1

SETS, SEQUENCES, AND SERIES

1.1 Sets

1.1.1 Definitions and basic properties

Definition 1.1.1 (set, union, and intersection). *[1, p. 1]*

- *A **set** is a collection of arbitrary objects. If x is an object in A, we write $x \in A$ and say x is an element of A. If x is not an object in A, we write $x \notin A$.*
- *If A and B are sets, the **union** of A and B, denoted by $A \cup B$, is the set*

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$

- *If A and B are sets, the **intersection** of A and B, denoted by $A \cap B$, is the set*

$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$

- *The **empty set** is the set containing no objects. We denote an empty set by $\emptyset$.*

Definition 1.1.2 (subset, difference, and complement). *[1, p. 3]*

- *If every elements in set A is an element in set B, we write $A \subset B$ and say set A is **contained** in B or that A is a **subset** of B.*
- *If A and B are two sets, the **difference** of A and B, denoted by $A - B$ or A B, is the set*

$$A - B = \{x | x \in A \text{ and } x \notin B\}.$$

- *If we are working in a fixed universe U, and $A \subset U$, we define the **complement** of A relative to U as*

$$A^c = U - A.$$

Example 1.1.1.

- Let $\mathbb{Z}$ denote the set of all integers. Then $4 \in \mathbb{Z}$ but $\frac{2}{3} \in \mathbb{Z}$.
- Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Then $A \cup B = \{1, 2, 3, 4\}$, $A \cap B = \{3\}$, and $A - B \in \{1, 2\}$.

Lemma 1.1.1 (algebra properties of sets). *[1, p. 4] Sets have the following Algebraic properties:*

- *Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$*
- *Associative: $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$*

- *Distributive:* $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$,$A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$

1.1.2 DeMorgan's Law

Lemma 1.1.2 (Demorgan's law). *[1, p. 3]Let S and T be sets, then*

1. $(S \cup T)^c = S^c \cap T^c$
2. $(S \cap T)^c = S^c \cup T^c$

Moreover, given a collection of sets indexed by I, we have

$$(\cup_{i \in I} A_i)^c = (\cap_{i \in I} A_i^c),$$

and

$$(\cap_{i \in I} A_i)^c = (\cup_{i \in I} A_i^c).$$

Lemma 1.1.3 (principle of inclusion exclusion). *Let $A_1, A_2, ..., A_n$ be sets, then*

$$\left|\cup_{i=1}^n A_i\right| = \sum_{i=1}^n (-1)^{i+1} S_i,$$

where

$$\begin{aligned} S_1 &= \sum_{i=1}^n |A_i| \\ S_2 &= \sum_{1 \le i < j \le n} \left|A_i \cap A_j\right| \\ \cdots &= \cdots \\ S_m &= \sum_{1 \le i_i < i_2 < ... < i_m \le n} \left|A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_m}\right|. \end{aligned}$$

More specifically,

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |C \cap B| - |A \cap C| + |A \cap B \cap C|.$$

1.1.3 Set equivalence and partition

Definition 1.1.3 (relation). *A relation on a set A is any statement which is either true or false for each ordered pair* (x, y) *of elements in A. Examples are* $x = y, x < y$

Definition 1.1.4 (equivalence relation). *[1] Let A and B be two sets and let f be a mapping of A into B. If there exist a 1-1 mapping of A* onto *B, we say that A and B can be put in 1-1 correspondence, or that A and B have the same* cardinal number*, or briefly, that A and B are* ***equivalent****, and we write* $A \sim B$*. The relation has the following properties:*

1. *It is reflective:* $A \sim A$.
2. *It is symmetric: if* $A \sim B$*, then* $B \sim A$.
3. *it is transitive: if* $A \sim B$ *and* $B \sim C$*, then* $A \sim C$.

Definition 1.1.5 (partitions of a set). *Let S be a set. A collection of (finitely or infinitely many) nonempty subsets* $A_1, A_2, ... \subseteq S$ *is called a partition of S if:*

- *These sets* A_i *are pairwise disjoint.*
- *The union of all subsets* $A_1 \cup A_2 ... = S$.

Theorem 1.1.1 (partition a set by equivalence). *Elements in a set X equivalent to each other form an equivalent class. All the equivalent classes of a set partition the set.*

Proof. We can show that any element cannot belong to two distinct equivalent classes using transitivity. □

1.1.4 Countability

Definition 1.1.6. *[1] For any positive integer n, let* P_n *be the set whose elements are the integers 1,2,...,n; let P be the set consisting of all positive integers. For any set A, we say:*

1. *A is* finite *if* $A \sim P_n$ *for some n.*
2. *A is* infinite *if A is not finite.*
3. *A is* countable *if* $A \sim P$ *(countable infinite)or A is finite.*
4. *A is* uncountable *if A is not countable*

Example 1.1.2.

- The integers $\mathbf{Z}$ form a countable set. The 1-1 mapping is given as $f(k) + 2k$ if $k >= 0$ and $f(k) = 2(-k) + 1$ if $k < 0$.

- The real number is uncountable set.

Lemma 1.1.4. *[1]Properties of countable sets:*

- *Any subset of a countable set is countable*
- *If A, B are countable sets, then $A \cup B$ is a countable set.*
- *The Cartesian product of two countable sets is countable.*

Proof. (1)(2) Straight forward. (3)Let S, T be the two countable sets. If they are finite, then $S \times T$ will be finite. Consider S, T have infinite elements, we can list $S \times T$ as a table and count them diagonally from a corner. This counting can count all the element in $S \times T$. □

Corollary 1.1.1.1. *Let k>1. Then the cartesian product of k countable sets is countable.*

1.2 Functions

1.2.1 Basic concepts

Definition 1.2.1 (function). *[1]If X and Y are sets. A* function *from X to Y is a subset f of $X \times Y$ satisfying:*

1. *Uniqueness of mapping: If (x, y) and (x, y') belong to f, then $y = y'$.*
2. *Completenss: If $x \in X$, then $(x, y) \in f$ for some $y \in Y$. Every element x in X must have a $y \in Y$.*

Definition 1.2.2. *[1]. For a function $f : X \to Y$, we have*

- *X is called the domain, Y is called the codomain. $f(X) = \{f(x) | x \in X\}$ is called the range.*
- *f is **onto** Y if $f(X) = Y$. Or equivalently, for any $y \in Y$, there exists $x \in X$(not necessarily unique) such that $f(x) = Y$.*
- *f is **one-to-one** if $f(x) = f(x') \Rightarrow x = x'$*
- *If f is one-to-one function, we can define f^{-1} as a function from $f(X)$ to X. Note that it is not from Y, but from $f(X)$.* Onto *is not required for the existence of f^{-1}.*
- *If f is one-to-one and onto, it is **bijective**.*
- *The* inverse image *of $B \subseteq Y$ under f is the set*

$$f^{-1}(B) = \{x | f(x) \in B\}$$

Definition 1.2.3 (inverse function). *Denote f as a function $f : X \to Y$.*

- *If f is one-to-one function, we can define f^{-1} as a function from $f(X)$ to X. Note that it is f^{-1} is not mapped from Y,*
- *If f is one-to-one and onto function, we can define f^{-1} as a function from Y to X.*

1.2.2 Inverse image vs. inverse function

Note 1.2.1. Note that *inverse image* and *inverse function* are fundamentally different. Inverse image always exist whereas inverse function requires 1-1 to exist.

