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1. (a)Compute the cosine of the angle between
$\vec{A}=\vec{i}+\vec{j}+z\vec{k}$ and $\vec{B}=-\vec{i}+z\vec{k}$,
and find a value of z for which the two vectors are perpendicular to each other.
(b)If a differentiable position vector $\vec{r}(t)$ has a constant length, show that $\frac{d}{dt}\vec{r}(t)$ is perpendicular to $\vec{r}(t)$, and interpret geometrically.

a) First note:

$\vec{A}\cdot\vec{B} = \|\vec{A}\|\|\vec{B}\|\cos\theta$

$\Rightarrow \cos\theta = \frac{\vec{A}\cdot\vec{B}}{\|\vec{A}\|\|\vec{B}\|}$

$A\cdot B = 1\cdot(-1) + 1\cdot 0 + z\cdot z = -1+z^2$

$\|A\| = \sqrt{1^2+1^2+z^2} = \sqrt{2+z^2}$

$\|B\| = \sqrt{(-1)^2+z^2} = \sqrt{1+z^2}$

$\cos\theta = \frac{-1+z^2}{\sqrt{2+z^2}\sqrt{1+z^2}}$

Note for perpendicular vectors:

$\cos\theta = 0$

$\Rightarrow \frac{-1+z^2}{\sqrt{2+z^2}\sqrt{1+z^2}} = 0$

$\Rightarrow -1+z^2 = 0$

$z^2 = 1$

$z = \pm 1$

b) First note that:

$|r(t)|$ is a constant

$\Rightarrow |r(t)|^2$ is a constant.

Then note:

$|r(t)|^2 = r(t)\cdot r(t)$ is a constant

Note the derivative of a constant is 0.

$\Rightarrow \frac{d}{dx}(r(t)\cdot r(t)) = 0$

Then note:

$\frac{d}{dx}(r(t)\cdot r(t)) = r'(t)r(t) + r(t)r'(t) = 0$

$\Rightarrow 2r'(t)r(t) = 0$

$r'(t)r(t) = 0$

Thus they are perpendicular.

On the geometric perspective:

Note that since the magnitude is constant.
Thus the point is moving on the surface of a sphere centered at the origin
Then note $r'(t)$ is the velocity, or the direction of the movement.
So with $r(t)$ and $\frac{d}{dt}r(t)$ perpendicular, it means that the point moves in a direction perpendicular to the origin line.

2. (a)Let $h = 8 - x^2 - 2x - y^2 - 2y$ denote the height on a mountain at position (x, y). Assume that the positive x axis points east and that the positive y axis points north. In what direction from (1, 1) is the steepest descent? What is your rate of change of elevation if you head northeast ? Use your knowledge of the gradient to locate the top of the mountain.

(b)Find the unit normal to the surface $x^2 - y^2 + z^2 = 6$ at the point (1,2,3).

a) First note the gradient vector is:

$$\nabla h = \vec{i}\frac{\partial h}{\partial x} + \vec{j}\frac{\partial h}{\partial y}$$

$$= \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$$

$$\frac{\partial h}{\partial x} = -2x-2$$

$$\frac{\partial h}{\partial y} = -2y-2$$

We have:

$$\nabla = (-2x-2, -2y-2)$$

Plug in (1,1):

$$\nabla(1,1) = (-2-2, -2-2) = \boxed{(-4,-4)}$$

Then note:

$$\nabla h = \left(\frac{1}{h_1}\hat{e}_1\frac{\partial}{\partial x}h, \frac{1}{h_2}\hat{e}_2\frac{\partial}{\partial y}h\right)$$

$$\nabla h(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \nabla(1,1)\cdot \nabla h(1,1) = (-4,-4)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= -\frac{4}{\sqrt{2}} - \frac{4}{\sqrt{2}} = \boxed{-\frac{8}{\sqrt{2}}}$$

Then note: $\nabla = 0$ at the top of the montain

$-2x-2=0 \rightarrow x=-1$

$-2y-2=0 \rightarrow y=-1$

The top of the montain is : $(-1,-1)$

b) Let $f(x,y,z) = x^2-y^2+z^2-6$ $\leftarrow$ evaluate this function at (1,2,3) gives the normal vector

Then $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2y, \frac{\partial f}{\partial z} = 2z$$

Then we have:

$$\nabla f = (2x, -2y, 2z)$$

$$\Rightarrow \nabla f(1,2,3) = (2,-4,6)$$

Since we want the "unit" normal vector:

$$|\nabla f(1,2,3)| = \sqrt{4+16+36} = \sqrt{56} = 2\sqrt{14}$$

$$\vec{n} = \left(\frac{2}{2\sqrt{14}}, -\frac{4}{2\sqrt{14}}, \frac{6}{2\sqrt{14}}\right) = \boxed{\left(\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)}$$

3. A scalar field is given by $h = f(x,y,z) = \frac{xyz}{x^2+y^2}$.

Find (the components of) $\vec{\nabla} h$ in (a) Cartesian, (b) cylindrical and (c) spherical coordinates.

a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\frac{\partial f}{\partial x} = \frac{yz(x^2+y^2)-2x^2yz}{(x^2+y^2)^2} = \frac{y^3z-x^2yz}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{xz(x^2+y^2)-2xy^2z}{(x^2+y^2)^2} = \frac{x^3z-xy^2z}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial z} = \frac{xy(x^2+y^2)}{(x^2+y^2)^2} = \frac{xy}{x^2+y^2}$$

$$\nabla f = \boxed{\left(\frac{y^3z-x^2yz}{(x^2+y^2)^2}, \frac{x^3z-xy^2z}{(x^2+y^2)^2}, \frac{xy}{x^2+y^2}\right)}$$

b) Note :

$x = r\cos\theta$

$y = r\sin\theta$

$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$

plug in x and y into f we have:

$$f = \frac{r\cos\theta\, r\sin\theta\, z}{r^2\cos\theta^2 + r^2\sin\theta^2} = z\cos\theta\sin\theta$$

$$\frac{\partial f}{\partial r} = 0$$

$$\frac{1}{r}\frac{\partial f}{\partial \theta} = \frac{1}{r} z(-\sin^2\theta + \cos^2\theta) = \frac{z}{r}\cos(2\theta)$$

$$\frac{\partial f}{\partial z} = \cos\theta\sin\theta$$

$$\nabla f = \boxed{\left(0, \frac{z}{r}\cos(2\theta), \cos\theta\sin\theta\right)}$$

c) Note:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Then we have:

$f(r,\theta,\phi) = r^3\sin^2\theta\cos\phi\sin\phi\cos\theta$

The note :

$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$

$$\frac{\partial f}{\partial r} = 3r^2\sin^2\theta\cos\phi\sin\phi\cos\theta$$

$$\frac{\partial f}{\partial \theta} = r^3(\cos\phi\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta))$$

$$\frac{\partial f}{\partial \phi} = r^3\sin^2\theta\cos\theta(\cos^2\theta - \sin^2\theta) = r^3\sin^2\theta\cos\theta\cos(2\phi)$$

$$\nabla f = \boxed{\left(3r^2\sin^2\theta\cos\phi\sin\phi\cos\theta, r^2(\cos\phi\sin\phi(2\sin\theta\cos^2\theta - \sin^3\theta)), r^2\sin\theta\cos\theta\cos(2\phi)\right)}$$