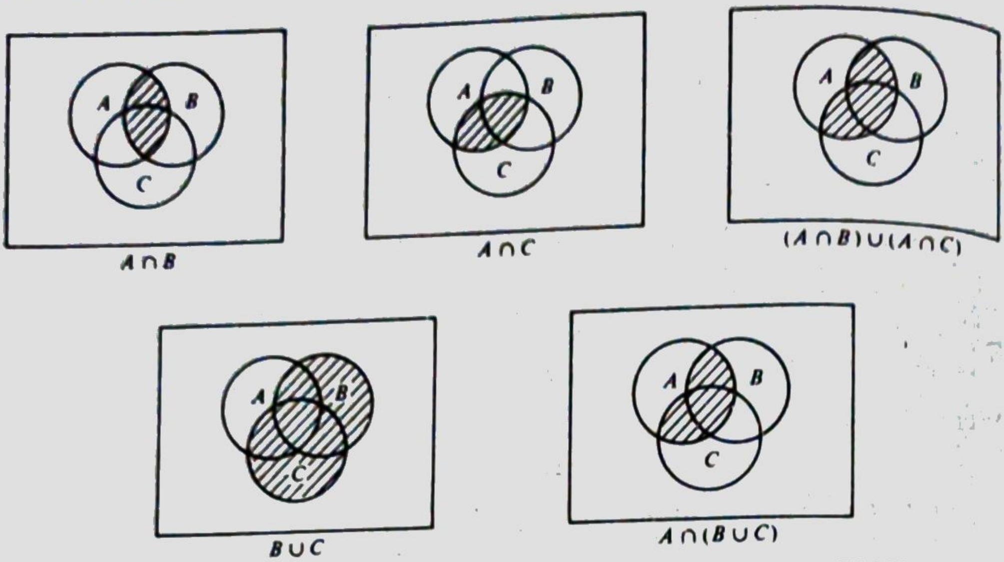


FIGURE 2-1.3

We shall now prove Eq. (3). For any x,

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in \{x \mid x \in A \vee x \in (B \cap C)\} \\ &\Leftrightarrow x \in \{x \mid x \in A \vee (x \in B \wedge x \in C)\} \\ &\Leftrightarrow x \in \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

EXERCISES 2-1.5

1 Prove Eq. (2).
2 Draw a Venn diagram to illustrate Eq. (3).

2-1.6 Some Basic Set Identities

Set operations such as union, intersection, complementation, etc., have been defined. With the help of these operations one can construct new sets from given sets. Capital letters have been used to denote definite sets. These letters have also been used as set variables. This practice is similar to the one employed in the statement calculus. Capital letters such as A, B, C, ... are used as set variables; they are not exactly sets, but *set formulas*. The operations on sets can also be extended to set formulas, so that $A \cup B$, $A \cap B$, $\sim A$, etc., are all set formulas. Any well-formed string involving set variables, the operations $\cap$, $\cup$, $\sim$, and parentheses is a set formula which will also be called a set for the sake of brevity. In fact, one obtains a set from a set formula by replacing the variables by definite sets. Two set formulas are said to be equal if they are equal as sets whenever the set variables appearing in both the formulas are replaced by any sets. It is assumed that any particular variable is replaced by the same set throughout both formulas. Since the equality of set formulas does not depend upon the sets which replace the variables, these equalities are called set *identities*. Some of the basic identities describe certain properties of the operations involved and are given special names. These properties describe an algebra called set algebra. We shall see in Chap. 4 that both the statement algebra and the set algebra are particular cases of an abstract algebra called a Boolean algebra. This fact also explains why one could see similarities between the operators in the statement calculus and the operations of set theory. For all the identities listed in this section, we have also listed the corresponding equivalences from the statement calculus. Similar equivalences hold for the predicate calculus.

Not all the identities listed here are independent. Some of the identities can be proved by assuming certain other identities. However, we have listed these identities in order to include all those identities which exhibit some basic and useful properties. Most of these identities have been proved earlier in this section, and the others can be proved either by using the same technique or by using the identities already known to be true.

