AMATH 503 HW8

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7.4.1. (a) Write out the Plancherel formula for the square wave pulse $f(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$

(b) What is $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$?

a) First note we have:

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x) e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ik} - e^{ik}}{-ik}$$

$$= \frac{1}{\sqrt{2\pi}} 2\sin(k)/k$$

Then note:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-1}^{1} 1\,dx = 2$$

And also note:

$$|\hat{f}(x)|^2 = \frac{1}{2\pi} 4\sin^2(k)/k^2 = \frac{2}{\pi}\frac{\sin^2(k)}{k^2}$$

Then by Plancherel theorem we have:

$$\boxed{2 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(k)}{k^2}\,dk}$$

b) Note the integral is even

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}\,dx = \boxed{\frac{\pi}{2}}$$

8.1.1. Find the solution to the heat equation $u_t = u_{xx}$ on the real line having the following initial condition at time $t = 0$. Then sketch graphs of the resulting temperature distribution at times $t = 0, 1$, and 5.

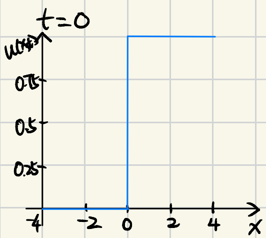
(a) e^{-x^2}, (b) the step function $\sigma(x)$, (c) $e^{-|x|}$, (d) $\begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$

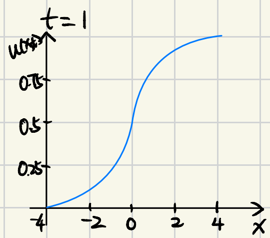
b) $\sigma(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

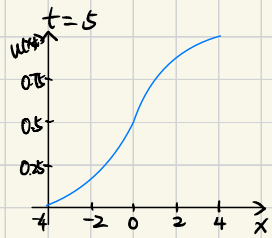
Then note we have:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-\xi)^2}{4t}} \, d\xi$$

$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right)$$







8.1.3. (*a*) Find the solution to the heat equation (8.6) whose initial data corresponds to a pair of unit heat sources placed at positions $x = \pm 1$. (*b*) Graph the solution at times $t = .1, .25, .5, 1$. (*c*) At what time(s) does the origin experience its maximum overall temperature? What is the maximum temperature at the origin?

a) First note for $x = \pm 1$:

$$u(x,0) = \delta(x-1) + \delta(x+1)$$

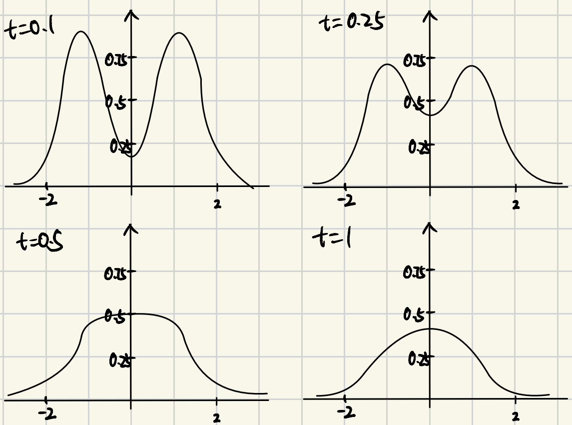
Then note we have:

$$u(x,0) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-a)^2}{4t}}$$

Hence:

$$\boxed{u(x,t) = \frac{1}{2\sqrt{\pi t}}\left(e^{-\frac{(x-1)^2}{4t}} + e^{-\frac{(x+1)^2}{4t}}\right)}$$

b)



c) Note we have:

$$u(t,0) = \frac{1}{\sqrt{\pi t}} e^{-\frac{1}{4t}}$$
$$= (\pi t)^{-\frac{1}{2}} e^{-(4t)^{-1}}$$

Then note

$$\frac{d}{dt}(u(t,0)) = -\frac{\pi}{2}(\pi t)^{-\frac{3}{2}} e^{-\frac{1}{4t}} + (\pi t)^{-\frac{1}{2}} \frac{e^{-\frac{1}{4t}}}{4t^2}$$

