AMATH 503 HW1

Tianbo Zhang 1938501

1.1. Classify each of the following differential equations as ordinary or partial, and equilibrium or dynamic; then write down its order. (a) $\frac{du}{dx} + xu = 1$, (b) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x$, (c) $u_{tt} = 9u_{xx}$, (d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$, (e) $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$, (f) $\frac{d^2 u}{dt^2} + 3u = \sin t$, (g) $u_{xx} + u_{yy} + u_{zz} + (x^2 + y^2 + z^2)u = 0$, (h) $u_{xx} = x + u^2$, (i) $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0$, (j) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y\,\partial z} = u$, (k) $u_{tt} = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$.

a) ① Ordinary ② Equilibrium ③ 1st order
b) ① Partial ② Dynamic ③ 1st order
c) ① Partial ② Dynamic ③ 2nd order
d) ① Partial ② Dynamic ③ 2nd order
e) ① Partial ② Equilibrium ③ 2nd order
f) ① Ordinary ② Dynamic ③ 2nd order
g) ① Partial ② Equilibrium ③ 2nd order
h) ① Ordinary ② Equilibrium ③ 2nd order
i) ① Partial ② Dynamic ③ 3rd order
j) ① Partial ② Equilibrium ③ 2nd order
k) ① Partial ② Dynamic ③ 4th order

1.17. Classify the following differential equations as either

(*i*) homogeneous linear; (*ii*) inhomogeneous linear; or (*iii*) nonlinear:

(a) $u_t = x^2 u_{xx} + 2x u_x$, (b) $-u_{xx} - u_{yy} = \sin u$; (c) $u_{xx} + 2y u_{yy} = 3$;

(d) $u_t + u u_x = 3u$; (e) $e^y u_x = e^x u_y$; (f) $u_t = 5u_{xxx} + x^2 u + x$.

a) i) homogeneous linear

b) i) homogeneous linear

c) ii) inhomogeneous linear

d) iii) nonlinear

e) i) homogeneous linear

f) ii) inhomogeneous linear

1.19. (a) Show that the following functions are solutions to the wave equation $u_{tt} = 4u_{xx}$:
(i) $\cos(x-2t)$, (ii) e^{x+2t}; (iii) $x^2+2xt+4t^2$.

i) $\cos(x-2t)$

$u_{tt} = \frac{\partial}{\partial t}(2\sin(x-2t)) = -4\cos(x-2t)$

$u_{xx} = \frac{\partial}{\partial x}(-\sin(x-2t)) = -\cos(x-2t)$

$\Rightarrow u_{tt} = 4u_{xx}$

Hence, it's a solution

ii) e^{x+2t}

$u_{tt} = 4e^{x+2t}$

$u_{xx} = e^{x+2t}$

$\Rightarrow u_{tt} = 4u_{xx}$

Hence, it's a solution

iii) $x^2+2xt+4t^2$

$u_{tt} = \frac{\partial}{\partial t}(2x+8t) = 8$

$u_{xx} = \frac{\partial}{\partial x}(2x+2t) = 2$

$\Rightarrow u_{tt} = 4u_{xx}$

Hence, it's a solution

1.22. (a) Prove that the Laplacian $\Delta = \partial_x^2 + \partial_y^2$ defines a linear differential operator.
(b) Write out the Laplace equation $\Delta[u] = 0$ and the Poisson equation $-\Delta[u] = f$.

a) ① $\Delta[u+v] = \partial_x^2(u+v) + \partial_y^2(u+v)$

$= \partial_x^2 u + \partial_y^2 u + \partial_x^2 v + \partial_y^2 v$

$= \Delta[u] + \Delta[v]$

② $\Delta[cu] = \partial_x^2(cu) + \partial_y^2(cv)$

$= c\partial_x^2 u + c\partial_y^2(v)$

$= c\Delta[u]$

Hence Δ is a linear differential operator. $\square$

b) The Laplace equation:

$\Delta[u] = \boxed{u_{xx} + u_{yy} = 0}$

The Poisson equation:

$-\Delta[u] = \boxed{-u_{xx} - u_{yy} = f}$

1.27. Solve the following inhomogeneous linear ordinary differential equations:
(a) $u' - 4u = x - 3$, (b) $5u'' - 4u' + 4u = e^x \cos x$, (c) $u'' - 3u' = e^{3x}$.

a) First note the integrating factor is:

$\mu(x) = e^{\int -4\,dx} = e^{-4x}$

Then multiply both side by $\mu(x)$ we have:

$e^{-4x}u' - 4e^{-4x}u = e^{-4x}x - 3e^{-4x}$

$\Rightarrow \frac{d}{dx}(e^{-4x}u) = (x-3)e^{-4x}$

$\Rightarrow \int \frac{d}{dx}(e^{-4x}u)\,dx = \int (x-3)e^{-4x}\,dx$

$\Rightarrow e^{-4x}u + C_1 = -(x-3)\frac{e^{-4x}}{4} - \frac{1}{16}e^{-4x} + C_2$

$\Rightarrow u = \boxed{Ce^{4x} - \frac{4x-11}{16}}$

b) First note the complementary equation is:

$5u'' - 4u' + 4u = 0$

$\Rightarrow$ root $= \frac{4 \pm \sqrt{16-80}}{10}$

$\Rightarrow \alpha = \frac{2}{5},\ \beta = \frac{4}{5}i$

$\Rightarrow u_c(x) = e^{\frac{2}{5}x}(C_1\cos(\frac{4}{5}x) + C_2\sin(\frac{4}{5}x))$

Then for particular solution we have:

$u_p(x) = e^x(A\cos(x) + B\sin(x))$

$u_p'(x) = -A\sin(x)e^x + A\cos(x)e^x + B\cos(x)e^x + B\sin(x)e^x$

$= (B-A)\sin(x)e^x + (A+B)\cos(x)e^x$

$u_p''(x) = (B-\cancel{A})\cos(x)e^x + (\cancel{B}-A)\sin(x)e^x - (A+\cancel{B})\sin(x)e^x + (\cancel{A}+B)\cos(x)e^x$

$= 2B\cos(x)e^x - 2A\sin(x)e^x$

$\Rightarrow 5\cdot 2B - 4\cdot(A+B) + 4\cdot A = 1$

$B = \frac{1}{6}$

$\Rightarrow \boxed{u(x) = \frac{1}{6}e^x\sin x + C_1e^{\frac{4}{5}x}\cos(\frac{4}{5}x) + C_2e^{\frac{4}{5}x}\sin(\frac{4}{5}x)}$

c) First note the complementary equation is:

$u'' - 3u' = 0$

$\Rightarrow$ root $= 0$ and 3

$\Rightarrow u_c(x) = C_1 + C_2e^{3x}$

Then for particular solution we have:

$u_p(x) = Axe^{3x}$

$u_p'(x) = 3Axe^{3x} + Ae^{3x}$

$u_p''(x) = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x}$

$\Rightarrow 9Ax + 3A + 3A - 9Ax - 3A = 1$

$A = \frac{1}{3}$

$\Rightarrow u(x) = \boxed{\frac{1}{3}xe^{3x} + C_2e^{3x} + C_1}$

2.2.2. Solve the following initial value problems and graph the solutions at times $t = 1, 2$, and 3:
(a) $u_t - 3u_x = 0$, $u(0, x) = e^{-x^2}$; (b) $u_t + 2u_x = 0$, $u(-1, x) = x/(1 + x^2)$;
(c) $u_t + u_x + \frac{1}{2}u = 0$, $u(0, x) = \tan^{-1} x$; (d) $u_t - 4u_x + u = 0$, $u(0, x) = 1/(1 + x^2)$.

a) $u_t - 3u_x = 0$, $u(0,x) = e^{-x^2}$

Suppose $(x(t), t(s))$ is a parametric curve in (x,t) plane

$\Rightarrow u(s) = (x(s), t(s))$

$\Rightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$

$\Rightarrow \begin{cases} \frac{dx}{ds} = -3 \\ \frac{dt}{ds} = 1 \end{cases}$ and $\frac{du}{ds} = 0$

Then by using the initial condition we have:

