AMATH 502 HW4

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1. For each of the following systems, find the fixed points, classify them, sketch (by hand, do not use any software such as pplane) the neighboring trajectories, and try to fill in the rest of the phase portrait.

   (a) $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

   (b) $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

a) We have: $\begin{cases} \dot{x} = x - y \\ \dot{y} = x^2 - 4 \end{cases}$

Set $\dot{x}, \dot{y} = 0$ we have:

$(x+2)(x-2) = 0 \to x = \pm 2$

plug in x into $\dot{x}$ we have $y = \pm 2$

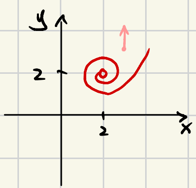
$\Rightarrow (x^*, y^*): (2,2), (-2,-2)$

Then we have the Jocobian matrix:

$$A = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$$

① $A|_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \to \lambda = \dfrac{1 \pm \sqrt{1-16}}{2} \to \lambda = \mathrm{Real}(\lambda_1) = \mathrm{Real}(\lambda_2) = \frac{1}{2} > 0$

$\Rightarrow (2,2)$ $\boxed{\text{unstable spiral}}$



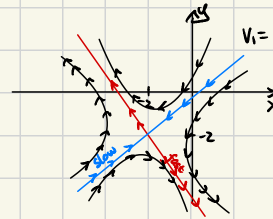
$A\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$

② $A|_{(-2,-2)} = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix} \to \lambda = \dfrac{1 \pm \sqrt{1+16}}{2}$

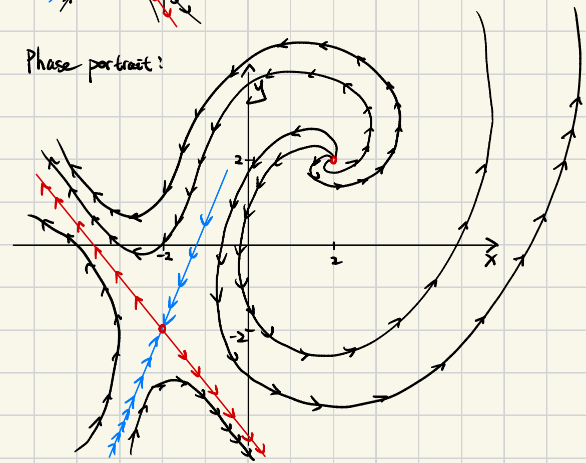
$\Rightarrow \lambda_1 > 0, \lambda_2 < 0$

$\Rightarrow (-2,-2)$ $\boxed{\text{unstable saddle point}}$

$V_1 = \begin{bmatrix} -\frac{\sqrt{17}}{8} - \frac{1}{8} \\ 1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} \frac{\sqrt{17}}{8} - \frac{1}{8} \\ 1 \end{bmatrix}$



Phase portrait:



b) We have: $\begin{cases} \dot{x} = 1+y-e^{-x} \\ \dot{y} = x^3 - y \end{cases}$

Set $\dot{x}, \dot{y} = 0$ we have:

$1+y-e^{-x} = 0$

$x^3 - y = 0$

$\Rightarrow 1 + x^3 - e^{-x} = 0$

$\rightarrow x = 0, \; y = 0$

$(x^*, y^*) : (0,0)$

Then we have the Jacobian matrix:

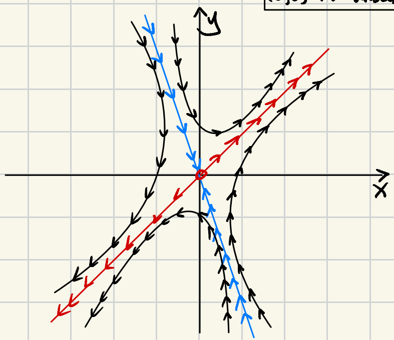
$A = \begin{bmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{bmatrix}$

$-1+\lambda-\lambda+\lambda^2 \quad \lambda^2-4$

$(1-\lambda)(-1-\lambda)-3$

$A\Big|_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \rightarrow \lambda = \pm 2, \; v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\boxed{(0,0) \text{ is unstable saddle point.}}$



2. Consider the ODE

$$\dot{x} = y - x^2 + 2,$$
$$\dot{y} = 2y^2 - 2xy.$$

Find each fixed point and classify it according to linear stability analysis. Is it stable or unstable? Can the linear stability analysis be trusted? Indicate when it can and when it cannot be trusted. Additionally, identify the stable and unstable manifold (locally) for any saddle points.