Example 1.2.1. Take $X = Y = \mathbb{R}$, let $f(x) = cos(x)$. Then

- inverse function f^{-1} does not exist
- $f^{-1}(1)$ technically make no sense since inverse image will only take subset as input
- $f^{-1}(\{1\}) = \{\text{all integer multiples of } 2\pi\}$
- $f^{-1}(\{1\}) = \emptyset$
- $f^{-1}([-1, 1]) = \mathbb{R}$

1.2.3 Set operations in function mapping

Lemma 1.2.1 (Preserving set operators in function mapping). *[1, p. 7] Let f be a function from X into Y. Let $\mathcal{A}$ be a collection of subsets of X, and let $\mathcal{G}$ be a collection of subsets of Y. Let $C \subset Y$.*

1. $f(\cup\mathcal{A}) = \cup\{f(A)|A \in \mathcal{A}\}$
2. $f^{-1}(\cup\mathcal{B}) = \{f^{-1}(C)|C \in \mathcal{G}\}$
3. $f^{-1}(\cup\mathcal{B}) = \{f^{-1}(C)|C \in \mathcal{G}\}$
4. $f^{-1}(C^C) = (f^{-1}(C))^C$

Remark 1.2.1. It is in general not true that $f(A \cap B) = f(A) \cap f(B)$, because maybe $A \cap B = \emptyset$. However, if f is a one-to-one function if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

because if $(A \cap B) = \emptyset$, then $f(A) \cap f(B) = \emptyset$.

1.2.4 Parameter change of function

A function $f : A \to B$ is a rule to associate an element $a \in A$ to an element $b \in B$. The exact expression of f depends on how we parameterize the set A. For example, consider $A = [0, 1]$, and we want to map every element $x \in A$ to $5x$, then we have $f(x) = 5x$. However, if we want to express/reparameterize A as $x = 5t, t \in [0, 0.2]$, then we introduce a new local coordinate system on A as $\phi(x) = x/5$. The functon on the new local coordinate system is given as $f \circ \phi^{-1}(t) = 25t$

1.3 Real numbers

1.3.1 Rational numbers

Definition 1.3.1 (rational number, irrational number). *[1, p. 21] The set of* ***rational number****, denoted* $\mathbb{Q}$*, is the set*

$$\{\frac{p}{q}|p,q \in \mathbb{Z}, \text{ and } q \neq 0\}.$$

A real number which is not rational is said to be ***irrational****.*

Lemma 1.3.1 (rational number and irrational number). *If r is a rational number, which can be represented by p/q and x is an irrational number, then*

- *$r + x$ is irrational.*
- *rx is irrational, provided that $r \neq 0$.*

Proof. (1) Suppose $r + x$ is rational, then it can be represented by m/n. Then $x = r + x - r = m/n - p/q$ will still be rational, which contradicts that x is a rational number. (2)Suppose rx is rational, then it can be represented by m/n. Then $x = rx/r = (m/n)/(p/q)$ will still be rational, which contradicts that x is a rational number. □

1.3.2 Dense subset

Definition 1.3.2 (dense subset in $\mathbb{R}$). *[2, p. 15] Let S be a subset of $\mathbb{R}$.*

- *We say S is a* ***dense subset*** *in $\mathbb{R}$ provided that every interval $I = (a,b), a < b$, contains a member of S.*
- *(alternative) We say S is a* ***dense subset*** *in $\mathbb{R}$ provided that for every number $r \in \mathbb{R}$ and any $\epsilon > 0$, there exists a member $s \in S$ such that $|s - r| < \epsilon$.*

Remark 1.3.1 (equivalence of the two definitions)**.** (1) implies (2): for $r \in \mathbb{R}$, there exists a number $s \in S, s \in (r - \epsilon, r + \epsilon)$ such that $|r - s| < \epsilon$. (2) implies (1): for any interval (a,b), we let $\epsilon = d = \frac{b-a}{2}$, then there exists $s \in S$ such that

$$\left|s - \frac{a+b}{2}\right| < \frac{b-a}{2} \implies s \in (a,b).$$

Theorem 1.3.1 (rational and irrational numbers are dense). *[1, p. 23] If a and b are real numbers with $a < b$, then there exists both a rational number and an irrational number between a and b.*

Proof. Let k be an integer smaller than a, and let n be an integer such that $n > \sqrt{2}/(b-a)$. Then

$$0 < 1/n < \frac{2}{n} < b - a.$$

So the sequence $k + 1/n, k + 2/n + ...$ will have at least one term falling into (a, b). Similarly, the sequence $k + \sqrt{2}/n, k + 2\sqrt{2}/n + ...$ will have at least one term falling into (a, b). □

1.3.3 Axiom of completeness

Definition 1.3.3 (bounded above, bounded below).

- *A nonempty subset X of $\mathbb{R}$ is said to be* ***bounded above*** *if there exists a $a \in \mathbb{R}$ such that $a \geq x, \forall x \in X$.*
- *A nonempty subset X of $\mathbb{R}$ is said to be* ***bounded below*** *if there exists a $b \in \mathbb{R}$ such that $b \leq x, \forall x \in X$.*

Definition 1.3.4 (Least upper bound, supremum). *[1, p. 5] Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then a number u is said to be the* ***supremum*** *of S if:*

- *u is an upper bound of S.*
- *if v is also an upper bound of S, then $v \geq u$.*

Definition 1.3.5 (greatest lower bound, infimum). *[1, p. 5] Let S be a nonempty subset of $\mathbb{R}$. If S is bounded above, then a number u is said to be the* ***infimum*** *of S if:*

- *u is an lower bound of S.*
- *if v is also an lower bound of S, then if $v \leq u$.*

Definition 1.3.6 (maximum of a set). *A real number a_0 is a maximum of the set A if $a_0 \in A$ and $a \leq a_0, \forall a \in A$, but it has a supremum of 1.*

Remark 1.3.2. The open interval $(0, 1)$ does not have a maximum.

