AMATH 502 HW8

Tianbo Zhang 1938501

Problem 1

a) First note each set A_n covers all the sets come after.

$\Rightarrow$ A_∞ is covered by each sets A_n.

$\Rightarrow$ Total area of A_∞ is smaller than total area of A_n, for any n.

Let L_n be the area of A_n we have:

$L_0 = 1$, $L_1 = \frac{8}{9}$, $L_2 = \frac{8}{9} \cdot \frac{8}{9} = \frac{64}{81}$ $\cdots$, $L_n = (\frac{8}{9})^n$

Then note since L_i is a geometric series.

And also note $r = \frac{8}{9} < 1$

$\Rightarrow n \to \infty \Rightarrow L_n \to 0 \Rightarrow \sum_{n=0}^{\infty} \frac{1}{9}(\frac{8}{9})^n = \frac{\frac{1}{9}}{1-\frac{8}{9}} = 1$

Thus measure of the resulting fractal is 0. $\square$

b) Note the set we have composed 8 copies of itself scaled down by a factor of 3 on each direction.

$\Rightarrow d = \frac{\ln 8}{\ln 3} \approx \boxed{1.893}$

c) Let $\epsilon = (\frac{1}{3})^n$

Then note S_1 is covered by $N = 8$ squares.

$\Rightarrow d = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln N(1/\epsilon)} = \frac{\ln(8^n)}{\ln(3^n)} = \frac{n\ln(8)}{n\ln(3)} = \frac{\ln(8)}{\ln(3)} \approx \boxed{1.893}$

d) First suppose we have the line segment at $y = 0.5$ for $x \in [0,1]$

Then let X denote the set of all real numbers between 0 and 1

Suppose X is countable than we have:

$X_1 = 0.x_{11}x_{12}x_{13}\cdots$

$X_2 = 0.x_{21}x_{22}x_{23}\cdots$

$\vdots$

Let r be a number between 0 and 1.

Let r have 1st digit $\neq x_{11}$

2nd digit $\neq x_{22}$

Then we have r not on the list.

$\Rightarrow$ X is uncountable

$\Rightarrow$ We have uncountable points (x,y) with a fixed value of y.

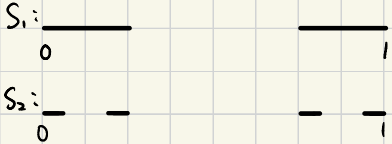
Then note Y is also uncountable if we set $x = 0.5$.

Hence since x and y are both uncountable.

We have uncountably many points on the interior. $\square$

Problem 2

a)



b) Note we have 2 copies scaled down by a factor of 4.

$\Rightarrow d = \dfrac{\ln(2)}{\ln(4)} = \boxed{\dfrac{1}{2}}$

c) Note each set S_n covers all the sets after.

Let L_n be the length of S_n.

$\Rightarrow L_0 = 1,\ L_1 = \frac{1}{2},\ L_2 = \frac{1}{2}\cdot\frac{1}{2} \cdots L_n = \left(\frac{1}{2}\right)^n$

Since $\frac{1}{2} < 1$

$\Rightarrow L_n \to 0$ as $n \to \infty$.

$\Rightarrow$ measure is 0.

Problem 3

a) First note the total length removed is:

$$S = \frac{2}{7} + 2\left(\frac{2}{7}\right)^2 + 4\cdot\left(\frac{2}{7}\right)^3 + \cdots$$

$$\Rightarrow S = \frac{2}{7}\sum_{n=0}^{\infty}\left(2^n\left(\frac{2}{7}\right)^n\right)$$

$$= \frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n$$

$$= \frac{2}{7}\cdot\frac{1}{1-\frac{4}{7}} = \frac{2}{3}$$

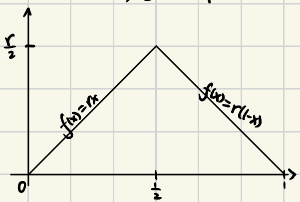
Then the measure is:

$$1-\frac{2}{3} = \boxed{\frac{1}{3}}$$

b) Note as the construction progresses,
the size of the intervals being removed changes
relative to the size of remaining interval.
Hence the scaling factor changes at each iteration.
Thus we can't find a uniform scaling factor to make
parts of fractal look like the whole.
$\Rightarrow$ It is not self similar.

