19

STOCHASTIC CALCULUS

19.1 Ito stochastic integral

19.1.1 Motivation

In this chapter, we introduce Ito stochastic integral, or simply Ito integral. Stochastic integral aims to evaluate integrals of taking form

$$\int_0^T g(W_t, t) dW_t,$$

where W_t is a Brownian motion and g is a function on W_t and t.

Stochastic integral allows us to model more complex stochastic processes via solving stochastic differential equations such as

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t.$$

The solution is given by

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s,$$

which consists of an ordinary Riemman integral and a stochastic integral. It is possible to define integrals with respect to other stochastic process X_t (e.g., Poisson jump process), i.e.,

$$\int_0^T g(W_t, t)dX_t,$$

however, it is out of scope of this book.

19.1.2 Construction of Ito integral

19.1.2.1 *Ito integral of a simple process*

Like constructing ordinary Riemman integral, we start with the definition of Ito integral with respect to simple stochastic processes.

Definition 19.1.1 (simple process). *The stochastic process $C_t, t \in [0, T]$ is said to be **simple** if: there exists a partition*

$$\tau_n : 0 = t_0 < t_1 < ... < t_n = T$$

and a sequence of random variables $Z_i, i = 1, 2, ..., n$ such that

$$C_t = \begin{cases} Z_n, if\ t = T, \\ Z_i, if\ t_{i-1} \leq t < t_i, i = 1, 2, ..., n \end{cases}$$

*The sequence Z_i is adapted to $\mathcal{F}_{t_{i-1}}$ and $E[Z_i^2] < \infty$, i.e., the sequence Z_i is a **previsible process**.* [a]

[a] The implication of previsiblity is that Z_i is independent from Brownian motion increment $W(s) - W_{t_{i-1}}, s > t_{t-1}$

Example 19.1.1.

- A constant function $f(t) = c \in \mathbb{R}$ is a simple process.
- The deterministic function

$$f(t) = \begin{cases} \frac{n-1}{n}, & \text{if } t = T \\ \frac{i-1}{n}, & \text{if } \frac{i-1}{n}T \leq t < \frac{i}{n}T, i = 1, \ldots, n \end{cases}$$

defined on $[0, T]$ is a simple process. The associated partition is

$$0 = t_0 < t_1 < \cdots < t_n = T$$

where $t_i = \frac{i}{n}T$.

Definition 19.1.2 (Ito integral for simple processes). *Let C_t be a simple process on $[0, T]$ and its associated partition be $\tau_n : 0 = t_0 < t_1 < ... < t_n = T$. The Ito integral is defined as*

$$\int_0^T C_s dW_s = \sum_{i=1}^{n} C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}).$$

Note that definition of Ito integral requires that simple function, C_t, is evaluated at the left-end point during each the interval $[t_i, t_{i+1}]$.

The integral $\int_0^T C_s dW_s = \sum_{i=1}^n C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$ can be interpreted naturally as a trading strategy in finance. Let C_t be a trading strategy (holding amount of a stock C_t) on time t and assume the trader can only change its position at the beginning of each interval. Let W_t be a stock price process. Then

$$I_t(C) = \int_0^t C_s dW_s$$

describes the total gain or loss during period $[0, T]$, since $C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$ is the gain or loss during interval $[t_{i-1}, t_i)$.

Note that the sequence of Ito integral

$$\int_0^{t_k} C_s dW_s, k = 0, 1, 2, ..., n$$

of a simple process C_s is indeed a martingale transform [Lemma 18.6.3] with respect to the Brownian filtration $\mathcal{F}_{t_k}$. Therefore the stochastic process $I_t(C) = \int_0^t C_s dW_s$ is a martingale with respect to the Brownian filtration $\mathcal{F}_t$.

Example 19.1.2. We consider the evaluation of the integral $I = \int_0^T cdW_t, c \in \mathbb{R}$ based on its definition. Let $= t_0 < t_1 < t_2 < \ldots < t_n = T$ be a partition of $[0, T]$, and define a simple process

$$X_t := \sum_{i=0}^{n-1} cI_{[t_i, t_{i+1})}(t),$$

where I is an indicator function. Based on the definition of Ito integral, we have

$$I = \int_0^T cdW_t = \int_0^T X_t^n dW_t = \sum_{i=0}^{n-1} c\left(W_{t_{i+1}} - W_{t_i}\right) = c(W_T - W_0).$$

Note that $W_0 = 0$ and we have

$$I = cW_T.$$

Now we discuss basic properties for Ito integral of a simple process

Theorem 19.1.1 (basic properties for Ito integral of a simple process). *Let C_s be a simple process. The Ito stochastic integral of this simple process $I_t = \int_0^t C_s dW_t$ satisfies the following properties:*

- *zero mean $E[I_t] = 0$.*
- *variance via isometry property:*

$$Var[I_t] = E[I_t^2] = E[(\int_0^t C_s dW_s)^2] = E[\int_0^t C_s^2 ds].$$

- *partition property:*

$$\int_S^T C_t dW_t = \int_S^u C_t dW_t + \int_u^T C_t dW_t$$

if $S < u < T$

- *linearity: Let C_t and D_t be two simple processes,*

$$\int_0^T (\alpha C_s + \beta D_s) dW_s = \alpha \int_0^T dW_s + \beta \int_0^T D_s dW_s,$$

where $\alpha, \beta \in \mathbb{R}$.

Proof. (1) Note the $C_{t_{i-1}}$ independent from Brownian motion increment $W(t_i) - W_{t_{i-1}}$ based on definition of simple processes. It has zero mean since

$$E[C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})] = E[C_{t_{i-1}}]E[(W_{t_i} - W_{t_{i-1}})] = 0.$$

(2) By definition, for simple processes:

$$\int_0^t C_s dW_s = \sum_{i=1}^n Z_i \Delta_i W$$

where $\Delta_i W = W(t_i) - W(t_{i-1})$. Then

$$\begin{aligned} E[(\int_0^t C_s dW_s)^2] &= E[\sum_{i=1}^n \sum_{j=1}^n Z_i \Delta_i W Z_j \Delta_j W] \\ &= E[\sum_{i=1}^n (Z_i)^2 (\Delta_i W)^2] \\ &= \sum_{j=1}^n E[(Z_i)^2](t_i - t_{i-1}) \\ &= \int_0^t E[C_s^2] ds = E[\int_0^t C_s^2 ds], \end{aligned}$$

where we use the fact $E[(\Delta_i W)^2] = t_i - t_{i-1}$. (3)(4) Directly from definition. □

19.1.2.2 *Ito integral of a general process*

Now we are in a position to generalize the definition of Ito integral of a simple process to Ito integral of a general process C.

The basic idea is to construct a sequence of simple processes $C^{(1)}, C^{(2)}, ...$ that converge to C in the mean squared sense (which implies convergence in probability and distribution, Theorem 11.10.4). Then we define

$$\int_0^T C dW_t = \lim_{n \to \infty} \int_0^T C_s^{(n)} dW_s.$$

More formally, we have the following definition.

Definition 19.1.3 (Ito integral of a general process). *Let C be a stochastic process that is adapted to Brownian filtration on $[0,T]$ and $\int_0^T E[C^2]dt$ is finite. Then*

$$\int_0^T CdW = \lim_{n\to\infty} \int_0^T C_s^{(n)} dW,$$

where $C_s^{(n)}$ is a sequence of simple process approximating C in the mean square sense.

The definition of $\int_0^T CdW$ relies on the existence of simple processes $C^{(1)}, C^{(2)}, ...$ that converge to C. The following theorem guarantees its existence.

Theorem 19.1.2 (existence of approximating simple process). *Let C be a stochastic process that is adapted to Brownian filtration on $[0,T]$ and $\int_0^T E[C^2]dt$ is finite. Then there exist a sequence of simple process $C^{(1)}, C^{(2)}, ...$ such that*

$$\lim_{n\to\infty} \int_0^T E[(C_s^{(n)} - C)^2]dt = 0,$$

which indicates C_s^n converges to C in the mean squared sense.

Proof. see[1, p. 109]. □

Remark 19.1.1 (understanding adapted process).

- Because C_t is an adapted process, so it cannot be a process depending on the future of Brownian motion $W_{t'}, t' > t$. For example, we cannot allow $C_t = W_T$.
- Here the process C_t is adapted to the Brownian motion filtration also indicates that C_t is independent of the increment $B_{t'} - B_t$ with $t' > t$.

Example 19.1.3. We now consider the evaluation of the integral $I = \int_0^T W_t dW_t$. Let $= t_0 < t_1 < t_2 < \ldots < t_n = T$ be a partition of $[0,T]$, and define a simple process

$$X_t = \sum_{i=0}^{n-1} W_{t_i} I_{[t_i, t_{i+1})}(t).$$

X_t^n is an adapted simple process satisfies $\lim_{n\to\infty} E[(X_t - W_t)^2] = 0$ as $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$.

Based on definition of Ito integral, we have

$$\begin{aligned}\int_0^T X_t dW_t &= \sum_{i=0}^{n-1} W_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right) \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\left(W_{t_{i+1}}^2 - W_{t_i}^2 - \left(W_{t_{i+1}} - W_{t_i}\right)^2\right) \\ &= \frac{1}{2}W_T^2 - \frac{1}{2}W_0^2 - \frac{1}{2}\sum_{i=0}^{n-1}\left(W_{t_{i+1}} - W_{t_i}\right)^2\end{aligned}$$

Note that $W_0 = 0$ and

$$\sum_{i=0}^{n-1}\left(W_{t_{i+1}} - W_{t_i}\right)^2 \to T$$

in the mean squared sense as $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$ [using quadratic variation property of Brownian motion, Theorem 18.3.5], we get

$$\int_0^T W_t dW_t = \lim_{n\to\infty}\int_0^T X_t^n dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

Example 19.1.4. Now consider the evaluation of the integral

$$\int_0^t e^{-cs} dW_s.$$

Let $= t_0 < t_1 < t_2 < \ldots < t_n = T$ be a partition of $[0, T]$, and define a simple process

$$X_t = \sum_{i=0}^{n-1} e^{-ct_i} I_{[t_i, t_{i+1})}(t).$$

Clearly X_t converges to e^{-cs} as $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$.

