HW 3

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1. Express each of the following in polar exponential form:
   (a) $-i$
   (b) $1+i$
   (c) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$

a) $x=0,\ y=-1$

$r=1$

$\cos\theta = \frac{x}{r} = 0$

$\Rightarrow \theta = \frac{3\pi}{2}$

$\boxed{-i = e^{i\frac{3\pi}{2}}}$

b) $x=1,\ y=1$

$r=\sqrt{1+1}=\sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

$\boxed{1+i = \sqrt{2}e^{i\frac{\pi}{4}}}$

c) $x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$

$r=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$\boxed{\frac{1}{2}+\frac{\sqrt{3}}{2}i = e^{i\frac{\pi}{3}}}$

2. Express each of the following in the form of a+bi, where a and b are real.
   (a) $e^{2+i\pi/2}$
   (b) $\frac{1}{1+i}$
   (c) $(1+i)^3$
   (d) $|3+4i|$
   (e) $\cos(i\pi/4+c)$, where c is real.

a) $e^{2+i\frac{\pi}{2}} = e^2 \cdot e^{i\frac{\pi}{2}} = e^2(\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})i)$

$= e^2(0+i)$

$e^{2+i\frac{\pi}{2}} = \boxed{0+e^2 i}$

b) $\frac{1}{1+i} = \frac{1}{1+i} \frac{1-i}{1-i} = \frac{1-i}{1+1} = \boxed{\frac{1}{2} - \frac{1}{2}i}$

c) $(1+i)^3 = (1+i)^2(1+i) = (1+2i+i^2)(1+i)$

$= 2i(1+i)$

$= 2i - 2$

$= \boxed{-2+2i}$

d) $|3+4i| = \sqrt{(3+4i)(3-4i)} = (9+16)^{\frac{1}{2}} = 5$

$= \boxed{5+0i}$

e) $\cos(i\frac{\pi}{4}+c) = \cos(z) = \cos(x+yi) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

$= \cos(c)\cosh(\frac{\pi}{4}) - i\sin(c)\sinh(\frac{\pi}{4})$

$= \boxed{\cos(c) \cdot \frac{e^{\frac{\pi}{4}} + e^{-\frac{\pi}{4}}}{2} - i\sin(c)\frac{e^{\frac{\pi}{4}} - e^{-\frac{\pi}{4}}}{2}}$

3. Solve for the roots of the following equation:
   (a) $z^3 = 4$
   (b) $z^4 = -1$

a) $z^3 = 4$

Note $4 = 4(\cos(0) + i\sin(0))$

Note the primary root is:

$z_1 = \sqrt[3]{4} = (4)^{\frac{1}{3}}$

Then note we have:

$z^3 = r^3(\cos(3\theta) + i\sin(3\theta))$

$\Rightarrow z = r^{\frac{1}{3}}\left(\cos\left(\frac{\theta+2k\pi}{3}\right) + i\sin\left(\frac{\theta+2k\pi}{3}\right)\right)$, for $k = 1, 2$ and $\theta = 0$

$z_2 = r^{\frac{1}{3}}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

$= 4^{\frac{1}{3}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$z_3 = r^{\frac{1}{3}}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

$= 4^{\frac{1}{3}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

We have:

$z_1 = 4^{\frac{1}{3}}$

$z_2 = 4^{\frac{1}{3}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$z_3 = 4^{\frac{1}{3}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

b) $z^4 = -1$

Note: $-1 = 1(\cos\pi + i\sin\pi)$

Again we have:

$z^4 = r^4(\cos(4\theta) + i\sin(4\theta))$

$z_k = r^{\frac{1}{4}}\left(\cos\left(\frac{\theta+2k\pi}{4}\right) + i\sin\left(\frac{\theta+2k\pi}{4}\right)\right)$ for $k = 0, 1, 2, 3$ and $\theta = \pi$

$z_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$

$= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$z_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$

$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

We have:

$z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_1 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$z_2 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$z_3 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

4. Evaluate $\oint_c f(z)dz$

where C is the unit circle centered at the origin, and f(z) is given by the following:

(a) e^{iz}

(b) e^{z^2}

(c) $\frac{1}{z-1/2}$

(d) $\frac{1}{z^2-4}$

(e) $\frac{1}{2z^2+1}$

(f) $\sqrt{z-4}$, where $0 \le \arg(z-4) < 2\pi$

a) Note: $\frac{d}{dz} e^{iz} = ie^{iz}$

Then we know e^{iz} is analytic everywhere

Also note e^{iz} can be evaluated at each point from 0 to 2π.

Thus we have:

$$\boxed{\oint_C e^{iz}\,dz = 0}$$

b) Note $e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$

Then note for $a_n = \frac{z^{2n}}{n!}$ we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{z^{2n}\cdot z^2}{(n+1)!} \cdot \frac{n!}{z^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{z^2}{n+1} \right| = 0$$

Thus by ratio test we know the power series converges for all $z \in \mathbb{C}$

Thus we have:

$$\boxed{\oint_C e^{z^2} dz = 0}$$

C) Note from lecture 6 we know:

$$\oint_C (z-z_0)^n = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n=-1 \text{ and } c \text{ encloses } z_0. \end{cases}$$

Note the unit circle enclose $z_0 = \frac{1}{2}$ since $\frac{1}{2} < 1$.

Thus we have $\boxed{\oint_C (z-\tfrac{1}{2})^{-1} dz = 2\pi i}$

d) Note $f(z) = \frac{1}{z^2-4} = \frac{1}{(z+2)(z-2)}$

Then note $z = \pm 2$ are the singularities for $f(z)$

Then note $|2| = |-2| > |1|$

Thus we know $f(z)$ is analytic inside and on the curve.

Hence $\boxed{\oint_C (z^2-4)^{-1} dz = 0}$

e) Note we have $\frac{1}{2z^2+1}$

Then we have the singularities at:

$$2z^2+1=0 \Rightarrow z = \pm i \frac{1}{\sqrt{2}}$$

Then note $|z| = \frac{\sqrt{2}}{2} < 1$

Thus we have to find the residues at each singularities.

Then we have: $f(z) = \frac{g(z)}{h(z)}$

$\Rightarrow h(z) = 2z^2+1$

$h'(z) = 4z$

Note the residue cancels for $\pm i \frac{\sqrt{2}}{2}$.

Thus $\boxed{\oint_C \frac{1}{2z^2+1} dz = 2\pi i \cdot 0 = 0}$

f). Note since $0 \le \arg(z-4) < 2\pi$

$\Rightarrow$ The branch cut is at π.

Then note $|z|=4 > |1|$

Thus $f(z)$ is analytic inside and on closed curve C.

Thus by Cauchy's Theorem we have:

$$\oint_C \sqrt{z-4}\, dz = 0.$$