Theorem 1.3.2 (axiom of completeness, existence of least upper bound). ***Axiom of completeness:*** *Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.*

Theorem 1.3.3 (existence of greatest lower bound). *[1, p. 15] Every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.*

Proof. Let X be a nonempty subset of $\mathbb{R}$ that is bounded below. Let Y be the set of lower bounds. Then Y is bounded above, and therefore exists a least upper bound, denoted by b. We want to show that b is the greatest lower bound of X. First $b \leq x, \forall x \in X$(if there exists $c \in X$, $c < b$, then $c \in Y$ contradicts that $b \geq y, \forall y \in Y$.) Second, for any other lower bound d, we have $d \leq b$, since $d \in Y$. □

Lemma 1.3.2 (uniqueness of least upper bound). *[1, p. 15] Let X be a subset of $\mathbb{R}$. If a and b are least upper bounds of X, then $a = b$.*

Proof. Let a be the least upper bound. Since b is also a upper bound, then from the definition of least upper bound, we have $a \leq b$; Similarly, we have $a \geq b$. Therefore, $a = b$. □

Lemma 1.3.3 (least upper bound is tight(has no hole)). *[1, p. 17][3, p. 17]*

- *Let X be a set of real numbers with least upper bound a. Then for any positive $\epsilon > 0$, there exists $x \in X$ such that $a - \epsilon < x \leq a$.*
- *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if for any choice of $\epsilon > 0$, there exist an element $a \in A$ such that $s - a < \epsilon$.*

Proof. (1) $x \leq a$ is directly from the definition of least upper bound. Suppose there does not exist $x \in X$, such that $x > a - \epsilon$, then we conclude all $x \in X, x \leq a - \epsilon$. Let $b = a - \epsilon$ be another upper bound, then we have $b < a$, contradicting the fact that a is the least upper bound. (2) Suppose $\sup A$ is the least upper bound, then we have $s \geq \sup A$. For contradiction purpose, assume $s > \sup A$. Let $\epsilon = 0.5 * (s - \sup A)$, then $s - \epsilon > \sup A$, and there does not exist an element $a \in A$ such that $s - \epsilon < a$, which is a contradiction. Therefore, we must have $s = \sup A$.

□

Remark 1.3.3. This lemma illustrates the idea of no-holes between an nonempty set and its least upper bound.

Lemma 1.3.4 (least upper bound of rational numbers). *Given a real number a, define $S = \{x | x \in \mathbb{Q}, x < a\}$. It follows that*

$$\sup S = a.$$

Proof. First any number greater than a cannot be the least supper bound However, a is smaller upper bound. Now suppose $\sup S < a$; however, there exists a rational number exists inside $(\sup S, a)$ since $\mathbb{Q}$ is dense subset(Theorem 1.3.1]. therefore $\sup S = a$. □

Theorem 1.3.4 (nested interval property). *[3, p. 20] For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. And we also have $a_1 \leq a_2 \leq ....$ and $b_1 \geq b_2 \geq ...$, then,*

$$I_1 \supseteq I_2...$$

has a nonempty intersection; that is $\cap^\infty I_n \neq \emptyset$.

Proof. Let $A = \{a_1, a_2, ...\}$, and let $x = \sup A$. Given any set $I_n = [a_n, b_n]$, we have $x \geq a_n$, because x is the least upper bound. Also $x \leq b_n$, because b_n is another upper bound for A. Since n is arbitrary, we have $x \in \cap^\infty I_n$. □

1.4 Sequence in $\mathbb{R}$

1.4.1 Basics

Definition 1.4.1 (sequence). *[1, p. 35] A* sequence *in $\mathbb{R}$ a function f maps from the set P of all positive integers to $\mathbb{R}$. If $f(n) = x_n$, for $n \in P$, it is customary to denote the sequence f by the symbol $\{x_n\}$.*

Definition 1.4.2 (convergence of a sequence). *A sequence $\{p_n\}$ in $\mathbb{R}$ is said to* converge *if there is a point $p \in \mathbb{R}$ with the following property. For every $\epsilon > 0$ there is an integer N such that $n > N, |p_n - p| < \epsilon$.*

Theorem 1.4.1 (uniqueness of limits). *[1, p. 35] A sequence in $\mathbb{R}$ can have at most one limit.*

Proof. If a sequence has two different limits, then when n is large enough, a_n has to be increasingly closer to to both limit, which is contradiction. □

Theorem 1.4.2 (Boundedness of a convergent sequence). *Every convergent sequence in $\mathbb{R}$ is bounded.*

Proof. Because the sequence is convergent to a number a, then when given $\epsilon = 1$, there is an N, such that $n \geq N, |x_n| \leq |x_n - a| + |a| < 1 + absa$. Then $|x_n| \leq \max(|x_1| ... |x_N|, 1 + absa)$. □

Lemma 1.4.1 (algebra of limits). *[1, p. 40] Let $\{a_n\}$ and $\{b_n\}$ be two real-valued sequences and $\lim_{n\to\infty} a_n = M$, $\lim_{n\to\infty} b_n = L$, we have: [1]*

- *linearity:* $\lim_{n\to\infty} \alpha a_n + \beta b_n = \alpha M + \beta L$
- *product rule:* $\lim_{n\to\infty} a_n b_n = ML$
- *quotient rule: if $L \neq 0$,*$\lim_{n\to\infty} 1/a_n = 1/L$
- *If $\{c_n\}$ is a bounded sequence, and $\lim_{n\to\infty} b_n = 0$, then*

$$\lim_{n\to\infty} c_n b_n = 0.$$

- *(absolute value rule)* $\lim_{n\to\infty} |a_n| = |M|$.

Proof. (1)linearity from triangle inequality;(2) $|a_n b_n - LM| = |a_n b_n + a_n M - a_n M - LM|$, then use triangle inequality and boundedness; (3) use boundedness. (4) Since $\{c_n\}$ is bounded. then there exists a number S such that $|c_n| \leq S, \forall n$. For any given $\epsilon > 0$, there exists a N such that for all $n > N, |b_n| \leq \epsilon/N$. Therefore,

$$|c_n b_n| \leq |c_n||b_n| \leq S\epsilon/S = \epsilon, \forall n > N.$$

(5) Note that for any given $\epsilon > 0$, there exists a N such that for all $n > N$,

$$\big||a_n| - |M|\big| \leq |a_n - M| \leq S\epsilon/S = \epsilon.$$

□

Note 1.4.1 (the reverse of absolute value rule is not true). Note that if $\{|a_n|\}$ converges, $\{a_n\}$ not necessarily converges. For example $\{(-1)^n\}$.

