2

# METRIC SPACE

## 2.1 Metric space

### 2.1.1 Definitions

**Definition 2.1.1 (metric, distance function).** *Let $M$ be a set $X$. A **metric** on $M$ is a function $d$ from $M \times M$ into $[0, \infty)$ which satisfies*

1. $d(p, q) > 0$ *if* $p \neq q$;
2. $d(p, q) = 0$ *if and only if* $p = q$;
3. $d(p, q) = d(q, p)$;
4. *(triangle inequality)*$d(p, q) \leq d(p, r) + d(r, q)$ *for any*$p, r, q \in M$

**Definition 2.1.2 (metric space).** *A **metric space** is an ordered pair $(M, d)$, where $M$ is a set and $d$ is a metric on $M$.*

**Definition 2.1.3 (metric space subspace).** *Let $(M, d)$ be a metric space. Let $A$ be a subset of $M$. Let $d_{X \times X}$ be the metric $d$ restricted to the domain $X \times X$. Then $(A, d_{X \times X})$ is called the **subspace** of metric space $(M, d)$.*

**Lemma 2.1.1 (the metric space $\mathbb{R}^n$ with Euclidean metric).** *The set $\mathbb{R}^n$ with the metric $d(x, y)$ defined by*

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2},$$

*where* $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$.

*Proof.* We can verify $d(x, y)$ satisfies

1. $d(p, q) > 0$ if $p \neq q$;
2. $d(p, q) = 0$ if and only if $p = q$;
3. $d(p, q) = d(q, p)$;

4. (triangle inequality) we have

$$
\begin{aligned}
d(x,z) = \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} &= \sqrt{\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 + 2(x_i - y_i)(y_i - z_i) + (y_i - z_i)^2} \\
&= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n} 2(x_i - y_i)(y_i - z_i) + \sum_{i=1}^{n}(y_i - z_i)^2} \\
&\leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2} + \sum_{i=1}^{n}(y_i - z_i)^2} \\
&= \sqrt{(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2})^2} \\
&= (\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}) \\
&= d(x,y) + d(y,z)
\end{aligned}
$$

where we Cauchy inequality for finite terms [Corollary 5.3.1.1] such that

$$\left|\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i)\right| \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2}$$

. □

**Lemma 2.1.2 (the metric space $l^2(\mathbb{N})$ with Euclidean metric).** *Let $l^2(\mathbb{N})$ denote the set of all real sequence $\{c_k\}$ such that $\sum_{k=0}^{\infty} c_k^2$ converges. The set $l^2(\mathbb{N})$ with the metric $d(x,y)$ defined by*

$$d(x,y) = \sqrt{\sum_{k=1}^{\infty}(x_k - y_k)^2},$$

*where $x = (x_1, ..., x_n, ...), y = (y_1, ..., y_n, ...)$.*

*Proof.* First, we want to show that $d(x,y)$ actually exists(i.e. converges):

$$\sum_{k=1}^{\infty}(x_k - y_k)^2 = \sum_{k=1}^{\infty} x_k^2 + \sum_{k=1}^{\infty} y_k^2 - 2\sum_{k=1}^{\infty} x_k y_k$$

where the right side three series converge; therefore the left side converges.

We can verify $d(x,y)$ satisfies

1. $d(p,q) > 0$ if $p \neq q$;
2. $d(p,q) = 0$ if and only if $p = q$;
3. $d(p,q) = d(q,p)$;
4. (triangle inequality) we have, similar to $\mathbb{R}^n$ case, for finite terms

$$\begin{aligned}\sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} &= \sqrt{\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2} \\ &\leq (\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}] \\ &= d(x,y) + d(y,z)\end{aligned}$$

Then take limit on both side and use the sequence-limit-inequity rule [Lemma 1.4.2], we have

$$d(x,y) \leq d(x,z) + d(z,y).$$

□

### 2.1.2 metric space vs. normed (vector) space vs. Banach space

**Note 2.1.1 (metric space, normed space and Banach space).**

- All normed spaces are metric spaces by defining $d(x,y) = \|x - y\|$. However, metric space is not necessarily normed space because the two properties:
  1. Translation invariance: $d(u+w, v+w) = d(u,v)$
  2. Scaling properties: $d(tu, tv) = |t|\, d(u,v)$

  not necessarily satisfied by the metric.
- Also, normed space requires the underlying space is vector space, but metric space does not require so.[1]
- A Banach space is a normed vector space which is also a complete metric space. (Complete with respect to the metric defined by the norm)

