AMATH 502 HW2

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2.6.1 Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can't oscillate.

First note from text we know:

There are no periodic solutions to $\dot{x} = f(x)$.

However what we have is:

$$\ddot{x} = -\frac{k}{m}x = f(x)$$

Let $v = \dot{x} \longrightarrow \dot{v} = \ddot{x}$, then we have

$$\dot{v} = -\frac{k}{m}x$$

Then note there are two variables.

Thus this is a 2-dimensional system.

Hence the simple harmonic oscillator can oscillate

3.1.1 $\dot{x} = 1 + rx + x^2$

Solve for fixed points:

$\dot{x} = x^2 + rx + 1 = 0$

$x = \dfrac{-r \pm \sqrt{r^2 - 4}}{2}$

Note fixed points exist iff:

$r^2 - 4 \geq 0 \rightarrow r^2 \geq 4$

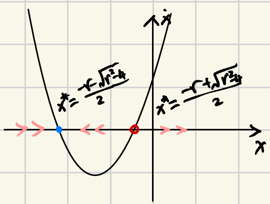
$\Rightarrow r \geq 2, \ r \leq -2$

Then note we have:

① If $r > 2$, we have:

$x_1^*, x_2^* < 0$

$\dot{x} = x^2 + rx + 1$

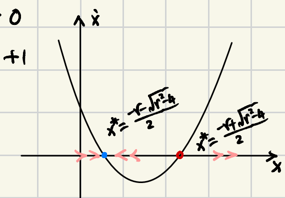


$1+1$

② If $r < -2$, we have:

$x_1^*, x_2^* > 0$

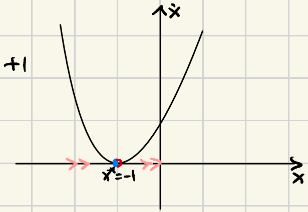
$\dot{x} = x^2 + rx + 1$



③ If $r = 2$, we have:

$x^* = -1$

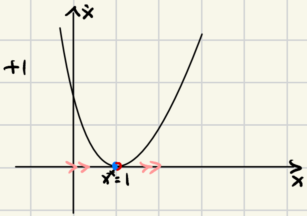
$\dot{x} = x^2 + rx + 1$



④ If $r = -2$, we have:

$x^* = 1$

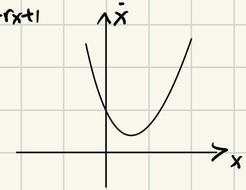
$\dot{x} = x^2 + rx + 1$



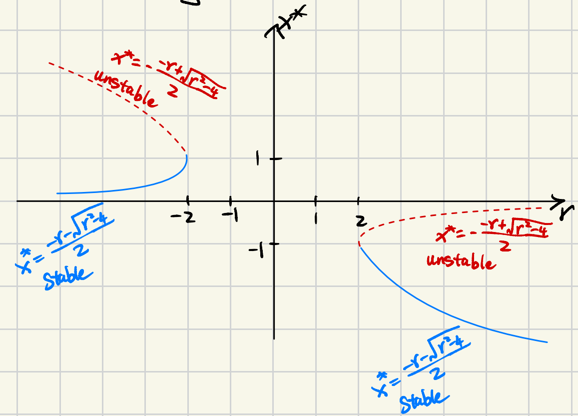
⑤ If $-2 \leq r \leq 2$,

Then there's no fixed point

$\dot{x} = x^2 + rx + 1$



Bifurcation Diagram:



Note saddle-node bifurcation occurs at $r = \pm 2$

3.1.5 (Unusual bifurcations) In discussing the normal form of the saddle-node bifurcation, we mentioned the assumption that $a = \partial f / \partial r|_{(x^*, r_c)} = 0$. To see what can happen if $a = \partial f / \partial r|_{(x^*, r_c)} \neq 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r.

(a) $\dot{x} = r^2 - x^2$

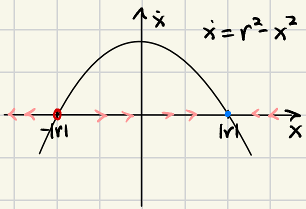
(b) $\dot{x} = r^2 + x^2$

a) First to find fixed points:

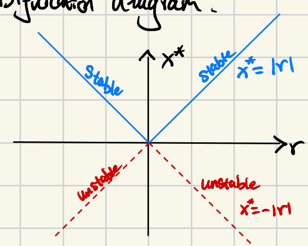
$\dot{x} = r^2 - x^2 = 0$

$\Rightarrow x = \pm |r|$

Then we have:



Bifurcation diagram:

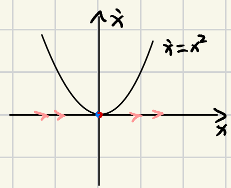


b) First to find fixed points:

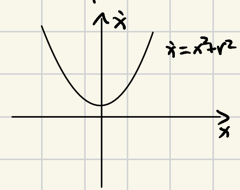
$\dot{x} = r^2 + x^2 = 0$

$\Rightarrow x = \pm ri \rightarrow \quad r=0 \rightarrow x=0$

Then we have no real fixed point except when $r=0$

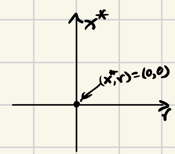


$r=0$



$r > 0$.