Lemma 1.4.2 (sequence limit inequality). *[1, p. 47] Let $\{a_n\}$ be a convergent sequence with limit L. It follows that*

- *If $a_n \geq M$ for all $n \geq 0$, then $L \geq M$.*
- *If $a_n \geq b_n$ for all $n \geq 0$, and $\lim_{n\to\infty} b_n = K$then $L \geq K$.*

Proof. (1) For contradiction purpose, we assume $L < M$. Denote $d = M - L, d > 0$. Since a_n converges to L, then for $\epsilon = d/2$, there exist a N such that

$$|a_N - L| < \epsilon = d/2,$$

which implies

$$a_N < d/2 + L = \frac{M-L}{2} + L = \frac{M+L}{2} < M.$$

This contradict the condition that $a_n \geq M$. (2) Note that $(a_n - b_n) \geq 0$. Using (1) we have

$$\lim_{n\to\infty} (a_n - b_n) = L - K \geq 0,$$

where we have used the algebraic properties of limits [Lemma 1.4.1]. □

Lemma 1.4.3 (squeeze theorem). *[1, p. 47] Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences such that*

$$a_n \leq b_n \leq c_n, \forall n \in \mathcal{N}.$$

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then

$$\lim_{n\to\infty} b_n = L.$$

Proof. Note that for any $\epsilon > 0$ there exists N_1 such that for all $n > N_1$ such that

$$L - \epsilon < a_n < L + \epsilon.$$

Similarly, there exists N_2 such that for all $n > N_1$ such that

$$L - \epsilon < c_n < L + \epsilon.$$

Then for $n > \max(N_1, N_2)$, we have

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon.$$

That is, $b_n \to L$ as $n \to \infty$. □

Corollary 1.4.2.1 (applications of squeeze theorem).

- *If* $\lim_{n\to\infty}|a_n| = 0$*, then* $\lim_{n\to\infty} a_n = 0$.
- *For any number c,*
$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$
- $$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Proof. (1) Note that

$$-|a_n| \leq a_n \leq |a_n|,$$

and

$$\lim_{n\to\infty} -|a_n| = \lim_{n\to\infty} |a_n|,$$

then use squeeze theorem to prove. (2)Choose k to be an integer such that $k \geq 2|c|$. Then if $n \geq K$

$$0 \leq \left|\frac{c^n}{n!}\right| \leq |c|^k \left(\frac{1}{2}\right)^{n-k} \to 0, asn \to \infty.$$

(3)

$$\frac{n!}{n^n} = \frac{1}{n}\frac{2}{n}\cdots\frac{n-1}{n}\frac{n}{n} \leq \frac{1}{n}\cdot 1\cdot 1\cdots 1 = 1/n \to 0, as\ n \to \infty.$$

□

1.4.2 Cauchy criterion

Definition 1.4.3 (Cauchy sequence in $\mathbb{R}$). *[1, p. 59] A sequence $\{a_n\}$ in $\mathbb{R}$ is called a Cauchy sequence, if for every $\epsilon > 0$, there exist an N such that for all $n, m \geq N$, we have*

$$|x_n - x_m| < \epsilon.$$

Remark 1.4.1 (interpretation). Informally, a sequence $\{x_k\}$ satisfies the Cauchy criterion if, by choosing k large enough, the distance between any two element x_m and x_l in the 'tail' of the sequence can be made as small as desired. A sequence satisfying Cauchy criterion is called a **Cauchy sequence**.

Lemma 1.4.4 (boundedness of Cauchy sequence). *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let $\epsilon = 1$, then there exists an N, such that for all $n, m \geq N$, we have

$$|x_n| \leq |x_n - X_N| + |X_N| < 1 + |X_N|$$

Then the Cauchy sequence is bounded by the maximum of $1 + |x_N|, |x_1|, ..., |x_{N-1}|$. □

Theorem 1.4.3 (Cauchy sequence is a convergent sequence, vice versa). *[3, p. 66] A sequence $\{x_k\}$ in $\mathbb{R}$ is Cauchy sequence if and only if it is a convergent sequence.*

Proof. (1) the converse part that a convergent sequence is a Cauchy sequence can be proved triangle inequality; (2) the forward part: Because Cauchy sequence is bounded, then there will be a subsequence x_{n_i} converge to a limit a(Theorem 1.6.2). Let $\epsilon > 0$, then there exist an N, such that for all $n, m, n_K > N, |x_{n_K} - a| < \epsilon$. We have

$$|x_n - a| \leq |x_n - x_{n_K}| + |x_{n_K} - a| < 2\epsilon$$

□

Remark 1.4.2 (Cauchy sequence might not be convergent in more general space). **In incomplete normed space, a Cauchy sequence will not converge.** Since $\mathbb{R}$ is complete, therefore the Cauchy sequence will converge.

1.4.3 Sequence characterization of a dense subset

Theorem 1.4.4. *[2, p. 36] Let S be a subset in $\mathbb{R}$. It follows that*

- *If for every number $x \in \mathbb{R}$ there exists a sequence $\{s_n\}, s_n \in S$ converging to x, then S is a dense subset.*
- *If S is a dense subset, then for every number $x \in \mathbb{R}$ there exists a sequence $\{s_n\}, s_n \in S$ converging to x.*

Proof. (1) If for every x there exists a sequence converging to it, then that means for every interval (a, b), there exists at least a number getting arbitrary closer to $(a + b)/2$, that is inside the interval (a, b). (2) Based on the definition of dense subset [Definition 1.3.2], we can construct a sequence s_n such that s_n lying inside the interval $(x - 1/2n, x + 1/2n)$. Such sequence $\{s_n\}$ will converge to x. □

Corollary 1.4.4.1 (sequential density of rations). *For every $x \in \mathbb{R}$ there exists a sequence $\{s_n\}, s_n \in \mathbb{Q}$ converging to x.*

Proof. Use the fact that $\mathbb{Q}$ is a dense subset in $\mathbb{R}$ [Definition 1.3.2]. □

Example 1.4.1. Consider the irrational number $\sqrt{2}$, the above theorem ensures that there exists a sequence of rational numbers converging to $\sqrt{2}$, even though $\sqrt{2}$ is irrational.

1.5 Monotone sequence

1.5.1 Fundamentals

Definition 1.5.1 (monotone sequence). *A sequence if monotone is it is either increasing(i.e., $a_{n+1} \geq a_n$) or decreasing.*

Theorem 1.5.1 (convergence of monotone sequence). *[1, p. 48] A monotone increasing sequence $\{a_n\}$ is convergent if and only if $\{a_n\}$ is bounded.*

If it is convergent, then its limit is $\sup_n a_n$.

Proof. (1)Suppose $\{a_n\}$ is bounded, then there exists a least upper bound [Theorem 1.3.2]. Let $A = \sup\{a_n\}$. Then given $\epsilon > 0$, there exists a_k such that $A - \epsilon < a_k < A$ [Lemma 1.3.3]; take $N = k$, then we prove A is the limit.(2) The converse is the direct result of boundedness of any convergent sequence. □

Note 1.5.1 (equivalence of sup and sequential limit for monotone sequence). For a convergent monotone increasing sequence $\{a_n\}$, we have

$$\sup_n a_n = \lim_{n\to\infty} a_n.$$

Usually directly taking sup can be difficult and we can seek how to find a monotone increasing sequence.