## 2.2 Sequences in metric space

**Definition 2.2.1.** *[2, p. 125] Let $\{a_n\}$ be a sequence in a metric space $(M, d)$. We say that $\{a_n\}$ converges to $L$, where $L \in M$, and write*

$$\lim_{n\to\infty} a_n = L$$

*if for every $\epsilon > 0$, there exists $N > 0$ such that if $n > N$, then $d(a_n, L) < \epsilon$.*

**Remark 2.2.1.** Note that when $M$ is equipped with different metric $d$, a same sequence might have different convergent properties. For example, consider $M = \mathbb{R}$,and $d_1$ is the Euclidean norm, $d_2(x, y) = 1 - \delta(x, y)$. Then a sequence convergent using $d_1$ will not be convergent using $d_2$.

**Theorem 2.2.1 (componentwise convergence condition for sequence in $\mathbb{R}^n$).** *[2, p. 126] Let $\{a^{(k)}\}_{k=1}^{\infty}, a^{(k)} \in \mathbb{R}^n$ be a sequence in $\mathbb{R}^n$ such that for each $k$, $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, ..., a_n^{(k)})$. Then $a^{(k)}$ converges to $a \in \mathbb{R}^n$ if and only if*

$$\lim_{k\to\infty} a_j^{(k)} = a_j, for\ j = 1, 2, ..., n.$$

*Proof.* (1) First suppose $a^{(k)}$ converges to $a$, which means for every $\epsilon > 0$, there exists a N such that $\forall k > N$,

$$\left\|a^{(k)} - a\right\|_2 < \epsilon.$$

Note that for each component $a_i^{(k)}$, we have

$$\left|a_i^{(k)} - a_i\right| \leq \left\|a^{(k)} - a\right\|_2 < \epsilon;$$

that is $\{a_i^{(k)}\}, i = 1, 2, ..., n$ converges to $a_i$. (2) Second suppose $\{a_i^{(k)}\}, i = 1, 2, ..., n$ converges to $a_i$, which means for every $\epsilon > 0$, there exists a $N_i$ such that $\forall k > N_i$,

$$\left|a^{(k)} - a\right|_2 < \epsilon/\sqrt{n}.$$

Let $N = \max(N_1, N_2, ..., N_n)$, we have

$$\begin{aligned}\left\|a^{(k)} - a\right\|_2 &= \sqrt{\sum_{i=1}^{n} (a_j^{(k)} - a_j)^2} \\ &< \sqrt{\sum_{i=1}^{n} \epsilon^2/n} \\ &= \epsilon;\end{aligned}$$

that is $\{a^{(k)}\}$ converges to $a$. $\square$

## 2.3 Closed sets & open sets in metric space

### 2.3.1 Closed set

**Definition 2.3.1 (closure point, closure, closed subset).** *[3, p. 25]*

- *A point $x \in X$ is said to be **a closure point** of a set $P$ if given $\epsilon > 0$, there is a point $p \in P$ satisfying $d(x, p) < \epsilon$. The collection of all closure points of $P$ called the **closure** of $P$, denoted as $\bar{P}$. It is clear that $P \subseteq \bar{P}$*
- *A set $P$ is **closed** if $P = \bar{P}$.*

**Definition 2.3.2 (limit point, closed set, alternative).** *[2, pp. 128–129]*

- *Let $(M, d)$ be a metric space, and let $X$ be a subset of $M$. We say that a point $x \in M$ is a limit point of $X$ if there is a sequence $\{x_n\}$ that $x_n \in X, \forall n$ and $\lim_{n\to\infty} x_n = x$.*
- *Let $M$ be a metric space, and let $X$ be a subset of $M$. If **every** convergent sequence in $X$ and its limit point belongs to $X$, we say that $X$ is closed in $M$.*

**Remark 2.3.1** (equivalence of the two definitions)**.** These definitions about closed sets consistent; all says that points in a closed set can be getting to arbitrarily close.

