AMATH 503 HW7

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7.1.1. Find the Fourier transform of the following functions:

(a) $e^{-(x+4)^2}$, (b) $e^{-|x+1|}$, (c) $\begin{cases} x, & |x|<1, \\ 0, & \text{otherwise}, \end{cases}$ (d) $\begin{cases} e^{-2x}, & x \geq 0, \\ e^{3x}, & x \leq 0, \end{cases}$

(e) $\begin{cases} e^{-|x|}, & |x| \geq 1, \\ e^{-1}, & |x| \leq 1, \end{cases}$ (f) $\begin{cases} e^{-x}\sin x, & x > 0, \\ 0, & x \leq 0, \end{cases}$ (g) $\begin{cases} 1-|x|, & |x| \leq 1, \\ 0, & \text{otherwise}. \end{cases}$

a) $f(x) = e^{-(x+4)^2}$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+4)^2 - ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+8x+16)-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2+(8+ik)x+(4+\frac{ik}{2})^2)+4ik-\frac{k^2}{2}} dx$$

$$= \frac{e^{4ik-\frac{k^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+4+\frac{ik}{2})^2} dx$$

$$= \boxed{\frac{1}{\sqrt{2}} e^{4ik-\frac{k^2}{2}}}$$

b) $f(x) = e^{-|x+1|}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} e^{x+1-ikx} dx + \int_{-1}^{\infty} e^{-x-1-iwx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{e^{-(1-ik)}}{1-ik} + \frac{e^{-(1+ik)}}{1+ik} \right]$$

$$= \boxed{\sqrt{\frac{2}{\pi}} \frac{e^{ik}}{k^2+1}}$$

f) $f(x) = e^{-x}\sin(x)$, $x \geq 0$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x-ikx} dx \cdot \int_0^{\infty} \sin(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{ikx}-e^{-ikx}}{2i} e^{-ikx} dx$$

$$= \boxed{\frac{1}{\sqrt{2\pi}(-k^2+2ik+2)}}$$

◇ 7.1.5. (a) Find the Fourier transform of $e^{\,i\,\omega\,x}$. (b) Use this to find the Fourier transforms of the basic trigonometric functions $\cos\omega x$ and $\sin\omega x$.

a) $f(x) = e^{i\omega x}$

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x - ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\omega-k)x}\,dx$$

$$= \boxed{\sqrt{2\pi}\,\delta(k-\omega)}$$

b) Note:

$$\cos(\omega x) = \frac{e^{i\omega x} + e^{-i\omega x}}{2}$$

$$\Rightarrow \hat{f}(\cos(\omega x)) = \frac{1}{2}\left(\hat{f}(e^{i\omega x}) + \hat{f}(e^{-i\omega x})\right)$$

$$= \boxed{\sqrt{\frac{\pi}{2}}\left[\delta(k+\omega) + \delta(k-\omega)\right]}$$

$$\sin(\omega x) = \frac{e^{i\omega x} - e^{-i\omega x}}{2i}$$

$$\Rightarrow \hat{f}(\sin(\omega x)) = \frac{1}{2i}\left(\hat{f}(e^{i\omega x}) - \hat{f}(e^{-i\omega x})\right)$$

$$= \boxed{i\sqrt{\frac{\pi}{2}}\left[\delta(k+\omega) - \delta(k-\omega)\right]}$$

◇ 7.1.10. If the Fourier transform of $f(x)$ is $\widehat{f}(k)$, prove that (a) the Fourier transform of $f(-x)$ is $\widehat{f}(-k)$; (b) the Fourier transform of the complex conjugate function $\overline{f(x)}$ is $\overline{\widehat{f}(-k)}$.

a) $$F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)\, e^{-ikx}\, dx$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(\hat{x})\, e^{ik\hat{x}}\, d\hat{x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\hat{x})\, e^{-i(-k)\hat{x}}\, d\hat{x}$$

$$= \hat{f}(-x)$$

$\square$

b) Let the complex conjugate of $f(x)$ be $\overline{f(x)}$

$$\Rightarrow F(\overline{f(x)}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-ikx}\, dx$$

