AMATH 503 HW5

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♡ 4.2.6. (a) Formulate the periodic initial-boundary value problem for the wave equation on the interval $-\pi \le x \le \pi$, modeling the vibrations of a circular ring. (b) Write out a formula for the solution to your problem in the form of a Fourier series. (c) Is the solution a periodic function of t? If so, what is the period? (d) Suppose the initial displacement coincides with that in Figure 4.6, while the initial velocity is zero. Describe what happens to the solution as time evolves.

a) $u_{tt} = c^2 u_{xx}$

$u(-\pi, t) = u(\pi, t)$

$u_x(-\pi, t) = u_x(\pi, t)$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

b) Note we have the ODE:

$v''(x) - \lambda v(x) = 0$

① Note $\lambda = 0$ has the solution:

$T_0(t) = At + B$.

② Note $\lambda < 0$ and complex λ is another solution:

Let $\lambda = -w^2$

Then we have: $v(x) = c_1\cos(wx) + c_2\sin(wx)$

BC1: $c_2\sin(-w\pi) = c_2\sin(w\pi)$

$\Rightarrow w = n\pi$, $\lambda = n^2$ for $n \in \mathbb{N}$

BC2: $c_1 n\sin(n\pi) = -c_1 n\sin(n\pi)$

$\Rightarrow T_n(t) = C_n\cos(cnt) + E_n\sin(cnt)$

Hence we have:

$$\boxed{u(x,t) = At + B + \sum_{n=1}^{\infty}\left[a_n\cos(cnt)\cos(nx) + b_n\cos(cnt)\sin(nx) + \frac{d_n}{nc}\sin(cnt) + \frac{e_n}{nc}\sin(cnt)\sin(nx)\right]}$$

C). The solution is periodic in Time:

$ct = \frac{2\pi}{T}t$

$\Rightarrow \boxed{T = \frac{2\pi}{c}}$

d). The solution will behave similarly at first.

But after splits in half, each half wave reflects off the boundary and move back towards middle until they meet and reproduce the same initial conditions.

The process repeats itself.

4.2.11. The initial-boundary value problem

$$u_{tt} = -u_{xxxx}, \qquad \begin{array}{ll} u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, & 0 < x < 1, \\ u(0,x) = f(x), \qquad u_t(0,x) = 0, & t > 0, \end{array}$$

models the vibrations of an elastic beam of unit length with simply supported ends, subject to a nonzero initial displacement $f(x)$ and zero initial velocity. (a) What are the vibrational frequencies for the beam? (b) Write down the solution to the initial-boundary value problem as a Fourier series. (c) Does the beam vibrate periodically (i) for all initial conditions? (ii) for some initial conditions? (iii) for no initial conditions?

a) Note by solving for solution formula we have:

$V''''(x) - \lambda V(x) = 0.$

Then for $\lambda < 0$ we have:

$V(x) = C_1 \cos(wx) + C_2 \sin(wx)$

$\Rightarrow V(0) = C_1 = 0$

$V''(0) \Rightarrow C_1 = 0$

$V(1) = C_1 \cos(w) + C_2 \sin(w)$

$V''(1) = -w^2 C_1 \cos(w) - w^2 C_2 \sin(w)$

$\Rightarrow (C_2 - w^2 C_2) \sin(w) = 0.$

$\lambda = -n^4\pi^4$

$\Rightarrow \boxed{w = n^2\pi^2}$

b) From part a) we know:

$V(x) = \sin(n\pi x)$

Then note we have $\lambda = -n^4\pi^4$

$\Rightarrow T''(t) + n^4\pi^4 T(t) = 0.$

$\Rightarrow T(t) = A\cos(wt) + B\sin(wt)$

$\Rightarrow T_n(t) = \cos(n^2\pi^2 t)$

$$\Rightarrow \boxed{\begin{array}{l} u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n^2\pi^2 t)\sin(n\pi x) \\ A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \end{array}}$$

c) Note since the viberational frequences are rational. Thus the solution is periodic for all initial conditions.

♡ 4.2.27. (a) Explain how to use d'Alembert's formula (4.77) to solve the periodic initial-boundary value problem for the wave equation given in Exercise 4.2.6.

(b) Do the d'Alembert and Fourier series formulae represent the same solution? If so, can you justify it? If not, explain why they are different.

a) Note we can perform the following:

Extend the initial displacement $f(x)$

and initial velocity $g(x)$

Where:

$f(x+l)=f(x)$ and $g(x+l)=g(x)$

Thus we have:

$f(x+2\pi)=f(x)$ and $g(x+2\pi)=g(x)$.

Then the overlapping waves of f and g can be interfered depending on their phases and amplitudes.

b) Since the initial conditions are periodic, the solution will be the same.

Because Fourier series calculate the coefficients for different terms of cos and sin.

In fact we can use F.S. formula to find the coefficients for $g(x)$ in d'Alembert's formula.

◇ 4.2.28. Show that the solution $u(t,x)$ to the wave equation on an interval $[0,\ell]$, subject to periodic boundary conditions $u(t,0)=u(t,\ell)$, $u_x(t,0)=u_x(t,\ell)$, is a periodic function of t if and only if there is no net initial velocity: $\int_0^\ell g(x)\,dx = 0$.

