AMATH 502 HW6

Tianbo Zhang 1938501

Problem 1

$\ddot{x} + x + \varepsilon(x^2-1)\dot{x}^3 = 0$ (1)

a) Then note we have:

$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$\Rightarrow x(t,\varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2)$

Then from lecture we have:

$\dot{x} = \frac{\partial x_0}{\partial \tau} + \varepsilon \frac{\partial x_0}{\partial T} + \varepsilon \frac{\partial x_1}{\partial \tau} + O(\varepsilon^2)$ ①

$\ddot{x} = \frac{\partial^2 x_0}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x_0}{\partial \tau \partial T} + \varepsilon \frac{\partial^2 x_1}{\partial \tau^2} + O(\varepsilon^2)$ ②

Then note we have:

$\varepsilon(x^2-1) = \varepsilon(x_0^2 + 2\varepsilon x_1 + \varepsilon^2 x_1^2 - 1) = \varepsilon(x_0^2-1) + O(\varepsilon^2)$

Then note after plug in the expression ① and ② into (1).

Then only terms of order ε^0 are:

$$\boxed{\frac{\partial^2 x_0}{\partial \tau^2} + x_0 = 0}$$

Then note the solution is:

$$\boxed{x_0 = R(T)\cos(\tau + \phi(T))}$$

b) Note collect power of ε^1 we have:

$$2\frac{\partial^2 x_0}{\partial \tau \partial T} + \frac{\partial^2 x_1}{\partial \tau^2} + x_1 + \left(\frac{\partial x_0}{\partial \tau}\right)^3 (x_0^2 - 1) = 0$$

$$\Rightarrow \frac{\partial^2 x_1}{\partial \tau^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \tau \partial T} - (x_0^2-1)\left(\frac{\partial x_0}{\partial \tau}\right)^3$$

Then plug in x_0 we have:

$$\frac{\partial^2 x_1}{\partial \tau^2} + x_1 = 2[R'\sin(\tau+\phi(T)) + R\phi'(T)\cos(\tau+\phi(T))] + R^3\sin^3(\tau+\phi)[R^2\cos^2(\tau+\phi(T)) - 1]$$

Note let $y = \tau + \phi(T)$ we have:

$$R^3\sin^3(y)[R^2\cos^2(y)-1] = -\frac{R^3}{16}[(12-2R^2)\sin(y) - (4+R^2)\sin(3y) + R^2\sin(5y)]$$

$$\Rightarrow \frac{\partial^2 x}{\partial \tau^2} + x_1 = \sin(y)\left[2R' - \frac{R^3}{16}(12-2R^2)\right] - \frac{R^3}{16}(4+R^2)\sin(3y) + \frac{R^3}{16}R^2\sin(5y) + 2R\phi'\cos(y)$$

Then note to avoid secular terms we have:

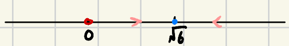
① $2R\phi' = 0 \Rightarrow \phi' = 0 \Rightarrow \phi(T) = \phi_0$

② $R = 0$

③ $2R' - \frac{R^3}{16}(12-2R^2) = 0$

$\Rightarrow R' = \frac{R^3}{16}(6-R^2)$

$\Rightarrow R = 0,\ R = \pm\sqrt{6}$



Note we have $R^* = \sqrt{6}$ is stable

$\Rightarrow R(T) \rightarrow \sqrt{6}$ as $T \rightarrow \infty$

$\Rightarrow x(t) = x_0 + \varepsilon x_1 + O(\varepsilon^2) = \sqrt{6}\cos(t+\phi_0) + O(\varepsilon)$

Thus $x(t)$ approaches a stable limit cycle of radius $r = \sqrt{6} + O(\varepsilon)$

Then the frequency is: $f = \varepsilon^2 t$

Problem 2

$\dot{x} = x(y-1)$

$\dot{y} = -y(x+a)+b$

where $a, b > 0$

Claim: Bifurcation occurs at $b=a$.