Remark 1.5.1 (application in lim sup). Let $\{a_n\}$ be a bounded real sequence(not necessarily convergent) and define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup_{k\geq n} a_k).$$

Then the sequence $\{\sup_{k\geq n} a_k\}_n^\infty$ is a monotone increasing sequence therefore $\limsup_{n\to\infty} a_n$ will have a limit.

1.5.2 Applications

Lemma 1.5.1 (limit of exponent).

- *If $a > 1$, then $\lim_{n\to\infty} a^{1/n} = 1$.*

- *If* $0 < a < 1$, *then* $\lim_{n\to\infty} a^{1/n} = 1$.

That is, for all $a > 0$, *we have* $\lim_{n\to\infty} a^{1/n} = 1$.

Proof. (1) When $a > 1$, $a^{1/n}$ is a decreasing sequence and bounded below by 1. Therefore $\{a^{1/n}\}$ is a convergent sequence. Let the limit be L. Then

$$\lim_{n\to\infty} a^{1/n} = L, \lim_{n\to\infty} a^{1/n}a^{1/n} = \lim_{n\to\infty} (a^2)^{1/n} = L^2.$$

Also note that $a^2 > 1$, then $\lim_{n\to\infty}(a^2)^{1/n} = L$.

So we have $L^2 = L \implies L = 1$.

(2) When $0 < a < 1$, let $b = 1/a$, then

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} (\frac{1}{b})^{1/n} = \lim_{n\to\infty} (\frac{1}{b^{1/n}}) = L.$$

where we use the quotient rule [Lemma 1.4.1). □

Lemma 1.5.2 (the limit to e). *[1, p. 53]*

- *The sequence*

$$\{a_n = (1 + \frac{1}{n})^{1/n}\}$$

is increasing and convergent.link. And we denote

$$e \triangleq \lim_{n\to\infty} (1 + \frac{1}{n})^{1/n}.$$

-

$$\lim_{n\to\infty} (1 - \frac{1}{n})^n = \frac{1}{e}$$

-

$$\lim_{n\to\infty} (1 + \frac{1}{n^2})^{n^2} = e, \lim_{n\to\infty} (1 + \frac{1}{n^2})^n = 1, \lim_{n\to\infty} (1 + \frac{1}{n})^{n^2} = \infty$$

- *Let* $c \in \mathbb{R}$, *then*

$$\lim_{n\to\infty} (1 + \frac{c}{n})^n = e^c$$

- *Let* $c \in \mathbb{R}$ *and* $\lim_{n\to\infty} q_n = 0$, *then*

$$\lim_{n\to\infty} (1 + \frac{c}{n} + \frac{q_n}{n})^n = e^c$$

Proof. (1) Let $x_1 = 1, x_2 = x_3 = \cdots = x_{n+1} = 1 + \frac{1}{n}$.

Then
$$(x_1x_2\cdots x_{n+1})^{1/n+1} < \frac{x_1+x_2+...+x_{n+1}}{n+1},$$
where we have used the geometric average inequality(can be showed using convexity). That is
$$(1+\frac{1}{n})^n < (\frac{1+n(1+\frac{1}{n})}{n+1})^{n+1} = (1+\frac{1}{n+1})^{n+1}.$$

To show it is bounded. We let $x_1 = 1, x_2 = x_3 = \cdots = x_{n+1} = 1-\frac{1}{n}$ and use the geometric average inequality to show
$$b_n \geq b_{n+1},$$
where $b_n = (1-\frac{1}{n})^{-n}$.

Further, we can show
$$b_n = a_{n-1}(1+\frac{1}{n-1}) \geq a_{n-1},$$
where $a_n = (1+\frac{1}{n})^n$.

Therefore, we have
$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1.$$

(2)
$$(1-\frac{1}{n})^n = \frac{1}{(\frac{n}{n-1})^n} = \frac{1}{(1+\frac{1}{n-1})^n},$$
and note that $\lim_{n\to\infty}(1+\frac{1}{n-1})^n = e$

(3) (a) this is a subsequence of (1). (b) note that
$$\lim_{n\to\infty}(1+\frac{1}{n^2})^n = \lim_{n\to\infty}\sqrt{1/n}(1+\frac{1}{n^2})^{n^2} = 1$$
where we use the factor for $a>0, \lim_{n\to\infty} a^{1/n} = 1$ in Lemma 1.5.1,since $(1+\frac{1}{n^2})^{n^2}$ will be bounded.

(4) Let $f(x) = x^a$. Then f is a continuous function. Therefore
$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n),$$
where $x_n = (1+\frac{c}{n})^{n/c}$. □

Lemma 1.5.3. *The sequence* $\{n^{1/n}\}, n \geq 3$ *is decreasing and* $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. Consider $f(x) = x^{1/x}$ and $g(x) = \ln f(x) = \frac{\ln x}{x}$. $f(x)$ and $g(x)$ are decreasing for $x \geq e$. Also, $\lim_{x\to\infty} g(x) = 0 \implies \lim_{x\to\infty} f(x) = 1$. □

1.6 Subsequence and limits

1.6.1 Subsequence

Definition 1.6.1 (subsequence). *[1, p. 39] Let $\{a_n\}$ be a sequence. Let f be a strictly increasing function for the positive integer set $\mathbb{P}$ to $\mathbb{P}$. The sequence $\{a_{f(n)}\}$ is called a* subsequence *of $\{a_n\}$.*

Remark 1.6.1.

- Note that subsequence also have infinite number of terms.
- Usually, we denote $n_k = f(k)$ as the kth term in the subsequence $\{a_{n_k}\}$, where $n_1 < n_2 < n_3 ...$, and $n_k \geq k$.

