AMATH 503 HW3

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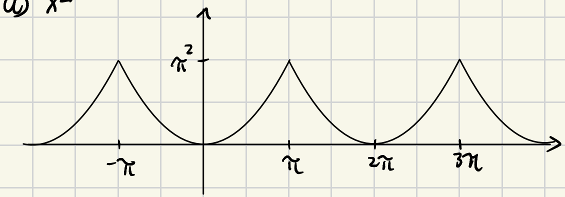
3.2.6. Graph the 2π–periodic extension of each of the following functions. Which extensions are continuous? Differentiable? (a) x^2, (b) $(x^2-\pi^2)^2$, (c) e^x, (d) $e^{-|x|}$, (e) $\sinh x$, (f) $1+\cos^2 x$; (g) $\sin \frac{1}{2}\pi x$, (h) $\dfrac{1}{x}$, (i) $\dfrac{1}{1+x^2}$.

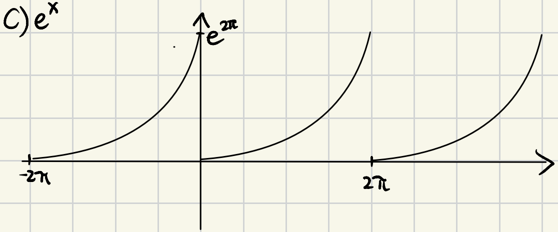
a) x^2



Continuous

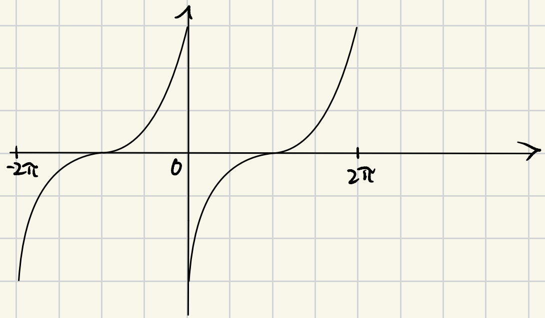
Not differentiable due to the jumps

c) e^x



Not continuous and not differentiable.

e) $\sinh(x)$



Not continuous

Not differentiable

3.2.14. Find the discontinuities and the jump magnitudes for the following piecewise continuous functions:

(a) $2\sigma(x) + \sigma(x+1) - 3\sigma(x-1)$, (b) $\operatorname{sign}(x^2 - 2x)$, (c) $\sigma(x^2 - 2x)$, (d) $|\,x^2 - 2x\,|$,
(e) $\sqrt{|\,x-2\,|}$, (f) $\sigma(\sin x)$, (g) $\operatorname{sign}(\sin x)$, (h) $|\sin x\,|$, (i) $e^{\sigma(x)}$, (j) $\sigma(e^x)$, (k) $e^{|\,x-2\,|}$.

a) Note the jump discontinuities happened at:

$x_k = 0 \rightarrow \sigma(x)$

$x_k = -1 \longrightarrow \sigma(x+1)$

$x_k = 1 \longrightarrow \sigma(x-1)$

① $x_k = 0$

magnitude $= 2$.

② $x_k = -1$

magnitude $= 1$

③ $x_k = 1$

magnitude $= -3$ or 3.

b) Note the jump discontinuities happened at:

$x_k = 0$ and $x_k = 2$

① $x_k = 0$

Magnitude: $f(x_k^+) - f(x_k^-) = -1 - 1 = -2$ or 2

② $x_k = 2$

Magnitude: $f(x_k^+) - f(x_k^-) = 1 - (-1) = 2$

e) The function is continuous.

3.2.16. Are the functions in Exercises 3.2.14 and 3.2.15 piecewise C^1? If so, list all corners.

a) The function is not differentiable at:

$x=0$, $x=-1$, $x=1$.

Hence it's piecewise C^1

No corners.

b) Not differentiable at:

$x=0$, $x=2$

It's piecewise C^1

No corners.

e) The function is piecewise C^1

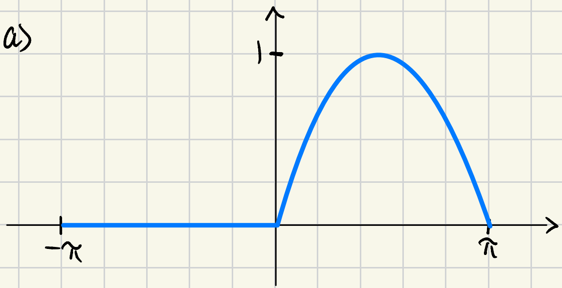
Corner: $x=2$

$f(2^-)=f(2^+)$

$f'(2^-)\neq f'(2^+)$

3.2.25. (a) Sketch the 2π–periodic half-wave $f(x) = \begin{cases} \sin x, & 0 < x \le \pi, \\ 0, & -\pi \le x < 0. \end{cases}$ (b) Find its Fourier series. (c) Graph the first five Fourier sums and compare with the function. (d) Discuss convergence of the Fourier series.