In our discussion here we assume that all the sets involved are subsets of a universal set E. Although such an assumption is not necessary for many of the identities, there is no loss of generality. Furthermore, some of the identities do require such an assumption, particularly those involving complementation.

Set Algebra	*Statement Algebra*	
	Idempotent laws	
$A \cup A = A$	$P \vee P \Leftrightarrow P$	(1)
$A \cap A = A$	$P \wedge P \Leftrightarrow P$	
	Associative laws	
$(A \cup B) \cup C = A \cup (B \cup C)$	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	(2)
$(A \cap B) \cap C = A \cap (B \cap C)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	
	Commutative laws	
$A \cup B = B \cup A$	$P \vee Q \Leftrightarrow Q \vee P$	(3)
$A \cap B = B \cap A$	$P \wedge Q \Leftrightarrow Q \wedge P$	
	Distributive laws	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (Q \vee R)$	(4)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	
$A \cup \varnothing = A$	$P \vee \mathbf{F} \Leftrightarrow P$	(5)
$A \cap E = A$	$P \wedge \mathbf{T} \Leftrightarrow P$	
$A \cup E = E$	$P \vee \mathbf{T} \Leftrightarrow \mathbf{T}$	(6)
$A \cap \varnothing = \varnothing$	$P \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$	
$A \cup \sim A = E$	$P \vee \neg P \Leftrightarrow \mathbf{T}$	(7)
$A \cap \sim A = \varnothing$	$P \wedge \neg P \Leftrightarrow \mathbf{F}$	
	Absorption laws	
$A \cup (A \cap B) = A$	$P \vee (P \wedge Q) \Leftrightarrow P$	(8)
$A \cap (A \cup B) = A$	$P \wedge (P \vee Q) \Leftrightarrow P$	
	De Morgan's laws	
$\sim(A \cup B) = \sim A \cap \sim B$	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	(9)
$\sim(A \cap B) = \sim A \cup \sim B$	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	
$\sim \varnothing = E$	$\neg \mathbf{F} \Leftrightarrow \mathbf{T}$	(10)
$\sim E = \varnothing$	$\neg \mathbf{T} \Leftrightarrow \mathbf{F}$	
$\sim(\sim A) = A$	$\neg\neg P \Leftrightarrow P$	(11)

All the identities just given are presented in pairs except for the identity (11). This pairing is done because a principle of duality similar to the one given for statement algebra (see Sec. 1-2.10) also holds in the case of set algebra. In fact, the principle of duality holds for any Boolean algebra. At present it is sufficient to note that given any identity of the set algebra, one can obtain another identity by interchanging $\cup$ with $\cap$ and E with $\varnothing$.

Assuming identities (4) to (6), we shall prove the absorption laws. First note that

$$A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B)$$

from the distributive and idempotent laws. Now

$$\begin{aligned} A \cup (A \cap B) &= (A \cap E) \cup (A \cap B) && \text{from (5)} \\ &= A \cap (E \cup B) && \text{from (4)} \\ &= A \cap E && \text{from (6)} \\ &= A && \text{from (5)} \end{aligned}$$

Alternatively one can prove it as follows. For any x,

$$\begin{aligned} x \in A \cup (A \cap B) &\Leftrightarrow x \in \{x \mid (x \in A) \vee ((x \in A) \wedge (x \in B))\} \\ &\Leftrightarrow x \in \{x \mid x \in A\} \\ &\Leftrightarrow x \in A \end{aligned}$$

using the absorption laws of predicate calculus.

