10

BASIC GAME THEORY

10.1 Static normal form game

10.1.1 Normal form game concepts

Definition 10.1.1 (normal-form game). *[1, p. 3]A (finite, n-player) normal-form game is a tuple (N, A, u), where*

- *N is the number of players, indexed by $i = 1, 2, ..., N$;*
- *$A = A_1 \times A_2 ... \times A_N$, where A_i is the action/strategy space for player i. An element $a = (a_1, a_2, ..., a_N) \in A$ is called the action profile;*
- *$u = (u_1, u_2, ..., u_n)$ where $u_i = u_i(a_1, a_2, ..., a_N)$ is the **utility/payoff function** for player i.*

In general, based on the specification of payoff, games can further be classified into **pure coordination or team games**, **pure competitive games**, and games in between. In a common-payoff game, there is no interest conflict between any pair of players and the goal of any single player is to achieve what is maximally beneficial to all. In a pure competitive game, one player's gain must come at the expense of the other player. We have more formal definitions given below.

Definition 10.1.2 (common-payoff games, coordinated games). *[1, p. 4]A common-payoff game is a normal-form game in which for **all** action profiles $a \in A$ and **any** pair of agents,i,j,$u_i(a) = u_j(b)$.*

Definition 10.1.3 (constant sum game, zero-sum game). *[1, p. 5] A two-player normal form game is constant-sum if there exists a constant c such that for each strategy profile $a \in A_1 \times A_2$ it is the case that $u_1(a) + u_2(a) = c$. When $c = 0$, it is called **zero-sum game**.*

10.1.2 Pure strategy and equilibrium

10.1.2.1 *Solution concepts*

Definition 10.1.4 (strictly dominated strategy, strictly inferior strategy). *[2, p. 5] In a normal-form game (N, A, u), let a'_i and a''_i denote two feasible strategies of player i. We say a'_i* ***is strictly dominated by (or strictly inferior to)*** *a''_i if*

$$u_i(a'_i, a_{-i}) < u_i(a''_i, a_{-i}), \forall a_{-i}.$$

That is, a'_i is strictly inferior to a''_i for player i if the payoff is strictly smaller for all other players' strategies.

Definition 10.1.5 (best response to other players' fixed strategies). *[1, p. 11] In a normal-form game (N, A, u), player i's* ***best response*** *to other players' fixed strategy profile a_{-i} is a strategy $a_i^*(a_{-i}), a_i^*(a_{-i}) \in A_i$ such that*

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \forall a_i \in A_i.$$

Definition 10.1.6 (dominant pure strategy for a single player).

- *In a normal-form game (N, A, u), if strategy $a_i^* \in A_i$ is called* ***dominant strategy*** *if*

  $$u(a_i^*, a_{-i}) \geq u(a_i(a_{-i}), a_{-i}), \forall a_{-i},$$

  where $a_i(a_{-i})$ is the best response for player i when facing other players strategy a_{-i}. In other words, ***a dominant strategy is the best strategy among all possible situations.***
- *A dominant strategy is called* ***strictly dominant strategy*** *if*

  $$u(a_i^*, a_{-i}) > u(a_i, a_{-i}), if\ a_i \neq a_i^*,$$

  $\forall a_{-i}$.

Note 10.1.1 (interpretation).

- The best response of player i is the best strategy he can do when facing other players' fixed strategies; it is the best among a particular situation.
- The dominant strategy of player i is the best strategy he can do when facing other players' arbitrary strategies; it is the best among all possible situations.
- **(existence issue)** The best response always exist; the dominant response not necessarily exists.

A **rational player** will maximize its utility in face of the all possible counteracts from its opponents. We have the following

Theorem 10.1.1 (Rational decision for a single player). *[2, p. 6]*

- *Any strictly inferior strategy will not be chosen by a rational player.*
- *A dominant strategy (if exists) will be chosen by a rational player.*

Proof. (1) A rational player will have no incentive to stay in a inferior strategy. (2) If a dominant strategy exists, then a rational player will choose it since it is the best among all possible situations. □