Bifurcation diagram:



3.2.3 $\dot{x} = x - rx(1-x)$

First to find fixed points:

$\dot{x} = x - rx(1-x) = 0$

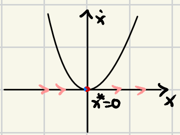
$x(1-r+rx) = 0$

$\Rightarrow 1-r-rx=0 \rightarrow \frac{1}{r} - 1 + x = 0 \rightarrow x = 1-\frac{1}{r}$

So fixed points at: $x=0$, $x = 1-\frac{1}{r}$

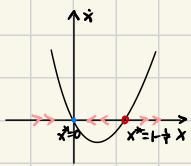
① $r=1$

$\dot{x} = x^2$



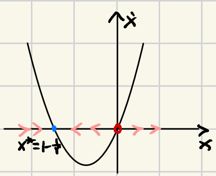
② $r>1$

$\dot{x} = x - rx(1-x)$



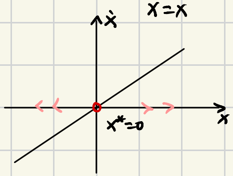
③ $0<r<1$

$\dot{x} = x - rx(1-x)$



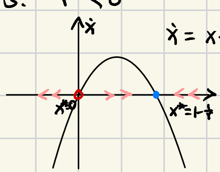
④ $r=0$

$\dot{x} = x$

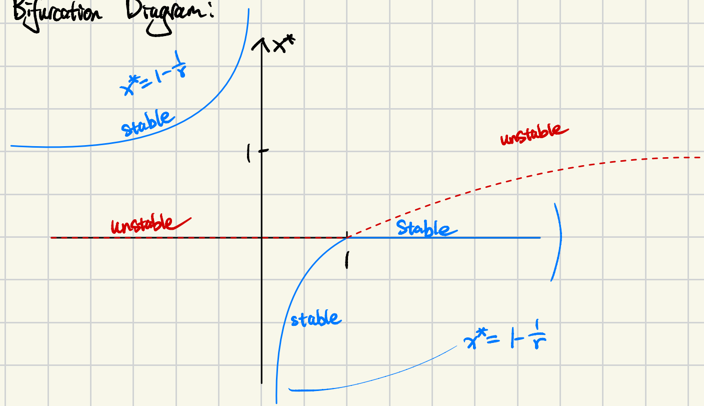


5. $r<0$

$\dot{x} = x - rx(1-x)$



Bifurcation Diagram:



3.4.11 (An interesting bifurcation diagram) Consider the system $\dot{x} = rx - \sin x$.

a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.

d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.

e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.

f) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

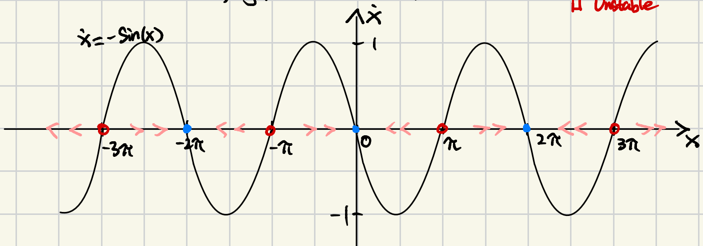
a) Let $r=0$ we have:

$\dot{x} = -\sin x = 0$

$\Rightarrow x = k\pi$ for $k \in \mathbb{Z}$ are the fixed points.

Then by stability analysis we have:

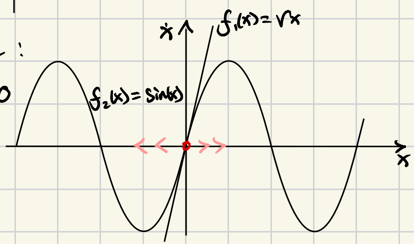
$$\dot{x} = -\cos(x) = \begin{cases} >0, & \text{for } k \text{ is odd} \\ <0, & \text{for } k \text{ is even and } k=0. \end{cases}$$



b) When $r > 1$ we have:

$\dot{x} = rx - \sin(x) = 0 \rightarrow x = 0$

Thus its $\boxed{\text{unstable}}$.