**Lemma 2.3.1 (closure is the smallest containing closed set).** *The closure of a set $X$ is the smallest closed set containing $X$.*

*Proof.* Let $L$ be the set of all limits of every convergent sequences in $X$. Then $\bar{X} = X \cup L$ is a closed set containing $X$. To show it is the smallest closed set, we need to show that for any other closed set $Y$ containing $X$, $\bar{X} \subset Y$. For any convergent sequence in $\bar{X}$, which is also in $Y$, the limit point is also in both $\bar{X}$ and $Y$, and therefore $\bar{X} \subset Y$. □

### 2.3.2 Open sets

**Definition 2.3.3 (open ball).** *[2, p. 132] Let $(M, d)$ be a metric space. Let $\epsilon > 0$ and let $x \in M$. We let*

$$B(\epsilon, x) = \{y \in M | d(x, y) < \epsilon\}$$

**Definition 2.3.4 (interior point,interior, open subset).** *[3, p. 24]*

- *Let $P$ be a subset of a metric space $(X, d)$. The point $p \in P$ is said to be* ***an interior point*** *of $P$ if there is an $\delta > 0$ such that $B(x, \delta)$ is a subset of $P$. The collections of all interior points is called* ***interior*** *of $X$, denoted as $int(P)$. It is clear that $int(P) \subseteq P$.*
- *A set $P$ is said to be open if $int(P) = P$.*

**Lemma 2.3.2 (interior as the largest open set contained inside, characterization).** *The interior of $X$ is the largest open set contained inside $X$.*

### 2.3.3 Further characterization and properties

**Lemma 2.3.3 (empty set, entire set, and singleton set are closed sets).** *[2, p. 130]* *The empty set $\emptyset$ and the entire space are closed sets; A singleton subset is a closed set.*

*Proof.* The closure of an empty set is still empty. From $P \subseteq \bar{P}$ we have the entire space must be closed. For a singleton subset, any element outside this subset, we can find a $\epsilon$ such that there is no point in the singleton set getting close. □

**Theorem 2.3.1 (empty set and entire space are open).** *The empty set and the entire space $X$ are open sets.*

*Proof.* The interior points of an empty set is empty set. The entire set $X$: for every point in $X$ that is near $x_0 \in X$ will be in $X$, therefore every point in $X$ is interior point. □

**Theorem 2.3.2.** *Let $M$ be a metric space and let $x \in M$. Then $M, \emptyset$ and $\{x\}$ (a singleton set) are closed subsets of $M$.*

**Remark 2.3.2.**

- The reason that a singleton set is closed because there exist a constant sequence converge to $x$.
- There maybe confusion on why $M$ is closed. Note that a limit point is defined for points in $M$. If $M = (0, 1)$, then limits points does not include 0 and 1, therefore $M$ is closed. However, if $M = \mathbb{R}$,$(0, 1)$ is not closed in $M$.
- Based on the theorem of complement of open set is closed. We know that $M$ is both open and closed. $\mathbb{R}$ is both open and closed. $\emptyset$ is both open and closed.

**Theorem 2.3.3 (the complementary characterization of open and closed sets).** *[4, p. 285] Let $A$ be a subset in metric space $(M,d)$.*

- *If $A$ is a closed subset, then its complement is an open subset.*
- *If $A$ is an open subset, then its complement is closed set.*

*Proof.* (1) Let $A$ be a closed set. (2) Let $A$ be an open set. □

**Remark 2.3.3 (intuition on closed set and open set).** Closed set means every point in it can be getting arbitrarily close to, which is useful in taking limits. Open set means every point in it can be surrounded by an open ball belong to the set.

**Theorem 2.3.4.** *[2, p. 134]Let $M$ be a metric space and let $X \subset M$. Then $X$ is open if and only if $X^C$ is closed.*

*Proof.* see the closed set subsection. □

**Theorem 2.3.5.** *[2, p. 130] Any finite union of closed sets is closed; Any arbitrary intersection of closed set is closed.*

*Proof.* (1) use induction from above lemma; (2) let $\mathcal{G}$ be the collection of closed sets, then let $M = \cap\mathcal{G}$. If $M = \emptyset$, then $M$ is a closed set. If $M \neq \emptyset$, then let $x \in M$, and $x$ is a limit point(since every element of a closed set is a limit point), then there exist a sequence in $M$ that converges to $x$ □

**Remark 2.3.4 (Why finite union).** consider the collection of sets like $[0, 1-1/n]$, where the union is $[0, 1)$

**Theorem 2.3.6.** *[2, p. 134] Let $M$ be a metric space, then any **finite** intersection of open subsets are open in $M$; And **arbitrary** union of open subsets is open in $M$.*

*Proof.* (1) Let $U_1, ...U_n$ be open sets. Let $M = \cup_i U_i$. Consider $x \in M$, then there exist open balls in $U_1, ..., U_n$ centered at $x$. Choose the smallest open ball $B$ and $B$ will be in $M$ and therefore $M$ is open. (2) Let $M = \cap_i U_i^C = (\cup U_i)^C$, then $M$ will be closed since arbitrary intersection of closed set is closed, and then $\cup U_i$ will be open since its complement is open. □

**Remark 2.3.5 (Why finite intersection?).** Consider the countable intersections of sets $(-1/n, 1/n)$, which will produce the set $\{0\}$, which is closed set.