Definition 10.1.7 (Nash equilibrium for pure strategies). *[1, p. 11][2, p. 8]*

- *A strategy profile $a^* = (a_1^*, a_2^*, ..., a_n^*)$ is called a* ***Nash equilibrium*** *if, for all players i, a_i^* is a best response to a_{-i}^*. In other words, for every player i, a_i^* solves*

$$\max_{a_i \in A_i} u_i(a_i, a_{-i}^*).$$

- *A* ***strict Nash equilibrium*** *strategy a satisfies: for all agents i and for all strategies $a_i' \neq a_i^*, u_i(a_i^*, a_{-i}^*) > u_i(a_i', a_{-i}^*)$; A* ***weak Nash equilibrium*** *strategy s satisfies: for all agents i and for all strategies $a_i' \neq a_i^*, u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*)$ and a is not strict Nash equilibrium.*

Theorem 10.1.2 (rational players will reach Nash equilibrium (if the equilibrium exists)). *[2, p. 9] Assume there exists a Nash equilibrium a^* in a normal-form game (N, A, u). All rational players will not have the incentive to deviate from the equilibrium strategy individually.*

Proof. If a player deviates individually whereas all other players remain the same. Then the player is deviating from his best response therefore he has no incentive to deviate. □

One common theme of game theory is to search and understand Nash equilibrium. It turns out that dominant pure strategies we defined previously[Definition 10.1.6] have a deep connection to it.

Theorem 10.1.3 (Dominant pure strategies and Nash equilibrium).

- *In a normal-form game (N, A, u), a strategy profile a consisting of dominant strategies for every player must be a Nash equilibrium;*
- *If the equilibrium is associated with strictly dominant strategies, then it must be a unique Nash equilibrium.*

Proof. (1) A dominant strategy is the best response for each individual player. When all the players play the dominant strategy, it is the Nash equilibrium by definition. (2) Suppose a' and a'' are both Nash equilibrium such that there exists at least one component $a'_i \neq a''_i$. Then $u(a'_i, a'_{-i}) > u(a''_i, a'_{-1})$ because a'_i is a strictly dominant strategy; however,we also have $u(a'_i, a'_{-i}) < u(a''_i, a'_{-1})$ because a''_i is a strictly dominant strategy. This leads to contradiction. □

Example 10.1.1 (the prisoners' dilemma). In the game of prisoners' dilemma showed below, we first identity the best response(which always exists) at different strategies chosen by the counter-player. The equilibrium is at (betray, betray), which are best responses of the two players at the same time.

		Player 2	
		Silent	Betray
Player 1	Silent	$-1, -1$	$-9, \underline{0}$
	Betray	$\underline{0}, -9$	$\underline{-6}, \underline{-6}$

Note 10.1.2 (existence and uniqueness).

- In a finite-player game where only pure strategies are allowed, there are no guarantees on both existence and uniqueness. See the following examples.
- However, In a finite-player game where mixed strategies are allowed, the existence but not the uniqueness can be guaranteed [Theorem 10.1.5].

Example 10.1.2 (the battle fo the sexes, existence of multiple Nash equilibrium). [2, p. 11] The battle fo the sexes game, the two players have two Nash equilibra.

		Player 2	
		Opera	Fight
Player 1	Opera	$\underline{2}, \underline{1}$	$0, 0$
	Fight	$0, 0$	$\underline{1}, \underline{2}$

Example 10.1.3 (matching pennies, nonexistence of Nash equilibrium for pure strategies). [2, p. 29] In the following matching pennies, we see that there does not exist Nash equilibrium.

		Player 2	
		Heads	Tails
Player 1	Heads	$-1, \underline{1}$	$\underline{1}, -1$
	Tails	$\underline{1}, -1$	$-1, \underline{1}$

10.1.3 Mixed strategy and equilibrium

Definition 10.1.8 (Mixed strategy and support). *[1, p. 7] Let (N, A, u) be a normal-form game, and for any set X let $\Pi(X)$ be the set of all probability distributions over X. Then the set of mixed strategies for player i is $S_i = \Pi(A_i)$. $s_i \in S_i$ is a probability measure on sample space A_i, with $s_i(a), a \in A_i$ represents the probability of taking a. The set of* ***Mixed strategy profiles*** *is $S = S_1 \times S_2 ... \times S_n$.*

The ***support*** *of a mixed strategy s_i for the player i is the set of the pure strategies $\{a_i | s_i(a_i) > 0\}$.*

Definition 10.1.9 (utility of a mixed strategy). *Given a normal form game (N, A, u), the expected utility u_i for player i is*

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{N} s_j(a_j) \tag{5}$$

where $s \in S, s = (s_1, s_2, ..., s_N), a = (a_1, a_2, ..., a_N)$

Remark 10.1.1.