In order to complete our discussion, we list some implications and certain set inclusions

$$(A \cup B \neq \varnothing) \Rightarrow (A \neq \varnothing) \vee (B \neq \varnothing) \tag{12}$$

$$(A \cap B \neq \varnothing) \Rightarrow (A \neq \varnothing) \wedge (B \neq \varnothing) \tag{13}$$

To prove the implication (12), let us assume that $A \neq \varnothing \vee B \neq \varnothing$ is *false*. This requires that $A \neq \varnothing$ is *false* and also that $B \neq \varnothing$ is *false*, that is, $A = B = \varnothing$. But then $A \cup B = \varnothing$, so that $A \cup B \neq \varnothing$ is also *false*. Hence the implication is proved. One could also have proved (12) by assuming that $A \cup B \neq \varnothing$ is *true* and showing that this assumption requires $A \neq \varnothing \vee B \neq \varnothing$ to be *true*. Implication (13) can be proved in a similar manner.

The following inclusions follow from the definition and have been proved earlier in this section.

$$A \cap B \subseteq A \qquad A \cap B \subseteq B \qquad A \subseteq A \cup B \qquad A - B \subseteq A \tag{14}$$

Let A be a family of indexed sets over an index set $\mathbf{I}$ such that $A = \{A_1, A_2, \ldots\} = \{A_i\}_{i \in \mathbf{I}}$. Then

$$\bigcup_{i \in \mathbf{I}} A_i = \{x \mid x \in A_i \text{ for some } i \in \mathbf{I}\} \tag{15}$$

$$\bigcap_{i \in \mathbf{I}} A_i = \{x \mid x \in A_i \text{ for every } i \in \mathbf{I}\} \tag{16}$$

The associative laws and the distributive laws can be generalized in the following manner.

$$B \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} (B \cup A_i) \tag{17}$$

$$B \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} (B \cap A_i)$$

The identities (17) can be proved using mathematical induction discussed later in Sec. 2-5.1. We now give some examples illustrating the above operations.

EXAMPLE 1 Verify the identities (17) for

$$A_1 = \{1, 5\} \qquad A_2 = \{1, 2, 4, 6\} \qquad A_3 = \{3, 4, 7\}$$

$$B = \{2, 4\} \qquad \text{and} \qquad I = \{1, 2, 3\}$$

SOLUTION

$$\bigcup_{i \in I} A_i = \{1, 2, 3, 4, 5, 6, 7\}$$

$$\bigcap_{i \in I} A_i = \varnothing$$

$$B \cup \left(\bigcap_{i \in I} A_i\right) = \{2, 4\}$$

$$\bigcap_{i \in I} (B \cup A_i) = \{1, 2, 4, 5\} \cap \{1, 2, 4, 6\} \cap \{2, 3, 4, 7\} = \{2, 4\}$$

$$B \cap \left(\bigcup_{i \in I} A_i\right) = \{2, 4\}$$

$$\bigcup_{i \in I} (B \cap A_i) = \varnothing \cup \{2, 4\} \cup \{4\} = \{2, 4\} \qquad ////$$

2-1.7 The Principle of Specification

The idea of a set was discussed in the beginning of this chapter, although we had used the notion earlier in Chap. 1 while discussing the universe of discourse or the domain of the object variable (see Sec. 1-5.5). The universe of discourse was defined as the set of all objects under consideration, and this set is the same as the universal set defined in Sec. 2-1.2.

A set is usually defined by means of a predicate. The connection between a predicate and a set defined by it is known as the *principle of specification*, which states that every predicate specifies a set which is a subset of a universal set. The subset specified by a predicate is called an *extension* of the predicate in the universal set. This method has been used extensively in defining sets. For example, if $P(x)$ is a predicate, then a set A is called an extension of $P(x)$ if

$$A = \{x \mid P(x)\}$$

A predicate can be considered as a condition, and any object of the universal set satisfying the condition is then an element of the set which is an extension of the predicate. Obviously, if two predicates are equivalent, then they have the same extension, and the two sets specified by equivalent predicates are equal. In other words, if $P(x) \rightleftharpoons Q(x)$, then $A = B$ where A and B are the extensions of $P(x)$ and $Q(x)$, respectively. We now have an analogy between the equality of sets and the equivalence of predicates. A similar analogy exists between set inclusion and implication. If $P(x) \Rightarrow Q(x)$, then $A \subseteq B$ where, again, A and B are extensions of $P(x)$ and $Q(x)$ respectively.