**Remark 2.3.6.** Let $\mathcal{J}$ be the collection of subsets of $M$, such that

1. $M \in \mathcal{J}$ and $\emptyset \in \mathcal{M}$.
2. if $O_1, O_2, ..., O_n \in \mathcal{J}$, then $O_1 \cap O_2 \cap ... O_n \in \mathcal{J}$(finite intersection).
3. if for each $\alpha \in \mathcal{I}$, $O_a \in \mathcal{J}$, then $\cup_{\alpha\in\mathcal{I}} O_\alpha \in \mathcal{J}$(arbitrary union).

$\mathcal{J}$ is called a topology on $M$.

### 2.3.4 Open and closed sets in $\mathbb{R}^n$

**Note 2.3.1 (the metric space $\mathbb{R}^n$).** In this section, we discuss the metric space $(\mathbb{R}^n, d)$, where the validity of the metric is discussed in Lemma 2.1.1.

**Definition 2.3.5 (open ball in $\mathbb{R}^n$).** *[4, p. 282] Let $u \in \mathbb{R}^n$ be a point in metric space $(\mathbb{R}^n, d)$. Let $r$ be a positive number, we call the set*

$$B_r(u) \triangleq \{v \in \mathbb{R}^n | d(u, v) < r\}$$

*the **open ball** with radius $r$ about $u$.*

**Definition 2.3.6 (interior and open sets in $\mathbb{R}^n$).** *[4, p. 282] Let $A$ be a subset of $\mathbb{R}^n$.*

- *The **interior** of $A$, denoted by int $A$ is the set of points in $\mathbb{R}^n$ such that each element in int$A$, there is an open ball about it contained in $A$.*
- *$A$ is called an open subset if int $A = A$.*

**Definition 2.3.7 (closed sets in $\mathbb{R}^n$).** *[4, p. 284] A subset $S \subset \mathbb{R}^n$ is closed provided that whenever $\{x_n\}, x_n \in \mathbb{R}^n$ is a sequence of points in $S$ that converges to a point $x$ in $\mathbb{R}^n$, then $x$ belongs to $\mathbb{R}^n$.*

**Lemma 2.3.4 (closed interval is closed subset in $\mathbb{R}$, open interval is not closed subset in $\mathbb{R}$).** *[4, p. 37]*

- *Let $[c, d], c < d$ be a closed interval. Then it is closed.*
- *Let $(c, d), c < d$ be an open interval. Then it is not closed.*

*Proof.* (1) Let $\{a_n\}$ be a convergent sequence such that every term lying in $[c,d]$. Based on the sequence-limit-inequality [Lemma 1.4.2], the limit must be in $[c,d]$. Therefore, $[c,d]$ is a closed set. (2) We can construct a convergent sequence $\{a_n\}$ that $x_n \in (c,d)$ for all $k$, but it converges to $d$. Then we can see that its limit point $d$ does not belong to the set $(c,d)$. □

**Note 2.3.2** (caution! closed set is a relative concept, specifying the metric space is critical).

- In the space $\mathbb{R}$, the interval $(0,1)$ is open.
- However, if the space is $(c,d)$, then $(c,d)$ is closed set in $(c,d)$. This is because the entire set is always closed [Lemma 2.3.3].

**Lemma 2.3.5 (rational number is not closed in $\mathbb{R}$ ).** *[4, p. 37] The $\mathbb{Q}$ of rational number is not closed in $\mathbb{R}$.(However, it is not necessarily open)*

*Proof.* Because of the sequential density property of the rationals [Corollary 1.4.4.1], there exists a rational number sequence converging to $\sqrt{2}$, which is not a rational number. Therefore, $\mathbb{Q}$ is not closed. □

**Theorem 2.3.7 (the complementary characterization of open and closed sets).** *[4, p. 285] A subset of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ if and only if its complement in $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.*

*Proof.* Theorem 2.3.3. □

*Example* 2.3.1. For example, the interval $(0,1)$ is open is because we can pick Some examples are:

1. some sets are neither open or closed, say $[0,1)$
2. Half-interval $[1,+\infty)$ is closed.
3. $\mathbb{R}^n$ and $\emptyset$ are both open and closed.