$\frac{d}{dt} u(x(t), t) = 0$

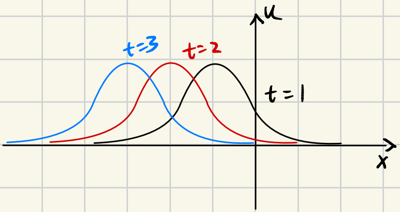
$\Rightarrow u(x(t), t) = u_0(\xi, 0) = e^{-\xi^2}$ for $\xi = x - ct$

$\Rightarrow$ Since $\frac{dx}{ds} = \frac{dx}{dt} = -3$

$\Rightarrow \int_{\xi}^{x} \frac{d\tilde{x}}{-3} = \int_0^t d\tilde{t} \Rightarrow t = -\frac{1}{3}(x - \xi)$

$\Rightarrow \xi = x + 3t$

$\Rightarrow \boxed{u(x,t) = e^{-(x+3t)^2}}$



c) $u_t + u_x + \frac{1}{2}u = 0$, $u(0,x) = \tan^{-1}(x)$

Suppose $(x(t), t(s))$ is a parametric curve in (x,t) plane

$\Rightarrow u(s) = (x(s), t(s))$

$\Rightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$

$\Rightarrow \begin{cases} \frac{dx}{ds} = -3 \\ \frac{dt}{ds} = 1 \end{cases}$ and $\frac{du}{ds} = -\frac{1}{2}u$

Then by similar step in part a we have:

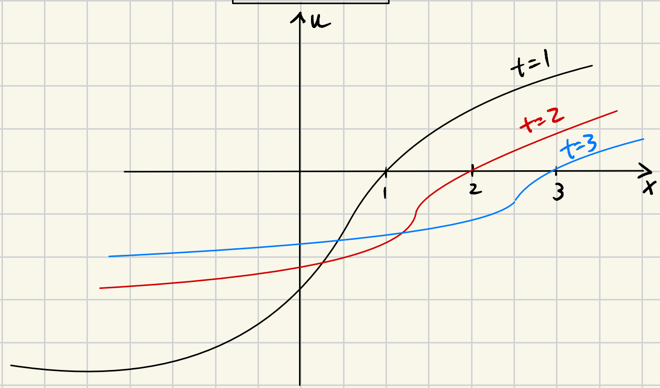
$\int_{\xi}^{x} d\tilde{x} = \int_0^t dt \Rightarrow \xi = x - t$

Then note from ODE we know:

the integrating factor is: $e^{\frac{1}{2}t}$

$\Rightarrow u(t,x) = e^{-\frac{1}{2}t} u_0(\xi, 0)$

$= \boxed{e^{-\frac{1}{2}t} \tan^{-1}(x - t)}$



2.2.3. Graph some of the characteristic lines for the following equations, and write down a formula for the general solution:

(a) $u_t - 3u_x = 0$, (b) $u_t + 5u_x = 0$, (c) $u_t + u_x + 3u = 0$, (d) $u_t - 4u_x + u = 0$.

b) $u_t + 5u_x = 0$

Suppose $(x(t), t(s))$ is a parametric curve in (x,t) plane

$\Rightarrow u(s) = (x(s), t(s))$

$\Rightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$

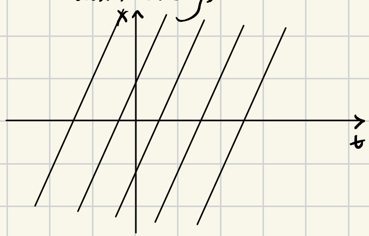
$\Rightarrow \begin{cases} \frac{dx}{ds} = 5 \\ \frac{dt}{ds} = 1 \end{cases}$ and $\frac{du}{ds} = 0$

$\Rightarrow \int_\xi^x \frac{d\tilde{x}}{5} = \int_0^t d\tilde{t} \rightarrow t = \frac{1}{5}(x - \xi)$

$\Rightarrow \xi = x - 5t$

$\Rightarrow$ general solution :

$u(t,x) = u_0(\xi)$ with $x = 5t + c$



2.2.4. Solve the initial value problem $u_t + 2u_x = 1$, $u(0,x) = e^{-x^2}$.

Hint: Use characteristic coordinates.