If $P(x)$ is identically *true* for all x in E, then the extension of $P(x)$ in the universal set is the universal set itself. Similarly, if $P(x)$ is identically *false* for all x in E, then the extension of $P(x)$ in E is the null set. Recall that the universal set and the null set were defined as extensions of $P(x) \vee \urcorner P(x)$ and $P(x) \wedge \urcorner P(x)$ respectively. However, any other identically *true* (valid) and *false* predicates could have been used to define them.

If A and B are extensions of the predicates $P(x)$ and $Q(x)$, respectively, in a universal set E, then it is easy to see that $A \cup B$ and $A \cap B$ are the extensions of $P(x) \vee Q(x)$ and $P(x) \wedge Q(x)$ respectively. Similarly $\sim A$ is the extension of $\urcorner P(x)$. The extension of $P(x) \rightarrow Q(x)$ is the set $\sim A \cup B$, and that of $P(x) \rightleftarrows Q(x)$ is the set $(\sim A \cup B) \cap (A \cup \sim B)$. Thus the new sets formed from the sets A and B can be interpreted in terms of extensions of formulas containing $P(x)$ and $Q(x)$.

From the above discussion it is clear that all the identities of set theory given in the previous section should follow from the corresponding equivalence of predicate formulas. Similarly, the inclusions of sets should follow from the corresponding implications of predicates. If we replace the predicates by their extensions—$\wedge$ by $\cap$, $\vee$ by $\cup$, and $\urcorner$ by $\sim$—in any predicate formula, then we obtain the corresponding formula of set theory. Also, the equivalences and implications are replaced by equality and inclusions of sets. In fact, this technique has often been used in proving the identities and other relations of set theory so far. For example, let us consider

$$\urcorner(P(x) \vee Q(x)) \Leftrightarrow \urcorner P(x) \wedge \urcorner Q(x)$$

If A and B denote the extensions of $P(x)$ and $Q(x)$ respectively, then we can write

$$\sim(A \cup B) = \sim A \cap \sim B$$

Similarly, from

$$P(x) \vee (Q(x) \wedge R(x)) \Leftrightarrow (P(x) \vee Q(x)) \wedge (P(x) \vee R(x))$$

we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad [C \text{ is the extension of } R(x)]$$

2-1.8 Ordered Pairs and n-tuples

So far we have been solely concerned with sets, their equality, and operations on sets to form new sets. We now introduce the notion of an ordered pair. Although it is possible to define ordered pairs rigorously, we shall give an intuitive definition.

An *ordered pair* consists of two objects in a given fixed order. Note that an ordered pair is not a set consisting of two elements. The ordering of the two objects is important. The two objects need not be distinct. We shall denote an ordered pair by $\langle x, y \rangle$. A familiar example of an ordered pair is the representation of a point in a two-dimensional plane in cartesian coordinates. Accordingly, the ordered pairs $\langle 1, 3\rangle$, $\langle 2, 4\rangle$, $\langle 1, 2\rangle$, and $\langle 2, 1\rangle$ represent different points in a plane.

The *equality* of two ordered pairs $\langle x, y\rangle$ and $\langle u, v\rangle$ is defined by

$$\langle x, y\rangle = \langle u, v\rangle \Leftrightarrow ((x = u) \wedge (y = v)) \tag{1}$$

so that $\langle 1, 2\rangle \neq \langle 2, 1\rangle$ and $\langle 1, 1\rangle \neq \langle 2, 2\rangle$. A distinction between ordered pairs and sets containing two elements will be clear from the following examples:

$$\{a, b\} = \{b, a\} = \{a, a, b\} \qquad \{a, a\} = \{a\} \qquad \langle a, b\rangle \neq \langle b, a\rangle \qquad \langle a, a\rangle \neq \{a\}$$

The idea of an ordered pair can be extended to define an ordered triple, and, more generally, an n-tuple.