- S_i is the **set** of all possible stochastic strategies for player i.
- A **pure** strategy is a 'degenerate' case of mixed strategy.

Definition 10.1.10 (best response of mixed strategy). *[1, p. 11] Player i's best response to the strategy profile s_{-i} is a mixed strategy $s_I^* \in S_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \forall s_i \in S_i$.*

Definition 10.1.11 (Nash equilibrium for mixed strategies). *[1, p. 11] A strategy profile $s = (s_1, s_2, ..., s_n)$ is a* ***Nash equilibrium*** *if, for all agents i, s_i is a best response to s_{-i}. A* ***strict Nash equilibrium*** *strategy s satisfies: for all agents i and for all strategies $s_i' \neq s_i, u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$;A* ***weak Nash equilibrium*** *strategy s satisfies: for all* *agents i and for all strategies $s_i' \neq s_i, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and s is not strict Nash equilibrium.*

Remark 10.1.2.

- At Nash equilibrium s^*, no agent has the incentive to change his own strategy, because every agent is already at the best response.
- At a strict Nash equilibrium, any change of individual's strategy will cause his utility loss.

Definition 10.1.12 (dominance). *[1, p. 20] s_i strictly dominates s_i' if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-1}) > u_i(s_i', s_{-1}).s_i$ weakly dominates s_i' if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-1}) \geq u_i(s_i', s_{-1})$.*

Definition 10.1.13 (dominant strategy). *If strategy s_i strictly(weakly) dominates all other strategies, then s_i is called strictly(weakly) dominant strategy.*

Remark 10.1.3.

- A dominant strategy is usually better than an usual Nash equilibrium strategy
- A dominant strategy associated Nash equilibrium might not exist.

Theorem 10.1.4 (Nash equilibrium of dominant strategies). *If there exist a strategy profile consisting of dominant strategies for every player, the strategy profile must be a Nash equilibrium; The equilibrium associated with a strictly dominant profile must be unique in the strategy profile.*

Remark 10.1.4. The proof is directly from definition since dominant strategy implies best response.

Theorem 10.1.5 (existence of Nash equilibrium). *[3, p. 117] Every game with a finite number of players and action profiles has at least one Nash equilibrium.*

10.1.4 Pareto optimality

Definition 10.1.14 (Pareto domination). *[1, p. 10] Strategy profile $s \in S$ Pareto dominates s' if for all $i \in N, u_i \geq u_i(s')$, and there exists some $j \in N$ for which $u_j(s) > u_j(s')$*

Definition 10.1.15 (Pareto optimal). *[1, p. 10] A strategy profile s is Pareto optimal if there does not exist s' that Pareto dominates s.*

Remark 10.1.5 (intuition and uniqueness)**.**

- A Pareto optimal is a strategy that if someone wants to make his utility better, he has to hurt other people's utility.
- Every game must have at least one Pareto optimal strategy, and it is not necessarily unique.
- A Pareto optimal strategy can be obtained by optimizing each agent's strategy one-by-one sequentially without hurting other agent's utility.
- In a zero-sum game, every pure strategies are Pareto optimal,i.e., any changes in the strategy will lead to the cost of other player's utility.

Remark 10.1.6 (Pareto optimal vs. Nash equilibrium)**.**

- A Pareto optimal strategy is not necessarily a Nash equilibrium, because there might exist someone to increase his own interest at the cost of other people's utility.A Nash equilibrium is not necessarily Pareto optimal.For example, in the canonical Prisoner's dilemma formulation, the Nash equilibrium is not Pareto optimal, because the Pareto optimal is both cooperate.
- Nash equilibrium is the optimal under unilateral changes; The total utility of a game can be improved from the Nash equilibrium by enabling multi-lateral moves.

10.2 Zero-sum matrix game

10.2.1 Fundamentals

Definition 10.2.1 (zero-sum matrix game). *A **zero-sum game** is defined by a **payoff matrix** A, where a_{ij} represents the payoff to the **row player R** if R chooses option i and **column players C** chooses option j. A number $a_{ij} > 0$ is the gain of row player;$a_{ij} < 0$ is the gain of the column player.*

Definition 10.2.2 (strategy in a zero-sum matrix game).

