AMATH 502 HW3

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1. For each of the following vector fields, find and classify all the fixed points, and sketch the phase portrait on the circle.

   (a) $\dot{\theta} = 1 + 2\cos\theta$.

   (b) $\dot{\theta} = \sin\theta + \cos\theta$.

   (c) $\dot{\theta} = \sin 4\theta$.

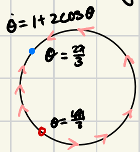
a) i. To identify the fixed points we have:

$\dot{\theta} = 1 + 2\cos\theta = 0 \longrightarrow 2\cos\theta = -1 \longrightarrow \cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ are f.p. for $\theta \in [0, 2\pi)$

Note $\dot{\theta} > 0$ for $\theta \in [0, \frac{2\pi}{3})$ and $(\frac{4\pi}{3}, 2\pi]$

$\dot{\theta} < 0$ for $\theta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$



$\theta = \frac{2\pi}{3}$ stable f.p.

$\theta = \frac{4\pi}{3}$ unstable f.p.

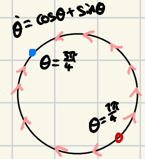
b) i. To identify the fixed points we have:

$\dot{\theta} = \sin\theta + \cos\theta = 0 \longrightarrow \cos\theta = -\sin\theta$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ are f.p. for $\theta \in [0, 2\pi)$

Note $\dot{\theta} > 0$ for $\theta \in (\frac{7\pi}{4}, 2\pi]$ and $[0, \frac{3\pi}{4})$

$\dot{\theta} < 0$ for $\theta \in (\frac{3\pi}{4}, \frac{7\pi}{4})$



$\theta = \frac{3\pi}{4}$ stable f.p.

$\theta = \frac{7\pi}{4}$ unstable f.p.

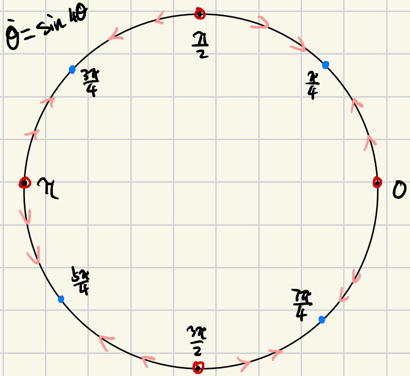
c). i. To identify the fixed points we have:

$\dot{\theta} = \sin 4\theta = 0 \longrightarrow \theta = \frac{n\pi}{4}$ for $n \in \mathbb{N}$

$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$ are f.p. for $\theta \in [0, 2\pi)$

Then note: $\dot{\theta} > 0$ for $\theta \in (0, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{3\pi}{4}), (\pi, \frac{5\pi}{4}), (\frac{3\pi}{2}, \frac{7\pi}{4})$

$\dot{\theta} < 0$ for $\theta \in (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{3\pi}{4}, \pi), (\frac{5\pi}{4}, \frac{3\pi}{2})$



$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ stable

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ unstable

2. (You can read more about the background of this problem in Strogatz, Section 4.5).

Fireflies are known to synchronize their flashing in certain settings. Before beginning this problem, I highly recommend watching a video of this incredible phenomena. The synchronization does not occur immediately or randomly. When the fireflies first group, they are out of sync. As the night progresses, they begin to sync up and flash in unison. The key to this behavior is that the fireflies influence one another. When a firefly synchronizes its flashing to an external stimulus (potentially another firefly), this is called *entrainment*. Hanson (1978) showed that fireflies would become entrained to external stimulus (a flashing light) if that stimulus was close enough to the firefly's natural flashing period (about 0.9 seconds).

