AMATH 502 HW1

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2.2.3 $\dot{x} = x - x^3$

To find fixed points:

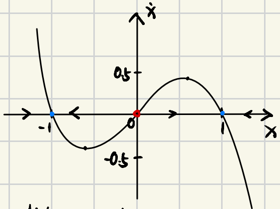
$\dot{x} = x - x^3 = 0$

$x(1-x^2) = 0$

$x = 0, \pm 1$ are fixed points

$\dot{x}(-2) = -2+8 = 10$, $\dot{x}(-\frac{1}{2}) = -\frac{1}{2} - \frac{1}{8} = -\frac{5}{8}$

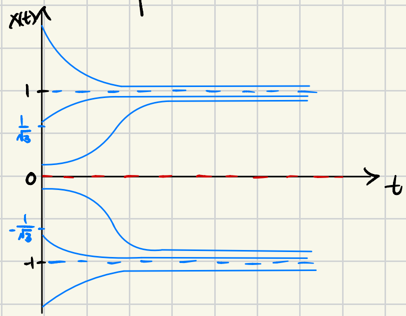
$\dot{x}(\frac{1}{2}) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$, $\dot{x}(2) = 2-8 = -6$



Note -1 and 1 are stable fixed points.

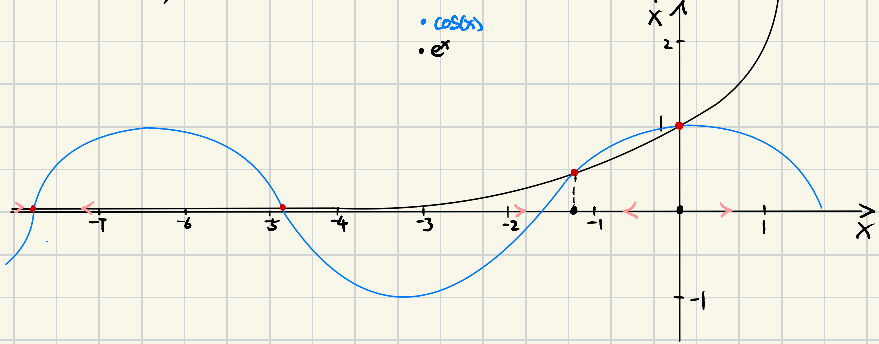
0 is unstable fixed point.

Solution Map:

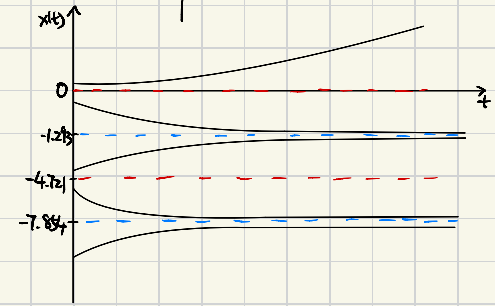


2.2.7 $\dot{x} = e^x - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.)

Set $\dot{x}_1 = e^x$, $\dot{x}_2 = \cos(x)$



Solution Map:



Note all the intersections between e^x and $\cos(x)$ are fixed points.

Since at the intersection, $e^x = \cos(x) \rightarrow e^x - \cos(x) = 0 = \dot{x}$

Then note there are infinitely many fixed points for $x < 0$

And they loop between stable and unstable fixed points

Note when $\cos(x) > e^x \longrightarrow \dot{x} < 0$

$\cos(x) < e^x \longrightarrow \dot{x} > 0$

2.2.8 (Working backwards, from flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too.)

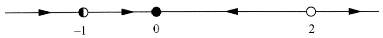


Figure 1

First note, we have $-1, 0, 2$ as fixed points.

Thus we have:

$X(0) = 0$, $x(2)=0$, $x(-1)=0$

Then note we have:

$x(t)$ flowing to the right for $t<0$

left for $0<t<2$

right for $t>2$

Take $\boxed{\dot{x} = (x+1)^2(x-2)x}$

$x(-2)>0$, $x(-\frac{1}{3})>0$, $x(1)<0$, $x(3)>0$

2.2.10 (Fixed points) For each of (a)–(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

a) Every real number is a fixed point.
b) Every integer is a fixed point, and there are no others.
c) There are precisely three fixed points, and all of them are stable.
d) There are no fixed points.
e) There are precisely 100 fixed points.

a) $f(x)=0$

Then $f(x)$ is a smooth function

And all real number is a fixed point.

b) $f(x)=\sin(\pi x)$

Then $f(x)$ is a smooth function

And all integer is a fixed point. and no others.

c) There's no such flow

If there are 3 stable fixed points

And there are only 3 fixed points

This means that the flow between fixed points has to different directions.

Hence impossible.

d) $f(x)=e^x$

Note $f(x)$ never equals to 0.

e) $f(x)=\prod_{i=1}^{100}(x-i)$

Has fixed point for $x=1,2,\cdots,100$.