- *In a zero-sum matrix game, a **strategy** for a player consists of a probability vector representing the probability of each option being chosen.*
- *Let be $A \in \mathbb{R}^{m\times n}$ be the payoff matrix. The strategy for the row player is a row vector $p \in \mathbb{R}^m$; the strategy for the column player is a column vector $q \in \mathbb{R}^m$;*
- *A **pure strategy** is represented by a standard basis vector e_i.*
- *A non-pure strategy is called **mixed strategy**.*

Example 10.2.1 (the Rock, Paper, Scissors game). In the game of Rock, Paper, Scissors we have the following bi-matrix representation showed below.

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

Alternatively, we can also use the following matrix representation.

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Definition 10.2.3 (expected value of a pair of strategies, utilities).

- *Let $A \in \mathbb{R}^{m\times n}$ be the payoff matrix. The expected value of a row strategy p and a column strategy q is given by*

$$E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q.$$

- *When the players are taking (p,q) strategies, the utility for the row player is represented by*

$$u_r(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j.$$

The utility for column player is represented by

$$u_c(p,q) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j.$$

That is, the expected gain the row player receives is what the column player loses.

Example 10.2.2. Consider a Rock-Paper-Scissor game with payoff matrix

	Rock (R)	Paper (P)	Scissors (S)
Rock (R)	0	-1	1
Paper (P)	1	0	-1
Scissors (C)	-1	1	0

.

If two players both take a uniform strategy, then we have

$$\begin{aligned} E = &\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ R, R }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ R, P }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ R, S })+ \\ &\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ P, R }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ P, P }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ P, S })+ \\ &\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ S, R }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ S, P }) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)A(\text{ S, S}) = 0 \end{aligned}$$

10.2.2 Optimal strategy and Nash equilibrium

Definition 10.2.4 (optimal strategies for zero-sum matrix game). *In the zero-sum matrix game, a pair of strategies (p^*, q^*) is called* ***optimal strategies*** *if for* ***any*** *strategies p and q such that*

$$E(p^*, q) \geq E(p^*, q^*) \geq E(p, q^*).$$

Lemma 10.2.1 (optimal strategies are Nash equilibrium). *Let a pair of strategies (p^*, q^*) be the optimal strategies in the zero-sum matrix game. The (p^*, q^*) is the Nash equilibrium. Conversely, if the pair of strategies (p^*, q^*) is Nash equilibrium, then they are optimal strategies.*

Proof. (1) To prove optimal strategies are Nash equilibrium we are going to show that the strategies are best responses for each player. For row player, $E(p^*, q^*) \geq E(p, q^*)$ implies $u_r(p^*, q^*) \geq u_r(p, q^*)$; that is p^* is the best response of row player when column player takes q^*. Similarly, $E(p^*, q) \geq E(p^*, q^*)$ implies $u_c(p^*, q^*) \geq u_c(p^*, q)$; that is q^* is the best response of column player when row player takes p^*. (2) Conversely, if (p^*, q^*) are Nash equilibrium, we have $u_c(p^*, q^*) \geq u_c(p^*, q)$ and $u_r(p^*, q^*) \geq u_r(p, q^*)$, which together implies

$$E(p^*, q) \geq E(p^*, q^*) \geq E(p, q^*).$$

□

Remark 10.2.1.

- $E(p^*, q^*) \geq E(p^*, q)$ shows that $E(p^*, q^*)$ is the maximum on the row player can gain when the column player plays q^*.
- $E(p^*, q) \geq E(p^*, q^*)$ shows that $E(p^*, q^*)$ is the minimum on the column player can lose when the the row player plays p^*.

Theorem 10.2.1 (fundamental theorem of matrix games, existence of Nash equilibrium/optimal strategies for zero-sum games). *A zero-sum matrix game always has at least one Nash equilibrium or at least one pair of optimal strategies.*

Proof. Theorem 10.1.5. □

Remark 10.2.2 (nonexistence of Nash equilibirum for pure strategies game). If we only allow pure strategies, the following payoff matrix does not have a Nash equilibrium

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

10.2.3 Saddle points as solutions

Definition 10.2.5 (saddle point in payoff matrix). *Let $A \in \mathbb{R}^{m\times n}$ be a payoff matrix. The element a_{rc} is called a **saddle point** if the following conditions are satisfied:*

- $a_{rj} \geq a_{rc}, \forall j$.
- $a_{ic} \leq a_{rc}, \forall i$.