Suppose we have an external periodic stimulus whose phase θ_s satisfies

$$\dot{\theta}_s = \Omega,$$

for some constant Ω. The stimulus flashes when $\theta_s = 0$. If the stimulus is flashing before the firefly, the firefly speeds up its flashing to match the stimulus. If the stimulus is flashing after the firefly, the firefly slows down to try to match the stimulus. In this problem we model the phase of the firefly's flashing, θ_f, as

$$\dot{\theta}_f = \omega + Af(\theta_s - \theta_f),$$

where $A > 0$, ω is the natural flashing frequency of the firefly, and f is a somewhat arbitrary function for us to define. In this problem we will take $\theta_s \in [-\pi, \pi)$ with $-\pi$ and π being identified to form a circle. This allows the flashing to occur in the middle of our symmetric domain. For $|\phi| < \pi$, define

$$f(\phi) = \begin{cases} 2\phi, & |\phi| \leq \pi/2 \\ 2\text{sgn}(\phi)\pi - 2\phi, & |\phi| > \pi/2, \end{cases} \tag{1}$$

with $f(\phi)$ defined periodically outside of the interval $[-\pi, \pi)$.

(a) Graph $f(\phi)$ on the domain $-\pi \leq \phi < \pi$.

(b) Write the dynamical system for the phase difference $\phi = \theta_s - \theta_f$ in terms of the dimensionless time $\tau = At$ and the dimensionless parameter $\mu = (\Omega - \omega)/A$.

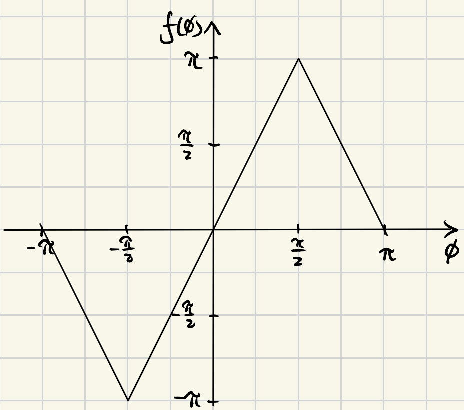
(c) Find the values of μ for which the firefly will be phase-locked to the stimulus i.e. any values for which the firefly approaches the same frequency as the stimulus. Note that the phase does not need to be the same in order to be phase-locked, only the frequency needs to be the same (e.g. the firefly does not need to flash at the same time as the stimulus, but it does need to flash with the same frequency).

(d) Using the definition of μ and your answer from part (c), find the range of frequencies of the stimulus Ω for which the firefly will be phase-locked to the stimulus. This is called the range of entrainment.

(e) What kind of bifurcation occurs at $\mu = \pm\pi$? Does this look like the usual form for this type of bifurcation? If not, why not?

(f) Assuming the firefly is phase-locked to the stimulus, find a formula for the phase difference ϕ^* (i.e. the stable fixed point).

a).



b) Let $\tau = At$, $\mu = (\Omega - \omega)/A$

$\Rightarrow d\tau = A\,dt \rightarrow \frac{d\tau}{dt} = A$

Then note we have:

$\phi = \theta_s - \theta_f \rightarrow \dot{\phi} = \dot{\theta}_s - \dot{\theta}_f$

$\Rightarrow \dot{\phi} = \Omega - \omega - Af(\phi)$

Then note: $\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \dot{\phi} \cdot \frac{1}{A}$

$\Rightarrow \frac{d\phi}{d\tau} = \frac{1}{A}(\Omega - \omega) - f(\phi) = \boxed{\mu - f(\phi)}$

c) Note that to be phase-locked we have to have ϕ must not change over time.

$\Rightarrow \frac{d\phi}{d\tau} = 0 = \mu - f(\phi)$

$\Rightarrow \mu = f(\phi)$

Note for $|\phi| \leq \frac{\pi}{2}$, $\max(f(\phi)) = \pm\pi$

$|\phi| > \frac{\pi}{2}$, $\max(f(\phi)) \pm\pi$

Thus $\mu \in [-\pi, \pi]$

d) Note $\mu A = \Omega - \omega \Rightarrow \Omega = \omega + \mu A$

Thus we have:

$\Omega \in [\omega - \pi A, \omega + \pi A]$

e) This is a saddle-node bifurcation.

f) Note for firefly and the stimulus phaselocked.