2.2.13 (Terminal velocity) The velocity $v(t)$ of a skydiver falling to the ground is governed by $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.

b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the *terminal velocity*. (Beware of bad jokes about the word *terminal* and parachutes that fail to open.)

c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

d) An experimental study (Carlson et al. 1942) confirmed that the equation $m\dot{v} = mg - kv^2$ gives a good quantitative fit to data on human skydivers. Six men were dropped from altitudes varying from 10,600 feet to 31,400 feet to a terminal altitude of 2,100 feet, at which they opened their parachutes. The long free fall from 31,400 to 2,100 feet took 116 seconds. The average weight of the men and their equipment was 261.2 pounds. In these units, $g = 32.2$ ft/sec^2. Compute the average velocity V_{avg}.

a) First note we have: $m\frac{dv}{dt} = mg - kv^2$

$$\frac{dv}{dt} = g - \frac{k}{m}v^2$$

$$\left(g - \frac{k}{m}v^2\right)^{-1} dv = dt$$

Then take integral we have:

$$\int_0^v \left(g - \frac{k}{m}v^2\right)^{-1} dv = \int dt$$

Note $\int_0^v \left(g - \frac{k}{m}v^2\right)^{-1} dv$

$$= \frac{1}{g}\int_0^v \frac{1}{\left(1-\left(\frac{k}{gm}\right)^{\frac{1}{2}}v\right)\left(1+\left(\frac{k}{gm}\right)^{\frac{1}{2}}v\right)} dv$$

$$= \frac{1}{g}\int_0^v \frac{1}{2\left(1-\left(\frac{k}{gm}\right)^{\frac{1}{2}}v\right)} dv + \int_0^v \frac{1}{2\left(1+\left(\frac{k}{gm}\right)^{\frac{1}{2}}v\right)} dv$$

$$= \frac{1}{2}\left(\frac{m}{kg}\right)^{\frac{1}{2}} \ln\left(\frac{1+\left(\frac{k}{gm}\right)^{\frac{1}{2}}v}{1-\left(\frac{k}{gm}\right)^{\frac{1}{2}}v}\right) = t$$

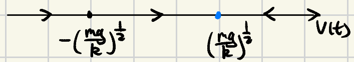
$$\Rightarrow \frac{1+\left(\frac{k}{gm}\right)^{\frac{1}{2}}v}{1-\left(\frac{k}{gm}\right)^{\frac{1}{2}}v} = e^{2\left(\frac{kg}{m}\right)^{\frac{1}{2}}t}$$

$$\Rightarrow \boxed{v(t) = \left(\frac{mg}{k}\right)^{\frac{1}{2}}\left[\frac{\exp\left(2\left(\frac{kg}{m}\right)^{\frac{1}{2}}t\right)-1}{\exp\left(2\left(\frac{kg}{m}\right)^{\frac{1}{2}}t\right)+1}\right] = \left(\frac{mg}{k}\right)^{\frac{1}{2}} \tanh\left(\left(\frac{gk}{m}\right)^{\frac{1}{2}}t\right)}$$

b) Note we have:

$$\lim_{t\to\infty} v(t) = \left(\frac{mg}{k}\right)^{\frac{1}{2}} \cdot \left(\frac{\infty}{\infty}\right) = \boxed{\left(\frac{mg}{k}\right)^{\frac{1}{2}}}$$

c) $\dot{v} = g - \frac{k}{m}v^2$



d) $v = \frac{\Delta x}{\Delta t} = \frac{31400 - 2100}{116} \approx \boxed{252.586\ ft/s}$

2.3.2 (Autocatalysis) Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production, a process called *autocatalysis*. This positive feedback process leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations by lowercase letters $x = [X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 a x - k_{-1} x^2$$

where k_1 and k_{-1} are positive parameters called rate constants.

a) Find all the fixed points of this equation and classify their stability.

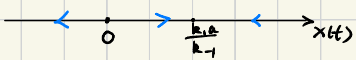
b) Sketch the graph of $x(t)$ for various initial values x_0.

a) We have $\dot{x} = k_1 a x - k_{-1} x^2$

Set $\dot{x} = 0$ we have:

$x(k_1 a - k_{-1} x) = 0$

$x = 0$ and $x = \frac{k_1 a}{k_{-1}}$ are the fixed points



Set $x = \frac{k_1 a + 1}{k_{-1}} > \frac{k_1 a}{k_{-1}}$

$\dot{x} = \frac{k_1^2 a^2 + k_1 a}{k^{-1}} - \frac{(k_1 a + 1)^2}{k^{-1}}$

$= \frac{(k_1 a + 1)(k_1 a - k_1 a - 1)}{k_{-1}} < 0$

Set $x = -1 < 0$

$\dot{x} = -k_1 a - k_{-1} < 0$

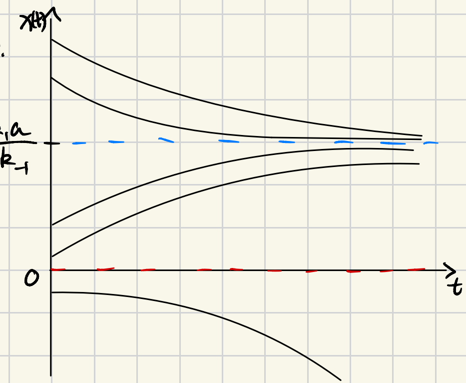
Set $0 < x < \frac{k_1 a}{k_{-1}}$

$\Rightarrow \dot{x} > 0$

Thus $x = 0$ is unstable fixed point

$x = \frac{k_1 a}{k_{-1}}$ is stable fixed point

b).



2.3.6 (Language death) Thousands of the world's languages are vanishing at an alarming rate, with 90 percent of them being expected to disappear by the end of this century. Abrams and Strogatz (2003) proposed the following model of language competition, and compared it to historical data on the decline of Welsh, Scottish Gaelic, Quechua (the most common surviving indigenous language in the Americas), and other endangered languages.

Let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = (1-x)P_{YX} - xP_{XY}$$

where $0 \le x \le 1$ is the current fraction of the population speaking X, $1-x$ is the complementary fraction speaking Y, and P_{YX} is the rate at which individuals switch from Y to X. This deliberately idealized model assumes that the population is well mixed (meaning that it lacks all spatial and social structure) and that all speakers are monolingual.

Next, the model posits that the attractiveness of a language increases with both its number of speakers and its perceived status, as quantified by a parameter $0 \le s \le 1$ that reflects the social or economic opportunities afforded to its speakers. Specifically, assume that $P_{YX} = sx^a$ and, by symmetry, $P_{XY} = (1-s)(1-x)^a$, where the exponent $a > 1$ is an adjustable parameter. Then the model becomes

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a.$$

a) Show that this equation for $\dot{x}$ has three fixed points.
b) Show that for all $a > 1$, the fixed points at $x = 0$ and $x = 1$ are both stable.
c) Show that the third fixed point, $0 < x^* < 1$, is unstable.

a) Note we have:

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a$$
$$= x(1-x)\left[sx^{a-1} - (1-s)(1-x)^{a-1}\right]$$

Then note $\dot{x} = 0$ when $x = 0$, $x = 1$ and

$$sx^{a-1} - (1-s)(1-x)^{a-1} = 0$$

$$\frac{s}{1-s} = \frac{(1-x)^{a-1}}{x^{a-1}} = \left[\frac{(1-x)}{x}\right]^{a-1}$$

$$\frac{1-x}{x} = \left[\frac{s}{1-s}\right]^{\frac{1}{a-1}}$$

$$\frac{1}{x} = \left[\frac{s}{1-s}\right]^{\frac{1}{a-1}} + 1$$

$$x = \left[\left[\frac{s}{1-s}\right]^{\frac{1}{a-1}} + 1\right]^{-1}$$

Thus $x = 0, 1, \left[\left[\frac{s}{1-s}\right]^{\frac{1}{a-1}} + 1\right]^{-1}$ are the fixed points

b) Note $\ddot{x} = s\left[-x^a + a(1-x)x^{a-1}\right] - (1-s)\left[(1-x)^a - ax(1-x)^{a-1}\right]$

Plug in $x = 0, 1$

Note $\ddot{x}(0) = -(1-s) < 0$

$\ddot{x}(1) = -s < 0$

Thus means that $\dot{x}$ is flowing towards 0 and 1.

Thus stable

c) Note from 2.2.10 we know

We can not have all the fixed points be stable.

This means the flow between 0 and $\left[\left[\frac{s}{1-s}\right]^{\frac{1}{a-1}} + 1\right]^{-1}$

also $\left[\left[\frac{s}{1-s}\right]^{\frac{1}{a-1}} + 1\right]^{-1}$ and 1 have two directions.

Which is impossible.

Thus the point is unstable.

2.4.7 $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

Note we have:

$\dot{x} = x(a - x^2) = 0$

$x = 0$, $x = \pm\sqrt{a}$ are fixed points

Then note we have:

$\ddot{x} = a - 3x^2$

Then plug in the fixed points we have:

$\ddot{x}(0) = a$

$\ddot{x}(\sqrt{a}) = a - 3a = -2a$

$\ddot{x}(-\sqrt{a}) = a - 3a = -2a$

Case 1: $a > 0$

Then 0 is unstable fixed point.

And $\pm\sqrt{a}$ are stable fixed points

Case 2: $a < 0$

Then 0 is stable fixed point

And $\pm\sqrt{a}$ are unstable fixed points

Case 3: $a = 0$

No conclusion from analysis

From graph 0 is stable fixed point