Note that

$$S = \sum_{i=1}^{n} e^{-ct_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right)$$

is a normal distribution since it is a sum of independent Brownian motion increments. It has zero mean and a variance of

$$Var[S] = \sum_{i=1}^{n} e^{-2ct_{i-1}} (t_i - t_{i-1}).$$

As $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$, the variance of S approach a Riemann integral

$$\int_0^t e^{-2cs} ds = \frac{1}{2c}\left(1 - e^{-2ct}\right).$$

Taken together,

$$\int_0^t e^{-cs} dW_s$$

is has a normal distribution given by $N(0, \frac{1}{2c}\left(1 - e^{-2ct}\right))$.

Lemma 19.1.1 (basic properties).

- *$\int_0^T c dW_t = cW_T$, where c is a constant.*
- *$I_s = \int_0^s W_t dW_t = 0.5(W_s^2 - s)$ is a martingale, and $E[I_s] = E[I_0] = 0$.*
- *Let g be an square-integrable adapted process to the Brownian filtration $\{\mathcal{F}_t\}$ generated by Brownian motion $W(s)$. Then $I(t) = \int_0^t g(s) dW(s)$ is a continuous square-integrable martingale.* [a]

[a] an adapted process means either deterministic process or stochastic process represented by $dg_t = \mu(g(t), t)dt + \sigma(g(t), t)dW(t)$.

Proof. (1) recognize that this is a Wiener integral [Theorem 19.1.4] on the left, which will produce a normal distribution of $N(0, \int_0^T c^2 dt)$. The Right side has the exact same distribution. (2) Let $Y_t = W_t^2$, and then

$$dY_t = 2W_t dW_t + dt$$

Integrate both sides, we have

$$W_T^2 = 2\int_0^T W_t dW_t + T.$$

(3)

$$dI_t = g(t)dW(t) + \int_0^t dg(s)dW(s) = g(t)dW(t)$$

where we ignore $\int_0^t dg(s)dW(s)$ since it is of order $(O(t))$.

□

Theorem 19.1.3 (Properties of Ito integral). *[2, p. 100][3] Let $f(W_t, t), g(W_t, t)$ be adapted processes and $c \in \mathbb{R}$, then we have*

1. *partition property:*
$$\int_S^T f dW_t = \int_S^u f dW_t + \int_u^T f dW_t$$
if $S < u < T$
2. *linearity:*
$$\int_S^T (cf + dg) dW_t = c\int_S^T f dW_t + d\int_S^T g dW_t$$
3. *zero mean:*
$$E[\int_S^T f(W_t, t) dW_t] = 0$$
4. *Isometry:*
$$E[(\int_a^b f(W_t, t) dW_t)^2] = E[\int_a^b f(W_t, t)^2 dt]$$
5. *Covariance:*
$$E[(\int_a^b f(W_t, t) dW_t)(\int_a^b g(W_t, t) dW_t)] = E[\int_a^b f(W_t, t) g(W_t, t) dt]$$

19.1.3 Ito integral with deterministic integrands (Wiener integral)

19.1.3.1 *Basics*

In this section, we discuss a special type of Ito integral given by

$$I = \int_0^t g(s) dW_t,$$

where $g(t)$ is a deterministic function. This type of integral is known as **Wiener integral**. The application of Wiener integral is ubiquitous in stochastic calculus. The result can be derived from the definition of Ito integral.

Theorem 19.1.4 (Wiener integral). *Suppose $g : [0, \infty) \to \mathbb{R}$ is a bounded, piece-wise continuous function in L^2. Let W_t be a Brownian motion and denote*

$$X_t = \int_0^t g(s) dW_s.$$

It follows that

- X_t *is a random variable which has a distribution*

$$N(0, \int_0^t g^2(s)ds).$$

- $\{X_t\}$ *is a zero mean Gaussian process with covariance structure*

$$Cov(X_t, X_s) = \int_0^{\min(t,s)} g^2(u)du.$$

- *In particular,*

$$\int_0^t dW_s = W_t \sim N(0, t).$$

Proof. (1) Let $= t_0 < t_1 < t_2 < \ldots < t_n = T$ be a partition of $[0, t]$, and define a simple process

$$S_t = \sum_{i=0}^{n-1} g(t_i) I_{[t_i, t_{i+1})}(t).$$

Clearly X_t converges to $g(t)$ as $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$.

Note that

$$M = \sum_{i=1}^{n} g(t_{i-1}) \left(W_{t_i} - W_{t_{i-1}} \right)$$

is a normal distribution since it is a sum of independent Brownian motion increments. It has zero mean and a variance of

$$Var[M] = \sum_{i=1}^{n} g(t_{i-1}) \left(t_i - t_{i-1} \right).$$

As $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$, the variance of M approach a Riemann integral

$$\int_0^t g(s)^2 ds.$$

(2) We have shown that $X(t) \sim N(0, \int_0^t g(s)^2 ds)$. Also note that the increment is independent and Gaussian; that is $X(t_1) - X(t_2)$ is independent of $X(t_2) - X(t_3), t_1 > t_2 > t_3$. Therefore, the random vector $(X(t_1), X(t_2), ..., X(t_n))$ is multivariate normal since it can be constructed by affine transformation of $(X(t_1) - X(t_2), X(t_2) - X(t_3), ..., X(t_n)$ [Theorem 14.1.1]. Therefore $X(t)$ is a Gaussian process. Its variance can be evaluated via

$$E[\int_0^t g(u)dW_u \int_0^s g(v)dW_v] = E[\int_0^{\min(t,s)} g(u)dW_u \int_0^{\min(t,s)} g(v)dW_v] = \int_0^{\min(t,s)} g^2(u)du.$$

(3) is directly from (1). □

Corollary 19.1.4.1 (Gaussian process stochastic differential equation). *Consider a stochastic process X_t governed by*

$$dX_t = a(t)dt + b(t)dW_t,$$

where W_t Brownian. It follows that

$$X(t) \sim N(\int_0^t a(s)ds, \int_0^t b(s)^2 ds)$$

and $X(t)$ is a Gaussian process.

Example 19.1.5. Consider stochastic process

$$X_t = \int_0^t \frac{1}{1-s} dW_s,$$

where W_t is the Wiener process. Then we have

- X_t is a Gaussian process.
- $E[X_t] = 0$.
- $Var[X_t] = E[X_t^2] - E[X_t]^2 = E[X_t^2]$, and

$$E[X_t^2] = \int_0^t \frac{1}{(1-s)^2} ds$$

  via Ito Isometry.

Caution!
We know that

$$Z_t = \int_0^t \alpha g(s) + \beta h(s) dW_s \sim N(0, \int_0^t (\alpha g(s) + \beta h(s))^2 dt)$$

however,

$$X_t = \alpha \int_0^t g(s) dW_s \sim N(0, \alpha^2 \int_0^t g(s)^2 ds), Y_t = \beta \int_0^t h(s) dW_s \sim N(0, \beta^2 \int_0^t h(s)^2 ds)$$

Note that X_t and Y_t are **not independent to each other because they are generated from the same Brownian motion.**

It is straight forward to arrive at the following linearity properties.

Theorem 19.1.5 (linearity of Wiener integral). *let $W(t)$ be a Brownian process, let $g, h, m : [0, \infty) \to \mathbb{R}$ be a bounded, piecewise continuous function in L^2. Then*

$$\alpha \int_0^t g(s)dW_s + \beta \int_0^t h(s)dW_s = \int_0^t \alpha g(s) + \beta h(s)dW_s.$$

In particular,

$$m(t)W_t - \int_0^t h(s)dW_s = \int_0^t (m(t) - h(s))dW_s \sim N(0, \int_0^t (m(t) - h(s))^2 ds)$$

19.1.3.2 *Integration by parts*

In this section, we discuss an import tool from Wiener integral - integration by parts. This tool enables us to analytically evaluate challenging integrals like

$$\int_0^t W(s)ds, \int_0^t s^n W(s)ds, \int_0^t f(s)W(s)ds.$$

Theorem 19.1.6 (Wiener integral: integration by part). *Let $g : [0, \infty) \to \mathbb{R}$ be a bounded, continuously differentiable function in $L^2([0, \infty))$. The integration-by-parts formula is given by*

$$\int_0^t g(s)dW_s = g(t)W_t - \int_0^t g'(s)W_s ds.$$

By re-arranging, we have

$$\int_0^t g'(s)W_s ds = g(t)W_t - \int_0^t g(s)dW_s.$$

Proof. Let $= t_0 < t_1 < t_2 < \ldots < t_n = T$ be a partition of $[0, t]$. First, we can write

$$\sum_{j=1}^n g\left(t_{j-1}\right)\left(W_{t_j} - W_{t_{j-1}}\right) = \sum_{j=1}^n g\left(t_{j-1}\right) W_{t_j} - \sum_{j=1}^n g\left(t_{j-1}\right) W_{t_{j-1}}$$

Because g is differentiable, the mean value theorem implies that there exists some value $t_j^* \in [t_{j-1}, t_j)$ such that

$$g'\left(t_j^*\right)\left(t_j - t_{j-1}\right) = g\left(t_j\right) - g\left(t_{j-1}\right)$$

Substituting this for $g\left(t_{j-1}\right)$ in the previous expression (9.2) gives

$$\begin{aligned}
&\sum_{j=1}^{n} g\left(t_{j-1}\right) W_{t_j} - \sum_{j=1}^{n} g\left(t_{j-1}\right) W_{t_{j-1}} \\
=&\sum_{j=1}^{n} g\left(t_j\right) W_{t_j} - \sum_{j=1}^{n} g'\left(t_j^*\right)\left(t_j - t_{j-1}\right) W_{t_j} - \sum_{j=1}^{n} g\left(t_{j-1}\right) W_{t_{j-1}} \\
=&\sum_{j=1}^{n}\left[g\left(t_j\right) W_{t_j} - g\left(t_{j-1}\right) W_{t_{j-1}}\right] - \sum_{j=1}^{n} g'\left(t_j^*\right)\left(t_j - t_{j-1}\right) W_{t_j} \\
=&g\left(t_n\right) W_{t_n} - g\left(t_0\right) W_{t_0} - \sum_{j=1}^{n} g'\left(t_j^*\right) W_{t_j}\left(t_j - t_{j-1}\right) \\
=&g(t)W_t - \sum_{j=1}^{n} g'\left(t_j^*\right) W_{t_j}\left(t_j - t_{j-1}\right)
\end{aligned}$$

where $t_n = t$ and $W_{t_0} = 0$. So far we have established

$$\sum_{j=1}^{n} g\left(t_{j-1}\right)\left(W_t, -W_{t_{j-1}}\right) = g(t)W_t - \sum_{j=1}^{n} g'\left(t_j^*\right) W_{t_j}\left(t_j - t_{j-1}\right).$$

