HW4

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1. Find and classify all singularities of the following functions. Be sure to justify your answer.

(a)
$$f(z) = \frac{e^{-z}}{z^3(z^2+1)}$$

(b)
$$f(z) = \frac{z+1}{z\sin z}$$

a) First note we have:

$$z^3(z^2+1)=0$$

$$\Rightarrow z=0,\ z=\pm i$$

Thus there are singularities around $z=0, \pm i$

Then let's do the Laurent expansion:

① Expansion around $z=0$

$$f(z) = \frac{e^{-z}}{z^3}\cdot\frac{1}{z^2+1}$$

Note we have:

$$\frac{1}{z^2+1} = 1 - z^2 + z^4 - z^6 + \cdots$$

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$$

Thus we have:

$$f(z) = \frac{1}{z^3}\left(1 - z + \frac{z^2}{2!} + \cdots\right)\left(1 - z^2 + z^4 - z^6 \cdots\right)$$

$$= \frac{1}{z^3} + 1 + \cdots$$

$\boxed{\text{Thus } z=0 \text{ is a pole of order 3.}}$

② Singularities at $z=\pm i$:

Note for $z=i$ we have:

$$a_{-1} = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-i)e^{-z}}{z^3(z^2+1)}$$

$$= \lim_{z\to i}\frac{e^{-z}}{z^3} \rightarrow \text{finite}$$

Note $z=-i$ share similar idea.

$\boxed{\text{Thus } z=\pm i \text{ is a simple pole}}$

b) First note we have:

$$z\sin(z)=0$$

Then we have $z=0$, $z=n\pi$ for $n\in\mathbb{Z}/\{0\}$

Thus there are singularitis around $z=0$, $z=n\pi$ for $n\in\mathbb{Z}/\{0\}$

① Expansion around $z=0$:

$$f(z) = \frac{z+1}{z}\cdot\frac{1}{\sin(z)}$$

Note the Laurant expansion for $\frac{1}{\sin(z)}$ is:

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots$$

Also note: $\frac{z+1}{z} = 1 + \frac{1}{z}$

Then we have:

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + \cdots$$

$\boxed{\text{Thus } z=0 \text{ is a pole of order 2}}$

② Expansion around $z=\pi$

Note $z = \pi + z - \pi$

$$\Rightarrow \sin(\pi + (z-\pi)) = -\sin(z-\pi)$$

$$f(z) = -\frac{z+1}{z}\cdot\frac{1}{\sin(z-\pi)}$$

Note the Laurant expansion for $\frac{1}{\sin(z)}$ is:

$$\frac{1}{\sin(z-\pi)} = \frac{1}{z-\pi} + \frac{z-\pi}{6} + \cdots$$

Then we will have:

$$(-z+1)\left(\frac{1}{z-\pi} + \frac{z-\pi}{6} + \cdots\right)\frac{1}{z}$$

Note the leading term for $z-\pi$ is: $\frac{1}{z-\pi}$

$\boxed{\text{Thus } z=\pi \text{ is a simple pole}}$

2. Evaluate

$$I = \oint_C \frac{(5z+3)}{z\,(z^2+2z-3)}\, dz$$

using the calculus of residues (i.e., the residue theorem). Here, C is the closed, counterclockwise circle of radius 2 centered at the origin, $|z| = 2$.

First note we have:

$z(z^2+2z-3)=0 \rightarrow z(z-1)(z+3)=0$

Thus the singularities are:

$z=0,\ z=1,\ z=-3$

Note that $z=-3$ is outside the contour $|z|=2$

Thus we only need to consider $z=0$ and $z=1$

Then note:

$$f(z) = \frac{5z+3}{z(z^2+2z-3)} = \frac{5z+3}{z(z-1)(z+3)} = \frac{5}{(z-1)(z+3)} + \frac{3}{z(z-1)(z+3)}$$

First we'll do the expansion around $z=0$:

Note we have the following Laurent expansion:

$$\frac{1}{z-1} = -1-z-z^2-z^3-\cdots\cdots$$

$$\frac{1}{z+3} = \frac{1}{3}\,\frac{1}{1+\frac{z}{3}} = \frac{1}{3}\left(1-\frac{z}{3}+\left(-\frac{z}{3}\right)^2+\left(-\frac{z}{3}\right)^3+\cdots\right)$$

Then we have:

$$f(z) = -\frac{1}{z}-\frac{5}{3}-\frac{10}{9}z-\cdots\cdots\cdots$$

Thus $z=0$ is a simple singularities.

Then will do the expansion around $z=1$:

Let $w=z-1 \rightarrow z=w+1$

$$\Rightarrow f(z) = \frac{5w+8}{(w+1)(w^2+4w)}$$

Then note we have the following expansion:

$$\frac{1}{w+1} = 1-w+w^2-w^3+\cdots$$

$$\frac{1}{w^2+4w} = \frac{1}{w(w+4)} = \frac{1}{4w}-\frac{1}{4w+16}$$

$$= \frac{1}{4w}-\frac{1}{16}+\frac{w}{64}-\frac{w^2}{256}+\cdots$$

Then note the leading term is: $\frac{2}{w}$

Thus $z=1$ is a simple pole.