*We say the **saddle point is strict** if the inequality holds.*

Remark 10.2.3 (existence, uniqueness and usage of saddle points)**.**

- For a given matrix A, the saddle point might not exists; if exists, there might exist multiple saddle points.
- When the saddle points exists, the optimal strategy can be readily found. See the following theorem [Theorem 10.2.2].
- When exists, the saddle point might not be unique.

Example 10.2.3.

- Consider the payoff matrix

$$\begin{bmatrix} 4 & 2 & 5 & 2 \\ 2 & 1 & -1 & -20 \\ 3 & 2 & 5 & 2 \\ -16 & 0 & 16 & 1 \end{bmatrix}.$$

The four circled 2 are saddle points in the payoff matrix. Clearly, at each of these saddle point (pure strategy), any unilateral deviation of the saddle point will end up with reduced utility.
- Consider the payoff matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 4 & 6 \\ 4 & -1 & 5 \end{bmatrix}$$

The circled 4 is the saddle point in the payoff matrix.
Clearly, at this saddle point (pure strategy), any unilateral deviation will end up with reduced utility.

- The following payoff matrix does not have a saddle point, although a pure strategy (3, 3) will be the Nash equilibrium.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Theorem 10.2.2 (saddle points implies optimal pure strategy and Nash equilibrium). *Let $A \in \mathbb{R}^{m\times n}$ be a payoff matrix with a saddle point a_{rs}. It follows that $e_r \in \mathbb{R}^m$ and $e_c \in \mathbb{R}^n$ will be the optimal strategy and Nash equilibrium.*

Proof. Note that the strategies (e_r, e_c) will give

$$a_{ic} \leq a_{rc}, \forall i \implies E(e_r, r_c) \geq E(p, e_c)$$

and

$$a_{rj} \geq a_{rc}, \forall j \implies E(e_r, q) \geq E(e_r, e_c)$$

That is

$$E(e_r, q) \geq E(e_r, e_c) \geq E(p, e_c).$$

Therefore, (e_r, e_c) are optimal strategies and thus the Nash equilibrium [Lemma 10.2.1]. □

10.2.4 Maxmin strategies and Nash equilibrium

Definition 10.2.6 (maxmin strategy). *[1, p. 15] The maxmin strategy for player i is* $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$

Remark 10.2.4.

- The maxmin strategy allows player to achieve good payoff in the worst case scenario.

Lemma 10.2.2 (minmax inequality lemma). *For any zero-sum matrix game with payoff matrix $A \in \mathbb{R}^{m\times n}$,*

$$\min_{q\in\mathbb{R}^n} \max_{p\in\mathbb{R}^m} p \cdot Aq \geq \max_{p\in\mathbb{R}^m} \min_{q\in\mathbb{R}^n} p \cdot Aq,$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1.$

Proof. For **any** $p \in \mathbb{R}^m, q \in \mathbb{R}^n$, we always have

$$p \cdot Aq \geq \min_{q'} p \cdot Aq'.$$

Note that both sides are a function of p. By maximizing both sides on p, we also have

$$\max_{p'} p' \cdot Aq \geq \max_{p'} \min_{q'} p' \cdot Aq'.$$

Note that left side is a function of q and the right side is a constant. Since the inequality holds for all q, we also have

$$\min_{q'} \max_{p'} p' \cdot Aq' \geq \max_{p'} \min_{q'} p' \cdot Aq'.$$

□

Theorem 10.2.3 (equivalence between minmax and maxmin). *For any zero-sum matrix game with payoff matrix* $A \in \mathbb{R}^{m\times n}$,

$$\min_{q\in\mathbb{R}^n} \max_{p\in\mathbb{R}^m} p \cdot Aq = \max_{p\in\mathbb{R}^m} \min_{q\in\mathbb{R}^n} p \cdot Aq.$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1.$