First note the d'Alembert formula states that:

$$u(t,x)=\frac{f(x-ct)+f(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Then by Lemma from lecture we have:

$$u(t+\tfrac{\ell}{c},x)-u(t,x)=\frac{f(x-ct-\ell)-f(x-ct)+f(x+ct+\ell)-f(x+ct)}{2}$$
$$+\frac{1}{2c}\left(\int_{x-ct-\ell}^{x+ct+\ell} g(z)\,dz-\int_{x-ct}^{x+ct} g(z)\,dz\right)$$
$$=\frac{1}{2c}\left(\int_{x+ct}^{x+ct+\ell} g(z)\,dz+\int_{x-ct-\ell}^{x-ct} g(z)\,dz\right)$$
$$=\frac{1}{c}\int_0^\ell g(z)\,dz$$

Then note for $\int_0^\ell g(x)\,dx=0$

$\Rightarrow u(t,x)$ is time periodic with $\frac{\ell}{c}$ period.

However if the condition is not true

$\Rightarrow u(t,x)$ will note be periodic function of t. $\square$

4.3.6. Write down the Dirichlet boundary value problem for the Laplace equation on the unit square $0 \leq x, y \leq 1$ that is satisfied by $u(x, y) = 1 + xy$.

First note we have:

$$\frac{\partial u}{\partial x} = y \quad , \quad \frac{\partial u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = x \quad , \quad \frac{\partial u}{\partial y^2} = 0$$

The the Laplacian of $u(x,y)$ is:

$$\Delta u = 0 + 0 = 0$$

Hence satisfies Laplace equation.

Then note:

① $x=0 \Rightarrow u(0,y) = 1$

② $x=1 \Rightarrow u(1,y) = 1+y$

③ $y=0 \Rightarrow u(x,0) = 1$

④ $y=1 \Rightarrow u(x,1) = 1+x$

Thus we have:

$$\boxed{\begin{array}{l} \Delta u = 0 \quad \text{for } 0 \leq x, y \leq 1 \\ \left.\begin{array}{l} u(0,y) = 1 \\ u(1,y) = 1+y \end{array}\right\rangle \text{for } 0 \leq y \leq 1 \\ \left.\begin{array}{l} u(x,0) = 1 \\ u(x,1) = 1+x \end{array}\right\rangle \text{for } 0 \leq x \leq 1 \end{array}}$$

4.3.15. Find the solution to the boundary value problem

$$\Delta u = 0, \qquad u(x,0) = 2\cos 7\pi x - 4, \quad u(x,1) = 5\cos 3\pi x, \qquad 0 < x, y < 1.$$
$$u_x(0,y) = u_x(1,y) = 0,$$

First note from the seperable solution for Laplace equition we know:

For $v'(0) = v'(1) = 0$ we have:

$$v(x) = \begin{cases} \cos(\omega x), & \lambda = -\omega^2 < 0 \\ x, & \lambda = 0 \\ \cosh(\omega x), & \lambda = \omega^2 > 0 \end{cases}$$

Then by applying the other initial condition we have:

① $y=0$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\sinh(n\pi(1-y)) = 2\cos(7\pi x) - 4$$

$$\Rightarrow A_0 = -4(1-y), \quad A_n = \frac{2}{\sin(7\pi)}$$

② $y=1$

$$u(x,1) = \sum_{n=1}^{\infty} B_n \cos(n\pi x)\sinh(n\pi)$$
$$= 5\cos(3\pi x)$$

$$\Rightarrow B_3 \sinh(3\pi) = 5$$

$$\Rightarrow B_3 = \frac{5}{\sinh(3\pi)} \text{ and } B_n = 0 \text{ for } n \neq 3$$

Thus we have:

$$\boxed{u(x,y) = -4(1-y) + \frac{2}{\sin(7\pi)}\cos(7\pi x)\sinh(7\pi(1-y)) + \frac{5}{\sinh(3\pi)}\cos(3\pi x)\sinh(3\pi y)}$$

4.3.29. Find the equilibrium temperature on a half-disk of radius 1 when the temperature is held to 1° on the curved edge, while the straight edge is insulated.

First note we have:

$T(1,\theta)=1$ for $0\leq\theta\leq\pi$

Then since insulated:

$T_r(0,\theta)=0$

Then note we have:

$\Theta''=-\lambda\Theta$

$r^2R''+rR'-\lambda R=0$

Then note we can solve the eigenvalue problem.

But note for $\lambda=n^2$ let $n=0$

$\Rightarrow$ $\Theta_0(\theta)=1$

$R_0(r)=1$

$\Rightarrow$ $\boxed{T(r,\theta)=1}$

4.3.35. Solve the following boundary value problems for the Laplace equation on the semi-annular domain $D = \{1 < x^2 + y^2 < 2,\ y > 0\}$:

(a) $u(x, y) = 0, \quad x^2 + y^2 = 1, \quad u(x, y) = 1, \quad x^2 + y^2 = 2, \quad u(x, 0) = 0;$

(b) $u(x, y) = 0, \quad x^2 + y^2 = 1$ or $2, \quad u(x, 0) = 0, \quad x > 0, \quad u(x, 0) = 1, \quad x < 0.$

a) First note we have:

$x = r\cos\theta$, $y = r\sin\theta$

Then note the Laplacian for $u(r,\theta)$ is:

$$u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Then we have:

$$\Theta''(\theta) = -\lambda\Theta \quad \text{and} \quad r^2 R''(r) + rR'(r) = \lambda R$$

Then note we can derive the condition:

$u(1,\theta) = 0$, $u(\sqrt{2},\theta) = 1$

$u(r,0) = 0$

Then note we have the solution of the form:

$R(r) = r^n - r^n$

$\Theta(\theta) = \sin(n\theta)$

And the general solution:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n r^n \sin(n\theta) - b_n r^{-n}\sin(n\theta)\right)$$

Then apply the BC. we have:

$$\boxed{u(r,\theta) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(r^{2j+1} - r^{-2j-1})\sin((2j+1)\theta)}{(2^{j+\frac{1}{2}} - 2^{-j-\frac{1}{2}})(2j+1)}}$$