Note $\overline{e^{-ikx}} = e^{ikx}$

$$\Rightarrow F(\overline{f}(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

$$= \overline{\hat{f}(x)}$$

Then using part a) we have:

$$F(\overline{f(x)}) = \overline{\hat{f}(x)}$$

$\square$

7.3.8. Use the Fourier transform to find an integral formula for a bounded solution to the *Airy differential equation* $-\dfrac{d^2u}{dx^2} = x\,u$.

First note we have:

$$k^2 \hat{u}(k) = i\frac{d\hat{u}}{dk}$$

$$\Rightarrow \hat{u}(k) = ce^{-ik^3/3}$$

Hence we have:

$$u(x) = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(kx-k^3/3)}\,dk$$

Note the red part is odd and the real part is even we have:

$$u(x) = c\sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos\left(kx+\frac{1}{3}k^3\right)dk$$

7.3.10. (a) Find the Fourier transform of the convolution $h(x) = f_e * g(x)$ of an even exponential pulse $f_e(x) = e^{-|x|}$ and a Gaussian $g(x) = e^{-x^2}$. (b) What is $h(x)$?

a) We have $f_e(x) = e^{-|x|}$, $g(x) = e^{-x^2}$

$$\Rightarrow \mathcal{F}(e^{-|x|}) = \sqrt{\frac{2}{\pi}} \frac{1}{k^2+1}$$

$$\Rightarrow \mathcal{F}(e^{-x^2}) = \frac{e^{-k^2/4}}{\sqrt{2}}$$

Then we have:

$$\mathcal{F}(f_e * g(x)) = \sqrt{2\pi}\, \mathcal{F}(f_e) \cdot \mathcal{F}(g(x))$$

$$= \boxed{\frac{\sqrt{2}}{k^2+1} e^{-k^2/4}}$$

b) Note:

$$h(x) = f_e * g(x)$$

$$\Rightarrow h(x) = \int_{-\infty}^{\infty} e^{-|k|} e^{-(x-k)^2} dk$$

$$= \int_{-\infty}^{0} e^{k} e^{-(x-k)^2} dk + \int_{0}^{\infty} e^{-k} e^{-(x-k)^2} dk$$

Then note let $v = x-u$ we have:

$$\int_{-\infty}^{0} e^{k} e^{-(x-k)^2} dk = e^{x} \int_{-\infty}^{x} e^{-v} e^{-v^2} dv$$

$$\int_{0}^{\infty} e^{-k} e^{-(x-k)^2} dk = e^{-x} \int_{-\infty}^{x} e^{v} e^{-v^2} dv$$

Then note:

$$\operatorname{erf}(v) = \frac{2}{\sqrt{\pi}} \int_0^v e^{-t^2} dt$$

Then we have:

$$\boxed{h(x) = \frac{\sqrt{\pi}\, e^{\frac{1}{4}}}{2} \left(e^{-x}\left[1 - \operatorname{erf}\left(\tfrac{1}{2} - x\right)\right] + e^{x}\left[1 - \operatorname{erf}\left(\tfrac{1}{2} + x\right)\right]\right)}$$

7.3.12. Find the function whose Fourier transform is $\widehat{f}(k) = (k^2+1)^{-2}$.

Note that the Fourier transform of $e^{-a|x|}$ is $\sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}$

Hence we have:

$g(x) = \sqrt{\frac{\pi}{2}}e^{-|x|}$ where $\hat{g}(x) = \frac{1}{k^2+1}$

Thus by Theorem 7.13 we have:

$$f(x) = \frac{g * g(x)}{\sqrt{2\pi}} = \boxed{\sqrt{\frac{\pi}{8}}(1+|x|)e^{-|x|}}$$