## 2.4 Compact sets

### 2.4.1 Basic concepts

**Definition 2.4.1 (open cover).** *[5][6, p. 98][2, p. 144]A collection $\mathcal{O}$ of opens sets is an* ***open cover*** *of $A$ if every point $x \in A$ is in some open set in the collection $\mathcal{O}$. A* ***finite subcover*** *of $A$ is a finite subcollection of open sets from the original open cover whose union still manages to completely contain $A$.*

**Definition 2.4.2 (compact via open set).** *A set $A$ is called* ***compact*** *if* ***every open cover*** *$\mathbb{O}$ of $A$ contains a finite subcollection of open sets which also covers $\mathbb{A}$.*

**Remark 2.4.1.** The definition requires **every** open cover contains a finite subcover.

*Example* 2.4.1. [6, p. 98]The open interval $(0,1)$ has a open cover $\cup_{n=1}^{\infty}(1/n,1)$ that does not have a finite subcover: we can always select a $\epsilon > 0$ which is small enough that cannot be covered.

**Definition 2.4.3 (compact set via sequence).** *Metric space $M$ is compact if every sequence in $M$ has a convergent subsequence.*

**Remark 2.4.2.** The equivalence between the two definitions are discussed in [2, p. 150]

**Theorem 2.4.1 (necessary condition for compact set).** *Compact sets are closed and bounded.*

*Proof.* (1) Suppose it is not bounded, then we can have a sequence $\{x_n\}$ with $x_n \geq n$ that does not have convergent sequence. (2) It is closed can be proved directly from definition of closed set that every convergent sequence has a limit in it. □

**Remark 2.4.3.** **Not every closed and bounded set in metric space is compact.** But a closed and bounded subset in $\mathbb{R}^n$ is compact.

#### 2.4.1.1 *closed set vs. compact set*

- the definition of the closed set says "every convergent sequence in $E$ has a limit in $E$"
- the definition of compact set says "every sequence has a convergent subsequence that has a limit in $E$"
- compact set has more restrictive requirement and therefore compact sets are always closed.

### 2.4.2 Compact sets in $\mathbb{R}^N$

**Definition 2.4.4 (compact sets in $\mathbb{R}$).** *[6, p. 96] A set $K \subseteq \mathbb{R}$ is compact if **every** sequence in $K$ that has a subsequence that converges to a limit in $K$.*

*Example* 2.4.2. The basic example is a closed interval. For every convergent sequence(hence bounded), there is a convergent subsequence (due to Bolzano-Weierstrass theorem Theorem 1.6.2]. Because it is closed, the convergent subsequence will have a limit in it. Therefore, the closed interval is compact.

**Definition 2.4.5 (bounded sets in $\mathbb{R}$).** *A set $A \subseteq \mathbb{R}$ is bounded if there exists $M > 0$ such that $|a| < M, \forall a \in A$.*

**Theorem 2.4.2 (compact set on $\mathbb{R}^n$).** *[6, p. 96]A set $S \subset \mathbb{R}$ is compact if and only if it is closed and bounded.*

*Proof.* (1) Suppose $K$ is compact, and assume $K$ is not bounded, that is given a positive $M > 0$, we can always find $x \in K$,$|x| > M$. Construct a sequence which consists of $\{x_n, x_n > n\}$, every subsequence of this sequence will be convergent and therefore bounded because of the assumption that $K$ is compact. However, we can find a subsequence(the sequence itself) is not bounded, therefore we have contradiction,and we prove a compact set must be bounded. To show a compact set is closed(every Cauchy sequence has a limit contained in it), because of the compactness, every Cauchy(therefore convergent) sequence will have subsequence converge to a limit in $K$. The limit of the subsequence must be the same limit of its master sequence. (2) The converse is straightforward, every sequence has a convergent subsequence; and because of the close set, the convergent subsequence must converge to a limit in $K$. $\square$

**Lemma 2.4.1 (compactness of $\mathbb{R}$).** *[2, p. 145] The space $\mathbb{R}$ is not compact.*

*Proof.* The open cover $\cup_{n=1}^{\infty}(-n, n)$ does not have a finite subcover. $\square$