Finally, sketch (by hand, do not use any software such as pplane) the local behavior of the phase diagram based off of the eigenvalues and eigenvectors at the fixed point. Label any basins of attraction or separatrices in this problem.

To find the fixed points, set $\dot{x}, \dot{y} = 0$ we have:

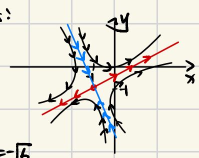
$$y - x^2 + 2 = 0$$
$$2y^2 - 2xy = 0$$
$$\Rightarrow y = x^2 - 2$$
$$\Rightarrow 2(x^2-2)^2 - 2x^3 + 4x$$
$$= 2(x+\sqrt{2})^2(x-\sqrt{2})^2 - 2x(x+\sqrt{2})(x-\sqrt{2})$$
$$= (x+\sqrt{2})(x-\sqrt{2})(2x^2 - 4 - 2x)$$
$$= 2(x+\sqrt{2})(x-\sqrt{2})(x+1)(x-2)$$

$x = \pm\sqrt{2}, -1, 2$, $y = 0, -1, 2$

(x^*, y^*): $(\sqrt{2}, 0), (-\sqrt{2}, 0), (-1,-1), (2,2)$

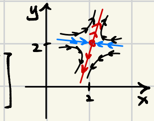
Then we have the Jacobian matrix:

$$A = \begin{bmatrix} -2x & 1 \\ -2y & 4y-2x \end{bmatrix}$$

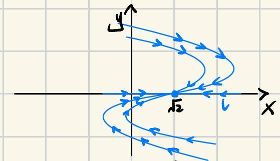


$$A\big|_{(-1,-1)} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix} \rightarrow \lambda_1 = \sqrt{6}, \lambda_2 = -\sqrt{6}$$
$$V_1 = \begin{bmatrix} \frac{\sqrt{6}+2}{2} \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} \frac{-\sqrt{6}+2}{2} \\ 1 \end{bmatrix}$$

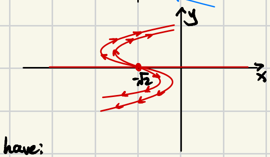
$$A\big|_{(2,2)} = \begin{bmatrix} -4 & 1 \\ -4 & 4 \end{bmatrix} \rightarrow \lambda_1 = 2\sqrt{3}, \lambda_2 = -2\sqrt{3}$$
$$V_1 = \begin{bmatrix} \frac{-\sqrt{3}+2}{2} \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} \frac{\sqrt{3}+2}{2} \\ 1 \end{bmatrix}$$



$$A\big|_{(\sqrt{2},0)} = \begin{bmatrix} -\sqrt{8} & 1 \\ 0 & -\sqrt{8} \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = -\sqrt{8}$$
$$V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$A\big|_{(-\sqrt{2},0)} = \begin{bmatrix} \sqrt{8} & 1 \\ 0 & \sqrt{8} \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = \sqrt{8}$$
$V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V_2 =$



Then from linear stability analysis we have:

① $(-1,-1)$ is unstable saddle point reliable

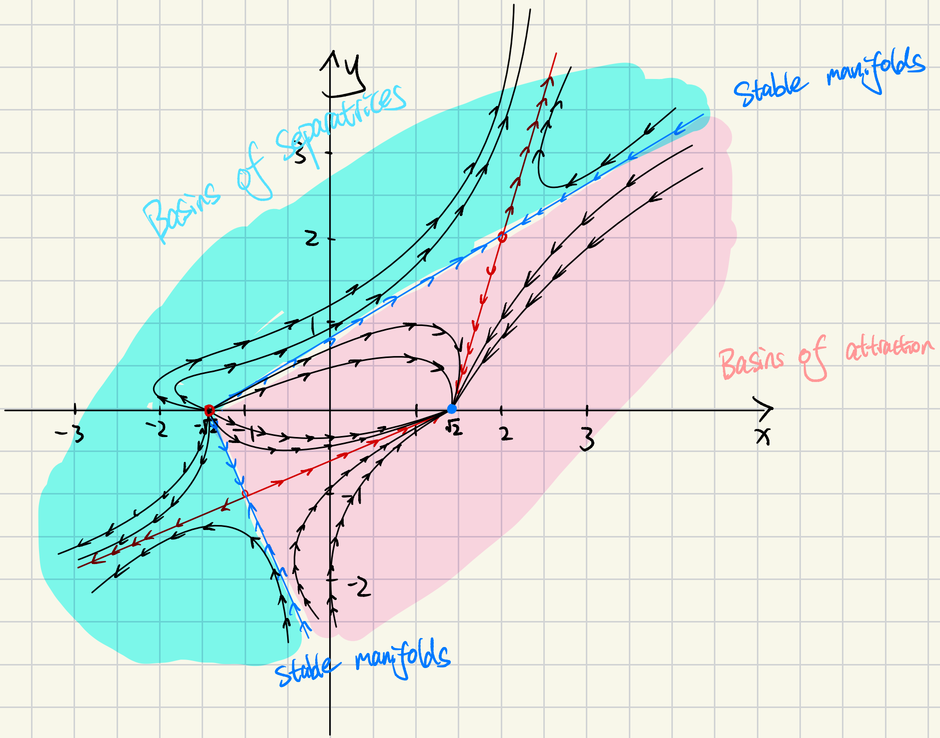
stable manifold: $\text{span}\left(\begin{bmatrix} \frac{-\sqrt{6}+2}{2} \\ 1 \end{bmatrix}\right)$, unstable manifold: $\text{span}\left(\begin{bmatrix} \frac{\sqrt{6}+2}{2} \\ 1 \end{bmatrix}\right)$