We have: $\dot{\phi} = 0$

$\Rightarrow \Omega - \omega - Af(\phi^*) = 0$

Then we have $\boxed{f(\phi^*) = \frac{\Omega - \omega}{A}}$

3. Plot the phase portrait and classify the fixed point of the following linear systems. Put the system in matrix form.

(a) $\dot{x} = y, \quad \dot{y} = -2x - 3y.$

(b) $\dot{x} = 3x - 4y, \quad \dot{y} = x - y.$

(c) $\ddot{x} + 2\dot{x} - x = 0.$

a) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \rightarrow Ax = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = -\lambda(-3-\lambda) + 2 = 3\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

$\lambda_1 = -1$ we have: $(A - \lambda_1 I)v = \vec{0}$

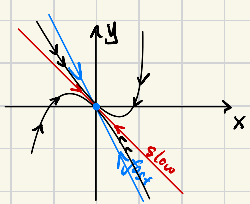
$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = -2$ we have: $(A - \lambda_2 I)v = \vec{0}$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2v_1 = -v_2 \Rightarrow v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

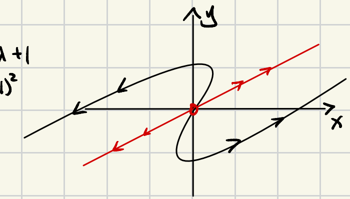
Note $\lambda_2 < \lambda_1 < 0$, then by 1.2 in Lecture 8 we have:
we know that the f.p. $(0,0)^T$ is a $\boxed{\text{stable node}}$

Note



b) $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 4 = -3 + \lambda - 3\lambda + \lambda^2 + 4 = \lambda^2 - 2\lambda + 1$$
$$= (\lambda - 1)^2$$



$\lambda = 1$ we have $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Since $\lambda_1 = \lambda_2 = 1 > 0$, by 1.4 in lecture 8 we have:
f.p. $(0,0)^T$ is $\boxed{\text{unstable}}$

c) Let $y = \dot{x} \Rightarrow \dot{y} = \ddot{x}$

we have: $\dot{y} = -2y + x$, $\dot{x} = y$

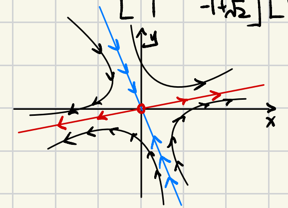
$$\begin{cases} \dot{x} = y \\ \dot{y} = -2y + x \end{cases}$$

$\lambda^2 + 2\lambda - 1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) - 1 = 2\lambda + \lambda^2 - 1 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$\lambda_1 = -1 + \sqrt{2}$, $\begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$ 0.4142

$\lambda_2 = -1 - \sqrt{2}$, $\begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}$ -2.4142



Note since $\lambda_1 > 0$, $\lambda_2 < 0$ we have:
$\boxed{(0,0)^T \text{ is unstable}}$

4. Consider an LRC circuit, which contains a capacitor, resistor, and inductor. The voltage across a capacitor is $V = Q/C$ where Q is the charge on the capacitor, and $C > 0$ is the capacitance. Similarly, the voltage across an inductor is given by $V = LdI/dt$, where $L > 0$ is the inductance. Finally, recall Ohm's law, namely that, for current flowing through a resistor, $V = IR$, where $R \geq 0$ is the resistance of that resistor. Putting all of this together with Kirchoff's loop rule (sum of voltage changes over any closed path is zero), we get to

$$Q/C + IR + LdI/dt = 0.$$

If we take a time-derivative of the above equation, and recall that $dQ/dt = I$ (current is flowing charge), we get to the equation of interest in this problem

$$L\ddot{I} + \dot{I}R + I/C = 0.$$

(a) Rewrite the second-order differential equation as a 2-D linear dynamical system