Theorem 1.6.1 (subsequences of a convergent sequence have the same limit). *[1, p. 39] If $\{a_n\}$ has a limit L, then every subsequence of $\{a_n\}$ has a limit L; If every subsequence of $\{a_n\}$ has a limit L and $\{a_n\}$ has a limit L.*

Proof. (1) For any $\epsilon > 0$, there exists a N such that $|a_k - L| < \epsilon, \forall k > N$. We have the terms in subsequence with $n_k \geq k > N$ will have $\left|a_{n_k} - L\right| < \epsilon$; (2) The converse is easy since the sequence itself is a subsequence. □

Example 1.6.1. [1, p. 39]The sequence $\{1/2^n\}$ and $\{1/n!\}$ are subsequences of $\{1/n\}$ and $\lim_{n\to\infty} 1/n = 0$. Therefore,

$$\lim_{n\to\infty} \frac{1}{2^n} = 0 = \lim_{n\to\infty} \frac{1}{n!}.$$

1.6.2 Bolzano-Weierstrass theorem

Theorem 1.6.2 (Bolzano-Weierstrass theorem). *[3, p. 62] Every bounded sequence has at least one convergent subsequence.*

Proof. We use nested interval property [Theorem 1.3.4] to prove. Let I_1 be the interval bounding the sequence. Let I_2 be one-half of I_1 that bounding infinite terms(if both halves have infinite terms, pick either one; there must one half containing infinite terms since a sequence has infinite terms). As we continue to partition, I_n becomes smaller. Note that the nested interval properties guarantee the existence a common element $a \in \cap^\infty I_n$ and the decreasing size of I_n guarantee the subsequence are sufficiently close to a. □

1.6.3 Subsequence limits

Definition 1.6.2 (limit point). *[4, p. 79] If $\{x_j\}$ is a sequence of real numbers and x a real number, we say x is a* ***limit point of the sequence*** *if for every $\epsilon > 0$, there are* ***infinite number of terms*** *x_j satisfying $\left|x_j - x\right| < \epsilon$.*

Remark 1.6.2 ($+\infty$ as a limit point). By convention, we say $+\infty$ is a limit point for $\{x_n\}$, if for any given $M > 0$, there exists infinitely many terms greater than M. If a sequence is unbounded above, then $+\infty$ is one limit point of the sequence.

Remark 1.6.3 (limit vs. limit point).

- For the definition of limit, we see a **stronger** statement "all terms beyond N"(therefore infinite many terms) should satisfy $|x_i - x| < \epsilon$.
- For the definition of limit, we see a **weaker** statement "infinitely many terms" should satisfy $|x_i - x| < \epsilon$.

Lemma 1.6.1 (relation between subsequence and limit point). *[4, p. 80] Let $\{x_n\}$ be a real sequence. A real number(even including $-\infty$ and ∞) is a limit point of $\{x_n\}$ if and only if there exists a subsequence $\{x_{n_k}\}$ such that*

$$\lim_{n_k \to \infty} x_{n_k} = x.$$

Proof. (1)(forward) If there exists a sequence, then for any given $\epsilon > 0$, there exists a $K > 0$, such that for all $n_k > K$(therefore infinitely many terms), $\left|x_{n_k} - x\right| < \epsilon$. Therefore x is a limit point. (2) (backward) Because x is a limit point, then given any given $\epsilon > 0$, there exists infinitely many terms, denoted by index set $\mathcal{K}$, in $\{x_n\}$, such that $|x_m - x| < \epsilon, \forall m \in \mathcal{K}$. Order the index in $\mathcal{K}$, then $\{x_n\}_K$ is a subsequence converging to x. □

Definition 1.6.3 ($\liminf$ and $\limsup$ for bounded sequences). *[1] Let $\{a_n\}$ be a bounded real sequence and let $\mathcal{L}_a$ denote the set of all different limits L_a of all the convergent subsequences, i.e.*

$$\lim_{k \to \infty} a_{n_k} = L_a$$

then we define

$$\limsup_{n \to \infty} a_n = \sup \mathcal{L}_a$$

$$\liminf_{n\to\infty} a_n = \inf \mathcal{L}_a$$

Definition 1.6.4 ($\liminf$ and $\limsup$ for bounded and unbounded sequences,alternative). *[5, p. 619][1, p. 69] Let $\{a_n\}$ be a bounded real sequence and let*

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, ...\}$$

and

$$c_n = \inf\{a_n, a_{n+1}, a_{n+2}, ...\}$$

then we define

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n.$$

If $\{a_n\}$ is not bounded above, then we define $\limsup_{n\to\infty} a_n = \infty$; If $\{a_n\}$ is not bounded above, then we define $\liminf_{n\to\infty} a_n = -\infty$.

Remark 1.6.4 (interpretation).

- $\{b_n\}$ and $\{c_n\}$ are nonincreasing and nondecreasing sequences. And since $\{a_n\}$ is bounded, both are monotone and bounded and therefore have limits.
- $c_n \leq a_n \leq b_n$.
- $\liminf$ and $\limsup$ for a sequence (no matter bounded or not) always exist.

Example 1.6.2.

- Let $\{a_n = n\}$. Then
$$\limsup_{n\to\infty} = \infty, \liminf_{n\to\infty} = \infty.$$
- Let $\{a_n\}$ be defined by
$$a_n = \begin{cases} -n, n \text{ is odd} \\ 0, n \text{ is even} \end{cases}.$$
Then,
$$\limsup_{n\to\infty} = 0, \liminf_{n\to\infty} = -\infty.$$

Theorem 1.6.3. *[1] Let $\{a_n\}$ be a bounded sequence, and let $L = \lim_{n\to\infty} \sup a_n$ and $M = \lim_{n\to\infty} \inf a_n$*

1. *If $\epsilon > 0$, there exist infinitely many positive integers n such that $L - \epsilon < a_n$ and there exist a positive interger N_1 such that if $n > N_1$, then $a_n < L + \epsilon$.*
2. *If $\epsilon > 0$, there exist infinitely many positive integers n such that $M + \epsilon > a_n$ and there exist a positive interger N_1 such that if $n > N_1$, then $a_n > M - \epsilon$.*

Proof. We only prove the first part. Suppose there exist only finite n such that $a_n > L - \epsilon$, then there will not exist a convergent subsequence approaching L; Suppose there are infinitely many n such that $a_n > L + \epsilon$, because of the Bolzano-Weierstrass theorem, there exist a convergent subsequence converge to a limit greater than $L + \epsilon$. □

Theorem 1.6.4 (limit from $\liminf$ **and** $\limsup$**).** *Let $\{a_n\}$ be a bounded sequence, then $\lim_{n\to\infty} a_n = L$ if and only if*

$$\limsup_{n\to\infty} a_n = L = \liminf_{n\to\infty} a_n.$$

Or equivalently, if and only $\{a_n\}$ has only one limit point.

Proof. (1) let

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, ...\}$$

and

$$c_n = \inf\{a_n, a_{n+1}, a_{n+2}, ...\}$$

and use $c_n \leq a_n \leq b_n$ to squeeze. (2) If $\{a_n\}$ converges and every subsequence will converge to the same limit. □

1.7 Infinite series

1.7.1 Fundamental results

Definition 1.7.1 (convergence of series). *[1] We denote $\sum_{n=1}^{\infty} a_n$ as infinite series, $s_n = \sum_{i=1}^{n} a_i$ as the partial sum. We say the infinite series $\sum_{n=1}^{\infty} a_n$ converges to L if $\sum_{n=1}^{\infty} a_n$ has sum L.*

Remark 1.7.1. Note that the convergence properties/limit properties of infinite series usually can be deduced from s_n.