An *ordered triple* is an ordered pair whose first member is itself an ordered pair. Thus an ordered triple can be written as $\langle\langle x, y\rangle, z\rangle$. From the definition of the equality of an ordered pair, we can arrive at the equality of ordered triples $\langle\langle x, y\rangle, z\rangle$ and $\langle\langle u, v\rangle, w\rangle$:

$$\langle\langle x, y\rangle, z\rangle = \langle\langle u, v\rangle, w\rangle \qquad \text{iff} \qquad \langle x, y\rangle = \langle u, v\rangle \wedge z = w$$

But, $\langle x, y\rangle = \langle u, v\rangle$ if $(x = u \wedge y = v)$. Therefore

$$\langle\langle x, y\rangle, z\rangle = \langle\langle u, v\rangle, w\rangle \Leftrightarrow ((x = u) \wedge (y = v) \wedge (z = w)) \tag{2}$$

From the above definition of equality of an ordered triple, we may write an ordered triple as $\langle x, y, z\rangle$ with an understanding that $\langle x, y, z\rangle$ stands for $\langle\langle x, y\rangle, z\rangle$. Note that

$$\langle x, y, z\rangle \neq \langle y, x, z\rangle \neq \langle x, z, y\rangle$$

An ordered quadruple can be defined as an ordered pair whose first member is an ordered triple. Thus, an ordered quadruple is written as $\langle\langle x, y, z\rangle, u\rangle$ which is actually $\langle\langle\langle x, y\rangle, z\rangle, u\rangle$. It is easy to show that two ordered quadruples $\langle\langle x, y, z\rangle, u\rangle$ and $\langle\langle p, q, r\rangle, s\rangle$ are equal provided that

$$(x = p) \wedge (y = q) \wedge (z = r) \wedge (u = s) \tag{3}$$

In view of this fact, we shall write an ordered quadruple as $\langle x, y, z, u\rangle$.

Continuing this process, an ordered n-tuple is defined to be an ordered pair whose first member is an ordered $(n - 1)$-tuple. We write an ordered n-tuple as $\langle\langle x_1, x_2, \ldots, x_{n-1}\rangle, x_n\rangle$. Further, given two ordered n-tuples $\langle\langle x_1, x_2, \ldots, x_{n-1}\rangle, x_n\rangle$ and $\langle\langle u_1, u_2, \ldots, u_{n-1}\rangle, u_n\rangle$, we have

$$\langle\langle x_1, x_2, \ldots, x_{n-1}\rangle, x_n\rangle = \langle\langle u_1, u_2, \ldots, u_{n-1}\rangle, u_n\rangle$$
$$\Leftrightarrow ((x_1 = u_1) \wedge (x_2 = u_2) \wedge \cdots \wedge (x_n = u_n))$$

Therefore, an ordered n-tuple will be written as $\langle x_1, x_2, \ldots, x_n\rangle$.

2-1.9 Cartesian Products

Definition 2-1.15 Let A and B be any two sets. The set of all ordered pairs such that the first member of the ordered pair is an element of A and the second member is an element of B is called the *cartesian product* of A

and B and is written as $A \times B$. Accordingly,

$$A \times B = \{\langle x, y\rangle \mid (x \in A) \wedge (y \in B)\}$$

EXAMPLE 1 If $A = \{\alpha, \beta\}$ and $B = \{1, 2, 3\}$, what are $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$?