Problem 4

$$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$



a) First note x_0 escape after 1 iteration if:

$f(x_0) > 1$

Then we have two cases:

Case 1: $0 \le x_0 \le \frac{1}{2}$

$f(x_0) = rx_0 > 1$

$\Rightarrow x_0 > \frac{1}{r}$

Note since $r > 2$ then $\frac{1}{r} < \frac{1}{2}$

Hence it's reasonable.

$\Rightarrow \boxed{\frac{1}{r} < x_0 \le \frac{1}{2}}$

Case 2: $\frac{1}{2} < x \le 1$

$f(x_0) = r(1-x_0) > 1$

$\Rightarrow 1 - x_0 > \frac{1}{r}$

$x_0 < 1 - \frac{1}{r}$

$\Rightarrow \boxed{\frac{1}{2} < x_0 < 1 - \frac{1}{r}}$

b) Note we have four cases:

Case 1: $0 \le x_0 \le \frac{1}{2}$, $0 \le x_1 \le \frac{1}{2}$

$\Rightarrow f(f(x_0)) = r(rx_0) > 1$

$\Rightarrow r^2 x_0 > 1$

$\Rightarrow x_0 > \frac{1}{r^2}$

$\Rightarrow \boxed{\frac{1}{r^2} < x_0 \le \frac{1}{2}}$

Case 2: $0 \le x_0 \le \frac{1}{2}$, $\frac{1}{2} < x_1 \le 1$

$\Rightarrow f(f(x_0)) = r(1 - rx_0) > 1$

$r - r^2 x_0 > 1$

$-r^2 x_0 > 1 - r$

$\Rightarrow x_0 < -\frac{1}{r^2} + \frac{1}{r}$

Note $\frac{1}{r^2} < \frac{1}{r}$ thus it's reasonable.

$\Rightarrow \boxed{0 \le x_0 < -\frac{1}{r^2} + \frac{1}{r}}$

Case 3: $\frac{1}{2} < x_0 \le 1$, $0 \le x_1 \le \frac{1}{2}$

$\Rightarrow f(f(x_0)) = r(r(1-x_0)) > 1$

$r^2 - r^2 x_0 > 1$

$-r^2 x_0 > 1 - r^2$

$x_0 < -\frac{1}{r^2} + 1$

$\Rightarrow \boxed{\frac{1}{2} < x_0 \le 1 - \frac{1}{r^2}}$

Case 4: $\frac{1}{2} < x_0 \le 1$, $\frac{1}{2} < x_1 \le 1$

$\Rightarrow f(f(x_0)) = r(1 - r(1-x_0)) > 1$

$r - r^2 + r^2 x_0 > 1$

$x_0 > \frac{1}{r^2} - \frac{1}{r} + 1$

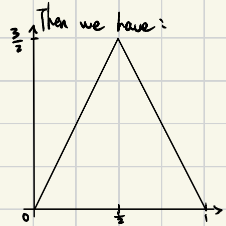
$\Rightarrow \boxed{\frac{1}{r^2} - \frac{1}{r} + 1 < x_0 \le 1}$

c) Note for $r = 3$.

Then we have the classic cantor set.

$$f(x) = \begin{cases} 3x, & 0 \le x \le \frac{1}{2} \\ 3 - 3x, & \frac{1}{2} < x \le 1 \end{cases}$$

Then we have:



Note that for each iteration those escaped are removed.

d). Note we have:

$$\varepsilon = \left(\frac{1}{r}\right)^n, \quad N = (r-1)^n$$

$$\Longrightarrow d = \lim_{\varepsilon \to 0} \frac{\ln (r-1)^n}{\ln \left(\frac{1}{r}\right)^n} = \boxed{\frac{\ln(r-1)}{\ln\left(\frac{1}{r}\right)}}$$