HW2 Tianbo Zhang 1938501

1. (a)Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface of the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = b$.
   (Bold letters are used to denote vectors).
   (b) Same as (a), but by computing the divergence and using the divergence theorem.

a) First note for this question we have to two seperate flux.
One is the flux over the side of cylinder. ①
Another one is the flux over the Top and Bottom of the cylinder.

① Using cyclindrical coordinates we have:

$x = a\cos(\theta)$, $y = a\sin(\theta)$, $z = z$

Then note $dS = h_1 du_1 h_2 du_2$

Let $u_1 = z$, $u_2 = \theta$

$h_1 = 1$, $h_2 = a$

$dS = a\, d\theta\, dz$

Then note we have:

$n_1 = \cos(\theta)i + \sin(\theta)j$

$F = x + y$

Then we have:

$$\int_0^b \int_0^{2\pi} (a\cos(\theta)^2 + a\sin^2\theta)\, a\, d\theta\, dz = a^2 \int_0^b \int_0^{2\pi} d\theta\, dz$$
$$= a^2 2\pi \int_0^b dz = 2ba^2\pi$$

② For the top: We have $z = b$

Then we have: $dS = r\,dr\,d\theta$

$$\int_0^a \int_0^{2\pi} b\, r\, dr\, d\theta = b\pi a^2$$

Note the bottom is 0

Total: $2\pi ba^2 + \pi ba^2 = \boxed{3\pi ba^2}$

b) First note by the definition of divergence we have:

$$\nabla F = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Then note by the Divergence Theorem we have:

$$\iiint_V \nabla F\, dv = \iint_{\partial V} \vec{F} \cdot \vec{n}\, ds$$

Then note $dv = r\,dr\,d\theta\,dz$

The note our cylinder have the following constraints.

$0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le b$

$$\Rightarrow \int_0^b \int_0^{2\pi} \int_0^a 3\, r\, dr\, d\theta\, dz = 3\int_0^b dz \int_0^{2\pi} d\theta\, \frac{1}{2} r^2 \Big|_0^a$$
$$= 3\int_0^b dz\, \pi a^2$$
$$= \boxed{3ba^2\pi}$$

2. Show that

$$\mathbf{F} = (\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x - y)\mathbf{k}$$

is a conservative field and calculate the value of its line integral from the origin to the point $(3, \pi/6, 2)$ along any path.

Note by definition of conservative field we have:

$F = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k$ is conservative

if $F = \nabla f$

Then all we need to proof is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$

$$\left.\begin{array}{l} \frac{\partial M}{\partial y} = \cos y \\ \frac{\partial N}{\partial x} = \cos y \end{array}\right\rangle \checkmark \qquad \left.\begin{array}{l} \frac{\partial N}{\partial z} = -1 \\ \frac{\partial P}{\partial y} = -1 \end{array}\right\rangle \checkmark \qquad \left.\begin{array}{l} \frac{\partial P}{\partial x} = 1 \\ \frac{\partial M}{\partial z} = 1 \end{array}\right\rangle \checkmark$$

Thus F is conservative. $\square$

Then to find f:

Note $\nabla f = \langle \underbrace{\sin y + z}_{\frac{\partial f}{\partial x}}, \underbrace{x\cos y - z}_{\frac{\partial f}{\partial y}}, \underbrace{x - y}_{\frac{\partial f}{\partial z}} \rangle$

$f = \sin y\, x + zx + C(y,z) \leftarrow \int \frac{\partial f}{\partial x}\,dx$

$f = x\sin y + zx - zy + C(z) \leftarrow$ additional part from $\int \frac{\partial f}{\partial y}\,dy$

$f = x\sin y + zx - zy$

Then note by the property of Conservative Fields we have:

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

Then we have:

$$f(3, \tfrac{\pi}{6}, 2) - f(0,0,0) = 3\sin(\tfrac{\pi}{6}) + 6 - \tfrac{\pi}{3} = \tfrac{3}{2} + 6 - \tfrac{\pi}{3} = \boxed{\tfrac{15}{2} - \tfrac{\pi}{3}}$$

3. A particle experiences a force field that may be written in spherical coordinates as

$\mathbf{F} = -4\theta \mathbf{e}_r + 1\mathbf{e}_\theta$.

Under the influence of this force field, this particle moves from the north pole to the equator on a sphere of radius 2. Its coordinates, as a function of time, are:

$r(t) = 2,\ \theta(t) = t,\ \varphi(t) = \pi/2,\ 0 \le t \le \pi/2$.

Compute the work done, $W = \int \mathbf{F} \cdot d\mathbf{r}$, by the force field in moving the particle by using spherical coordinates and unit vectors.

First note that the differential displacement for dr is:

$dr = e_r dr + e_\theta r d\theta + e_\varphi r \sin\theta d\varphi$

Then note $d\rho = d\varphi = 0$

Then we have:

$dr = e_\theta 2 d\theta$

Then plug this in we have:

$W = \int_0^{\frac{\pi}{2}} (-4\theta e_r + e_\theta)(2 e_\theta d\theta)$

orthogonal

$= \int_0^{\frac{\pi}{2}} (-4\theta e_r e_\theta + 2 e_\theta^2) d\theta$

$= \int_0^{\frac{\pi}{2}} 2\, d\theta = 2 \cdot \frac{\pi}{2} = \boxed{\pi}$

4. Same as Problem 3, but write out the position vector **r**(t) and the force field **F**(t) in Cartesian coordinates. Compute the work done using Cartesian unit vectors.

$r(t) = 2,\ \theta(t) = t,\ \varphi(t) = \frac{\pi}{2}$

First note to transform spherical coordinate to Cartesian coordinate we have:

$x = 2\sin(t)\cos(\frac{\pi}{2}) = 0 \rightarrow dx = 0$

$y = 2\sin(t)\sin(\frac{\pi}{2}) = 2\sin(t) \rightarrow dy = 2\cos(t)$

$z = 2\cos(t) \rightarrow dz = -2\sin(t)$

Then the position vector is:

$r(t) = \boxed{2\sin(t)j + 2\cos(t)k}$

$dr = 2\cos(t)j - 2\sin(t)k$

Then note:

$e_r = \sin\theta\cos\varphi\, i + \sin\theta\sin\varphi\, j + \cos\theta\, k$

$e_\theta = \cos\theta\cos\varphi\, i + \cos\theta\sin\varphi\, j - \sin\theta\, k$

Note $\phi(t) = \frac{\pi}{2} \rightarrow \cos\varphi = 0,\ \sin\varphi = 1$

$e_r = \sin(t)j + \cos(t)k$

$e_\theta = \cos(t)j - \sin(t)k$

Then we have:

$F(t) = -4t(\sin(t)j + \cos(t)k) + \cos(t)j - \sin(t)k$

$= (\cos(t) - 4t\sin(t))j + (-4t\cos(t) - \sin(t))k$

Then note:

$$W = \int_0^{\frac{\pi}{2}} (\cos(t) - 4t\sin(t))j + (-4t\cos(t) - \sin(t))k\,(2\cos(t)j - 2\sin(t)k)\,dt$$

$$= \int_0^{\frac{\pi}{2}} [\overbrace{2\cos^2(t) - 8t\cancel{\cos(t)\sin(t)} + 8t\cancel{\cos(t)\sin(t)} + 2\sin^2(t)}^{2}]\,dt$$

$$= \int_0^{\frac{\pi}{2}} 2\,dt = 2t\Big|_0^{\frac{\pi}{2}} = \boxed{\pi}$$