Risidue at $z=0$:

$$\lim_{z\to 0} z\,\frac{5z+3}{z(z^2+2z-3)} = \lim_{z\to 0}\frac{5z+3}{z^2+2z-3} = -\frac{3}{3} = -1$$

Risidue at $z=1$:

$$\lim_{z\to 1}(z-1)\frac{5z+3}{z(z-1)(z+3)} = \lim_{z\to 1}\frac{5z+3}{z(z+3)} = \frac{5+3}{4} = 2$$

Hence by the residue theorem we have:

$$\oint_C f(z)dz = 2\pi i(-1+2) = \boxed{2\pi i}$$

3. Evaluate the following integral using contour integration and the calculus of residues. Be sure to discuss the behaviour of your integral on each part of your contour in the complex plane.

$$I = \int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)^2}\, dx \; .$$

① Identify the singularities:

$x^2+2x+2=0$

$$x = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$$

② Expansion around $x=-1+i$.

Set $w = x-(-1+i) \rightarrow x = w-1+i$

$$f(w) = \frac{w-1+i}{((w-1+i)^2+2(w-1+i)+2)^2}$$

$$= \frac{\frac{1}{4}-\frac{i}{4}}{w^2} + \frac{i}{4w} + \cdots\cdots$$

Thus $x=-1+i$ is a pole of order 2

③ Choose the semicircle contour in the upper plane

Note $-1+i$ will be covered but not $-1-i$.

④ Residue at $x=-1+i$:

$$\lim_{x\to-1+i}\left((x+1-i)^2 \frac{x}{(x+1-i)^2(x+1+i)^2}\right)\frac{d}{dx} = \frac{4i^2-(-1+i)\cdot 2(2i)}{(2i)^4}$$

$$= \lim_{x\to-1+i}\left(\frac{x}{(x+1+i)^2}\right)\frac{d}{dx} = \frac{-4-(-4i-4)}{16}$$

$$= \lim_{x\to-1+i} \frac{(x+1+i)^2 - x\cdot 2(x+1+i)}{(x+1+i)^4} = \frac{i}{4}$$

Thus we have:

$$I = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx + \int_{C_R^+} f(x)dx$$

$$= \oint_C f(x)dx$$

Thus by Residue's Theorem we have:

$$I = 2\pi i \cdot \frac{i}{4} = \boxed{-\frac{1}{2}\pi}$$

4. Evaluate

$$I = \int_0^{2\pi} (\cos^3\theta + \sin^2\theta)\, d\theta$$

using the calculus of residues. (Hint: After you reformulate your problem as a complex integral, you can expand out your integrand algebraically to produce a Laurent series. You can then find the residue by inspection.)

To reformulate I to a complex integral we have:

$z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$\cos(\theta) = \frac{1}{2}(z + \frac{1}{z})$

$\sin(\theta) = \frac{1}{2i}(z - \frac{1}{z})$

$\frac{z^2+1}{z}$ $\frac{z^2-1}{z}$ $\frac{z^2-}{}$

Then we have:

$$I = \int_0^{2\pi}(\cos^3\theta + \sin^2\theta)\,d\theta = \oint_C \left(\frac{1}{8}(z+\frac{1}{z})^3 - \frac{1}{4}(z-\frac{1}{z})^2\right)\frac{dz}{iz}$$

$$= \oint_C \left(\frac{1}{8}(z^2+2+\frac{1}{z^2})(z+\frac{1}{z}) - \frac{1}{4}(z^2-2+\frac{1}{z^2})\right)\frac{dz}{iz}$$

$$= \oint_C \left(\frac{1}{8}\left(\frac{z^2+1}{z}\right)^3 - \frac{1}{4}\left(\frac{z^2-1}{z}\right)^2\right)\frac{dz}{iz}$$

$$= \oint_C \left(\frac{1}{8}\left(\frac{z^6+3z^4+3z^2+1}{z^3}\right) - \frac{1}{4}\left(\frac{z^4-2z^2+1}{z^2}\right)\right)\frac{dz}{iz}$$

$$= \oint_C \frac{1}{8}\left(\frac{z^6+3z^4+3z^2+1-2z^5+4z^3-2z}{z^3}\right)\frac{dz}{iz}$$

$$= \oint_C \frac{1}{8i}\left(\frac{z^6-2z^5+3z^4+4z^3+3z^2-2z+1}{z^4}\right)dz$$

Note $z=0$ is a pole of order 4.

Then we have:

$$\mathrm{Res}(z=0) = \lim_{z\to 0}\frac{1}{3!}\frac{d^3}{dz^3}\left[f(z)\,z^4\right] = \frac{1}{6}\cdot\frac{1}{8i}\cdot\lim_{z\to 0}\frac{d^3}{dz^3}(z^6-2z^5+3z^4+4z^3+3z^2-2z+1)$$

$$= \frac{1}{48i}\cdot\lim_{z\to 0}(120z^3-120z^2+72z+24) = \frac{1}{48i}\cdot 24 = \frac{1}{2i}$$

Thus we have $I = 2\pi i \cdot \mathrm{Res}(z=0)$

$$= 2\pi i \cdot \frac{1}{2i}$$

$$= \boxed{\pi}$$