AMATH 503 HW2

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2.4.2. (a) Solve the wave equation $u_{tt} = u_{xx}$ when the initial displacement is the box function $u(0,x) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$ while the initial velocity is 0.
(b) Sketch the resulting solution at several representative times.

a) First note by Theorem 2.15 we have:

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz.$$

Note $u_t(0,x) = g(x) = 0$.

Then note we have:

$$f(x-t) = \begin{cases} 1, & 1+t < x < 2+t \\ 0, & \text{otherwise} \end{cases}$$

$$f(x+t) = \begin{cases} 1, & 1-t < x < 2-t \\ 0, & \text{otherwise} \end{cases}$$

Then note we have two possibilities:

① $t < \frac{1}{2}$

Then we have:

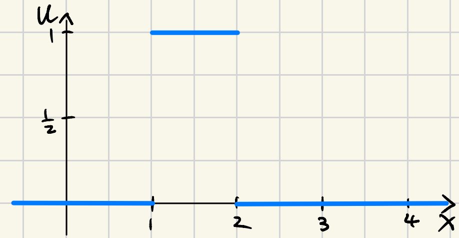
$$u(t+x) = \begin{cases} 1, & 1+t < x < 2-t \\ \frac{1}{2}, & 1-t < x < 1+t \text{ or } 2-t < x < 2+t \\ 0, & \text{otherwise.} \end{cases}$$

② $t \geq \frac{1}{2}$

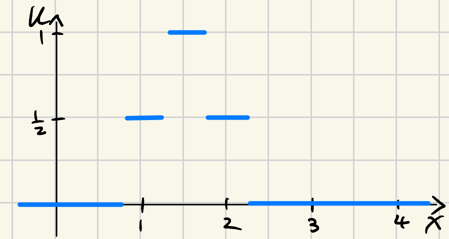
Then we have:

$$u(t+x) = \begin{cases} \frac{1}{2}, & 1-t < x < 2-t \text{ or } 1+t < x < 2+t \\ 0, & \text{otherwise.} \end{cases}$$

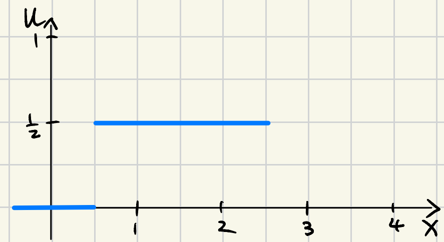
b). $t=0$



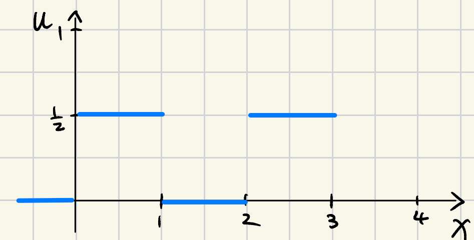
$t=0.25$



$t=.5$



$t=1$



2.4.4. Write the following solutions to the wave equation $u_{tt} = u_{xx}$ in d'Alembert form (2.82).
Hint: What is the appropriate initial data?
(a) $\cos x \cos t$, (b) $\cos 2x\ \sin 2t$, (c) e^{x+t}, (d) $t^2 + x^2$, (e) $t^3 + 3tx^2$.

b) Note the initial displacement is:

$$u(x,0) = 0$$

Note the intial velocity is:

$$\frac{\partial u}{\partial t}(x,0) = 2\cos(2x)\cos(0) = 2\cos(2x)$$

$$\Rightarrow u(t,x) = \frac{1}{2}\int_{x-ct}^{x+ct} 2\cos(2z)\,dz = \boxed{\frac{\sin(2x+2t) - \sin(2x-2t)}{2}}$$

d) Note the initial displacement is:

$$u(x,0) = x^2$$

Note the intial velocity is:

$$\frac{\partial u}{\partial t}(x,0) = x^2$$

$$\Rightarrow u(t,x) = \frac{(x+t)^2 + (x-t)^2}{2} + \int_{x-ct}^{x+ct} z^2\,dz \qquad \frac{1}{3}z^3$$

$$= \boxed{\frac{(x+t)^2 + (x-t)^2}{2} + \frac{(x+t)^3 - (x-t)^3}{3}}$$

2.4.8. Consider the initial value problem $u_{tt} = 4u_{xx} + F(t,x)$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$. Determine (a) the domain of influence of the point $(0,2)$; (b) the domain of dependence of the point $(3,-1)$; (c) the domain of influence of the point $(3,-1)$.

a) $c^2 = 4 \Rightarrow c = 2$

Note the domain of influence is the area lie between the charisteristic curve: $y - ct$ and $y + ct$