Definition 1.7.2 (absolute convergence). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. If $\sum_{n=1}^{\infty} |a_n|$, we say $\sum_{n=1}^{\infty} a_n$ converge absolutely. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges, we say $\sum_{n=1}^{\infty} a_n$ converge conditionally.*

Example 1.7.1. $\sum_{n=1}^{\infty} (-1)^n/n$ converge conditionally.

Theorem 1.7.1 (important results of convergence). *[1, p. 76]*

- *If the $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*
- ***(absolute convergence implies convergence)****If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converge.*
- *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converge absolutely.*
- *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n a_{n+k}$ converge absolutely, where k is a fixed integer.*
- *(linearity)If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent with limits A and B, then*

$$\sum_{n=1}^{\infty} ka_n + mb_n = kA + mB$$

Note: *If $\sum_{n=1}^{\infty} a_n$ only converge conditionally, then $\sum_{n=1}^{\infty} a_n^2$ might diverge. For example $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$ converge conditionally and $\sum_{n=1}^{\infty} 1/n$ diverges.*

Proof. (1)$s_{n+1} - s_n = a_n$, using the algebraic properties of limits of s_n to prove. (2)Construct $\{p_n = max(a_n, 0)\}$ and $\{q_n = max(-a_n, 0)\}$. Then the partial sum $\{p_n\}$ and $\{q_n\}$ is bounded monotone sequence, thus converges; since $\{a_n = p_n - q_n\}$, then using algebraic properties of limits to prove. (3)We know that $\lim_{n\to\infty} a_n = 0$ from (1), then there exist an N such that for all $n \geq N, |a_n| < 1$, therefore $a_n^2 < |a_n|$. From comparison test, we know that $\sum_{n=N}^{\infty}|a_n|$ will converge, therefore $\sum_{n=1}^{\infty} a_n^2$ converges absolutely [Theorem 1.7.3].(4) use the fact that

$$|a_n a_{n+k}| \leq \frac{1}{2}(a_n^2 + a_{n-k}^2).$$

Then the series $\sum|a_n a_{n+k}|$ is bounded above since $\sum a_n^2$ and $\sum a_{n+k}^2$ are finite due to (3). (5)Directly from algebraic property of sequences. □

Remark 1.7.2. Some theorems (e.g. in Fourier series, time series analysis)simply state the assumptions of absolute convergence. Then from this theorem, we know that squared convergence is implied.

Remark 1.7.3 (product of two series). [1, p. 76]

- It is possible that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge but the series $\sum_{n=1}^{\infty} a_n b_n$ diverges. For example, $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n}$ converge conditionally and $\sum_{n=1}^{\infty} 1/n$ diverges.
- Even if $\sum_{n=1}^{\infty} a_n$ converges to M and $\sum_{n=1}^{\infty} b_n$ converges to N, and $\sum_{n=1}^{\infty} a_n b_n$ converges, the sum limit in general will not be MN.

Theorem 1.7.2 (Cauchy criterion analog). *The series $\sum_{n=1}^{\infty} a_n$ is convergent if and only for any $\epsilon > 0$, an N can be found such that*

$$|a_{n+1} + a_{n+2} + ... + a_m| < \epsilon, \forall m > n > N$$

1.7.2 Tests for convergence

For a comprehensive treatment, see [6].

Theorem 1.7.3 (comparison test for convergence). *If $|a_n| \leq b_n, \forall n = 1, 2, ...,$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is absolutely converge.*

Proof. The partial sum sequence for $\sum_{n=1}^{\infty}|a_n|$ is bounded (by $\sum_{n=1}^{\infty} b_n$) and monotone sequence. Then we use monotone convergence theorem [Theorem 1.5.1]. □

Example 1.7.2. $\sum_{n=1}^{\infty} 1/(n2^n)$ is convergent because $\sum_{n=1}^{\infty} 1/2^n$ is convergent.

Theorem 1.7.4 (geometric series). *The geometric series $\sum_{i=0}^{\infty} ar^i$ converges for $-1 < r < 1$, and diverges for $|r| \geq 1$.*

Theorem 1.7.5 (ratio test). *[1, p. 86] For a series $\sum_{n=1}^{\infty} a_n$, if $a_n \neq 0, \forall n$, and*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

then we have: if $L < 1$, absolutely converges; if $L = 1$, the test not applicable; if $L > 1$, divergent.

Proof. To prove absolute convergence, we prove the $\sum_{n=1}^{\infty} |a_n|$ is bounded first and then use monotone convergence property. To prove boundedness, let $L < r < 1, b_n = |a_{n+1}/a_n|$, then there exists N such that $0 < b_n < r, n > N$, then rewrite:

$$|a_{N+1}| + |a_{N+2}| + ... = |a_{N+1}| \left(1 + b_{N+1} + ...\right) < |a_{N+1}| \sum_{i=0} r^i.$$

The right hand side is a convergent series. Based on comparison test [Theorem 1.7.3], $\sum_{n=1}^{\infty} a_n$ will converge absolutely. □

Example 1.7.3. The series $\sum_{n=1}^{\infty} 1/n!$ converges since the ratio

$$\frac{1/(n+1)!}{1/n!} = \frac{1}{n+1}$$

converges to 0 as $n \to \infty$.

Theorem 1.7.6 (root test). *[1, p. 85] For a series $\sum_{n=1}^{\infty} a_n$, let*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = R$$

Then we have: if $R < 1$, absolutely converges; if $R > 1$, diverges; if $R = 1$, the test not applicable.

Proof. Suppose $L < 1$. Choose a number $L < M < 1$. Then there exists a N such that for $n \geq N$ then

$$|a_n|^{1/n} < M \Leftrightarrow |a_n| < M^n.$$

Since the geometric series $\sum_{n=1}^{\infty} M^n$ converges, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely based on comparison test [Theorem 1.7.3]. □

Lemma 1.7.1 (common applications of convergence test).

- *The series* $\sum_{n=1}^{\infty} n!/n^n$ *converges.*
- *The series* $\sum_{n=1}^{\infty} M^n/n!$ *converges for all* $M > 0$.