Now we take $n \to \infty$ and $\max_i |t_{i+1} - t_i| \to 0$, and we have

$$\sum_{j=1}^{n} g\left(t_{j-1}\right)\left(W_t, -W_{t_{j-1}}\right) \to \int_0^t g(s) dW_s$$

and

$$\sum_{j=1}^{n} g'\left(t_j^*\right) W_{t_j}\left(t_j - t_{j-1}\right) \to \int_0^t g'(s) W_s ds.$$

□

Corollary 19.1.6.1. *Let $W(t)$ be a Brownian motion, then*

$$\int_0^t W(s)ds = W(t)t - \int_0^t s dW_s \sim N(0, \int_0^t (t-s)^2 ds).$$

Proof. Using integration by parts theorem [Theorem 19.1.6] and setting $g(x) = x$, we have

$$\int_0^t W(s)ds = W(t)t - \int_0^t sdW_s \sim N(0, \int_0^t (t-s)^2 ds).$$

where we have used linearity of Wiener integral [Theorem 19.1.5]. □

Example 19.1.6. Let $W(t)$ be the Wiener process, then

- $$\int_0^1 W(s)ds = W(1) - \int_0^1 sdW_s \sim N(0, \int_0^1 (1-s)^2 ds) = N(0, \frac{1}{3}).$$
- $$\int_0^T W(s)ds = \frac{T}{\sqrt{3}} W(T) \sim N(0, \frac{T^3}{3})$$
- $$\int_0^t g'(s)W_s ds \sim N(0, \int_0^t [g(t) - g(s)]^2 ds)$$
- $$\int_0^1 s^n W_s ds \sim N(0, \frac{2}{(2n+3)(n+2)}), n = 0, 1, 2, ...$$

  (1) is straight forward. (2) Since

$$d(sW(s)) = W_s ds + sdW_s,$$

we have

$$TW_T) = \int_0^T W_s ds + \int_0^T sdW_s.$$

Rearrange, we have

$$\int_0^T W_s ds = \int_0^T (T-s)dW_s \sim N(0, \int_0^T (T-s)^2 ds) = N(0, \frac{T^3}{3}).$$

(3) Let $f(W_t, t) = g(t)W_t$. (4) is direct result from (2).

19.2 Stochastic differential equations

19.2.1 Ito Stochastic differential equations

Definition 19.2.1 (Ito SDE). *[1, p. 137]An Ito stochastic differential equation is defined as*

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

which could be interpret as

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s$$

where the first integral is Riemann integral, the second is Ito integral.

A straight forward interpretation of SDE is: For each sample path of $X_t(\omega)$, its change in temporal direction $dX_t = X_{t+dt} - X_t$ is comprised of a drift term $a(t, X_t)dt$ and Brownian motion term $dW_t = W_{t+dt} - W_t$ scaled by $b(t, X_t)$.

The following theorem ensure the existence of a solution. Note that in this book, we limit our discussion to weak solutions, which roughly require the SDE holds in the sense of distribution, as opposed to strong solutions, which roughly require the SDE to hold in the sense of sample path. To see the distinction, consider two different Brownian motions W_1, W_2. $W_1(t)$ is the weak solution of $dX_t = dW_1(t)$ and $dX_t = dW_2(t)$. However, $W_1(t)$ is the strong solution of $dX_t = dW_1(t)$ but not the strong solution to $dX_t = dW_2(t)$ since W_1 and W_2 have different sample paths.

Theorem 19.2.1 (existence). *[1, p. 138] Assume the initial condition X_0 has a finite second moment: $EX_0^2 < \infty$, and is independent of $(W_t, t \geq 0)$. Assume that, for all $t \in [0, T], x, y \in \mathbb{R}$, the coefficient functions $a(t, x)$ and $b(t, x)$ satisfy the following conditions:*

- *They are continuous*
- *They satisfy a Lipschitz condition with respect to the second variable:*

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|$$

Then the Ito stochastic differential equation has a unique solution X on $[0, T]$.

Theorem 19.2.2 (linear stochastic differential equation).

$$X_t = X_0 = \int_0^t (c_1 X_s + c_2) ds + \int_0^t (\sigma_1 X_s + \sigma_2) dW_s$$

for constants c_i and σ_i is called linear SDE. The Linear SDE has an unique solution.

Proof. It is easy to show that the continuous condition and Lipschitz condition are satisfied. □

19.2.2 Ito's lemma

Based on definition and basic algebra properties of Ito calcalus, we can solve SDE of forms

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

Its solution is

$$X_t = S_0 = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s.$$

But for SDE of more complex forms, such as

$$S_t - S_0 = \int_0^t \mu_s S_s ds + \int_0^t \sigma_s S_s dW_s,$$

and

$$S_t - S_0 = \int_0^t k(\mu - S_s) ds + \int_0^t \sigma_s dW_s,$$

evaluation of stochastic integral from definition is tedious. Like the chain rule in ordinary calculus, we now develop a similar tool to convert them to simpler SDEs.

Theorem 19.2.3 (Ito's lemma). *Let W_t be a Brownian motion on $[0, T]$ and suppose $f(x)$ is a twice continuously differentiable function on* **R**. *Then for any $t \leq T$ we have*

$$f(W_t) = f(0) + \frac{1}{2} \int_0^t f_{WW}(W_s)\, ds + \int_0^t f_W(W_s)\, dW_s,$$

which is equivalently written as

$$df = \frac{\partial f}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 f}{\partial W_t^2} dt.$$

Let $f(W_t, t)$ be a function of Brownian motion W_t and time t, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial W_t^2}dt.$$

Proof. Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = t$ be a partition of $[0, t]$. Denote $f_W = \frac{\partial f}{\partial W_t}, f_{WW} = \frac{\partial^2 f}{\partial W_t \partial W_t}$. Clearly

$$f(W_t) = f(0) + \sum_{i=0}^{n-1}\left(f\left(W_{t_{i+1}}\right) - f\left(W_{t_i}\right)\right).$$

Using Taylor's Theorem, we have each term in the summation given by

$$f\left(W_{t_{i+1}}\right) - f\left(W_{t_i}\right) = f_W\left(W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) + \frac{1}{2}f_{WW}\left(\theta_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)^2$$

for some $\theta_i \in \left(W_{t_i}, W_{t_{i+1}}\right)$. Now we have

$$f(W_t) = f(0) + \sum_{i=0}^{n-1} f_W\left(W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) + \frac{1}{2}\sum_{i=0}^{n-1} f''\left(\theta_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)^2$$

If we let $n \to \infty$ and $\delta = \max_i |t_{i+1} - t_i| \to 0$, then

$$\sum_{i=0}^{n-1} f_W\left(W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) \to \int_0^T f_W(s)dW_s$$

and

$$\frac{1}{2}\sum_{i=0}^{n-1} f_{WW}\left(\theta_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)^2 \to \int_0^t f_{WW}(s)ds.$$

where we have used the quadratic variation of Brownian motion

$$\left(W_{t_{i+1}} - W_{t_i}\right)^2 \to (t_{i+1} - t_i)$$

in the mean squared sense [Theorem 18.3.6]. □

We can similarly extend Ito's lemma for more general Ito stochastic process.

Theorem 19.2.4 (extended Ito's lemma). *Let $f(X_t, t)$ be a function of stochastic process X_t governed by $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$, then*

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 \\ &= (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}\sigma^2 + \frac{\partial f}{\partial X_t}\mu)dt + \frac{\partial f}{\partial X_t}\sigma dW_t \end{aligned}$$

Example 19.2.1. To evaluate $I = \int_0^T W_t dW_t$, we let $Y_t = W_t^2$, and then

$$dY_t = 2W_t dW_t + dt.$$

Integrate both sides, we have

$$Y_T - Y_0 = 2\int_0^T W_t dW_t + T,$$

which gives

$$W_T^2 = 2\int_0^T W_t dW_t + T.$$

In other words,

$$\int_0^T W_t dW_t = \frac{1}{2}(W_T^2 - T).$$

Example 19.2.2. To evaluate integral $\int_s^t W_s^2 dW_s$, we let $Y_t = W_t^3$ and use Ito's Lemma to get

$$dY_t = 3W_t^2 dW_t + 3W_t dt$$

$$W_t^3 - W_0^3 = 3\int_s^t W_s^2 dW_s + 3\int_0^t W_s ds$$

The solution to $\int_0^t W_s ds$ is addressed in Theorem 19.1.6.

Example 19.2.3. $X_t = W_t^3$, then $dX_t = 3W_t^2 dW_t + 3W_t dW_t dW_t = 3W_t^2 dW_t + 3W_t dt$.

Example 19.2.4. $Y_t = \ln(W_t)$, then $dY_t = dW_t/W_t - \frac{1}{2}dt/W_t^2$.

We can also extend Ito's lemma to multi-dimensional stochastic process.

Theorem 19.2.5. *Let $f(W_{1,t}, W_{2,t}, ..., W_{n,t}, t)$ be a function of Brownian motion $W_{1,t}, W_{2,t}, ..., W_{n,t}$, then*

$$df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial W_{i,t}} dW_{i,t} + \sum_i \sum_j \frac{1}{2} \frac{\partial^2 f}{\partial W_{i,t} \partial W_{j,t}} D_{ij} dt$$

where we assume $E[dW_{i,t}dW_{j,t}] = D_{ij}dt$.

19.2.3 Useful results of Ito's lemma

19.2.3.1 *Product rule and quotient rule*

Lemma 19.2.1 (product rule and quotient rule). *Consider*

$$\begin{aligned} dX_t/X_t &= r_1 dt + \sigma_1 dW_1 \\ dY_t/Y_t &= r_2 dt + \sigma_2 dW_2 \\ dW_1 dW_2 &= \rho dt \end{aligned}$$

It follows that

- *Given $Z_t = X_t Y_t$, we have*

$$\begin{aligned} dZ_t = d(X_t Y_t) &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t Y_t ((r_1 + r_2 + \rho\sigma_1\sigma_2)dt + (\sigma_1 dW_1 + \sigma_2 dW_2)) \end{aligned}$$

- *Given $Z_t = X_t/Y_t$, we have*

$$\begin{aligned} dZ_t = d(X_t/Y_t) &= dX_t/Y_t - X_t dY_t/(Y_t)^2 - dX_t dY_t/(Y_t)^2 + X_t(dY_t)^2/(Y_t)^3 \\ &= (X_t/Y_t)((r_1 - r_2 - \rho\sigma_1\sigma_2 + \sigma_2^2)dt + (\sigma_1 dW_1 - \sigma_2 dW_2)) \end{aligned}$$

- *Given $Z_t = 1/X_t$, we have*

$$\begin{aligned} dZ_t = d(1/X_t) &= -dX_t/(X_t)^2 + (dX_t)^2/(X_t)^3 \\ &= (1/X_t)((-r_1 + \sigma_1^2)dt - \sigma_1 dW_1) \end{aligned}$$

Note that we have to calculate the Hessian for $f(x,y) = x/y$, and there are two terms for the cross-term.