C) Note bifurcation occurs at $r=1$. And it is a subcritical pitchfork bifurcation.

For $0<r<1$ there will be more and more intersects points as r decrease.

These intersection will create a set of saddle node bifurcations.

d) Note:

$\ddot{x} = r - \cos(x)$

Then note bifurcation happens when $\ddot{x} = 0$:

$\ddot{x} = r - \cos(x) = 0 \longrightarrow r = \cos(x)$

Then note $r << 1 \longrightarrow \cos(x) << 1$

$\Longrightarrow \cos(x) \approx 0 \longrightarrow x^* \approx \frac{\pi}{2} + k\pi$

Then note from $\dot{x}$ we have:

$rx - \sin(x) = 0 \longrightarrow rx = \sin(x) \longrightarrow r = \frac{\sin(x)}{x}$

Note $\sin(\frac{\pi}{2} + k\pi) = \begin{cases} 1 & \text{if } k \text{ even} \\ -1 & \text{if } k \text{ odd} \end{cases}$

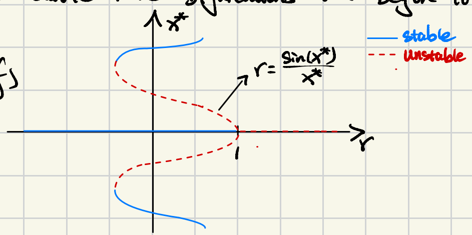
$\Longrightarrow r = (-1)^k (\frac{\pi}{2} + k\pi)^{-1}$

Note $r > 0$, thus we have:

$\boxed{r = (\frac{\pi}{2} + k\pi)^{-1}}$

e) Saddle node bifurcations will begin to dissapear.

f)



3.5.8 (Nondimensionalizing the subcritical pitchfork) The first-order system $\dot{u} = au + bu^3 - cu^5$, where $b, c > 0$, has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

where $x = u/U$, $\tau = t/T$, and U, T, and r are to be determined in terms of a, b, and c.

Let $x = u/U$, $\tau = t/T$

Then taking the derivative we have:

$dx = \frac{1}{U} du \longrightarrow \frac{du}{dx} = U$

$d\tau = \frac{1}{T} dt \longrightarrow \frac{d\tau}{dt} = \frac{1}{T}$

Then note:

$\frac{du}{dt} = \frac{du}{dx} \frac{dx}{d\tau} \frac{d\tau}{dt}$

$= \frac{U}{T} \frac{dx}{d\tau}$

$\Longrightarrow \frac{dx}{d\tau} = \frac{T}{U} \frac{du}{dt}$

Then we have:

$\frac{dx}{d\tau} = \frac{T}{U} a(Ux) + \frac{T}{U} b(Ux)^3 - \frac{T}{U} c(Ux)^5$

$= Tax + TU^2 b x^3 - TU^4 c x^5$

① $TU^2 b = 1 \rightarrow T = \frac{1}{U^2 b}$

② $TU^4 c = \frac{1}{U^2 b} U^4 c = \frac{U^2}{b} c = 1$

Note from ② we have

$U = \pm \left(\frac{b}{c}\right)^{\frac{1}{2}}$

Then substitute U to T we have:

$T = \frac{c}{b^2}$

Lastly substitute T into Ta we have:

$r = \frac{ca}{b^2}$

Then we have:

$\frac{dx}{d\tau} = rx + x^3 - x^5$

Where $U = \pm \left(\frac{b}{c}\right)^{\frac{1}{2}}$, $T = \frac{c}{b^2}$, $r = \frac{ca}{b^2}$

3.7.3 (A model of a fishery) The equation $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)

a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1-x) - h,$$

for suitably defined dimensionless quantities x, τ, and h.

b) Plot the vector field for different values of h.

c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.

d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model—the population can become negative! A better model would have a fixed point at zero population for all values of H. See the next exercise for such an improvement.

a) Let $\tau = t/T \rightarrow \frac{d\tau}{dt} = \frac{1}{T}$

$x = N/K \rightarrow \frac{dN}{dx} = K$

Then note:

$$\frac{dN}{dt} = \frac{dN}{dx}\frac{dx}{d\tau}\frac{d\tau}{dt} = K/T\,\frac{dx}{d\tau}$$

$$\Rightarrow \frac{dx}{d\tau} = \frac{T}{K}\frac{dN}{dt}$$

Then we have:

$$\frac{dx}{d\tau} = \frac{T}{K} rKx(1-x) - \frac{T}{K}H = Trx(1-x) - \frac{T}{K}H$$