**Lemma 2.4.2.** *[6, p. 99]Let K be a subset of $\mathbb{R}$. If $K$ is non-empty and compact, then $\sup K$ and $\inf K$ both exist and are elements of $K$*

*Proof.* Because $K$ is closed and bounded, then $\sup K$ and $\inf K$ both exist due to the completeness of $\mathbb{R}$. Suppose $\sup K$ in $K^C$, which is an open set, then there exist an open neighborhood around $\sup K$ in which some points are upper bound but smaller then $\sup K$, which is a contradiction. $\square$

### 2.4.3 The Hine-Borel Theorem and boundedness of continuous function

**Theorem 2.4.3.** *[2, p. 113] Let $\mathcal{F}$ be a collection of open intervals such that*

$$[a, b] \subset \cup\mathcal{F}$$

*Then there exist a finite subset $\{I_1, I_2, ..., I_n\}$ of $\mathcal{F}$ such that*

$$[a, b] \subset \cup_{i=1}^{n} I_i$$

## 2.5 Completeness of metric space

### 2.5.1 Sequence and completeness

**Definition 2.5.1 (Cauchy sequence).** *[2, p. 159] Let $(M, d)$ be a metric space. A sequence $\{x_n\}$ in $M$ is Cauchy sequence if for every $\epsilon > 0$, there exists a $N > 0$ such that if $m, n > N$, then $d(x_m, d_n) < \epsilon$.*

**Theorem 2.5.1 (every convergent sequence in metric space is Cauchy sequences).** *Every convergent sequence in a metric space is Cauchy sequence.*

*Proof.* Using triangle inequality, similar to Theorem 1.4.3. □

**Definition 2.5.2 (completeness of metric space).** *[2, p. 159]Let $M$ be a metric space. If every Cauchy sequence in $M$ is convergent to a point in $M$, then $M$ is a* ***complete*** *metric space.*

*Example* 2.5.1. The metric space $(0, 2)$ with the Euclidean metric is not complete metric space because we can construct a Cauchy sequence $\{2 - \frac{1}{n}\}$, whose every term is in $(0, 2)$, but converges to 2(outside of $(0, 2)$).

**Remark 2.5.1.** Even if the Cauchy sequence converge in a complete metric space, the limit point might not lie in the metric space. If the space is **closed**, then the limit lie in the metric space.

**Remark 2.5.2 (compared with Cauchy sequence in $\mathbb{R}$).** In real line, **every Cauchy sequence is convergent, since real line is complete.**Theorem 1.4.3 For general metric space, this is not true.

### 2.5.2 Completeness of $\mathbb{R}^n$

**Theorem 2.5.2 (metric space $\mathbb{R}^n$ with Euclidean metric is complete).** *[4, p. 323]*

- *The metric space $\mathbb{R}^n$ with the Euclidean metric is a complete metric space.*

- *Every closed subset of $\mathbb{R}^n$ is a complete metric space.*

*Proof.* (1) From the definition of complete metric space [Definition 2.5.2], we want to show that every Cauchy sequence in $\mathbb{R}^n$ is convergent to a point in $\mathbb{R}^n$. Now consider a Cauchy sequence $\{a^{(k)}\}, a^{(k)} \in \mathbb{R}^n$, because

$$\left|a_i^{(m)} - a_i^{(n)}\right| \leq \left\|a^{(m)} - a^{(n)}\right\|, i = 1, 2, ..., n,$$

therefore each component sequence $\{a_i^{(k)}\}$ is a Cauchy sequence and will converge to an element in $\mathbb{R}$. The componentwise convergence will further imply the convergence of $\{a^{(k)}\}$ in $\mathbb{R}^n$. [Theorem 2.2.1]. (2) Note that based on the definition of closedness, every convergent sequence will have its limit point in the subset; therefore, any closed subset of $\mathbb{R}^n$ is also a complete metric space. □

**Remark 2.5.3.** Closed subset of real line is complete metric space, however, open subset is not. For example, open interval $(0, 1)$ is not complete, because the Cauchy sequence $\{1/n\}$ not converge to an element in $(0, 1)$.