$u_t + 2u_x = 0$, $u(0,x) = e^{-x^2}$

Suppose $(x(t), t(s))$ is a parametric curve in (x,t) plane

$\Rightarrow u(s) = (x(s), t(s))$

$\Rightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$

$\Rightarrow \begin{cases} \frac{dx}{ds} = 2 \\ \frac{dt}{ds} = 1 \end{cases}$ and $\frac{du}{ds} = 1$

Then by using the ~~initial~~ condition we have:

$\frac{d}{dt} u(x(t), t) = 1$

$\frac{du}{ds} = 1$

$\Rightarrow u(x(t), t) = u_0(\xi, 0) + t = e^{-\xi^2} + t$ for $\xi = x - ct$

$\Rightarrow$ Since $\frac{dx}{ds} = \frac{dx}{dt} = 2$

$\Rightarrow \int_{\xi}^{x} \frac{d\tilde{x}}{2} = \int_0^t d\tilde{t} \Rightarrow t = \frac{1}{2}(x - \xi)$

$\Rightarrow \xi = x - 2t$

$\Rightarrow \boxed{u(x,t) = e^{-(x-2t)^2} + t}$

2.2.20. Consider the linear transport equation $u_t + (1+x^2)u_x = 0$. (a) Find and sketch the characteristic curves. (b) Write down a formula for the general solution. (c) Find the solution to the initial value problem $u(0,x) = f(x)$ and discuss its behavior as t increases.

a) $u_t + (1+x^2)u_x = 0$

Suppose $(x(t), t(s))$ is a parametric curve in (x,t) plane

$\Rightarrow u(s) = (x(s), t(s))$

$\Rightarrow \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial s}$

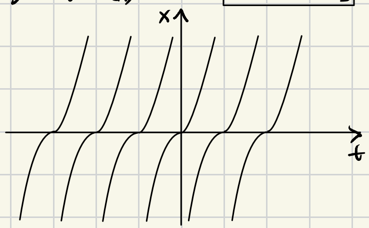
$\Rightarrow \begin{cases} \frac{dx}{ds} = 1+x^2 \\ \frac{dt}{ds} = 1 \end{cases}$ and $\frac{du}{ds} = 0$

Then we have:

$\frac{1}{1+x^2}dx = ds$

Then integrate both sides we have the characteristic curve:

$s = \tan^{-1}(x) \Rightarrow \boxed{x = \tan(t+k)}$



b) Note $\frac{du}{ds} = \frac{du}{dt}$ is constant

$\Rightarrow \tan^{-1}(x) - t$

Let $g(\xi)$ be an arbitrary function

$\Rightarrow u(x,t) = \boxed{g(\tan^{-1}(x) - t)}$

c) Then note we solve for:

$\frac{dx}{dt} = 1+x^2$

$\Rightarrow \frac{1}{1+x^2}dx = dt \Rightarrow \int_{\xi}^{x} \frac{1}{1+x^2}dx = \int_0^t dt \Rightarrow \tan^{-1}(x) - \tan^{-1}(\xi) = t$

$\Rightarrow \tan^{-1}(\xi) = \tan^{-1}(x) - t$

$\Rightarrow \xi = \tan(\tan^{-1}(x) - t)$

$\Rightarrow u(x,t) = \boxed{f(\tan(\tan^{-1}(x) - t))}$

Note the solution is not defined for:

$x < \tan(t - \frac{\pi}{2})$ where $0 < t < \pi$

and $t \geq \pi$, $\forall x$

Then when t increases to π, the wave $\to \infty$

Thus the solution disappears.

2.4.1. Solve the initial value problem $u_{tt} = c^2 u_{xx}$, $u(0,x) = e^{-x^2}$, $u_t(0,x) = \sin x$.

From the derivation in lecture and Theorem 2.15 on textbook we have:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$ d'Alembert's solution

$$\Rightarrow u(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(z)\,dz$$

$$\boxed{= \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{2c}\left(\cos(x-ct) - \cos(x+ct)\right)}$$