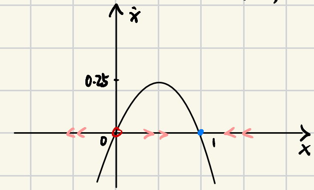
$$\Rightarrow Tr = 1 \rightarrow T = \frac{1}{r}$$

Thus we have $\tau = rt$, $h = \frac{H}{rK}$, $x = \frac{N}{K}$

Then we will have: $\frac{dx}{d\tau} = x(1-x) - h$

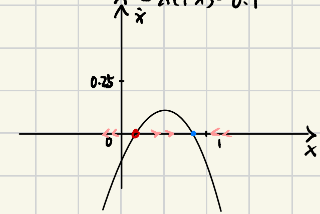
b) ① $h=0$ we have:

$\dot{x} = x(1-x) \rightarrow x_1^* = 0,\ x_2^* = 1$



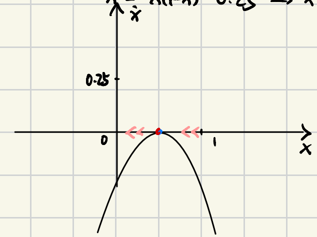
② $h = 0.1$

$\dot{x} = x(1-x) - 0.1$



③ $h = 0.25$

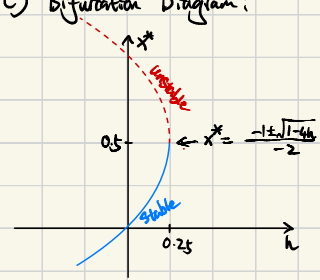
$\dot{x} = x(1-x) - 0.25 \rightarrow x^* = 0.5$



$\dot{x} = -x^2 + x - h$

$$x^* = \frac{-1 \pm \sqrt{1-4h}}{-2}$$

c) Bifurcation Diagram:



There's a saddle node bifurcation at $h_c = 0.25$

d) For $h > h_c$, the population will face extinction.

For $h < h_c$, extinction may still happen by reasons other than fishery, such as polution.

3.4.8 $\dot{x} = rx - \dfrac{x}{1+x^2}$

Note from the graph we know $\boxed{\text{subcritical pitchfork bifurcation}}$ happens at $r = 1$.

First to find the fixed points we have:

$$\dot{x} = rx - \frac{x}{1+x^2} = 0$$

$$rx(1+x^2) - x = 0$$

$$x(r + rx^2 - 1) = 0$$

$$\Rightarrow x^2 = \frac{1}{r} - 1 \rightarrow x = \pm\sqrt{\frac{1}{r} - 1}$$

$x = 0$ — fixed points.

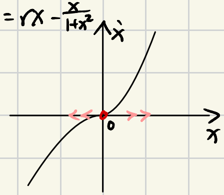
Note $\frac{1}{r} - 1 \geq 0$ for x_2^*, x_3^* to exists:

$$\frac{1}{r} \geq 1 \quad \rightarrow 0 < r \leq 1$$

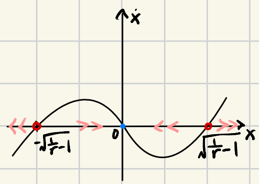
Vector field diagram:

① $r \geq 1$

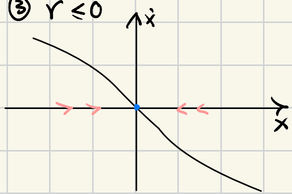
$\dot{x} = rx - \frac{x}{1+x^2}$



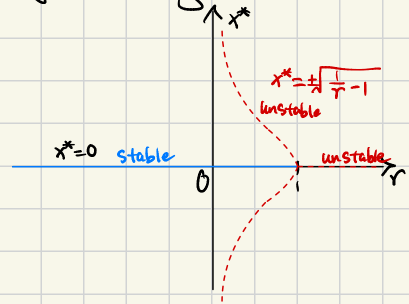
② $0 < r < 1$



③ $r \leq 0$



Bifurcation Diagram:



Expansion:

Note the tylor series expansion for $(1+x^2)^{-1}$ is:

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - \cdots$$

$$-(1+x^2)^{-1} = -1 + x^2 - x^4 \cdots$$

Then we will have:

$$f(x, r) = rx - x(1+x^2)^{-1} = rx - x + x^3 - x^5 \cdots$$

$$= \boxed{(r-1)x + x^3 - O(x^5)}$$

Match the normal form for subcritical pitchfork bifurcation