$\Rightarrow$ Domain of influence:

$$\boxed{\{(t,x) \mid 2-2t \leq x \leq 2+2t,\ 0 \leq t\}}$$

b) Then note by Formula 2.95 we have:

$D(t,x) = \{(s,y) \mid x - c(t-s) \leq y \leq x + c(t-s),\ 0 \leq s \leq t$

Then plug in x, t, c we have:

$D(s,y) = \{(s,y) \mid -1 - 2(3-s) \leq y \leq -1 + 2(3-s),\ 0 \leq s \leq 3\}$

$= \boxed{\{(s,y) \mid -7 + 2s \leq y \leq 5 + 2s,\ 0 \leq s \leq 3\}}$

3.1.2. Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on the interval $0 \le x \le \pi$ subject to (a) homogeneous Dirichlet boundary conditions $u(t,0) = 0$, $u(t,\pi) = 0$;
(b) mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$;
(c) Neumann boundary conditions $u_x(t,0) = 0$, $u_x(t,\pi) = 0$.

First if we consider the ansatz:

$u(x,t) = e^{\lambda t} v(x)$

Then we have the eigenequation:

$v''(x) = \lambda v(t)$

a) ① $\lambda = w^2 > 0$

$V(x) = C_1 e^{-wx} + C_2 e^{wx}$

$$\left.\begin{aligned} V(0) &= C_1 + C_2 = 0 \\ V(\pi) &= C_1 e^{-\pi w} + C_2 e^{\pi w} = 0 \end{aligned}\right\rangle C_1, C_2 = 0$$

② $\lambda = 0$

$V(x) = \alpha + \beta x$

$$\left.\begin{aligned} V(0) &= \alpha \\ V(\pi) &= \alpha + \beta\pi \end{aligned}\right\rangle \alpha, \beta = 0$$

② $\lambda = -w^2 < 0$

$V(x) = C_1 \cos(wx) + C_2 \sin(wx)$

We have: $V(0) = V(\pi) = 0$

$\Rightarrow C_1 \cos(0) + C_2 \sin(0) = 0 \Rightarrow C_1 = 0$

$C_1 \cos(\pi w) + C_2 \sin(\pi w) = 0 \Rightarrow \sin(\pi w) = 0$

$\Rightarrow w = 1, 2, 3, \dots$

Hence we have $\boxed{u(x,t) = e^{-n^2 t} \sin(nx) \text{ for } n \in \mathbb{N}}$

b) ① $\lambda = 0$

$\Rightarrow \alpha, \beta = 0$

② $\lambda = w^2 > 0$

$\Rightarrow C_1 = C_2 = 0$

③ $\lambda = -w^2 < 0$

We have: $V(0) = 0$, $V'(\pi) = 0$

$\Rightarrow C_1 \cos(0) + C_2 \sin(0) = 0 \Rightarrow C_1 = 0$

$-C_1 w \sin(w\pi) + C_2 w \cos(w\pi) = 0 \Rightarrow \cos(w\pi) = 0$

$\Rightarrow w\pi = (n + \frac{1}{2})\pi$, for $n \in \mathbb{N}$

Hence we have $\boxed{u(x,t) = e^{-(n+\frac{1}{2})^2 t} \sin((n+\tfrac{1}{2})x) \text{ for } n \in \mathbb{N}}$

c) ① $\lambda = 0$

$\Rightarrow \alpha, \beta = 0$

② $\lambda = w^2 > 0$

$\Rightarrow C_1 = C_2 = 0$

③ $\lambda = -w^2 < 0$

We have: $V'(0) = 0$, $V'(\pi) = 0$

$\Rightarrow -C_1 w \sin(0) + C_2 w \cos(0) = 0 \Rightarrow C_2 = 0$

$-C_1 w \sin(\pi w) + C_2 w \cos(\pi w) = 0 \Rightarrow \sin(\pi w) = 0$

Hence we have: $\boxed{u(x,t) = e^{-n^2 t} \cos(nx), \text{ for } n \in \mathbb{N}}$

3.1.4. Find all separable eigensolutions to the following partial differential equations:
(a) $u_t = u_x$, (b) $u_t = u_x - u$, (c) $u_t = x\,u_x$.

a) $u_t = u_x$

Assume the ansatz:

$u(x,t) = e^{\lambda t} v(x)$

Then we have:

$u_t = \lambda e^{\lambda t} v(x)$

$u_x = e^{\lambda t} v'(x)$

$\Rightarrow v'(x) = \lambda v(x)$

$\Rightarrow v(x) = e^{\lambda x}$

$\Rightarrow \boxed{u(x,t) = e^{\lambda(x+t)}}$

b) $u_t = u_x - u$

Assume the ansatz:

$u(x,t) = e^{\lambda t} v(x)$

Then we have:

$u_t = \lambda e^{\lambda t} v(x)$

$u_x = e^{\lambda t} v'(x)$

$\Rightarrow \lambda e^{\lambda t} v(x) = e^{\lambda t} v'(x) - e^{\lambda t} v(x)$

$\Rightarrow (\lambda+1) e^{\lambda t} v(x) = e^{\lambda t} v'(x)$

$\Rightarrow v(x) = e^{(\lambda+1)x}$

$\Rightarrow u(x,t) = e^{\lambda t} e^{(\lambda+1)x} = \boxed{e^{\lambda t + (\lambda+1)x}}$

3.2.1. Find the Fourier series of the following functions: (a) $\operatorname{sign} x$, (b) $|x|$, (c) $3x-1$, (d) x^2, (e) $\sin^3 x$, (f) $\sin x \cos x$, (g) $|\sin x|$, (h) $x\cos x$.

b) Note:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\begin{cases} \frac{x^2}{2} & \text{for } x \geq 0 \\ -\frac{x^2}{2} & \text{for } x \leq 0 \end{cases} = \pi$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)\,dx$$

Let $u = x$, $dv = \cos(kx)\,dx$

$du = dx$, $v = \frac{1}{n}\sin(nx)$

$$a_k = \frac{2}{\pi}\left[\frac{x}{k}\sin(kx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{k}\sin(kx)\,dx\right]$$

$$= \frac{2}{\pi}\frac{1}{k^2}\left[\cos(kx)\Big|_0^{\pi}\right] = \frac{2}{\pi}\left[\frac{1}{k^2}\cos(k\pi) - \frac{1}{k^2}\right]$$ Note equal to 0 for k even, 2 for k odd

$$= -\frac{4}{\pi}\left[\frac{1}{(2j+1)^2}\right],\ j \in \mathbb{N}$$

$b_k = 0$

$$\Rightarrow \boxed{|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos(2j+1)x}{(2j+1)^2}}$$

f) $\sin x \cos x$

Note that $\sin x \cos x = \boxed{\frac{1}{2}\sin(2x)}$

Which is the Fourier Series.

3.2.2. Find the Fourier series of the following functions:

(a) $\begin{cases} 1, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise}, \end{cases}$ (b) $\begin{cases} 1, & \frac{1}{2}\pi < |x| < \pi, \\ 0, & \text{otherwise}, \end{cases}$ (c) $\begin{cases} 1, & \frac{1}{2}\pi < x < \pi, \\ 0, & \text{otherwise}, \end{cases}$

(d) $\begin{cases} x, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise}, \end{cases}$ (e) $\begin{cases} \cos x, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise}. \end{cases}$

b) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\cdot\left[2\cdot\left(\pi-\frac{\pi}{2}\right)\right] = 1$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(k\pi)dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}}\cos(kx)dx + \int_{\frac{\pi}{2}}^{\pi}\cos(kx)\,dx\right]$$

$$= \frac{1}{\pi}\frac{1}{k}\left[\sin(kx)\Big|_{-\pi}^{-\frac{\pi}{2}} + \sin(kx)\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\frac{1}{k}\left[-\sin\left(\frac{k\pi}{2}\right)+\sin(k\pi)\right] \text{ for } k\in\mathbb{N} \quad \text{Let } k=2j+1$$

$$= \frac{2}{\pi}\frac{(-1)^j}{2j+1} \quad \text{for } j\in\mathbb{N}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(k\pi)dx = 0$$

Thus we have:

$$\boxed{f(x) \sim \frac{1}{2} - \frac{2}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j x\cos(2j+1)}{2j+1}}$$

3.2.3. Find the Fourier series of $\sin^2 x$ and $\cos^2 x$ without directly calculating the Fourier coefficients. *Hint*: Use some standard trigonometric identities.

Note $\sin^2(x) = \dfrac{1-\cos(2x)}{2} = \boxed{\frac{1}{2} - \frac{1}{2}\cos(2x)}$

where $a_0 = \frac{1}{2}$, $a_k = -\frac{1}{2}$, $k = 2$

Then note $\cos^2(x) = 1 - \sin^2(x)$

$\Rightarrow \cos^2(x) = \boxed{\frac{1}{2} + \frac{1}{2}\cos(2x)}$

where $a_0 = \frac{1}{2}$, $a_k = \frac{1}{2}$, $k = 2$