② $(2,2)$ is unstable saddle point reliable

stable manifold: $\text{span}\left(\begin{bmatrix} \frac{\sqrt{3}+2}{2} \\ 1 \end{bmatrix}\right)$, unstable manifold: $\text{span}\left(\begin{bmatrix} \frac{\sqrt{3}+2}{2} \\ 1 \end{bmatrix}\right)$

③ $(\sqrt{2}, 0)$ is stable degenerate node unreliable

④ $(-\sqrt{2}, 0)$ is unstable degenerate node unreliable



Basins of separatrices
Stable manifolds
Basins of attraction
Stable manifolds
y
x
3
2
1
-1
-2
-3
-2
$-\sqrt{2}$
-1
$\sqrt{2}$
2
3

3. Consider a glider flying at speed v at an angle θ to the horizontal. Its motion is governed approximately by the dimensionless equation

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2,$$

where the trigonometric terms represent the effects of gravity and the v^2 terms represent the effects of drag and lift.

(a) First suppose that there is no drag, $D = 0$.

i. Suppose there is no drag ($D = 0$). Show that $v^3 - 3v\cos\theta$ is a conserved quantity in this case.

ii. Sketch (by hand, do not use any software such as `pplane`) the complete phase portrait for the case $D = 0$. This means you need to find fixed points, calculate stability etc. Do not worry about everything looking perfect. You should classify and label each fixed point so that we do not rely entirely on your drawing for grading. For each fixed point classification, determine if we can or cannot trust the result of linearized system stability analysis.

(b) We now want to examine the case of positive drag, $D > 0$.

i. What is the effect of the drag on the fixed points? Does this agree with your intuition?

a) i) We have: $E = v^3 - 3v\cos\theta$

$\Rightarrow \dot{E} = 3v^2\cdot\dot{v} - 3\dot{v}\cos\theta + 3v\sin\theta\cdot\dot{\theta}$

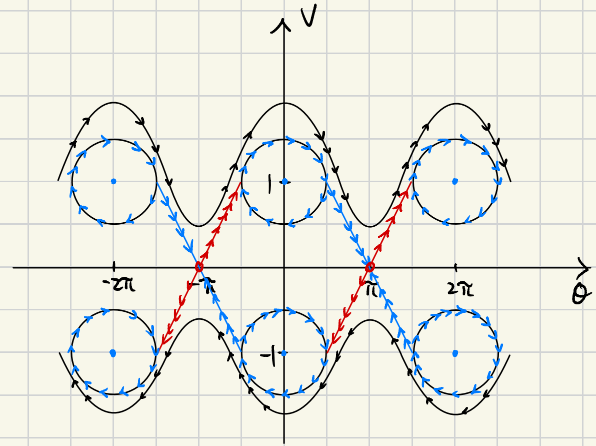
Then plug in $\dot{v} = -\sin\theta$ and $\dot{\theta} = (-\cos\theta + v^2)/v$

$\Rightarrow \dot{E} = \cancel{-3v^2\sin\theta} + \cancel{3\sin\theta\cos\theta} - \cancel{3\sin\theta\cos\theta} + \cancel{3v^2\sin\theta}$

$= 0$

Hence it is a conserved quantity.