Proof. (1)

$$dZ_t = \frac{\partial Z_t}{\partial X_t} dX_t + \frac{\partial Z_t}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 Z_t}{\partial X_t^2} dX_t dX_t + \frac{1}{2}\frac{\partial^2 Z_t}{\partial Y_t^2} dY_t dY_t + \frac{\partial^2 Z_t}{\partial X_t \partial Y_t} dX_t dY_t.$$

(2)(3) Same as (1). □

19.2.3.2 *Logarithm and exponential*

Lemma 19.2.2 (Ito lemma applied to logorithm and exponential). *Let $X(t)$ be an Ito stochasti process.*

- *If $Y_t = \exp(X(t))$, then*

$$dY_t = Y_t dX_t + \frac{1}{2} Y_t dX_t dX_t.$$

- *If $Z_t = \ln(X(t))$, then*

$$dZ_t = \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2} dX_t dX_t.$$

Proof. (1)

$$\begin{aligned} dY_t &= \exp(X_t) dX_t + \frac{1}{2}\exp(X_t) dX_t dX_t \\ &= Y_t dX_t + \frac{1}{2} Y_t dX_t dX_t. \end{aligned}$$

(2)

$$dZ_t = \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2} dX_t dX_t.$$

□

19.2.3.3 *Ito integral by parts*

Lemma 19.2.3 (Ito integral by parts). *Let $X(t), Y(t)$ be two Ito processes. Then*

$$\int_s^u Y(t)dX(t) = X(u)Y(u) - X(s)Y(s) - \int_s^u X(t)dY(t) - \int_s^u dX(t)dY(t)$$

Proof. From the product rule, we have

$$\begin{aligned} \int_s^u d[X(t)Y(t)] &= \int_s^u Y(t)dX(t) + \int_s^u X(t)dY(t) + \int_s^u dX(t)dY(t) \\ X(u)Y(u) - X(s)Y(s) &= \int_s^u Y(t)dX(t) + \int_s^u X(t)dY(t) + \int_s^u dX(t)dY(t) \\ \int_s^u Y(t)dX(t) &= X(u)Y(u) - X(s)Y(s) - \int_s^u X(t)dY(t) - \int_s^u dX(t)dY(t) \end{aligned}$$

Note that This integral-by-part formula is the same as Riemann integral except for the extra term $\int_s^u dX(t)dY(t)$. □

19.2.3.4 *Differentiate integrals of Ito process*

Lemma 19.2.4 (integrand is an Ito stochastic process). *Let $r(t)$ be an Ito stochastic process.*

- *If $X_t = \int_0^t r(s)ds$, then*
$$dX_t = r(t)dt.$$
- *If $Y_t = \exp(X_t)$, then*
$$dY_t = Y_t r(t)dt.$$

Proof. (1) Let Ω be the sample space associated with the stochastic process $r(t)$. Then for each sample path $\omega \in \Omega$, we have $X_t(\omega) = \int_0^t r(s,\omega)ds$ and $dX_t(\omega) = r(t,\omega)ds$. Since $dX_t(\omega) \triangleq \lim_{dt\to 0} X(t+dt,\omega) - X(t,\omega)$ and $r(t,\omega)ds$ are both random variables for fixed t, if they are equal for each $\omega \in \Omega$, we can write

$$dX_t = r(t)dt.$$

(2)

$$dY_t = \exp(X_t)dX_t = \exp(X_t)r(t)dt = Y_t r(t)dt.$$

□

Remark 19.2.1 (common pitfalls).

- Note that when $X_t = \int_0^t r(s)ds$ and $r(t)$ is an Ito stochastic process, X_t is not an Ito integral process.
- Similarly, for $Y_t = \exp(X_t)$, Y_t is not an Ito integral, and the Ito lemma does not apply.

Lemma 19.2.5 (Ito lemma applied to integral of Ito processes). *Let $X(t)$ be an Ito stochasti process. Let $r(t)$ be a deterministic function.*

- *If $Y_t = \int_0^t r(s)dX(s)$, then*
$$dY_t = r(t)dX(t).$$
- *If $Z_t = \exp(Y_t)$, then*
$$dZ_t = Z_t r(t)dX(t) + \frac{1}{2}Z_t r(t)^2 dX(t)dX(t).$$

Proof. (1) by definition. (2) Using Ito rule [Lemma 19.2.2], we have

$$\begin{aligned}
dZ_t &= Z_t dY_t + \frac{1}{2}Z_t dY_t dY_t \\
&= Z_t r(t)dX(t) + \frac{1}{2}Z_t r(t)^2 dX(t)dX(t).
\end{aligned}$$

□

19.3 Linear SDE

19.3.1 State-independent linear arithmetic SDE

Lemma 19.3.1 (state independent/general arithmetic SDE). *The solution X_t of the stochastic differential equation*

$$dX_t = a(t)dt + b(t)dW(t)$$

is given by

$$X_t = X_0 + \int_0^t a(s)ds + \int_0^t b(s)dW(s),$$

which is a ***Gaussian distribution*** *with mean $X_0 + \int_0^t a(s)ds$ and variance $\int_0^t b^2(s)ds$.*

Moreover, X_t is a Gaussian process [Corollary 19.1.4.1].

Proof. The integral form is

$$X_t - X_0 = \int_0^t a(s)ds + \int_0^t b(s)dW(s).$$

X_t is a Gaussian because it is a deterministic term plus a Gaussian random process $\int_0^t b(s)dW(s)$. The mean is

$$E[X_t] = X_0 + \int_0^t a(s)ds$$

where the fact of expectation of Ito integral is zero is used. For the calculation of variance, we use

$$E[(\int_0^t b(s)dW(s))^2] = \int_0^t b^2(s)ds$$

via Ito isomery. □

19.3.2 State-independent linear geometric SDE

Lemma 19.3.2 (general geometric SDE). *Consider the SDE*

$$dX(t) = \mu(t)X(t)dt + \sigma(t)X(t)dW(t).$$

It follows that

- *It has the equivalent form*

$$Y_t = \ln X_t$$
$$dY_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t$$

- *The solution for $X(t)$ is given by*

$$X(t) = X(0)\exp(\int_0^t [\mu(s) - \frac{1}{2}\sigma(s)^2]ds + \int_0^t \sigma(s)dW(s)).$$

- *Particularly, if $\mu(t) = 0$ and $\sigma(t)$ is a constant, then*

$$X(t) = X(0)\exp(-\frac{1}{2}\sigma^2 t + \sigma B(t))$$

is a martingale.

Proof. (1)(2) use $Y_t = f(X_t)\ln(X_t)$ and Ito rule, we have

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}dX_t dX_t \\ &= \frac{1}{X_t}dX_t - \frac{1}{2}\frac{1}{X_t^2}(\sigma X_t)^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned}$$

Then Y_t will have solution

$$Y_t = Y_t + \int_0^t (\mu - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dW_s.$$

(3) We want to prove $E[X(t)|\mathcal{F}_s] = X(s)$, where $\mathcal{F}_t$ is the filtration associated with Brownian motion. See Lemma 18.6.2. □

Corollary 19.3.0.1 (state independent geometric SDE, conversion to driftless SDE). *Consider SDE for X*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

with constant μ, σ, and let $Y = \exp(-\mu t)X$, then the SDE for Y is

$$dY = \sigma Y(t)dW(t)$$

with solution of $Y(t)$ being an exponential martingale as

$$Y(t) = Y(0)\exp(-\frac{1}{2}\sigma^2 t + \sigma B(t)).$$

Then, $X(t)$ is given by

$$X(t) = X(0)\exp(\mu t - \frac{1}{2}\sigma^2 t + \sigma B(t))$$

Proof. From Ito lemma, we have

$$dY = -\mu\exp(-\mu t)Xdt + \exp(-\mu t)dX = \sigma Y(t)dW(t).$$

The rest can be proved using above lemma. □

Corollary 19.3.0.2 (mean and variance of a state-independent geometric SDE). *Consider SDE for X*

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$$

with constant μ, σ. Then,

- $E[X(t)] = X(0)e^{\mu t}$
- $Var[X(t)] = X(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$

Proof. Note that

$$\ln(\frac{X(t)}{X(0)}) = (\mu - \frac{\sigma^2}{2})t + \sigma W_t.$$

That is, $\frac{X(t)}{X(0)} \sim LN((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$.Then we can use Definition 12.1.9. □

19.3.3 Multiple dimension extension

Lemma 19.3.3 (multi-dimensional state independent/general arithmetic SDE). *[2, p. 146][4, p. 116]Consder a N dimensional stochastic differential equation(SDE) given by*

$$dX_i = a_i(t)dt + b_i(t)dW_i(t),$$

where $E[dW_i dW_j] = \rho_{ij}dt, E[dWdW^T] = \Sigma dt$. It follows that

- *The solution for $X_i(t), i = 1, 2, ..., N$ is given by*

$$X_i(t) = X_i(0) + \int_0^t a_i(s)ds + \int_0^t b_i(s)dW_i(s),$$

which is a ***Gaussian distribution*** *with mean $X_i(0) + \int_0^t a_i(s)ds$ and variance $\int_0^t b_i^2(s)ds$.*

- *The covariance structure for different $X_i(t), X_j(s), s \geq t$ is given by*

$$Cov(X_i(t), X_j(s)) = \int_0^t b_i(u)b_j(u)\rho_{ij}du.$$

Proof. (1) See Lemma 19.3.1. (2)

$$\begin{aligned} Cov(X_i(t), X_j(s)) &= \int_0^t \int_0^s b_i(u)b_j(v)dW_i(u)dW_j(v) \\ &= \int_0^t \int_0^s b_i(u)b_j(v)\rho_{ij}\delta(u-v)du \\ &= \int_0^t b_i(u)b_j(u)\rho_{ij}du \end{aligned}$$

□

Lemma 19.3.4 (general multi-dimensional geometric SDE). *[4, p. 116] Consider a N-dimensional SDE*

$$dX_i(t) = \mu_i(t)X_i(t)dt + \sigma_i(t)X_i(t)dW_i(t),$$

where It follows that

- *The solution for $X_i(t), i = 1, 2, ..., N$, is given by*

$$X_i(t) = X_i(0)\exp(\int_0^t [\mu_i(s) - \frac{1}{2}\sigma_i(s)^2]ds + \int_0^t \sigma_i(s)dW_i(s)).$$

- *Particularly, if $\mu_i(t) = 0$ and $\sigma_i(t)$ is a constant, then*

$$X_i(t) = X_i(0)\exp(-\frac{1}{2}\int_0^t \sigma_i^2(s)ds + \int_0^t \sigma_i(s)dW_i(s))$$

is a martingale.