a)



b) $a_0 = \frac{1}{\pi}\left[\int_0^{\pi} \sin(x)\,dx\right] = \frac{1}{\pi}(-\cos(x))\Big|_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(kx)\,dx = \frac{1}{2\pi}\int_0^{\pi}\left[\sin(x+kx) + \sin(x-kx)\right]$$

$$= \frac{1}{2\pi}\left[\left(-\frac{1}{1+k}\cos((1+k)x)\right)\Big|_0^{\pi} + \left(-\frac{1}{1-k}\cos((1-k)x)\right)\Big|_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{1+k}\cos((1+k)\pi) + \frac{1}{1+k} - \frac{1}{1-k}\cos((1-k)\pi) + \frac{1}{1-k}\right]$$

$$= \frac{4}{(1-k^2)\pi} \text{ for } k \text{ even}, \ 0 \text{ for } k \text{ odd}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(kx)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi}\cos((1-k)x)\,dx + \int_0^{\pi}\cos((1+k)x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-k}\sin((1-k)x)\Big|_0^{\pi} + \frac{1}{1+k}\sin((1+k)x)\Big|_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-k}\sin(\pi-k\pi) + \frac{1}{1+k}\sin(\pi+k\pi)\right]$$

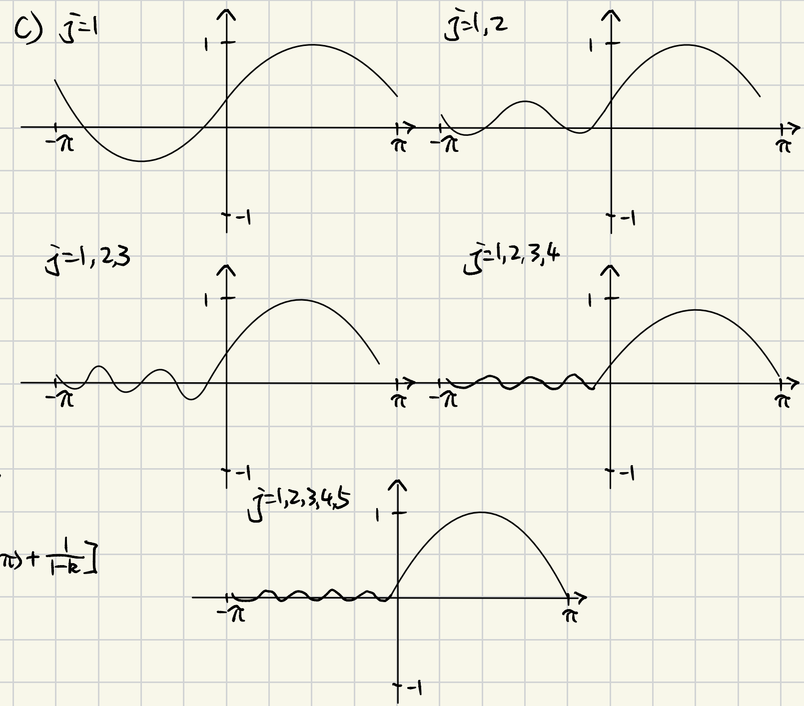
Note for $k=1$: $b_1 = \frac{1}{2}$

$k \ne 1$: $b_k = 0$.

Then we have:

$$\boxed{f(x) \sim \frac{1}{\pi} + \frac{\sin(x)}{2} - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\cos(2jx)}{4j^2-1}}$$

c)



The graph become more and more closer to the original function

d) $f(x) = \begin{cases} \sin(x) & , \quad x \in [2k\pi, (2k+1)\pi] \\ 0 & , \quad x \in [(2k-1)\pi, 2k\pi] \end{cases}$

3.2.31. Are the following functions even, odd, or neither?

(a) x^2, (b) e^x, (c) $\sinh x$, (d) $\sin \pi x$, (e) $\frac{1}{x}$, (f) $\frac{1}{1+x^2}$, (g) $\tan^{-1} x$.

