AMATH 502 HW5

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1. Decide whether the system $\dot{x} = 2x$, $\dot{y} = 8y$ is a gradient system. If so, find V and sketch the phase portrait.

① First note let $f(x,y) = 2x$, $g(x,y) = 8y$

Then we have:

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial x}$$

Thus this is a gradient system.

② Then to compute V we have:

$$-\frac{\partial V}{\partial x} = 2x \implies -V(x,y) = \int 2x\,dx + g(y) = x^2 + g(y)$$

$$-\frac{\partial V}{\partial y} = 8y \implies -V(x,y) = \int 8y\,dy + f(x) = 4y^2 + f(x)$$

Then combine we have:

$$\boxed{V(x,y) = -x^2 - 4y^2}$$

③ To find the fixed points set $\dot{x}, \dot{y} = 0$

$\implies (0,0)$ is the fixed point.

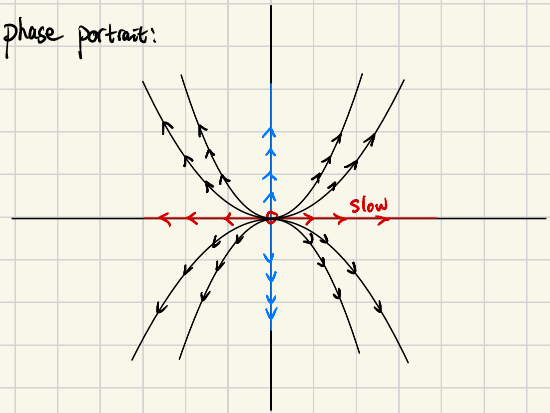
④ We have the Jacobian matrix:

$$J = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \to \lambda_1 = 2,\ \lambda_2 = 8$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have unstable node

phase portrait:



2. Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

Let $V = ax^2 + by^2$ then we have:

$\dot{V} = 2ax\dot{x} + 2by\dot{y}$

Then substitute $\dot{x}$ and $\dot{y}$ we have:

$\dot{V} = 2ax(y-x^3) + 2by(-x-y^3)$

$= 2axy - 2ax^4 - 2byx - 2by^4$

$= 2xy(a-b) - 2(ax^4 + by^4)$

Set $a = b = 1$ we have:

$\dot{V} = -2(x^4+y^4)$

Note $x^4+y^4 > 0$ for $\forall\, x, y$ except at fixed point $(0,0)$

$\Rightarrow \dot{V} = -2(x^4+y^4) < 0$ except $(0,0)$

Thus V is a Liapunov function and there could not be closed orbits. $\square$

3. Show that the dynamical system

$$\dot{x} = ye^{-2x} + 3$$
$$\dot{y} = -e^{-2x}(x + y - x^2 - y^2)$$

has no closed trajectories. **Hint:** Consider the Bendixon's theorem.

Let $f(x,y) = ye^{-2x} + 3$
$g(x,y) = -e^{-2x}(x+y-x^2-y^2)$

Then note we have:

$$\frac{\partial f}{\partial x} = -2ye^{-2x}$$
$$\frac{\partial f}{\partial y} = -e^{-2x} + 2ye^{-2x}$$

Then we have:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -e^{-2x} \neq 0$$

Then note by definition we have:

$$e^{-2x} > 0 \implies -e^{-2x} < 0$$

Thus $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ does not change sign.

Hence by Bendixon's Theorem the dynamical system has no closed trajectories. $\square$

4. (Based on Strogatz 7.3.1) Consider $\dot{x} = x - y - x(x^2+5y^2)$, $\dot{y} = x + y - y(x^2+y^2)$.

(a) Classify the fixed point at the origin.

(b) Rewrite the system in polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2}$. **Hint:** Note $-6r^4\cos^2(\theta)\sin^2(\theta) = -r^4\cos^2(\theta)\sin^2(\theta) - r^4\cos^2(\theta)\sin^2(\theta) - 4r^4\cos^2(\theta)\sin^2(\theta)$. You should end up with the following expression for r, $\dot{r} = r - r^3 - r^3\sin^2(2\theta)$.

(c) Determine the circle of maximum radius r_1 centered on the origin such that all trajectories have a radially outward components on it.

(d) Determine the circle of minimum radius r_2 centered on the origin such that all trajectories have a radially inward component on it.