Proof. Because the zero-sum matrix game always has Nash equilibrium (p^*, q^*) [Theorem 10.1.5], we have

$$p^* \cdot Aq^* \geq p \cdot Aq^*, \forall p \in \mathbb{R}^m,$$

and

$$p^* \cdot Aq \geq p^* \cdot Aq^*, \forall q \in \mathbb{R}^n.$$

Combine together we have

$$p^* \cdot Aq \geq p \cdot Aq^*, \forall p, q.$$

Minimizing both sides on q, we have

$$\min_q p^* \cdot Aq \geq p \cdot Aq^*, \forall p.$$

Note that the inequality holds for all p, we therefore have

$$\min_q p^* \cdot Aq \geq \max_p p \cdot Aq^*. (*)$$

We then have (using eq.(*) in the second line)

$$\begin{aligned}\max_p \min_q p\cdot Aq &\geq \min_q p^*\cdot Aq\\ &\geq \max_p p\cdot Aq^*\\ &\geq \min_q \max_p p\cdot Aq\end{aligned}$$

From above lemma [Lemma 10.2.2] we have

$$\min_{q\in\mathbb{R}^n}\max_{p\in\mathbb{R}^m} p\cdot Aq \geq \max_{p\in\mathbb{R}^m}\min_{q\in\mathbb{R}^n} p\cdot Aq.$$

Put together, we have

$$\min_{q\in\mathbb{R}^n}\max_{p\in\mathbb{R}^m} p\cdot Aq = \max_{p\in\mathbb{R}^m}\min_{q\in\mathbb{R}^n} p\cdot Aq.$$

□

Theorem 10.2.4 (minmax(maxmin) solution is Nash equilibrium for zero-sum matrix game). *For any zero-sum matrix game with payoff matrix $A\in\mathbb{R}^{m\times n}$. The solution (p^*,q^*) solving*

$$\min_{q\in\mathbb{R}^n}\max_{p\in\mathbb{R}^m} p\cdot Aq$$

or

$$\max_{p\in\mathbb{R}^m}\min_{q\in\mathbb{R}^n} p\cdot Aq,$$

where $p_i\geq 0, \sum_{i=1}^m p_i = 1, q_i\geq 0, \sum_{i=1}^n q_i = 1$, is the Nash equilibrium.

Proof. Denote the Nash equilibrium by (p^*,q^*). We have in the above theorem proof [Theorem 10.2.3]

$$\begin{aligned}\max_p \min_q p\cdot Aq &\geq \min_q p^*\cdot Aq\\ &\geq \max_p p\cdot Aq^*\\ &\geq \min_q \max_p p\cdot Aq\end{aligned}$$

Because $\min_{q\in\mathbb{R}^n}\max_{p\in\mathbb{R}^m} p\cdot Aq = \max_{p\in\mathbb{R}^m}\min_{q\in\mathbb{R}^n} p\cdot Aq$, therefore

$$\begin{aligned}\max_p \min_q p\cdot Aq &= \min_q p^*\cdot Aq\\ &= \max_p p\cdot Aq^*\\ &= \min_q \max_p p\cdot Aq.\end{aligned}$$

We have q^* solves $\min_q p^* \cdot Aq$; that is, q^* is the best response of the column player when the row player plays p^*. Similarly, we have p^* solves $\max_q p \cdot Aq^*$; that is, q^* is the best response of the row player when the column player plays q^*. □

10.2.5 Linear programming approach to optimal strategy

Definition 10.2.7 (optimization problem for zero-sum matrix game). *From the equivalence of minmax solution to Nash equilibrium [Theorem 10.2.4], we can define the following optimization problem:*

- *The row player's optimization problem is*

$$\max_{p\in\mathbb{R}^m}\min_{q\in\mathbb{R}^n} p^T Aq,$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1.$
- *The column player's optimization problem is*

$$\min_{q\in\mathbb{R}^n}\max_{p\in\mathbb{R}^m} p^T Aq.$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1.$

Lemma 10.2.3.

- *For any* $q \in \mathbb{R}^n$, *we have*

$$\max_{p\in\mathbb{R}^m} p^T Aq = \max_{1\leq i\leq m} e_i^T Aq,$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1.$
- *For any* $p \in \mathbb{R}^m$, *we have*

$$\min_{q\in\mathbb{R}^n} p^T Aq = \min_{1\leq i\leq n} p^T Ae_i,$$

where $q_i \geq 0, \sum_{i=1}^n q_i = 1.$

Proof. (1)(a) It is easy to see that we always have $\max_{p\in\mathbb{R}^m} p^T Aq \geq \max_{1\leq i\leq m} e_i^T Aq$. (b) Let i^* be the index maximize $e_i^T Aq$, then $p_i e_i Aq \leq p_i e_{i^*} Aq, i = 1, 2, ..., m$ then

$$pAq = \sum_{i=1}^m p_i e_i^T Aq = p^T Aq \leq \sum_{i=1}^m p_i e_{i^*} Aq = e_{i^*} Aq = \max_{1\leq i\leq m} e_i^T Aq.$$

Therefore,

$$\max_{p\in\mathbb{R}^m} p^T Aq = \max_{1\leq i\leq m} e_i^T Aq.$$

(2) Similar to (1). $\square$

Theorem 10.2.5 (alternative formulation for zero-sum matrix game optimization problems).