- *The covariance structure for different $X_i(t), X_j(s), s \geq t$ is given by*

$$\begin{aligned}&Cov(X_i(t), X_j(s))\\=&X_i(0)X_j(0)\exp(m_i(t) + m_j(s) + \frac{1}{2}(\Sigma_{ii}(t,t) + \Sigma_{jj}(s,s)))(\exp(\Sigma_{ij}(t,s)) - 1),\end{aligned}$$

where

$$m_i(t) = \int_0^t [\mu_i(u) - \frac{1}{2}\sigma_i(u)^2]du,$$

$$\Sigma_{ij}(t,s) = \int_0^t \sigma_i(u)\sigma_j(u)dt.$$

- *Particularly, if $\mu_i(t) = 0$ and $\sigma_i(t)$ is a constant, then*

$$Cov(X_i(t), X_j(s)) = X_i(0)X_j(0)\exp(\Sigma_{ij}(t,s)) - 1.$$

$$E(X_i(t), X_j(s)) = X_i(0)X_j(0)\exp(\Sigma_{ij}(t,s)),$$

Proof. (1)(2) See Lemma 19.3.2. (3) Lemma 12.1.21 (4) Note that when $\mu_i = 0$, we have $m_i(t) + \frac{1}{2}\Sigma_{ii}(t,t) = 0$. □

19.3.4 Exact SDE

Definition 19.3.1 (exact SDE). *[2, p. 151] The SDE*

$$dX_t = a(t, W_t)dt + b(t, W_t)dW_t$$

is called exact if there is a differentiable function $f(t, W_t)$ such that

$$a(t, W_t) = f_t + \frac{1}{2}f_{WW}, b(t, W_t) = f_W$$

Lemma 19.3.5. *The solution to an exact SDE is given as*

$$X_t = f(t, W_t) + C$$

Proof. Use Ito's lemma, we have

$$dX_t = df = f_t dt + f_W dW_t + \frac{1}{2}f_{WW}dt$$

□

Remark 19.3.1. Not every SDE is exact. With a, b given, we can try to first solve for f(not necessarily solvable). If we can get f then obtain an easy way to solve SDE.

Example 19.3.1. We have

$$dX_t = e^t(1+W_t^2)dt + (1+2e^tW_t)dW_t$$

We can find $f(t, W_t) = W_t + e^tW_t^2$

Theorem 19.3.1 (exact SDE criterion,necessary condition). *[2, p. 152] If SDE is exact, then*

$$a_x = b_t + \frac{1}{2}b_{xx}$$

19.3.5 Calculation mean and variance from SDE

Given a SDE

$$X_t = X_0 = \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s$$

we are just interested in the mean and variance of X_t. We can use the fact that **expectation of Ito integral is zero** to simplify our calculation:

$$E[X_t] = X_0 + \int_0^t E[a(X_s, s)]ds$$

where the integral of expectation and integral is justified by **Fubini's theorem**.

Theorem 19.3.2 (Fubini's theorem). *[5, p. 53]Let $X(t)$ be a stochastic process with continuous sample paths, then*

$$\int_0^T E[|X(t)|]dt = E[\int_0^T |X(t)|\, dt]$$

furthermore if this quantity is finite, then

$$\int_0^T E[X(t)]dt = E[\int_0^T X(t)dt]$$

Using the fundamental theorem of calculus, we know that

$$\frac{dE[X_t]}{dt} = E[a(X_t, t)].$$

Lemma 19.3.6 (mean and variance dynamics). *Let* $dX_t = a(t)X_t dt + c(t)dt + b(t)dW(t)$*, then*

$$E[X_t] = \Phi_1(t,0)X_0 + \int_0^t \Phi_1(t,\tau)c(\tau)d\tau,$$

and

$$Var[X_t] = \int_0^t \Phi_2(t,\tau)b(\tau)^2 d\tau,$$

where

$$\Phi_1(t,s) = \exp(\int_s^t a(u)du), \quad \Phi_2(t,s) = \exp(\int_s^t 2a(u)du)$$

Proof. It is easy to find the governing equation for $E[X_t]$ is

$$dE[X_t]/dt = a(t)E[X_t] + c(t),$$

then use solution methods in linear dynamical system to solve the equation [Theorem 17.3.6].

Let $Y_t = X_t^2$, then

$$dY_t = 2X_t dX_t + b(t)^2 dt = 2aY_t dt + b^2(t)dt + bdW(t),$$

use the above lemma, we have

$$E[Y_t] = E[X_t^2] = \Phi_2(t,0))X_0^2 + \int_0^t \Phi_2(t,\tau)b(\tau)^2 d\tau$$

Use $Var[X_t] = E[X_t^2] - E[X_t]^2 = \int_0^t \Phi_2(t,\tau)b(\tau)^2 d\tau$. □

Remark 19.3.2. We can also obtain the result using solutions to Ornstein-Uhlenbeck process Lemma 19.4.1.

19.3.6 Integrals of Ito SDE

Lemma 19.3.7 (Integral of state independent arithmetic SDE). *Let* X_t *be governed by stochastic differential equation*

$$dX_t = a(t)dt + b(t)dW(t).$$

Further define a integral

$$I(t,T) = \int_t^T X(s)ds.$$

It follows that

- $$X_s = X_t + \int_t^s a(u)du + \int_t^s b(u)dW(u),$$
  which is a Gaussian distribution with mean $X_0 + \int_0^t a(s)ds$ *and variance* $\int_0^t b^2(s)ds$.
- $I(t,T)$ *has explicit form*
  $$I(t,T) = X_t(T-t) + \int_t^T (T-u)a(u)du + \int_t^T (T-u)b(u)dW(u).$$
- $I(t,T)$ *is a Gaussian distribution with mean and variance given by*
  $$M(t,T) = X_t(T-t) + \int_t^T (T-u)a(u)du,$$
  $$V(t,T) = \int_t^T (T-u)^2 b^2(u)du.$$
- *If* $b(u) = b_0, a(u) = a_0$, *then*
  $$M(t,T) = X_t(T-t) + \frac{1}{2}a_0(T-t)^2,$$
  $$V(t,T) = \frac{1}{3}b_0(T-t)^2.$$

Proof. (1) See Lemma 19.3.1. (2)

$$\begin{aligned}
&\int_t^T X_s ds \\
=&\int_t^T X_t ds + \int_t^T \int_t^s a(u)duds + \int_t^T \int_t^s b(u)dW(u)ds \\
=&X_t(T-t) + \int_t^T \int_u^T a(u)dsdu + \int_t^T \int_u^T a(u)dsdW(u) \\
=&X_t(T-t) + \int_t^T (T-u)a(u)du + \int_t^T (T-u)b(u)dW(u)
\end{aligned}$$

where we changed the order of integral. (3)(4) Use Lemma 19.3.1 again, we can see that $I(t,T)$ is actually a Gaussian process. □

Lemma 19.3.8 (Integral of sum of two state independent arithmetic SDE). *Let $X_1(t), X_2(t)$ be governed by stochastic differential equations*

$$\begin{aligned} dX_1(t) &= a_1(t)dt + b_1(t)dW_1(t) \\ dX_2(t) &= a_2(t)dt + b_2(t)dW_2(t) \\ E[dW_1 dW_2] &= \rho dt \end{aligned}$$

Further define a integral

$$I(t,T) = \int_t^T X_1(s) + X_2(s)ds.$$

It follows that

- $$X_1(s) + X_2(s) = X_1(t) + X_2(t) + \int_t^s a_1(u) + a_2(u)du + \int_t^s b_1(u) + b_2(u)dW(u),$$

- *$I(t,T)$ has explicit form*

  $$I(t,T) = (X_1(t) + X_2(t))(T-t) + \int_t^T (T-u)a(u)du + \int_t^T (T-u)b(u)dW(u),$$

  where

  $$a(u) = a_1(u) + a_2(u), b(u) = \sqrt{b_1(u)^2 + b_2(u)^2 + 2\rho b_1(u)b_2(u)}$$

- *$I(t,T)$ is a Gaussian distribution with mean and variance given by*

  $$M(t,T) = X_t(T-t) + \int_t^T (T-u)a(u)du,$$

  $$V(t,T) = \int_t^T (T-u)^2 b^2(u)du.$$

- *If $b_1(u) = b_{10}, b_2(u) = b_{20}, a_1(u) = a_{10}, a_2(u) = a_{20}$, then*

  $$M(t,T) = X_t(T-t) + \frac{1}{2}a_0(T-t)^2,$$

  $$V(t,T) = \frac{1}{3}b_0(T-t)^2,$$

  where

  $$a_0 = a_{10} + a_{20}, b_0 = \sqrt{b_{10}^2 + b_{20}^2 + 2\rho b_{10}b_{20}}$$

Proof. Note that

$$d(X_1(t)+X_2(t)) = (a_1(t)+a_2(t))dt + b_1(t)dW_1(t) + b_2(t)dW_2(t)$$
$$dZ(t) = a(t)dt + \sqrt{b_1(u)^2 + b_2(u)^2 + 2\rho b_1(u)b_2(u)}dW_3$$

where $Z(t) \triangleq X_1(t)+X_2(t)$, the W_3 is a new Brownian motion. We arrive at

$$b_1(t)dW_1(t) + b_2(t)dW_2(t) = \sqrt{b_1(u)^2 + b_2(u)^2 + 2\rho b_1(u)b_2(u)}dW_3,$$

via the fact that two independent Gaussian random variable will sum to another Gaussian random variable. Then we use Lemma 19.3.7. □