① Find fixed points:

$\dot{x} = x(y-1) = 0 \longrightarrow x=0,\ y=1$

$\dot{y} = -y(x+a)+b = 0$

i) Plug in $x=0$ into $\dot{y}$:

$\dot{y} = -ya+b=0$

$\Rightarrow y = \frac{b}{a}$

ii) Plug in $y=1$ into $\dot{y}$:

$\dot{y} = -x-a+b=0$

$\Rightarrow x = b-a$

We have fixed point: $(0, \frac{b}{a})$

Then note we have the Jacobian matrix:

$$J = \begin{bmatrix} y-1 & x \\ -y & -x-a \end{bmatrix}$$

Then we have:

$$J_{(0,\frac{b}{a})} = \begin{bmatrix} \frac{b}{a}-1 & 0 \\ -\frac{b}{a} & -a \end{bmatrix} \Rightarrow \left(\frac{b}{a}-1-\lambda\right)(-a-\lambda)=0$$

$\Rightarrow \lambda_1 = \frac{b}{a}-1,\ \lambda_2 = -a$

Then note the sign of λ_1 changes with a, b:

when $a>b \Rightarrow \lambda_1 < 0$

$a<b \Rightarrow \lambda_1 > 0$

Thus a bifurcation happens at $a=b$.

Note since $a>0$ we have:

$\lambda_2 < 0$

Then when $a>b \Rightarrow \lambda_1 < 0$

We have a stable fixed point.

When $a<b \Rightarrow \lambda_1 > 0$

We have a unstable fixed point.

Thus we have $\boxed{\text{saddle-node bifurcation}}$

Problem 3

$$\dot{x} = -ax + y + x(x^2+y^2) - a\frac{x^3}{\sqrt{x^2+y^2}}$$
$$\dot{y} = -x - ay + y(x^2+y^2) - a\frac{x^2y}{\sqrt{x^2+y^2}}$$

a) First note we have the Jacobian matrix:

$$J = \begin{bmatrix} -a+3x^2+y^2 - a(2x^4+3y^2x^2)/(x^2+y^2)^{\frac{3}{2}} & 1+2yx - ax^3y/(y^2+x^2)^{\frac{3}{2}} \\ -1+2xy - a(yx^3+2y^3x)/(x^2+y^2)^{\frac{3}{2}} & -a+x^2+3y^2 - ax^4/(y^2+x^2)^{\frac{3}{2}} \end{bmatrix}$$

$$\Rightarrow J_{(0,0)} = \begin{bmatrix} -a & 1 \\ -1 & -a \end{bmatrix} \Rightarrow (-a-\lambda)^2+1 = a^2+2a\lambda+\lambda^2+1 = \lambda^2+2a\lambda+a^2+1$$

$$\lambda = \frac{-2a \pm\sqrt{4a^2-4a^2-4}}{2} = -a \pm i$$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{if } a>0 \text{ then } (0,0) \text{ is unstable} \\ \text{if } a<0 \text{ than } (0,0) \text{ is stable}\end{array}}$

b) Hopf bifurcation may occur at $a_c=0$
Sice $(0,0)$ changes stability.

c) Let $f(x,y) = \dot{x}$ and $g(x,y)=\dot{y}$

Then $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -2a+4x^2+4y^2 - a(3x^4+3y^2x^2)/(y^2+x^2)^{\frac{3}{2}} \neq 0$

$= -2a + 4x^2 + 4y^2 - 3ax^2(x^2+y^2)\cdot(x^2+y^2)^{-\frac{3}{2}}$

$= -2a + 4(x^2+y^2) - 3ax^2(x^2+y^2)^{-\frac{1}{2}}$

Note for $a<0$:

Since all x and y are to the even powers and $-2a>0$, $-a>0$

$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0$ and the sign is consistent

$\Rightarrow$ $\boxed{\text{There are no closed trajectories for } a<0}$

d) Let $r_1^2 = |a_c + a| = |a|$

$r_2^2 = |a_c + a + 1| = |a+1|$

Let $a > a_c \Rightarrow a>0$

Then we have:

① $r_1^2 = |a|$

$\dot{r} = r_1^3 - r_1^3 - ax^2$ $\qquad r_1^3 - ar_1^2 - ax^2 > 0$

$= -ax^2$

Note $a>0$ and $x^2>0$

$\Rightarrow \dot{r} < 0$

Problem 3

$$\dot{x} = -ax + y + x(x^2+y^2) - a\frac{x^3}{\sqrt{x^2+y^2}}$$
$$\dot{y} = -x - ay + y(x^2+y^2) - a\frac{x^2y}{\sqrt{x^2+y^2}}$$
$$\dot{r} = r^3 - ar - ax^2$$

a) Note $x = r\cos(\theta)$, $y = r\sin(\theta)$

Then we have:

$$\dot{r} = r^3 - ar - ar^2\cos^2(\theta)$$

Then note from HW5 we have:

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = r^{-2}\Big[-x^2 - \cancel{axy} + xy(x^2+y^2) - ax^3y/\sqrt{x^2+y^2}$$
$$+\cancel{axy} - y^2 - xy(x^2+y^2) + ax^3y/\sqrt{x^2+y^2}\Big]$$
$$= -r^{-2}(x^2+y^2) = -r^{-2}r^2 = -1$$

Then we have the Jacobian matrix:

$$J = \begin{bmatrix} 3r^2 - a - 2ar\cos^2(\theta) & 2ar^2\cos(\theta)\sin(\theta) \\ 0 & 0 \end{bmatrix}$$

Then note for $x=y=0$ we have:

$r=0$, $\theta=\theta$

$$\Rightarrow J_{(0,\theta)} = \begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$$

$\Longrightarrow \lambda_1 = -a$, $\lambda_2 = 0$

$\Longrightarrow$ $\boxed{\begin{array}{l}\text{if } a>0 \text{ then } (0,0) \text{ is stable} \\ \text{if } a<0 \text{ then } (0,0) \text{ is unstable}\end{array}}$

b) Hopf bifurcation may occur at $a_c=0$ since $(0,0)$ changes stability.

c) Let $f(r,\theta)=\dot{r}$, $g(r,\theta)=\dot{\theta}$

$\Rightarrow \frac{\partial f}{\partial r} = 3r^2 - a - 2ar\cos^2(\theta)$

$\frac{\partial g}{\partial \theta} = 0$

$\frac{\partial f}{\partial r} + \frac{\partial g}{\partial \theta} = 3r^2 - a - 2ar\cos^2(\theta) \neq 0$

Then note r by definition: $r \geq 0$

Then set $a < a_c \Rightarrow a < 0$

$\Rightarrow \frac{\partial f}{\partial r} + \frac{\partial g}{\partial \theta} \geq 0$

$\Rightarrow$ The sign is consistent.

$\Rightarrow$ D does not contain hole.

$\Rightarrow$ $a<0$ does not contain closed trajectories.

d) Let $r_1^2 = |a_c + a| = |a|$

Let $a > a_c \Rightarrow a > 0$

Then we have:

① $r_1^2 = |a|$

$\dot{r} = r_1^3 - r_1^3 - ax^2$

$= -ax^2$

Note $a>0$ and $x^2>0$

$\Rightarrow \dot{r} < 0$

Let $r_2 = |a_c + a + 1| = |a+1|$

Let $a > a_c \Rightarrow a > 0$

Then we have:

② $r_2 = |a+1|$

$\dot{r} = (a+1)^3 - a(a+1)^2 - ax^2$

$= (a+1-a)(a+1)^2 - ax^2$

$= (a+1)^2 - ax^2$

Note $r^2 = x^2+y^2 = (a+1)^2 \geq x^2$

$\Rightarrow \dot{r} > 0$.

We formed the trapping zone where it does not contain fixed point.

Hence by P.B. Theorem there must be closed trajectory for $a > 0$.

e) Note from part a we have:

the fixed point which is unstable than gained stability.

Thus it's a $\boxed{\text{subcritical bifurcation}}$