ii) We have: $\begin{cases} \dot{\theta} = (-\cos\theta + v^2)/v \\ \dot{v} = -\sin\theta \end{cases}$



Then note we have:

$\dot{v} = 0 = -\sin\theta \longrightarrow \theta = n\pi$

$\dot{\theta} = 0 \rightarrow v^2 = \cos\theta \longrightarrow v = \pm\sqrt{\cos\theta}$

Then we have the Jacobian matrix:

$A = \begin{bmatrix} \frac{\sin\theta}{v} & \cos\theta\, v^{-2} + 1 \\ -\cos\theta & 0 \end{bmatrix}$

Then note we have 2 cases:

Case 1: n is odd

$\Rightarrow \cos\theta = -1$

$\Rightarrow v = \pm i$

Let $\theta = \pi$, $v = \pm i$

$A|_{(\pi, \pm i)} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \rightarrow \lambda_1 = \sqrt{2}, \lambda_2 = -\sqrt{2}$ unstable saddle point. trustable

Case 2: n is even or 0.

$\Rightarrow \cos\theta = 1$

$v = \pm 1$

Let $\theta = 0$, $v = \pm 1$

$A|_{(0, \pm 1)} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \rightarrow \lambda_1 = \sqrt{2}i, \lambda_2 = -\sqrt{2}i$ neutrally stable center

untrustable

b) First note we have:

$$E = v^3 - 3v\cos\theta$$

$$\begin{cases} \dot\theta = (-\cos\theta + v^2)/v \\ \dot v = -\sin\theta - Dv^2 \end{cases}$$

Then we have:

$$\dot E = 3v^2\cdot\dot v - 3\dot v\cos\theta + 3v\sin\theta\,\dot\theta$$

$$= 3v^2(-\sin\theta - Dv^2) - 3(-\sin\theta - Dv^2)\cos\theta + 3v\sin\theta\left(-\frac{\cos\theta}{v} + v\right)$$

$$= \cancel{-3v^2\sin\theta} - 3Dv^4 + \cancel{3\sin\theta\cos\theta} + 3Dv^2\cos\theta - \cancel{3\sin\theta\cos\theta} + \cancel{3v^2\sin\theta}$$

$$= -3Dv^4 + 3Dv^2\cos\theta$$

Set $\dot v, \dot\theta = 0$ we have:

$\dot v = 0 \to Dv^2 = -\sin\theta \to v^2 = \frac{-\sin\theta}{D} \to v = \pm\sqrt{\frac{-\sin\theta}{D}}$

$\dot\theta = 0 \to v^2 = \cos\theta \to v = \pm\sqrt{\cos\theta}$

$\Rightarrow \sqrt{\cos\theta} = \sqrt{\frac{-\sin\theta}{D}} \to D = -\tan\theta \to \boxed{\theta^* = -\tan^{-1} D}$

Note $\cos(\arctan(x)) = (1+x^2)^{-\frac{1}{2}}$

$\Rightarrow \boxed{v^* = \pm(1+D^2)^{-\frac{1}{4}}}$

Then we have the Jacobian matrix:

$$A = \begin{bmatrix} \frac{\sin\theta}{v}\ ① & \cos\theta\, v^{-2} + 1\ ③ \\ -\cos\theta\ ② & -2Dv\ ④ \end{bmatrix}$$

$\Rightarrow A\big|_{(\theta^*, v^*)} =$ ① $-D(1+D^2)^{-\frac{1}{2}}(1+D^2)^{\frac{1}{4}} = -D(1+D^2)^{-\frac{1}{4}}$

② $-(1+D^2)^{-\frac{1}{2}}$

③ $(1+D^2)^{-\frac{1}{2}}\cdot(1+D^2)^{\frac{1}{2}} + 1 = 2$

④ $-2D(1+D^2)^{-\frac{1}{4}}$

Then we have: $\Delta = (2D^2+2)(1+D^2)^{-\frac{1}{2}}$, $\mathrm{Tr}(A) = -3D(1+D^2)^{-\frac{1}{4}}$

$$\mathrm{Tr}(A)^2 - \Delta = 9D^2(1+D^2)^{-\frac{1}{2}} - (2D^2+2)(1+D^2)^{-\frac{1}{2}}$$

$$= (7D^2 - 2)(1+D^2)^{-\frac{1}{2}}$$

$\Rightarrow 7D^2 = 2$

$D = \sqrt{\frac{2}{7}} \to$ boardline

① $D > \sqrt{\frac{2}{7}}$ we have stable node

② $D < \sqrt{\frac{2}{7}}$ we have stable spiral.