SOLUTION

$$A \times B = \{\langle\alpha, 1\rangle, \langle\alpha, 2\rangle, \langle\alpha, 3\rangle, \langle\beta, 1\rangle, \langle\beta, 2\rangle, \langle\beta, 3\rangle\}$$
$$B \times A = \{\langle 1, \alpha\rangle, \langle 2, \alpha\rangle, \langle 3, \alpha\rangle, \langle 1, \beta\rangle, \langle 2, \beta\rangle, \langle 3, \beta\rangle\}$$
$$A \times A = \{\langle\alpha, \alpha\rangle, \langle\alpha, \beta\rangle, \langle\beta, \alpha\rangle, \langle\beta, \beta\rangle\}$$
$$B \times B = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3. 3\rangle\}$$

$(A \times B) \cap (B \times A) = \varnothing$ ////

EXAMPLE 2 If $A = \varnothing$ and $B = \{1, 2, 3\}$ what are $A \times B$ and $B \times A$?

SOLUTION

$$A \times B = \varnothing = B \times A$$ ////

Before we consider the cartesian product of more than two sets let us consider the expressions $(A \times B) \times C$ and $A \times (B \times C)$. From the definition it follows that

$$\begin{aligned}(A \times B) \times C &= \{\langle\langle a, b\rangle, c\rangle \mid (\langle a, b\rangle \in A \times B) \wedge (c \in C)\}\\ &= \{\langle a, b, c\rangle \mid (a \in A) \wedge (b \in B) \wedge (c \in C)\}\end{aligned} \tag{1}$$

The last step follows from our definition of the ordered triple given in Sec. 2-1.8. Next,

$$A \times (B \times C) = \{\langle a, \langle b, c\rangle\rangle \mid (a \in A) \wedge (\langle b, c\rangle \in B \times C)\}$$

Here $\langle a, \langle b, c\rangle\rangle$ is not an ordered triple. If we consider $(A \times B) \times C$ as an ordered pair, then the first member is an ordered pair and the second member is an element of C. On the other hand, $A \times (B \times C)$ is an ordered pair in which the first member is an element of A while the second member is an ordered pair. This fact shows that

$$\underline{(A \times B) \times C \neq A \times (B \times C)}$$

Before defining the cartesian product of any finite number of sets, we shall show that the cartesian product satisfies the following distributive properties.

For any three sets A, B, and C

$$\checkmark\left\{\begin{aligned}A \times (B \cup C) &= (A \times B) \cup (A \times C)\\ A \times (B \cap C) &= (A \times B) \cap (A \times C)\end{aligned}\right. \tag{2}$$

We now prove the first of these two identities.

$$\begin{aligned} A \times (B \cup C) &= \{\langle x\ y\rangle \mid (x \in A) \wedge (y \in B \cup C)\} \\ &= \{\langle x, y\rangle \mid (x \in A) \wedge ((y \in B) \vee (y \in C))\} \\ &= \{\langle x, y\rangle \mid ((x \in A) \wedge (y \in B)) \vee ((x \in A) \wedge (y \in C))\} \\ &= (A \times B) \cup (A \times C) \end{aligned}$$

& Prove with predicate use

The second equality in Eq. (2) can be proved in a similar manner.

Let $A = \{A_i\}_{i \in I_n}$ be an indexed set and $\mathbf{I}_n = \{1, 2, \ldots, n\}$. We denote the cartesian product of the sets $A_1, A_2, \ldots, A_n$ by

$$\bigtimes_{i \in \mathbf{I}_n} A_i = A_1 \times A_2 \times \cdots \times A_n$$

which is defined by

$$\bigtimes_{i \in \mathbf{I}_1} A_i = A_1 \qquad \text{and} \qquad \bigtimes_{i \in \mathbf{I}_m} A_i = \Big(\bigtimes_{i \in \mathbf{I}_{m-1}} A_i\Big) \times A_m \qquad \text{for } m = 2, 3, \ldots, n$$

According to the above definition,

$$A_1 \times A_2 \times A_3 = (A_1 \times A_2) \times A_3$$

and

$$\begin{aligned} A_1 \times A_2 \times A_3 \times A_4 &= (A_1 \times A_2 \times A_3) \times A_4 \\ &= ((A_1 \times A_2) \times A_3) \times A_4 \end{aligned}$$

Our definition of cartesian product of n sets is related to the definition of n-tuples in the sense that

$$\begin{aligned} A_1 \times A_2 \times \ldots \times A_n = \{\langle x_1, x_2, \ldots, x_n\rangle \mid (x_1 \in A_1) \\ \wedge (x_2 \in A_2) \wedge \cdots \wedge (x_n \in A_n)\} \end{aligned}$$

The cartesian product $A \times A$ is also written as A^2, and similarly $A \times A \times A$ as A^3, and so on.