- *The row player's optimization problem given by*

$$\max_{p\in\mathbb{R}^m} \min_{q\in\mathbb{R}^n} p^T Aq,$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1$, *can be formulated by*

$$\begin{aligned} &\max_{v,p\in\mathbb{R}^m} v \\ &s.t.\ \ v \leq p^T Ae_j, j = 1,2,...,n \\ &p_i \geq 0, i = 1,2,...,m \\ &\sum_{j=1}^n p_i = 1. \end{aligned}$$

- *The column player's optimization problem given by*

$$\min_{q\in\mathbb{R}^n} \max_{p\in\mathbb{R}^m} pAq.$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1$, *can be formulated by*

$$\begin{aligned} &\min_{v,q\in\mathbb{R}^n} v \\ &s.t. v \geq e_i^T Aq, i = 1,2,...,m \\ &q_j \geq 0, j = 1,2,...,n \\ &\sum_{j=1}^n q_j = 1. \end{aligned}$$

Proof. (1) The constraints

$$v \leq pAe_j, j = 1,2,...,n$$

imply

$$v \leq \min_{1\leq j\leq n} p^T Ae_j = \min_{q\in\mathbb{R}^n} p^T Aq,$$

where the second equality is from above lemma. Now we are going to show

$$\max_{p,v} v, s.t.\ v \leq \min_q p^T Aq$$

is equivalent to

$$\max_p \min_q \min_q p^T Aq.$$

Since $v \leq \min_q p^T Aq \forall v, p$, we have (by maximizing both sides on p)

$$\max_p v \leq \max_p \min_q p^T Aq \implies v \leq \max_p \min_q p^T Aq.$$

Note that the right side is a constant and the left side is a function on v, we have(by maximizing on v)

$$\max_v v = v \leq \max_p \min_q p^T Aq.$$

If we maximize v first and then maximize p, we get the same result. Eventually, we can have the equivalence.

Therefore, the row player problem can be written as

$$\max_{p\in\mathbb{R}^m} \min_{q\in\mathbb{R}^n} p^T Aq,$$

where $p_i \geq 0, \sum_{i=1}^m p_i = 1, q_i \geq 0, \sum_{i=1}^n q_i = 1$, can be formulated by

$$\begin{aligned} &\max_{v,p\in\mathbb{R}^m} v \\ s.t.\ & v \leq \min_{q\in\mathbb{R}^n} p^T Aq \\ & p_i \geq 0, i = 1, 2, ..., m \\ & \sum_{j=1}^n p_i = 1. \end{aligned}$$

(2) Similar to (1). □

Example 10.2.4. Consider a zero-sum two player game with following payoff

$$\mathbf{A} = \begin{bmatrix} -15 & -35 & 10 \\ -7 & 11 & 0 \\ -12 & -36 & 20 \end{bmatrix}$$

The linear program problem for the first player, the row player, is

$$\begin{aligned}
& \max v \\
\text{s.t.} \quad & -15x_1 - 7x_2 - 12x_3 - v \geq 0 \\
& -35x_1 + 11x_2 - 36x_3 - v \geq 0 \\
& 10x_1 + 20x_3 - v \geq 0 \\
& x_1 + x_2 + x_3 = 1 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

The linear program problem for the second player, the column player, is

$$\begin{aligned}
& \min v \\
\text{s.t.} \quad & -15y_1 - 35y_2 + 10y_3 - v \leq 0 \\
& -7y_1 + 11y_2 - v \leq 0 \\
& -12y_1 - 36y_2 + 20y_3 - v \leq 0 \\
& y_1 + y_2 + y_3 = 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

10.3 Notes on bibliography

Good sources on introduction level game theory include [2][4][1]

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Part III

CLASSICAL STATISTICAL METHODS