a) $\boxed{\text{Even}}$ $\Rightarrow$ $x^2 = (-x)^2$

b) $\boxed{\text{Neither}}$ $\Rightarrow$ $e^{-x} \neq -e^x$ and $e^{-x} \neq e^x$

c) $\boxed{\text{Odd}}$ $\Rightarrow$ $-\sinh(x) = \sinh(-x)$

3.2.34. If $f(x)$ is odd, is $f'(x)$ (*i*) even? (*ii*) odd? (*iii*) neither? (*iv*) could be either?

Note if $f(x)$ is odd

$\Rightarrow f(-x) = -f(x)$

$\Rightarrow -f'(-x) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$

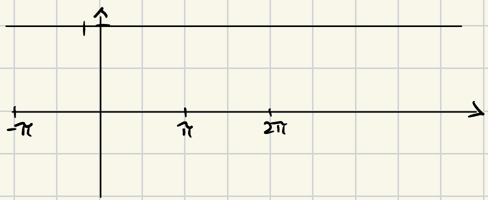
Hence $f'(x)$ is $\boxed{(i) \text{ even}}$

3.2.41. Find the Fourier sine and cosine series of the following functions. Then graph the function to which the series converges. (a) 1, (b) $\cos x$, (c) $\sin^3 x$, (d) $x(\pi - x)$.

a) $f(x) = 1$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{1}{\pi}\left(\frac{1}{k}\sin(kx)\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi}\left(\frac{1}{k}\sin(k\pi) - \frac{1}{k}\sin(k\pi)\right) = 0$$

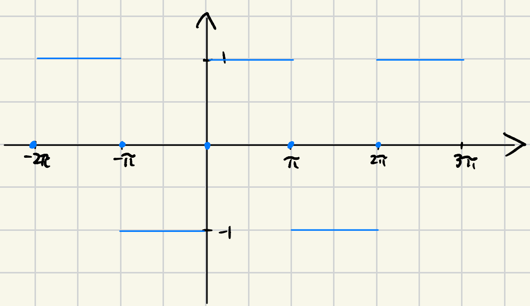
Cosine series $= \boxed{1}$



$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin(kx)\,dx = \frac{2}{k\pi}\left[-\cos(kx)\Big|_0^{\pi}\right] = \frac{2}{k\pi}\left[-\cos(k\pi) + 1\right]$$

$$= \begin{cases} \frac{4}{k\pi} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

Sin Series: $\boxed{\frac{4}{\pi}\sum_{j=0}^{\infty} \frac{\sin(2j+1)x}{2j+1}}$



C). $f(x) = \sin^3(x)$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^3(x)\cos(kx)\,dx = 0$$

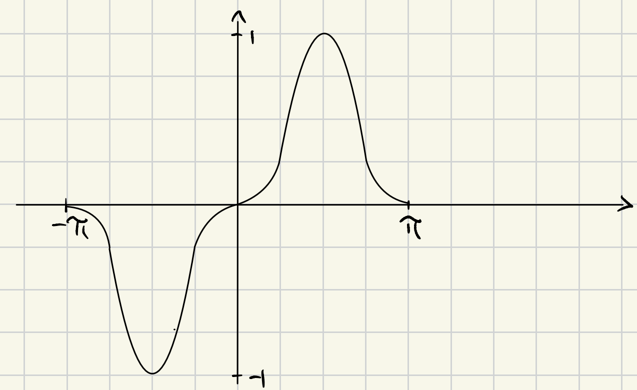
Cosin Series $= \boxed{0}$

Note we have:

$$\sin^3(x) = \sin(x)\,\frac{1-\cos(2x)}{2} = \frac{\sin(x)}{2} - \frac{\sin(x)\cos(2x)}{2}$$

$$= \frac{\sin(x)}{2} - \frac{\sin(3x) - \sin(x)}{4}$$

$$= \boxed{\frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)}$$



3.2.51. Find the complex Fourier series of the following functions: (a) $\sin x$, (b) $\sin^3 x$, (c) x, (d) $|x|$, (e) $|\sin x|$, (f) $\operatorname{sign} x$, (g) the ramp function $\rho(x) = \begin{cases} x, & x \geq 0, \\ 0, & x \leq 0. \end{cases}$

b) Note we know:

$$\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$$

$$= \frac{3}{4}\,\frac{e^{ix} - e^{-ix}}{2i} - \frac{1}{4}\,\frac{e^{i3x} - e^{-i3x}}{2i}$$

$$= \boxed{\frac{3e^{ix} - 3e^{-ix} - e^{i3x} + e^{-i3x}}{8i}}$$

e) $$c_k = 2\int_0^\pi \sin(x)e^{ikx}\,dx = 2\int_0^\pi \frac{e^{2ikx} - e^0}{2i}\,dx$$

$$= \frac{1}{i}\left[\int_0^\pi e^{2ikx}\,dx - \int_0^\pi 1\,dx\right]$$

$$= \frac{1}{i}\left[\frac{1}{2ik}\,e^{2ikx}\Big|_0^\pi - \pi\right]$$

$$= \frac{1}{i}\left[\frac{1}{2ik}\left[e^{2ikx} - 1\right] - \pi\right]$$

$$= \boxed{-\frac{e^{2ikx} - 1}{2k} - \frac{\pi}{i}}$$