(b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

(c) Classify the fixed point at the origin depending on whether $R^2C - 4L$ is positive, negative, or zero.

a) Let $x_1 = I$, $x_2 = \dot{I}$

Then we have: $\dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{I}$

Then plug x_1, x_2 into the equation we have:

$L\dot{x}_2 + x_2 R + x_1/C = 0$

$\Rightarrow \dot{x}_2 = -\frac{R}{L}x_2 - \frac{x_1}{LC}$

Thus we have the system:

$$\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{R}{L}x_2 - \frac{x_1}{LC} \end{cases}}$$

b). Note we have $A = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix}$

Then note we have:

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L}-\lambda \end{vmatrix} = -\lambda\left(-\frac{R}{L}-\lambda\right) + \frac{1}{LC} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

$$\Rightarrow \lambda = \frac{-\frac{R}{L} \pm \sqrt{R^2/L^2 - \frac{4}{LC}}}{2}$$

① $R=0$

Note if $R=0$ we have

$R^2/L^2 - 4/LC < 0$, since $L, C > 0$

Then we have complex roots for λ_1, λ_2

Then $\alpha = \text{Real}(\lambda_1) = \text{Real}(\lambda_2) = 0$

Then by Theorem 2.1 from lecture we have f.p. origin is naturally stable

② $R > 0$

Case ①: $R^2/L^2 = \frac{4}{LC}$

$\lambda_1 = \lambda_2 = -\frac{R}{2L} < 0 \Rightarrow$ stable

Case ②: $R^2/L^2 < \frac{4}{LC}$

λ_1 and λ_2 are complex $\Rightarrow \alpha = \text{Real}(\lambda_1) = \text{Real}(\lambda_2) = -\frac{R}{2L} < 0 \Rightarrow$ stable

Case ③: $R^2/L^2 > \frac{4}{LC}$

Note $\frac{R}{L} = \sqrt{R^2/L^2} > \sqrt{R^2/L^2 - 4/LC} \Rightarrow \lambda_1, \lambda_2 < 0$ and $\lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow$ stable

Thus asymptotically stable

c) Note $R^2C-4L=0$

$$\Rightarrow \quad R^2-\frac{4L}{C}=0$$

$$\frac{R^2}{L^2}-\frac{4}{LC}=0$$

Same for positive and negative since $L, C>0$

① $R^2C-4L=0$

$\lambda_1=\lambda_2=-\frac{R}{2L}$

Case ①: $R>0$, $\boxed{\text{origin is stable}}$

Case ②: $R<0$, $\boxed{\text{origin is unstable}}$

② $R^2C-4L>0$

$\frac{R}{L}=\sqrt{R^2/L^2}>\sqrt{R^2/L^2-4/LC} \Rightarrow \lambda_1, \lambda_2<0$

$\boxed{\text{Origin is stable}}$.

③ $R^2C-4L<0$

λ_1, λ_2 are complex

$\alpha=\text{Real}(\lambda_1)=\text{Real}(\lambda_2)$

① $R<0$, $\alpha>0$

$\boxed{\text{origin unstable}}$

② $R>0$, $\alpha<0$

$\boxed{\text{origin stable}}$

5. (You can read more about the background of this problem in Strogatz, Section 5.3)

In this problem we are going to model the feelings of affection between R and J. Let

$$R(t) = \text{R's feelings for J at time } t,$$
$$J(t) = \text{J's feelings for R at time } t.$$

When R or J is positive, the feeling is a positive feeling (endearment). When R or J is negative, the feeling is a negative feeling (animus). Consider the following dynamical system governing the relationship between R and J,

$$\dot{R} = aR + J$$
$$\dot{J} = -R - aJ,$$

where $a > 0$ is a constant.