EXERCISES 2-1

1 Give examples of sets A, B, C such that $A \cup B = A \cup C$, but $B \neq C$.

2 Write the sets

$$\varnothing \cap \{\varnothing\} \qquad \{\varnothing\} \cap \{\varnothing\} \qquad \{\varnothing, \{\varnothing\}\} - \varnothing$$

3 Write the members of $\{a, b\} \times \{1, 2, 3\}$.

4 Write $A \times B \times C$, B^2, A^3, $B^2 \times A$, and $A \times B$ where $A = \{1\}$, $B = \{a, b\}$, and $C = \{2, 3\}$.

5 Show by means of an example that $A \times B \neq B \times A$ and $(A \times B) \times C \neq A \times (B \times C)$.

6 Show that for any two sets A and B

$$\begin{aligned} \rho(A) \cup \rho(B) &\subseteq \rho(A \cup B) \\ \rho(A) \cap \rho(B) &= \rho(A \cap B) \end{aligned}$$

Show by means of an example that

$$\rho(A) \cup \rho(B) \neq \rho(A \cup B)$$

7 Prove the identities

$$A \cap A = A \qquad A \cap \varnothing = \varnothing \qquad A \cap E = A \qquad \text{and } A \cup E = E$$

8 Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

9 Prove that

$$(A \cap B) \cup (A \cap \sim B) = A$$

and

$$A \cap (\sim A \cup B) = A \cap B$$

10 Show that $A \times B = B \times A \Leftrightarrow (A = \varnothing) \vee (B = \varnothing) \vee (A = B)$.

11 Show that $(A \cap B) \cup C = A \cap (B \cup C)$ iff $C \subseteq A$.

12 Draw Venn diagrams showing

$$A \cup B \subset A \cup C \quad \text{but} \quad B \not\subseteq C$$
$$A \cap B \subset A \cap C \quad \text{but} \quad B \not\subseteq C$$
$$A \cup B = A \cup C \quad \text{but} \quad B \neq C$$
$$A \cap B = A \cap C \quad \text{but} \quad B \neq C$$

13 Draw Venn diagrams and show the sets

$$\sim B \qquad \sim(A \cup B) \qquad B - (\sim A) \qquad \sim A \cup B \qquad \sim A \cap B$$

where $A \cap B \neq \varnothing$.

14 Show that $(A + B) + C = A + (B + C)$.

15 Prove that $A + A = \varnothing$ and $A + \varnothing = A$.

16 Show that $(A - B) - C = (A - C) - (B - C)$.

17 Prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

2-2 REPRESENTATION OF DISCRETE STRUCTURES

A number of applications involving discrete mathematical structures will be discussed throughout this book. Under discrete mathematical structures we include sets, ordered sets, and other structures such as trees and graphs which will be discussed in Chap. 5. These applications require that discrete structures be represented in some suitable manner. This will be the topic of the present section.

2-2.1 Data Structures

If the representation for a discrete structure does not exist in the programming language being used, then the program for a particular algorithm may be quite complex. For example, in a payroll (data processing) application a treelike representation of information for an employee such as in Fig. 2-2.1 may be required. This structure does not exist in certain programming languages, such as FORTRAN. It does not mean that we cannot program a payroll application in FORTRAN. It could be done by writing programs to construct and manipulate trees, but the programs would be complex. It would be more suitable to use a data processing language such as COBOL in this case. Ideally, the programming language chosen for the implementation of an algorithm should possess the particular representations chosen for the discrete structures in the problem being