(e) Show that the trapping region does not contain any fixed points. So, the system has a limit cycle somewhere in the trapping region $r_1 \le r \le r_2$.

a). we have: $\begin{cases} \dot{x} = x - y - x^3 - 5xy^2 \\ \dot{y} = x + y - x^2y - y^3 \end{cases}$

Then we have the Jacobian matrix:

$$J = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$$

Then we have:

$$J_{(0,0)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rightarrow (1-\lambda)^2+1=0 \rightarrow 1-2\lambda+\lambda^2+1=0$$

$$\Rightarrow \lambda^2-2\lambda+2=0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm 2i$$

Since $\text{Real}(\lambda_1) = \text{Real}(\lambda_2) = 1 > 0$

We have $\boxed{\text{unstable spiral for } (0,0)}$

b) Note we have:

$x = r\cos(\theta)$, $y = r\sin(\theta)$

plug in to $\dot{x}$ and $\dot{y}$ we have:

$$\dot{x} = r\cos(\theta) - r\sin(\theta) - r\cos\theta(r^2\cos^2\theta + 5r^2\sin^2\theta)$$
$$= r\cos(\theta) - r\sin(\theta) - r\cos\theta(r^2(\cos^2\theta + \sin^2\theta) + 4r^2\sin^2\theta)$$
$$= r\cos(\theta) - r\sin(\theta) - r^3\cos\theta - 4r^3\cos\theta\sin^2\theta$$

$$\dot{y} = r\cos(\theta) + r\sin(\theta) - r\sin\theta(r^2(\cos^2\theta + \sin^2\theta))$$
$$= r\cos(\theta) + r\sin(\theta) - r^3\sin\theta$$

Then plug in $r\dot{r}$ we have:

$$r\dot{r} = r^2\cos^2\theta - r^2\cancel{\cos\theta\sin\theta} - r^4\cos^2\theta - 4r^4\cos^2\theta\sin^2\theta + r^2\cancel{\sin\theta\cos\theta} + r^2\sin^2\theta - r^4\sin^2\theta$$
$$= r^2 - r^4(\cos^2 + \sin^2\theta) - 4r^4\cos^2\theta\sin^2\theta$$
$$= r^2 - r^4 - r^4\sin^2(2\theta)$$
$$\Rightarrow \boxed{\dot{r} = r - r^3 - r^3\sin^2(2\theta)}$$

Then plug in $\dot{\theta}$ we have:

$$\dot{\theta} = r^{-2}\left[r^2\cos^2\theta + r^2\cancel{\cos\theta\sin\theta} - r^4\cancel{\cos\theta\sin\theta} - r^2\cancel{\cos\theta\sin\theta} + r^2\sin^2\theta + r^4\cancel{\cos\theta\sin\theta} + 4r^4\cos\theta\sin^3\theta\right.$$
$$= \boxed{1 + 4r^2\cos\theta\sin^3\theta}$$

c) First note we have:

$\dot{r} = r - r^3 - r^3\sin^2(2\theta) > 0$

Then note the effect of $\sin^2(2\theta)$ is maximized at

$\sin^2(2\theta) = 1$.

Thus set $\sin^2(2\theta) = 1$ we have

$\dot{r} = r - 2r^3$

Set $\dot{r} = 0$ we have:

$r(1-2r^2) = 0$

$r = 0,\ r = \pm\sqrt{\frac{1}{2}}$

Then we have:

$0 < r < \sqrt{\frac{1}{2}} \implies \dot{r} > 0$

Then we have $\boxed{r_1 = \sqrt{\frac{1}{2}} - \varepsilon}$ for small positive ε

d) Note we have

$\dot{r} = r - r^3 - r^3\sin^2(2\theta) < 0$

Then we want to find $\sin^2(2\theta)$ where the effect in minimum.

Choose $\sin^2(2\theta) = 0$ we have:

$\dot{r} = r - r^3 = r(1-r^2) < 0$

$\implies r > 1$

Set $\boxed{r_2 = 1 + \varepsilon}$ for small positive ε

e) Note our fixed point is $(0,0)$

Which is not in the trapping region.

Thus by Bendixson's Theorem.

A limited cycle exists in $r_1 \leq r \leq r_2$. $\square$