Proof. (1) note that the ratio

$$\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n+1}{(n+1)^{n+1}}n^n = \frac{1}{(1+1/n)^n}$$

converges to $1/e < 1$. Then we use ratio test [Theorem 1.7.5]. (2) note that the ratio

$$\frac{M^{n+1}/(n+1)!}{M^n/n!} = \frac{M}{n+1}$$

converges to zero. Then we use ratio test [Theorem 1.7.5]. □

1.7.3 Inequalities and l_2 series

1.7.3.1 *Holder's and Minkowski's inequality*

Lemma 1.7.2 (Holder's inequality and Minkowski's inequality for finite real-valued terms). *Consider any real-valued sequence* $\{x_n\}$ *and* $\{y_n\}$. *Then*

- *(Holder's inequality) For any number* $p, q \in (1, \infty)$ *with* $1/p + 1/q = 1$, *we have*

$$\sum_{k=1}^{n} |x_k y_k| \leq (\sum_{k=1}^{n} |x_k|^p)^{1/p} (\sum_{k=1}^{n} |y_k|^q)^{1/1}$$

- *(Minkowski's inequality) For any number* $p \in [1, \infty)$

$$(\sum_{k=1}^{n} |x_k + y_k|^p)^{1/p} \leq (\sum_{k=1}^{n} |x_k|^p)^{1/p} + (\sum_{k=1}^{n} |y_k|^p)^{1/p}$$

Theorem 1.7.7 (Holder's inequality and Minkowski's inequality for series). *Consider any real-valued sequence* $\{x_n\}$ *and* $\{y_n\}$. *Then*

- *(Holder's inequality) For any number* $p, q \in (1, \infty)$ *with* $1/p + 1/q = 1$, *we have*

$$\sum_{k=1}^{\infty} |x_k y_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} (\sum_{k=1}^{\infty} |y_k|^q)^{1/1}$$

- *(Minkowski's inequality) For any number $p \in [1, \infty)$*

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{1/p} \le (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} + (\sum_{k=1}^{\infty} |y_k|^p)^{1/p}$$

Remark 1.7.4 (application of Minkowski's inequality). The Minkowski's inequality is mainly used to prove the triangle equality for the norm, that is,

$$\|x + y\| \le \|x\| + \|y\|,$$

where we define $\|x\| = (\sum_{k=1}^{\infty} |y_k|^p)^{1/p}$.

1.7.3.2 *Cauchy-Schwarz inequality*

Lemma 1.7.3 (Cauchy-Schwarz inequality for finite real-valued terms, recap). *[1, p. 120] Let $a_1, a_2, ..., a_n \in \mathbb{R}$ and $b_1, b_2, ..., b_n \in \mathbb{R}$. Then*

$$\left| \sum_{k=1}^{n} a_k b_k \right| \le \sqrt{(\sum_{k=1}^{n} a_k^2) \sum_{k=1}^{n} b_k^2)}$$

Proof. See Corollary 5.3.1.1 and simply use Holder's inequality with $p = q = 2$. □

Lemma 1.7.4 (absolute convergence of product series). *[1, p. 123] Let l^2 denote the set of all real sequence $\{c_k\}$ such that $\sum_{k=0}^{\infty} c_k^2$ converges. If $\{a_k\}$, $\{b_k\} \in l^2$, then*

$$\sum_{k=1}^{\infty} a_k b_k$$

converges absolutely.

Proof. For every positive integer n, therefore

$$\begin{aligned} \sum_{k=1}^{n} |a_k b_k| &\le \sqrt{(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)} \\ &\le \sqrt{(\sum_{k=1}^{\infty} a_k^2)(\sum_{k=1}^{\infty} b_k^2)}. \end{aligned}$$

Note that the right side is bounded because $\sum_{k=1}^{\infty} a_k^2, \sum_{k=1}^{\infty} b_k^2$ converge. Finally, the monotone convergence theorem [Theorem 1.5.1] ensures that $\sum_{k=1}^{\infty} |a_k b_k|$. Or we simply use comparison test [Theorem 1.7.3]. □

Theorem 1.7.8 (Cauchy-Schwarz inequality for l_2 series). *[1, p. 123] Let l^2 denote the set of all real sequence $\{c_k\}$ such that $\sum_{k=0}^{\infty} c_k^2$ converges. If $\{a_k\}, \{b_k\} \in l^2$, then*

$$\sum_{k=1}^{\infty} a_k b_k$$

converges absolutely and

$$\left|\sum_{k=1}^{\infty} a_k b_k\right| \leq \sqrt{(\sum_{k=1}^{\infty} a_k^2) \sum_{k=1}^{\infty} b_k^2)} = M$$

Proof. Let $B_n = \sum_{k=1}^{n} a_k b_k$. We note that B_n converges due to $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely [Theorem 1.7.1]. Using the absolute-value rule [Lemma 1.4.1], we have that $|B_n|$ converges. Let

$$\lim_{n\to\infty} |B_n| = L.$$

Because $|B_n| \leq M$ for every n(Cauchy inequality Corollary 5.3.1.1], the limit-inequality rule [Lemma 1.4.2] gives that $L \leq M$. □

1.7.4 Alternating series

Theorem 1.7.9 (alternating series test). *[1, p. 25] Let $\{a_n\}$ be a **decreasing** sequence such that $\lim_{n\to\infty} a_n = 0$.Then*

- $$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

  converges.
- *Let L be the limit and s_n be the partial sum, then*

  $$|s_n - L| \leq a_{n+1}$$

Proof. (1)Note that

$$s_{2n} = (a_1 - a_2) + (a_3 - a_4) + ... + (a_{2n-1} - a_{2n})$$

is a monotone sequence in terms of n. Moreover,

$$s_{2n} = a_1 - (a_2 - a_3) - ... - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1.$$

Therefore, s_{2n} is monotone and bounded; therefore converges [Theorem 1.5.1) . Similarly, s_{2n-1} will converge since $s_{2n-1} = s_{2n} + a_{2n}$ and algebraic property of limits [Lemma 1.4.1). Since s_{2n} and s_{2n-1} converge, s_n must converge.

(2)

$$\begin{aligned} s_n - L &= s_n - \sum_{i=1}^{\infty} (-1)^{i+1} a_i \\ &= (-1)^{n+2} a_{n+1} + (-1)^{n+3} a_{n+2} + ... \\ &\begin{cases} \leq a_{n+1}, \; if \; n \; is \; even \\ \geq -a_{n+1}, \; if \; n \; is \; odd \end{cases} \end{aligned}$$

where we use analog result from(1) that $\lim_{n\to\infty} s_{2n} \leq a_1, \lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n} \leq a_1$ and $\lim_{n\to\infty} s_n \leq a_1$. □

Remark 1.7.5 (the oscillation mode). Note that s_n will oscillate as a function of n.

Remark 1.7.6 (application in approximation). Given an alternating series, we can approximate the limit using partial sum s_n, which contains finite terms and the error is bounded by a_{n+1}.

Corollary 1.7.9.1. *[1, p. 25] If $s > 0$, then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s}$$

converges.

1.8 Notes on bibliography

The key references for this chapter are intermediate level real analysis textbooks[1][3][7][8].

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