(a) (Not graded) Try to explain how R and J respond to their own feelings and to each other's feelings. What role does the parameter a play? This should be a text-response, not calculations.

(b) Do the following for $a > 1$ and $a < 1$.

i. Determine whether the origin is asymptotically stable, neutrally stable, unstable and attracting, or unstable and not attracting. Also classify the origin as a saddle point, node, center, or spiral. If the origin is a saddle point, identify (either by highlighting or explaining with words/formulae) the stable and unstable manifolds.

ii. Verify what you found in part (i) with a plot of the phase portrait (using pplane). You may choose a particular value of a for plotting. Interpret the long-time behavior of the system in terms of R and J's relationship.

(c) Summarize and interpret what you found in part (b). How does the size of a play out in J and R's relationship?

a) Note that R has a positive coefficients means that R's own feeling will grow more endearments over time. However for J, J will grow the feeling of animus overtime.

For each others feeling, R's positive feeling increases as J's positive feeling increases. And J's positive feeling decreases as R's positive feeling increases.

The role of a act as the rate where there own feelings amplify (for R) or dampen (for J)

b) i) We have the following matrix:

$$A = \begin{bmatrix} a & 1 \\ -1 & -a \end{bmatrix}$$

$$\begin{vmatrix} a-\lambda & 1 \\ -1 & -a-\lambda \end{vmatrix} = (a-\lambda)(-a-\lambda)+1 = -a^2+a\lambda-a\lambda+\lambda^2+1$$
$$= \lambda^2 - a^2 + 1$$
$$\Rightarrow \lambda = \pm\sqrt{a^2-1}$$

① $a > 1$

We have $\lambda_1 > 0$, $\lambda_2 < 0$.

Then by 1.3 from Lecture 8 we have:

$\boxed{\text{Origin is a unstable saddle point with unstable manifolds.}}$

Because as time moves on the trajectories move away from the origin. However as we go backwards it will converge to origin.

② $a < 1$

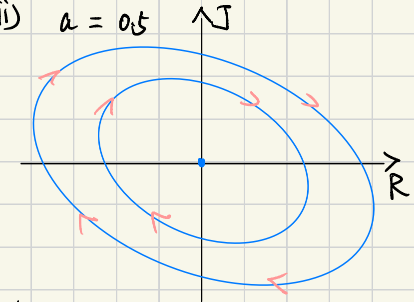
Which means that $a^2 - 1 < 0$

$\Rightarrow \lambda_1, \lambda_2$ are complex

$\sigma = \text{real}(\lambda_1) = \text{real}(\lambda_2) = 0$

Then origin is a $\boxed{\text{neutrally stable center.}}$

ii) $a = 0.5$

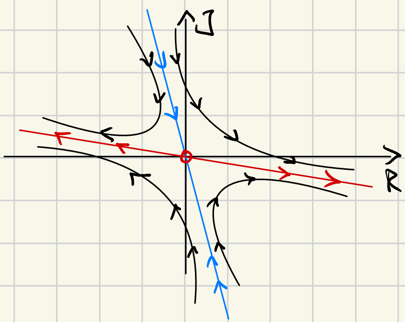


This plot shows that for $a<1$, there's a closed orbit around the origin.
Means that a oscillatory relationship between R and J.
There are no certain affection dominates in the long term.

$a = 2$

$\lambda_1 = \sqrt{3}$, $\lambda_2 = -\sqrt{3}$

$V_1 = \begin{bmatrix} 1 \\ \sqrt{3}-2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -\sqrt{3}-2 \end{bmatrix}$



Note this graph shows that the trajectories are moving away from the origin. Hence the relationship is unstable, small distance from origin caus the significant departures.

c) I found that for $a>1$ we have the relationship unstable.
Small changes will have huge effect to the relationship.
And R, J may experience increasing affecton, animus.

for $a<1$, we have the relationship stable.
Where they enter a cycle where feelings don't grow or decay over-time.