## 2.6 Topology space

### 2.6.1 Definitions

**Definition 2.6.1 (topology space).** *[2, p. 136] A topological space is a set $X$ together with a set $\mathcal{J}$ of subsets of $X$, such that*

1. $X \in \mathcal{J}$
2. $\emptyset \in \mathcal{J}$
3. *The intersection of any* ***finite*** *elements in $\mathcal{J}$ will be an element of $\mathcal{J}$; that is, if $O_1, O_2, ..., O_n \in \mathcal{J}$, then $O_1 \cap O_2 \cap ...O_n \in \mathcal{J}$*
4. *if for each $\alpha \in \mathcal{I}$, $O_a \in \mathcal{J}$, then $\cup_{\alpha \in \mathcal{I}} O_\alpha \in \mathcal{J}$.*

*$\mathcal{J}$ is called a topology on $X$.*

**Remark 2.6.1 (Why finite intersection).** Note that **only finite intersection is allowed, countable intersection is not allowed**, since we can countably intersect a nested interval to create a singleton, which is not an open set.

*Example* 2.6.1.

- The real line $\mathbb{R}$ with its topology consist of open sets given by $(a, b) = \{x \in \mathbb{R}, a < x < b\}$ for any real numbers $a$ and $b$

**Definition 2.6.2 (Open sets).** *Let $(X, \mathcal{J})$ be any topological space. Then the member of $\mathcal{J}$ are called open sets. Then in a* ***topological space****, we have*

1. *$X, \emptyset$ are open sets*
2. *the union of any(finite or infinite) number of open sets is an open set*
3. *the intersection of any finite number of open sets is an open set*

**Remark 2.6.2.** Note that open set in topology is an abstract concept. Different specifications or definitions of open set will give different topology space. [7]The simplest example is in metric spaces, where open sets can be defined as those sets which contain an open ball around each of their points, for example in $\mathbb{R}$, $(a, b)$ is defined as open sets.

**Definition 2.6.3 (Hausdorff, Hausdorff space).** *For any distinct $x_1, x_2 \in X$, there exist open sets $O_1$ and $O_2$ such that $x_1 \in O_1, x_2 \in O_2$, and $O_1 \cap O_2 = \emptyset$. Any topological space $X$ that satisfies this is call* Hausdorff space. *Or we say it is* ***Hausdorff****.*

**Remark 2.6.3.** Note: The concept of Hausdorff implies that around any two distinct points on $X$, we can find two non-overlapping open sets. [8]

**Definition 2.6.4 (Neighborhood).** *Given a topological space $(X, \mathcal{J})$, a subset $N$ of $X$ is called a neighborhood of a point $a \in X$ if $N$ contains an open set that contains $a$.*

**Remark 2.6.4.** Neighborhood aournd a point is not necessarily an open set, but it must contain an open set.

### 2.6.2 Continuous function, Homeomorphism in topological space

**Definition 2.6.5 (continuous function).** *A function $f : X \to Y$ between two topological space $X, Y$ is continuous if for every open set $V \in Y$, the pre-image $f^{-1}(V)$ is an open set in $X$.*

**Remark 2.6.5.** consider a function on real line with some 'jumps', this function is not continuous because at one end of the discontinuity, it is closed.

**Definition 2.6.6 (homemorphism).** *Suppose $f : X \to Y$ is a bijective function between topological spaces $X, Y$. If both $f, f^{-1}$ are continuous, then $f$ is called a* homeomorphism*. Two topological spaces $X, Y$ are said to be* homeomorphic *if there exist a homeomorphism between them.*

### 2.6.3 Subspaces of topological space

**Definition 2.6.7.** *[9]Let $(X, \mathcal{J}), (Y, \mathcal{J}')$ be topological spaces. The topological space $Y$ is called a* subspace *of $X$ if $Y \subset X$ and if all the open subsets of $Y$ are precisely the subsets $O'$ of the form*

$$O' = O \cap Y$$

*for some open subset $O$ of $X$. If $Y$ is the subspace $X$, we may say that each open subset $O'$ of $Y$ is the restriction to $Y$ of an open subset $O$ of $X$.Note that a subset $O'$ that is open in $Y$ is often called* relatively open*. Relatively open subsets of $Y$ are in general not open in $X$, because it does not contain its neighborhood in $Y$.*

*Example* 2.6.2.

- Consider $\mathbb{R}$ as a subspace in $\mathbb{R}^2$, the subsets $\mathbb{R}$ is just relatively open in $\mathbb{R}^2$. It is not open in $\mathbb{R}^2$, because it does not contain its neighborhood in $\mathbb{R}$.

## 2.7 Notes on bibliography

The key references for this chapter are intermediate level real analysis textbooks[2][6][10][11].

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