Lemma 19.3.9 (Integral of sum of multiple state independent arithmetic SDE). *Let $X_1(t), X_2(t), ..., X_n$ be governed by stochastic differential equations*

$$\begin{aligned} dX_1(t) &= a_1(t)dt + b_1(t)dW_1(t) \\ dX_2(t) &= a_2(t)dt + b_2(t)dW_2(t) \\ &\cdots\cdots \\ dX_n(t) &= a_n(t)dt + b_n(t)dW_n(t) \\ E[dW_i dW_j] &= \rho_{ij}dt \end{aligned}$$

Further define a integral

$$I(t,T) = \int_t^T X_1(s) + X_2(s) + \cdots + X_n(s)ds.$$

It follows that

- $$X_1(s) + X_2(s) + \cdots + X_n(s) = \sum_{i=1}^n X_i(t) + \int_t^s \sum_{i=1}^n a_i(u)du + \int_t^s \sum_{i=1}^n b_i(u)dW(u),$$
- *$I(t,T)$ has explicit form*

  $$I(t,T) = (\sum_{i=1}^n X_i(t))(T-t) + \int_t^T (T-u)a(u)du + \int_t^T (T-u)b(u)dW(u),$$

  where

  $$a(u) = \sum_{i=1}^n a_i(u), b(u) = \sqrt{\sum_{i=1}^n b_i(u)^2 + 2\sum_{1\leq i<j\leq n} \rho_{ij}b_i(u)b_j(u)}$$

- *$I(t,T)$ is a Gaussian distribution with mean and variance given by*

$$M(t,T) = X_t(T-t) + \int_t^T (T-u)a(u)du,$$

$$V(t,T) = \int_t^T (T-u)^2 b^2(u)du.$$

19.4 Ornstein-Uhlenbeck(OU) process

19.4.1 OU process

19.4.1.1 *Constant coefficient OU process*

Definition 19.4.1 (Ornstein-Uhlenbeck process). *A stochastic process*

$$X_t = e^{-at}x_0 + \sigma \int_0^t e^{-a(t-s)} dW_s,$$

where a, σ, x_0 are constant parameters and W_t is the Brownian motion, is called Ornstein-Uhlenbeck process with parameter (a, σ) and initial value x_0.

The differential form of the OU process is given by

$$dX_t = \sigma dW_t - aX_t dt, X_0 = x_0.$$

Lemma 19.4.1 (OU process solution). *Consider the SDE*

$$dX_t = \sigma dW_t - aX_t dt$$

with $\sigma > 0, a > 0$, and initial condition $X_0 = x_0$.

It follows that

- *It has the solution*

$$X_t = x_0 e^{-at} + \int_0^t \exp(-a(t-s))\sigma dW_s.$$

- *X_t has Gaussian distribution, i.e.,*

$$X_t \sim N(x_0 e^{-at}, \frac{\sigma^2(1-e^{-2at})}{2a}).$$

- *X_t has the stationary distribution given by*

$$X_t \sim N(0, \frac{\sigma^2}{2a}).$$

Proof. (1)(2) Use $Y_t = X_t e^{at}$, then Ito rule gives

$$dY_t = aY_t + e^{at} dX_t = e^{at}\sigma dW_t$$

We have

$$Y_T - Y_0 = \int_0^T e^{at}\sigma dW_t \Leftrightarrow X_T = \exp(-aT)X_0 + \int_0^T e^{-a(T-t)} dW_t.$$

Use Theorem 19.1.4, we have

$$Y_T - Y_0 \sim N(0, \int_0^T (e^{at}\sigma)^2 dt).$$

Then

$$X^T \sim e^{-aT} N(X_0, \int_0^T (e^{at}\sigma)^2 dt)$$

simplifies to

$$X^T \sim N(X_0, e^{-2aT} \int_0^T (e^{at}\sigma)^2 dt).$$

(3) Take $t \to \infty$ will get the result. □

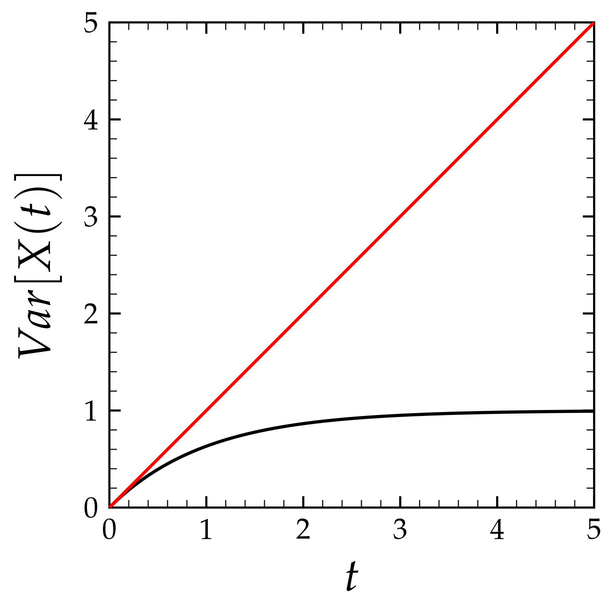

Figure 19.4.1: The variance function $Var[X(t)]$ for Brownian motion (red) and OU process(black) with $a = 0.5, \sigma = 1$.

Lemma 19.4.2 (constant shifted OU process). *Consider the constant shifted OU process*

$$dX_t = \sigma dW_t - a(X_t - \mu)dt$$

with $\sigma > 0, a > 0$*, and initial condition* $X_0 = x_0$*.*

- *It has the solution*

$$X_t \sim N((x_0 - \mu)e^{-at} + \mu, \frac{\sigma^2(1 - e^{-2at})}{2a})$$

and the stationary distribution is given as

$$X_t \sim N(\mu, \frac{\sigma^2}{2a}).$$

- *the constant shifted OU process can be re-written as*

$$\begin{aligned} X_t &= Z_t + \mu \\ dZ_t &= \sigma dW_t - aZ_t dt \end{aligned}$$

Proof. (1) Use $Y_t = (X_t - \mu)e^{at}$. The rest is similar to Lemma 19.4.1. (2) Note that $dZ_t = dX_t + d\mu = dX_t$. Therefore

$$\begin{aligned} & dZ_t = \sigma dW_t - aZ_t dt \\ \implies & dX_t = \sigma dW_t - a(X_t - \mu)dt \end{aligned}$$

It can also be verified that:

$$X_t = \mu + Z_t, x_0 = \mu + z_0, Z_t \sim N(z_0 e^{-at}, \frac{\sigma^2(1 - e^{-2at})}{2a})$$

gives

$$X_t \sim N(\mu + (x_0 - \mu)e^{-at}, \frac{\sigma^2(1 - e^{-2at})}{2a}).$$

□

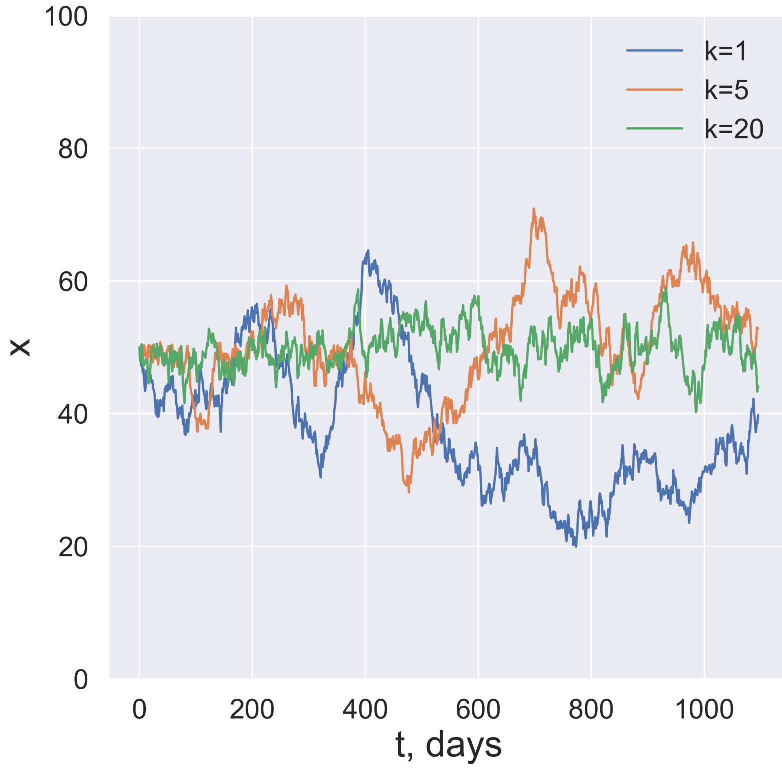

Figure 19.4.2: Representative trajectories from three OU processes with different k. k has the unit of inverse year. Mean level $\mu = 50$ and volatility $\sigma = 20$.

Lemma 19.4.3 (scaling property of OU process). *Consider the SDE*

$$dX(t) = \sigma dW(t) - a(X(t) - \mu)dt,$$

and let $X(t)$ be the solution. Then $Y(t) = \lambda X(mt)$ is the solution for

$$dY(t) = \sqrt{m}\lambda\sigma dW(t) - ma(Y(t) - \lambda\mu)dt.$$

Note that we interpret m as the time scaling factor and λ the spatial scaling factor.

Proof. Note that $X(mt)$ will satisfy

$$dX(mt) = \sigma dW(mt) - a(X(mt) - \mu)dmt,$$

or equivalently

$$dX(mt) = \sqrt{m}\sigma dW(t) - ma(X(mt) - \mu)dt.$$

Multiply both sides by λ, we have

$$d\lambda X(mt) = \sqrt{m}\lambda\sigma dW(t) - ma(\lambda X(mt) - \lambda\mu)dt.$$

Plug in $\lambda X(mt) = Y(t)$, we have

$$dY(t) = \sqrt{m}\lambda\sigma dW(t) - ma(Y(t) - \lambda\mu)dt.$$

□

Remark 19.4.1 (applications of scaling property). Suppose we have the dynamics of an asset with time unit day and value unit dollar, we can use the scaling property to find out the coefficients associated with time unit year and value unit JPY.

