HW 5

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Problem 1. Evaluate

$$I=\int_{-\infty}^{\infty}\frac{x^3\sin x}{(x^2+1)^2}dx$$

Be sure to justify closing the contour on the upper or lower half plane.

Set up I to be:

$$I = \operatorname{Im}\int_{-\infty}^{\infty}\frac{z^3 e^{iz}}{(z^2+1)^2}dz$$

Then we have

$$f(z)=\frac{z^3}{(z^2+1)^2}$$

Note $|f(z)| \longrightarrow 0$ as $z \to \infty$

Thus by Jordan's Lemma we have:

$$I = \frac{1}{2}\operatorname{Im}\oint_{\Delta}\frac{z^3 e^{iz}}{(z^2+1)^2}dz$$

Then note there are two poles:

$z=\pm i$ each of order 2.

Then note only $z=i$ is in the UHP

Thus we have:

$$I = \operatorname{Im}\left\{2\pi i \cdot \text{Res of } \frac{z^3 e^{iz}}{(z^2+1)^2} \text{ at } z=i\right\}$$

$$\text{Residual} = \lim_{z\to i}\frac{d}{dz}\left(\frac{z^3 e^{iz}}{(z+i)^2\cancel{(z-i)^2}}\cancel{(z-i)^2}\right)$$

$$= \lim_{z\to i}\frac{(3z^2e^{iz}+z^3e^{iz})(z+i)^{\cancel{2}}-z^3e^{iz}\cdot 2\cancel{(z+i)}}{(z+i)^{\cancel{4}\,3}}$$

$$= -\frac{1}{8i}\left[(-3e^{-1}-ie^{-1})(2i)+2ie^{-1}\right]$$

$$= -\frac{1}{4}(-2e^{-1}-ie^{-1})$$

$$= \frac{1}{4}(2+i)e^{-1} = \frac{2+i}{4e}$$

Then we have:

$$I = \operatorname{Im}\left\{2\pi i\left(\frac{2}{4e}+\frac{i}{4e}\right)\right\}$$

$$= \operatorname{Im}\left\{\frac{2\pi i}{e}-\frac{\pi}{2e}\right\}$$

$$= \boxed{2\pi i e^{-1}}$$

Problem 2. Show that

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{\pi}{2}(1-\frac{1}{e}).$$

If you want to use Jordan's Lemma, justify the use of the principle value integral and closing of the contour in the upper half plane.

Note by principle value integral we have:

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z(z^2+1)} dz$$

Then note we have:

$$f(z) = \frac{1}{z(z^2+1)}$$

Note that as $z \to \infty$, $|f(z)| \to 0$

Note $z=0$ is a simple pole

$$\Rightarrow \operatorname{Res}(z=0) = \lim_{z\to 0} z \frac{e^{iz}}{z(z^2+1)} = 1$$

Note $z=i$ is a simple pole

$$\Rightarrow \operatorname{Res}(z=i) = \lim_{z\to i} (z-i) \frac{e^{iz}}{z(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{e^{iz}}{z(z+i)} = \frac{e^{-1}}{-2}$$

Then note $z=0$ is on the contour, thus we have:

$$I = \frac{1}{2} \operatorname{Im} \oint_{\Delta} \frac{e^{iz}}{z(z^2+1)} dz$$

$$= \frac{1}{2} \operatorname{Im} \{ 2\pi i \cdot \text{Res of } \frac{e^{iz}}{z(z^2+1)} \text{ at } z=0 + \pi i \cdot \text{Res of } \frac{e^{iz}}{z(z^2+1)} \text{ at } z=0 \}$$

$$\Rightarrow I = \frac{1}{2} \operatorname{Im} \{ 2\pi i \cdot (\frac{1}{2} - \frac{e^{-1}}{2}) \} = \frac{1}{2}(\pi - \pi e^{-1}) = \boxed{\frac{\pi}{2}(1-e^{-1})}$$

Problem 3. Find the Fourier transform of

$$f(t)=\begin{cases}1 \text{ for } -a<t<a\\ 0 \text{ otherwise}\end{cases}$$

Then, do the inverse transform using techniques of contour integration, e.g. Jordan's lemma, principal values, etc.

For $-a<t<a$:

$$F(\lambda)=\int_{-a}^{a} e^{i\lambda t}\,dt$$
$$=\frac{1}{i\lambda}e^{i\lambda t}\Big|_{-a}^{a}$$
$$=\boxed{\frac{1}{i\lambda}\left(e^{i\lambda a}-e^{-i\lambda a}\right)}$$

For $t<-a$ or $t>a$ we have:

$F(\lambda)=0$

Then note we have:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\left(e^{i\lambda a}-e^{-i\lambda a}\right)/(i\lambda)\,d\lambda.$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{(a-t)i\lambda}-e^{-(t+a)i\lambda}\right)/(i\lambda)\,d\lambda$$
$$=\frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\frac{e^{(a-t)i\lambda}}{i\lambda}\,d\lambda-\int_{-\infty}^{\infty}\frac{1}{e^{(t+a)i\lambda}\,i\lambda}\,d\lambda\right)$$

Note $z=0$ is a simple pole in both integration.

Let $J_1 = P\oint_{\Delta}\frac{e^{(a-t)i\lambda}}{i\lambda}\,d\lambda = \pi i\,\mathrm{Res}(0) = \pi i/i = \pi$

Let $J_2 = -P\oint_{\triangledown}\frac{1}{e^{(t+a)i\lambda}\,i\lambda}\,d\lambda = \pi i\,\mathrm{Res}(0) = -\pi i/i = -\pi$

we have: $f(t)=\frac{1}{2\pi}\cdot(\pi-(-\pi))=\boxed{1}$