The drag changes the position of fixed points. And also change the stability and types of fixed points

The drag introduces force oppose the motion which reduces the velocity overtime. This will lead to a halt in motion. Since our fixed points become stable, thus this does agree with my intuition.

4. a) we have: $\frac{dP}{dt} = \alpha(D-S)$

$\frac{dD}{dt} = \beta(P_d - P)(1-\beta_1(P_d-P)^2)$

$\frac{dS}{dt} = -\gamma(P_d - P) + \delta(D-S)$

Let ① $p = P - P_d$, ② $q = D - S$

From ① we have:

$\Rightarrow \frac{dp}{dt} = \frac{dP}{dt} - \frac{dP_d}{dt} = \frac{dP}{dt}$

$\Rightarrow \frac{dp}{dt} = \alpha q$

From ② we have:

$\Rightarrow \frac{dq}{dt} = \frac{dD}{dt} - \frac{dS}{dt}$

$\Rightarrow \frac{dD}{dt} = -\beta p(1-\beta_1 p^2)$

$\frac{dS}{dt} = \gamma p + \delta q$

$\Rightarrow \frac{dq}{dt} = -\beta p(1-\beta_1 p^2) - \gamma p - \delta q$

Thus we have:

$$\begin{cases} \frac{dp}{dt} = \alpha q \\ \frac{dq}{dt} = -\beta p(1-\beta_1 p^2) - \gamma p - \delta q \end{cases}$$

b) Set $\delta = 0$ we have:

$$\begin{cases} \dot{p} = \alpha q & ① \\ \dot{q} = -\beta p(1-\beta_1 p^2) - \gamma p & ② \end{cases}$$

Integrat ① with respect to q we have:

$H = \frac{1}{2}\alpha q^2 + f(p)$

Integrat ② with respect to p we have:

$H = \frac{1}{2}\beta p^2 - \frac{1}{4}\beta\beta_1 p^4 + \frac{1}{2}\gamma p^2 + C$

Combine we have: $\boxed{H = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 - \frac{1}{4}\beta\beta_1 p^4 + \frac{1}{2}\gamma p^2 + C}$

c) Set $\delta = 0$ we have:

$$\begin{cases} \dot{p} = \alpha q \\ \dot{q} = -\beta p(1-\beta_1 p^2) - \gamma p \end{cases}$$

Set $\dot{p}, \dot{q} = 0$ we have:

$\dot{p} = \alpha q = 0 \rightarrow q = 0$

$\dot{q} = -\beta p(1-\beta_1 p^2) - \gamma p$

$= p(\beta\beta_1 p^2 - \beta - \gamma)$

① $p = 0$

② $\beta\beta_1 p^2 - \beta - \gamma = 0$

$\Rightarrow p^2 = (\beta+\gamma)/\beta\beta_1$

$p = \pm[\beta_1^{-1} + \gamma(\beta\beta_1)^{-1}]^{\frac{1}{2}}$

We have the fixed points:

$(0,0)$, $([\beta_1^{-1} + \gamma(\beta\beta_1)^{-1}]^{\frac{1}{2}}, 0)$

Then we have the Jacobian matrix:

$$A = \begin{bmatrix} 0 & \alpha \\ 3\beta\beta_1 p^2 - \beta - \gamma & 0 \end{bmatrix}$$

$$A|_{(0,0)} = \begin{bmatrix} 0 & \alpha \\ -(\beta+\gamma) & 0 \end{bmatrix}$$

$tr(A) = 0$, $\Delta = \alpha(\beta+\gamma)$

Note $\Delta > 0$, then since $tr(A) = 0$

$(0,0)$ is $\boxed{\text{neutrally stable center}}$ untrustable

$$A|_{(p^*,0)} = \begin{bmatrix} 0 & \alpha \\ 2\beta+2\gamma & 0 \end{bmatrix}$$

$tr(A) = 0$, $\Delta = -\alpha(2\beta + 2\gamma)$

Note $\Delta < 0$ we have:

$(p^*, 0)$ is a $\boxed{\text{unstable saddle point}}$ trustable

Then note we have the condition:

$p > -p_d$

$\Rightarrow \beta_1^{-1} + \gamma(\beta\beta_1)^{-1} > p_d^2$

$\Rightarrow \boxed{\dfrac{\beta+\gamma}{\beta\beta_1} > p_d^2}$

d)