Lemma 19.4.4 (Stationary Gaussian process). *An Ornstein-Uhlenbeck process (a, σ) with Gaussian initial distribution $\eta \sim N(0, \sigma^2/2a)$(i.e., stationary distribution) is a strictly/weakly stationary Gaussian process.*

Proof. (1)

$$E[X_t] = E[e^{-at}\eta + \sigma\int_0^t e^{-a(t-s)}dW_s] = 0$$

since $E[\eta] = 0$ and $\int_0^t e^{-a(t-s)}dW_s$ is Ito integral [Theorem 19.1.3]. (2) Let $s < t$, we have

$$\begin{aligned}
cov(X_t, X_s) = E[X_tX_s] &= e^{-a(s+t)}E[\eta^2] + \sigma^2 E[\int_0^s e^{-a(t-s)}dW_u \int_0^s e^{-a(t-m)}dW_m] \\
&= e^{-a(s+t)}\frac{\sigma^2}{2a} + \sigma^2\int_0^t e^{-2a(t-s)}dt \\
&= e^{-a(s+t)}\frac{\sigma^2}{2a} + \frac{\sigma^2}{2a}(e^{-2as} - 1) \\
&= e^{-a(s+t)}e^{-2as}\frac{\sigma^2}{2a} = \frac{\sigma^2}{2a}e^{-a(t-s)}
\end{aligned}$$

Note that a weakly stationary Gaussian process is strictly Gaussian process [Lemma 18.2.2].

□

19.4.1.2 *Time-dependent coefficient OU process*

Definition 19.4.2 (Time-dependent coefficient Ornstein-Uhlenbeck process). *A stochastic process with differential form*

$$dX_t = (\phi(t) - \lambda X_t)dt + \sigma dW_t,$$

where $\psi(t)$ is time dependent coefficient, a, σ, x_0 are constants, and W_t is Brownian motion., is called time-dependent coefficient Ornstein-Uhlenbeck process with parameter (a, σ) and initial distribution x_0.

Lemma 19.4.5. *Consider a stochastic process with differential form*

$$dX_t = (\psi(t) - \lambda X_t)dt + \sigma dW_t, X_t = x_0$$

where $\psi(t)$ is time dependent coefficient, a, σ are constants, and W_t is Brownian motion. It follows that

- *It has the equivalent form*

$$\begin{aligned} X_t &= Y_t + \int_0^t \exp(-a(t-s))\psi(s)ds \\ dY_t &= -aY_t dt + \sigma dW_t \end{aligned}$$

- *It has solution*

$$X_t = x_0 \exp(-at) + \int_0^t \exp(-a(t-s))\psi(s)ds + \int_0^t \sigma \exp(-a(t-s))dW_t.$$

- *X_t has mean and covariance given by*

$$E[X_t] = x_0 \exp(-at) + \int_0^t \exp(-a(t-s))\psi(s)ds$$

$$Var[X_t] = \frac{\sigma^2(1-e^{-2at})}{2a}$$

- *X_t has Gaussian distribution at any t, we have*

$$X_t \sim N(x_0 \exp(-at) + \int_0^t \exp(-a(t-s))\psi(s)ds, \frac{\sigma^2(1-e^{-2at})}{2a})$$

- *$X_t, t \to \infty$ is generally not a stationary process since its mean depends on t.*

Proof. (1)Note that

$$\frac{d}{dt}\int_0^t \exp(-a(t-s))\psi(s)ds = -a\int_0^t \exp(-a(t-s)) + \psi(t);$$

Therefore,

$$dX_t = dY_t + \frac{d}{dt}\int_0^t \exp(-a(t-s))\psi(s)ds = (\psi(t) - aX_t)dt + \sigma dW_t.$$

(2)(3) Note that Y_t has solution and distribution

$$Y_t = x_0 e^{-at} + \int_0^t \exp(-a(t-s))\sigma dW_s,$$

$$Y_t \sim N(x_0 e^{-at}, \frac{\sigma^2(1-e^{-2at})}{2a}).$$

Then we use relation in (1). □

19.4.1.3 *Integral of OU process*

Lemma 19.4.6 (integral of OU process). *Consider an OU process given by*

$$dx(t) = -ax(t)dt + \sigma dW(t), x(0) = x_0$$

where $a\sigma$ are constants,W is a Brownian motion. For each t, T, the random variable

$$I(t,T) = \int_t^T x(u)du$$

conditioned to the σ field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \frac{1-\exp(-a(T-t))}{a} x(t),$$

and variance

$$V(t,T) = \frac{\sigma^2}{a^2}(T - t + \frac{2}{a}\exp(-a(T-t)) - \frac{1}{2a}\exp(-2a(T-t)) - \frac{3}{2a}).$$

Proof. See the proof of Lemma 19.4.7. □

Lemma 19.4.7 (integral of sum of two OU process). *[6, p. 145][7, p. 64] Consider two OU processes given by*

$$dx_1(t) = -a_1 x_1(t)dt + \sigma_1 dW_1(t), x_1(0) = x_{10}$$
$$dx_2(t) = -a_2 x_2(t)dt + \sigma_2 dW_1(t), x_2(0) = x_{20}$$

where $a_1, a_2, \sigma_1, \sigma_2$ are constant, $\psi(t)$ is a time-dependent function, and W_1, W_2 are correlated Brownian motion such that

$$dW_1(t)dW_2(t) = \rho dt.$$

For each t, T, the random variable

$$I(t,T) = \int_t^T (x_1(u) + x_2(u))du$$

conditioned to the σ field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \frac{1-\exp(-a_1(T-t))}{a_1}x_1(t) + \frac{1-\exp(-a_2(T-t))}{a_2}x_2(t),$$

and variance

$$\begin{aligned}
V(t,T) = & \frac{\sigma_1^2}{a_1^2}(T-t+\frac{2}{a_1}\exp(-a_1(T-t)) - \frac{1}{2a_1}\exp(-2a_1(T-t)) - \frac{3}{2a_1}) \\
& + \frac{\sigma_2^2}{a_2^2}(T-t+\frac{2}{a_2}\exp(-a_1(T-t)) - \frac{1}{2a_2}\exp(-2a_2(T-t)) - \frac{3}{2a_2}) \\
& + \frac{2\rho\sigma_1\sigma_2}{a_1a_2}(T-t+\frac{\exp(-a_1(T-t))-1}{a_1} + \frac{\exp(-a_1(T-t))-1}{a_1} \\
& + \frac{\exp(-(a_1+a_2)(T-t))-1}{a_1+a_2})
\end{aligned}$$

Proof. (1)Note that given the observation $x_1(t)$ at t, we have

$$x_1(u) = x_1(t)\exp(-a_1(u-t)) + \int_t^u \sigma\exp(-a_1(u-s))dW(s)$$

Therefore,

$$\begin{aligned}
& \int_t^T x_1(u)du \\
& = \int_t^T x_1(t)\exp(-a_1(u-t))du + \int_t^T\int_t^u \sigma\exp(-a_1(u-s))dW(s)du \\
& = x_1(t)\frac{1-\exp(-a_1(T-t))}{a_1} + \int_t^T\int_s^T \sigma\exp(-a_1(u-s))dudW(s) \\
& = x_1(t)\frac{1-\exp(-a_1(T-t))}{a_1} + \int_t^T \frac{\sigma_1}{a_1}(1-\exp(-a_1(T-s)))dW_1(s)
\end{aligned}$$

where we changed the order of integration. From this, we note that

$$E[\int_t^T x_1(u)du] = x_1(t)\frac{1-\exp(-a_1(T-t))}{a_1}.$$

Similarly, we can get eh expectation for $\int_t^T x_2(u)du$.

(2) To get the variance, we have

$$\begin{aligned}
&Var[\int_t^T x_1(u) + x_2(u)du] \\
=&Var[\int_t^T x_1(u)du] + Var[\int_t^T x_1(u)du] + 2Cov(\int_t^T x_1(u)du, \int_t^T x_1(u)du).
\end{aligned}$$

For $Var[\int_t^T x_1(u)du]$, we have

$$\begin{aligned}
&Var[\int_t^T x_1(u)du \\
&= E[\int_t^T \frac{\sigma_1}{a_1}(1-\exp(-a_1(T-s)))dW_1(s)\int_t^T \frac{\sigma_1}{a_1}(1-\exp(-a_1(T-s)))dW_1(s)] \\
&= \frac{\sigma_1^2}{a_1^2}(\int_t^T ds + \int_t^T \exp(-2a_1(T-s))ds - 2\int_t^T \exp(-a_1(T-s))ds) \\
&= \frac{\sigma_1^2}{a_1^2}(T-t+\frac{2}{a_1}\exp(-a_1(T-t)) - \frac{1}{2a_1}\exp(-2a_1(T-t)) - \frac{3}{2a_1}).
\end{aligned}$$

We can similarly evaluate $Var[\int_t^T x_2(u)du]$.

For $Cov[\int_t^T x_1(u)du, \int_t^T x_2(u)du]$, we have

$$\begin{aligned}
&Cov[\int_t^T x_1(u)du, \int_t^T x_2(u)du] \\
&= E[\int_t^T \frac{\sigma_1}{a_1}(1-\exp(-a_1(T-s)))dW_1(s)\int_t^T \frac{\sigma_1}{a_1}(1-\exp(-a_1(T-s)))dW_2(s)] \\
&= \frac{\rho\sigma_1\sigma_2}{a_1a_2}(\int_t^T (1-\exp(-a_1(T-s) - \exp(-a_1(T-s)) + \exp(-(a_1+a_2)(T-s))))ds \\
&= \frac{2\rho\sigma_1\sigma_2}{a_1a_2}(T-t+\frac{\exp(-a_1(T-t))-1}{a_1} + \frac{\exp(-a_2(T-t))-1}{a_2} \\
&+ \frac{\exp(-(a_1+a_2)(T-t))-1}{a_1+a_2}).
\end{aligned}$$

