AMATH 502 HW1

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Problem 1

$x_{n+1} = f(x_n)$

$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}$

a) Let x^* be a simple roots of $g(x)$.

Then we have:

$g(x^*) = 0$

$g'(x^*) \neq 0$

Then note from Newton's method we have:

$$f(x^*) = x^* - \frac{g(x^*)}{g'(x^*)}$$
$$= x^* - \frac{0}{g'(x^*)}$$
$$= x^*$$

Thus no matter the number of iterations,

We have $f(x^*) = x^*$

$\Rightarrow x^*$ is a fixed point. $\square$

b) First note we have:

$$f'(x) = \frac{d}{dx}\left(x - \frac{g(x)}{g'(x)}\right)$$
$$= 1 - \left[\frac{g'(x)g'(x) - g(x)g''(x)}{g'(x)^2}\right]$$
$$= 1 - \left[\frac{g'(x)^2 - g(x)g''(x)}{g'(x)^2}\right]$$
$$= 1 - \left[1 - \frac{g(x)g''(x)}{g'(x)^2}\right]$$
$$= \frac{g(x)g''(x)}{g'(x)^2}$$

Then plug in $g(x^*) = 0$ and $g'(x^*) \neq 0$ we have

$f'(x^*) = 0$

Thus these fixed points are superstable. $\square$

Problem 2

$x_{n+1} = 3x_n - x_n^3$

a) Find fixed points:

We set up the following equation:

$x_{n+1} = 3x_n - x_n^3 = x_n$

$\Rightarrow 2x_n - x_n^3 = 0$

$x_n(2 - x_n^2) = 0$

$\Rightarrow \boxed{x^* = 0, \pm\sqrt{2}}$

To classify stability:

Note we have:

$f'(x) = 3 - 3x^2$

Then note:

$|f'(0)| = 3 > 1$

$|f'(\sqrt{2})| = |3-6| = 3 > 1$

$|f'(-\sqrt{2})| = |3-6| = 3 > 1$

Thus from lecture we know:

$\boxed{0, \pm\sqrt{2} \text{ are all unstable fixed points}}$

b) First note we have:

$f'(x) = \frac{d}{dx}(3x - x^3)$

$= 3 - 3x^2$

① Then note maxima/minima happens at $f'(x) = 0$

$\Rightarrow f'(x) = 3 - 3x^2 = 0$

$\Rightarrow 3(1 - x^2) = 0$

$\Rightarrow x = \pm 1$

Then note we have:

$f(1) = 3 - 1 = 2 \qquad f(2) = 6 - 8 = -2$

$f(-1) = -3 + 1 = -2 \qquad f(-2) = -6 + 8 = 2$

Since local maxima / minima has $|f(x)| \le 2$

And the endpoints $x = -2, 2$ has $|f(x)| \le 2$

We conclude $\boxed{\text{if } |x| \le 2 \Rightarrow |f(x)| \le 2}$

$\square$

② Then note we have

$f(x) = 3x - x^3 = x(3 - x^2)$

Note if $|x| > 2$ by definition we have:

$3 - x^2 > 1$ or $3 - x^2 < -1 \Rightarrow |3 - x^2| > 1$

Thus we have for $|x| > 2$:

$|f(x)| = |x| \cdot |3 - x^2| > |x|$

$\Rightarrow \boxed{\text{For } |x| > 2 \Rightarrow |f(x)| > |x|}$

$\square$

c) First note we have:

① Iteration from 2:

$f(2) = 6 - 8 = -2$

② Iteration from -2:

$f(-2) = -6 + 8 = 2$

$\Rightarrow x_{n+2} = x_n$

$\Rightarrow (2, -2)$ forms a 2-cycle

We would call it fractals

Problem 3

$\dot{x} = f(x), \ x \in \mathbb{R}$

$x_{n+1} = x_n + h f(x_n)$

a) First note to find the fixed points we have:

$\dot{x} = f(x) = 0$

$x_{n+1} = x_n$

Let x^* be fixed point of $\dot{x}$, then we have:

$\dot{x} = f(x^*) = 0$

Then plug in $f(x^*)$ into x_{n+1} we have:

$x_{n+1} = x^* + 0 = x^*$

Thus fixed points of $\dot{x}$ is also fixed points of x_{n+1}.

$\square$

b) Let $g(x) = x + h f(x)$

Then the stability condition is:

$g'(x) = 1 + h f'(x)$

$\ddot{x} = f'(x)$

If $\ddot{x} > 0 \Rightarrow f'(x^*) > 0$ then the fixed point is unstable

$f'(x^*) < 0$ then the fixed point is stable

Suppose $f'(x^*) < 0$ and $f'(x^*) < -\frac{2}{h}$

Then we have: $h \cdot f'(x^*) < -2$

$\Rightarrow |g'(x^*)| = |1 + h f'(x^*)| > 1$

Then x^* is stable in $\dot{x}$ and unstable in x_{n+1}.

Thus stability does not necessarily agree. $\square$

c) Note for stable fixed points we have:

$|g'(x^*)| = |1 + h f'(x^*)| < 1$

$\Rightarrow -1 < 1 + h f'(x^*) < 1$

$\Rightarrow -2 < h f'(x^*) < 0$

Thus we have the condition that:

$-2 < h f'(x^*) < 0$ guarantees stability.

Note $f'(x^*)$ has to be negative.

And we have to choose h small enough where:

$\boxed{0 < h < -\frac{2}{f'(x^*)}}$ since $f'(x^*) < 0$

d) First note the solution is oscillating when:

$g'(x^*) < 0$

Then we have the condition:

$1 + h f'(x^*) < 0$

$\Rightarrow h f'(x^*) < -1$

$\Rightarrow \boxed{h > -\frac{1}{f'(x^*)}}$ since $f'(x^*) < 0$

Thus note from part c we found the result of oscillations happened $\boxed{\text{when } h \text{ is too large}}$.

This made it common to see oscillations

e) First note plug in the expression for $\dot{x}$ we have:

$x_{n+1} = x_n + h(kx_n) = (1+hk)x_n$

Then note we have:

$x_{n+2} = (1+hk)x_{n+1} = (1+hk)^2 x_n$

Then note 2 cycles satisfies:

$x_{n+2} = (1+hk)^2 x_n = x_n$

$\Rightarrow (1+hk)^2 = 1$

$\Rightarrow 1 + 2hk + h^2k^2 = 1$

$\Rightarrow hk(hk+2) = 0$

$\Rightarrow hk = -2$ or $hk = 0$

Note $x_{n+2} = (1+hk)^2 x_n$

plug in $hk = -2$

$\Rightarrow x_{n+2} = x_n$

$\Rightarrow$ The 2 cycle is neutrally stable

This means that with any initial conditions. The numerical solution will oscillate between 2 values. Hence leads to the system never settle into steady state or diverge.