Then set:

$$\frac{d}{dt}(u(t,0)) = 0$$
$$\Longrightarrow \boxed{t = \frac{1}{2}}$$

$$\boxed{u(\tfrac{1}{2},0) = (2/e\pi)^{\frac{1}{2}}}$$

9.1.8. Compute the adjoint of the derivative operator $v = D[u] = u'$ under the weighted inner products $\langle u, \tilde{u} \rangle = \int_0^1 e^x u(x)\, \tilde{u}(x)\, dx$, $\langle\langle v, \tilde{v} \rangle\rangle = \int_0^1 (1+x)\, v(x)\, \tilde{v}(x)\, dx$. Clearly state any boundary conditions that you are imposing.

First note that:

$$\langle u, v\rangle = \int_0^1 e^x u(x)\overline{v(x)}\,dx$$

$$\langle u, w\rangle = \int_0^1 (1+x) v(x)\overline{w(x)}\,dx$$

Then note that:

$$\langle D[u], v\rangle = \int_0^1 (1+x) u'(x)\overline{v(x)}dx$$

$$= [(1+x)u(x)\overline{v(x)}]_0^1 - \int_0^1 u(x)\tfrac{d}{dx}((1+x)\overline{v(x)})dx$$

$$= 2u(1)\overline{v(1)} - u(0)\overline{v(0)} - \int_0^1 u(x)\overline{v(x)}dx - \int_0^1 u(x)(1+x)\overline{v'(x)}\,dx$$

Then we have:

$$\boxed{D^*[v](x) = -e^{-x}(v(x) + (1+x)v'(x))}$$

And the initial condition:

$$\boxed{u(0)=0,\ u(1)=0}$$

◇ 9.1.14. Let $L, M\colon U \to V$ be linear operators on the same inner product spaces. Prove that

(a) $(L+M)^* = L^* + M^*$, (b) $(cL)^* = cL^*$ for $c \in \mathbb{R}$.

a) First note we have:

$$\langle (L+M)u, v\rangle = \langle Lu + Mu, v\rangle$$
$$= \langle Lu, v\rangle + \langle Mu, v\rangle$$

Then note:

$$\langle Lu, v\rangle = \langle u, L^*v\rangle$$
$$\langle Mu, v\rangle = \langle u, M^*v\rangle$$

Then we have:

$$\langle (L+M)u, v\rangle = \langle u, L^*v\rangle + \langle u, M^*v\rangle$$
$$= \langle u, (L^*+M^*)v\rangle$$

Thus we have:

$$(L+M)^* = L^* + M^*$$

b) $$\langle u, (cL)^*[v]\rangle = \langle\langle (cL)[u], v\rangle\rangle$$
$$= c\,\langle\langle L[u], v\rangle\rangle$$
$$= c\langle u, L^*[v]\rangle$$
$$= \langle u, cL^*[v]\rangle$$

$\square$

9.1.22. Analyze the periodic boundary value problem

$$-u'' = f(x), \qquad u(0) = u(2\pi), \qquad u'(0) = u'(2\pi),$$

along the same lines as in Example 9.12. Characterize the forcing functions for which the problem has a solution. Explain why the constraints, if any, are in accordance with the Fredholm Alternative. Write down a forcing function $f(x)$ that satisfies all your constraints, and then find all corresponding solutions.

First note we have:

$$u(x) = ax + b + \int_0^x \left(\int_0^y f(z)\,dz \right) dy$$

Then note for $u'(0)=0$ we have:

$$a = \frac{1}{2\pi} \int_0^{2\pi} \left[\int_0^y f(z)\,dz \right] dy$$

Then for $u'(\ell) = 0$ we have:

$$\langle f, 1 \rangle = \boxed{\int_0^{2\pi} f(z)\,dz = 0}$$

Let $f(x) = \sin(x)$

$$\Rightarrow u(x) = b + \sin(x)$$