□

Lemma 19.4.8 (integral of sum of multiple OU process). *[6, p. 145][7, p. 64] Consider n OU processes given by*

$$
\begin{aligned}
dx_1(t) &= -a_1 x_1(t)dt + \sigma_1 dW_1(t), x_1(0) = x_{10} \\
dx_2(t) &= -a_2 x_2(t)dt + \sigma_2 dW_1(t), x_2(0) = x_{20} \\
&\cdots\cdots \\
dx_n(t) &= -a_n x_2(t)dt + \sigma_n dW_1(t), x_2(0) = x_{n0}
\end{aligned}
$$

where $a_1, ..., a_n, \sigma_1, ..., \sigma_n$ are constants, and $W_1, W_2, ..., W_n$ are correlated Brownian motions such that

$$dW_i(t)dW_j(t) = \rho_{ij}dt.$$

For each t, T, the random variable

$$I(t,T) = \int_t^T (x_1(u) + x_2(u) + ... + x_n(u))du$$

conditioned on the σ field $\mathcal{F}_t$ is normally distributed with mean $M(t,T)$ and variance $V(t,T)$, respectively given by

$$M(t,T) = \sum_{i=1}^n \frac{1-\exp(-a_i(T-t))}{a_i} x_i(t),$$

and variance

$$
\begin{aligned}
V(t,T) = &\sum_{i=1}^n \frac{\sigma_i^2}{a_i^2}(T - t + \frac{2}{a_i}\exp(-a_i(T-t)) - \frac{1}{2a_i}\exp(-2a_i(T-t)) - \frac{3}{2a_i}) \\
&+ \sum_{1\leq i<j\leq n} \frac{2\rho\sigma_i\sigma_j}{a_i a_j}(T - t + \frac{\exp(-a_i(T-t)) - 1}{a_i} + \frac{\exp(-a_j(T-t)) - 1}{a_j} + \frac{\exp(-(a_i+a_j)(T-t}{a_i + a_j}
\end{aligned}
$$

19.4.2 Exponential OU process

Definition 19.4.3 (exponential constant coefficient Ornstein-Uhlenbeck process). *A stochastic process with differential form*

$$d(\ln X_t) = -a \ln X_t dt + \sigma dW_t, X_0 = x_0,$$

where a, σ, x_0 are constant parameters and W_t is the Brownian motion, is called exponential constant coefficient Ornstein-Uhlenbeck process with parameter (a, σ) and initial distribution x_0.

It also has the equivalent form

$$\begin{aligned} X_t &= \exp(Y_t) \\ dY_t &= -aY_t dt + \sigma dW_t, Y_0 = \ln x_0 \end{aligned}$$

Lemma 19.4.9 (exponential OU process solution). *Consider the SDE*

$$d(\ln X_t) = -a \ln X_t dt + \sigma dW_t, X_0 = x_0,$$

with $\sigma > 0, a > 0$, and initial condition $X_0 = x_0$.

It follows that

- *It has the solution*

$$X_t = \exp(Y_t), Y_t \sim\sim N(\ln(x_0)e^{-at}, \frac{\sigma^2(1-e^{-2at})}{2a}).$$

and the stationary distribution is given as

$$X_t = \exp(Y_t), Y_t \sim N(0, \frac{\sigma^2}{2a}).$$

- *(mean and variance property)*

$$\begin{aligned} E[X_t] &= \exp(\mu_Y + \sigma_Y^2/2) \\ Var[X_t] &= (\exp(\sigma_Y^2) - 1)\exp(2\mu_Y + \sigma_Y^2) \end{aligned}$$

where

$$\mu_Y = \ln(x_0)e^{-at}, \sigma_Y^2 = \frac{\sigma^2(1-e^{-2at})}{2a}.$$

Proof. (1) Let $Y_t = \ln X_t$, then we have

$$dY_t = -aY_t dt + \sigma dW_t, Y_0 = \ln x_0.$$

From Lemma 19.4.1, we know that

$$Y_t \sim N(Y_0 e^{-at}, \frac{\sigma^2(1-e^{-2at})}{2a}).$$

(2) Use the property of log normal distribution [Lemma 12.1.18] □

Remark 19.4.2 (sanity check with Ito rule). Note that the SDE

$$d(\ln X_t) = -a \ln X_t dt + \sigma dW_t, X_0 = x_0,$$

will give the SDE for X_t via the equivalent form

$$X_t = f(Y_t) = \exp(Y_t)$$
$$dY_t = -aY_t dt + \sigma dW_t, Y_0 = \ln x_0.$$

Using Ito rule, we have

$$\begin{aligned} dX_t &= \frac{\partial f}{\partial Y_t} dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial Y_t^2} dY_t dY_t \\ &= \exp(Y_t) dY_t + \frac{1}{2}\exp(Y_t)\sigma^2 dt \\ \implies dX_t/X_t &= dY_t + \frac{1}{2}\sigma^2 dt \\ dX_t/X_t &= (-a\ln X_t + \frac{1}{2}\sigma^2)dt + \sigma dW_t \end{aligned}$$

19.4.3 Parameter estimation for OU process

Note 19.4.1. The OU process

$$dX_t = k(\theta - X_t)dt + \sigma dW_t,$$

can be discretized at times $n\Delta t, n = 1, 2, ..., \infty$ which gives

$$X_{k+1} - X_k = k\theta\Delta t - kX_k\Delta t + \sigma(W_{k+1} - W_k),$$

or equivalently,

$$X_{k+1} = k\theta\Delta t - (k\Delta t - 1)X_k + \sigma\sqrt{\Delta t}\epsilon_k,$$

where $\epsilon_k \sim WN(0, 1)$.

The discrete-time form can be viewed as an AR(1) process, and least square method can be used to estimate k, θ, σ.

19.4.4 Multiple factor extension

Definition 19.4.4 (two-factor OU process). *The two-factor OU process is given by the following SDE*

$$\begin{aligned} r(t) &= x_1(t) + x_2(t) + \psi(t), r(0) = r_0 \\ dx_1(t) &= -a_1 x_1(t)dt + \sigma_1 dW_1(t), x_1(0) = x_{10} \\ dx_2(t) &= -a_2 x_2(t)dt + \sigma_2 dW_1(t), x_2(0) = x_{20} \end{aligned}$$

where $a_1, a_2, \sigma_1, \sigma_2, r_0$ are constant, $\psi(t)$ is a time-dependent function, and W_1, W_2 are correlated Brownian motion such that

$$dW_1(t)dW_2(t) = \rho dt.$$

Lemma 19.4.10 (basic properties). *Consider a The two-factor OU process is given by the following SDE*

$$\begin{aligned} r(t) &= x_1(t) + x_2(t) + \psi(t), r(0) = r_0 \\ dx_1(t) &= -a_1 x_1(t)dt + \sigma_1 dW_1(t), x_1(0) = x_{10} \\ dx_2(t) &= -a_2 x_2(t)dt + \sigma_2 dW_1(t), x_2(0) = x_{20} \end{aligned}$$

where $a_1, a_2, \sigma_1, \sigma_2, r_0$ are constant, $\psi(t)$ is a time-dependent function, and W_1, W_2 are correlated Brownian motion such that

$$dW_1(t)dW_2(t) = \rho dt.$$

It follows that

- *It has solution given by*

$$\begin{aligned} r(t) &= x_{10}\exp(-a_1 t) + x_{20}\exp(-a_2 t) \\ &\quad + \sigma_1 \int_0^t \exp(-a_1(t-s))dW_1(s) + \sigma_2 \int_0^t \exp(-a_2(t-s))dW_2(s) + \psi(t). \end{aligned}$$

- $$E[r(t)] = x_{10}\exp(-a_1 t) + x_{20}\exp(-a_2 t) + \psi(t).$$

$$\begin{aligned} &Var[r(t)] \\ &= \frac{\sigma_1^2}{2a_1}(1 - \exp(-2a_1 t)) + \frac{\sigma_2^2}{2a_2}(1 - \exp(-2a_2 t)) + \frac{2\rho\sigma_1\sigma_2}{a_1 + a_2}(1 - \exp(-(a_1 + a_2)t)). \end{aligned}$$

- *$r(t)$ has Gaussian distribution; that is,*

$$r(t) \sim N(E[r(t)], Var[r(t)]).$$

Proof. (1) From single factor OU process result [Lemma 19.4.1], we know that

$$x_1(t) = x_{10}\exp(-a_1 t) + \sigma_1 \int_0^t \exp(-a_1(t-s))dW_1(s).$$

Similarly, we can evaluate $x_2(t)$ and eventually $r(t)$. (2) The expectation can be easily evaluated based on the fact that Ito integral has zero mean. To evaluate the variance we have

$$\begin{aligned}
Var[r(t)] &= Var[\sigma_1 \int_0^t \exp(-a_1(t-s))dW_1(s)] + Var[\sigma_2 \int_0^t \exp(-a_2(t-s))dW_2(s)] \\
&+ 2Cov(\sigma_1 \int_0^t \exp(-a_1(t-s))dW_1(s), \sigma_2 \int_0^t \exp(-a_2(t-s))dW_2(s)) \\
&= \int_0^t \sigma_1^2 \exp(-2a_1(t-s))ds + \int_0^t \sigma_2^2 \exp(-2a_1(t-s))ds + \int_0^t \sigma_1\sigma_2\rho \exp(-a_1(t-s))\exp(-a_2(\\
&= \frac{\sigma_1^2}{2a_1}(1-\exp(-2a_1 t)) + \frac{\sigma_2^2}{2a_2}(1-\exp(-2a_2 t)) \\
&+ \frac{2\rho\sigma_1\sigma_2}{a_1+a_2}(1-\exp(-(a_1+a_2)t))
\end{aligned}$$

where we use Ito isometry in the evaluation, for example,

$$\begin{aligned}
&E[\sigma_1 \int_0^t \exp(-a_1(t-s))dW_1(s) \cdot \sigma_2 \int_0^t \exp(-a_2(t-s))dW_2(s)] \\
&= E[\sigma_1\sigma_2 \int_0^t \int_0^t \exp(-a_1(t-s))\exp(-a_2(t-u))dW_1(s)dW_2(u)] \\
&= E[\sigma_1\sigma_2 \int_0^t \int_0^t \exp(-a_1(t-s))\exp(-a_2(t-u))\rho dt\delta(u-s)] \\
&= E[\sigma_1\sigma_2\rho \int_0^t \exp(-(a_1+a_2)(t-s))\rho dt\delta(u-s)] \\
&= \frac{\rho\sigma_1\sigma_2}{a_1+a_2}(1-\exp(-(a_1+a_2)t))
\end{aligned}$$

(3) The random variable $r(t) = x_1(t) + x_2(t)$ is a Gaussian process has been discussed in Theorem 18.4.1. □

19.5 Notes on bibliography

For treatment on Stratonovich integral, see [1].

For treatment on calculating mean and variance from SDE, see [2].

For treatment on the techniques for solving SDE, see [2][1][4].

For finance related treatment, see [4].

See [8] for treatment on Girsanov theory and Feynman-Kac connection.

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