AMATH 503 HW6

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4.3.35. Solve the following boundary value problems for the Laplace equation on the semi-annular domain $D = \{1 < x^2 + y^2 < 2,\ y > 0\}$:
(a) $u(x, y) = 0, \quad x^2 + y^2 = 1, \quad u(x, y) = 1, \quad x^2 + y^2 = 2, \quad u(x, 0) = 0;$
(b) $u(x, y) = 0, \quad x^2 + y^2 = 1$ or $2, \quad u(x, 0) = 0, \quad x > 0, \quad u(x, 0) = 1, \quad x < 0.$

b) First note:

$x = r\cos(\theta)$, $y = r\sin(\theta)$

Then we have:

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Then applying $u(r,\theta) = R(r)\,\Theta(\theta)$ we have:

$$\frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

① Solve for $\Theta''(\theta) + \lambda\Theta(\theta) = 0$, $\Theta(0) = 0$, $\Theta(\pi)$

$\Theta(\theta) = A\cos(\omega\theta) + B\sin(\omega\theta)$

i) $\Theta(0) = A = 0$

ii) $\Theta(\pi) = B\sin(n\pi^2) = B = 1$.

② Solve for $r^2R''(r) + rR'(r) - \lambda R(r) = 0$ $\quad R(1) = 0$, $R(\sqrt{2}) = 0$

$R(r) = C_1 r^{\omega} + C_2 r^{-\omega}$

i) $R(1) = C_1 + C_2 = 0$

ii) $R(\sqrt{2}) = C_1 2^{\frac{\omega}{2}} + C_2 2^{-\frac{\omega}{2}} = 0$

Then we have the general form:

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n (r^n - r^{-n}) \sin(n\pi\theta) \text{ for odd } n.$$

$$\Rightarrow a_n = \frac{4}{n\pi(r^{n\pi} - r^{-n\pi})} \quad \text{for } n \text{ odd.}$$

$$\Rightarrow u(r,\theta) = \boxed{\sum_{j=1}^{\infty} \frac{4}{(2j-1)\pi} \sin((2j-1)\pi\theta)}$$

6.1.1. Evaluate the following integrals: (a) $\int_{-\pi}^{\pi} \delta(x)\cos x\,dx$, (b) $\int_{1}^{2} \delta(x)\,(x-2)\,dx$, (c) $\int_{0}^{3} \delta_1(x)\,e^x\,dx$, (d) $\int_{1}^{e} \delta(x-2)\log x\,dx$, (e) $\int_{0}^{1} \delta\left(x-\frac{1}{3}\right)x^2\,dx$, (f) $\int_{-1}^{1} \frac{\delta(x+2)\,dx}{1+x^2}$.

Note by definition we have:

$$\int_a^b u(x)\,\delta(x-c)\,dx = \begin{cases} 0 & \text{if } c \notin (a,b) \\ u(c) & \text{if } c \in (a,b) \end{cases}$$

a) $\int_{-\pi}^{\pi} \delta(x-0)\cos(x) = \cos(0) = \boxed{1}$

e) $\int_0^1 \delta(x-\frac{1}{3})\,x^2\,dx = (\frac{1}{3})^2 = \boxed{\frac{1}{9}}$

◇ 6.1.7. Explain why the Gaussian functions $g_n(x) = \dfrac{n}{\sqrt{\pi}}\, e^{-n^2x^2}$ have the delta function $\delta(x)$ as their limit as $n \to \infty$.

First note:

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

Then note:

$$\lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} = \begin{cases} ① \; x \neq 0 \Rightarrow 0 \\ ② \; x = 0 \Rightarrow \infty \end{cases}$$

Then we have:

$$\int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} = \frac{n}{\sqrt{\pi}} \frac{\sqrt{\pi}}{n} = 1$$

$\square$

6.2.1. Let $c > 0$. Find the Green's function for the boundary value problem $-cu'' = f(x)$, $u(0) = 0$, $u'(1) = 0$, which models the displacement of a uniform bar of unit length with one fixed and one free end under an external force. Then use superposition to write down a formula for the solution. Verify that your integral formula is correct by direct differentiation and substitution into the differential equation and boundary conditions.

$$-cu'' = f(x), \quad u(0) = 0, \quad u'(1) = 0$$

First note by direct integration we have:

$$u(x) = -\frac{\rho(x-\xi)}{c} + ax + b$$

$$u'(x) = -\frac{\sigma(x-\xi)}{c} + a$$

Then note:

$$u(0) = b = 0$$

$$u'(1) = -\frac{1}{c} + a = 0$$

Then we have the Green's function:

$$\boxed{G(x;\xi) = \begin{cases} \frac{x}{c}, & x \le \xi \\ \frac{\xi}{c}, & x \ge \xi \end{cases}}$$

Then by superposition formula we have:

$$u(x) = \int_0^1 G(x;\xi) f(\xi)\, d\xi = \boxed{\frac{1}{c}\int_0^x \xi f(\xi)\, d\xi + \frac{x}{c}\int_x^1 f(\xi)\, d\xi}$$

Then note:

$$u'(x) = x f(x) - x f(x) + \frac{1}{c}\int_x^1 f(\xi)\, d\xi = \frac{1}{c}\int_x^1 f(\xi)\, d\xi$$

$$u''(x) = -\frac{1}{c} f(x)$$

Then we have:

$$\boxed{\begin{aligned} u(0) &= 0 \\ u'(1) &= 0 \end{aligned}}$$

6.2.3. A point 2 cm along a 10 cm bar experiences a displacement of 1 mm under a concentrated force of 2 newtons applied at the midpoint of the bar. How far does the midpoint deflect when a concentrated force of 1 newton is applied at the point 2 cm along the bar?

First note:

a force of 2 N is applied at $x=5cm$

The let $\delta_{5\to2}$ be deflection at $x=2$

Then note we have:

$$F\cdot\delta_{5\to2} = 2\cdot\delta_{5\to2} = 1mm$$

$$\Rightarrow \delta_{5\to2} = \frac{1}{2}mm$$

Then note by symmetry of the Green's function we have:

$$\delta_{2\to5} = \delta_{5\to2} = \frac{1}{2}mm$$

Then note by linearity of the Green's function we have:

for $F = 1N \Rightarrow F\delta_{2\to5} = 1\cdot\delta_{2\to5} = 1N\cdot\frac{1}{2}mm.$

$$\Rightarrow \boxed{\frac{1}{2}mm}$$

♡ 6.2.4. The boundary value problem $-\dfrac{d}{dx}\left(c(x)\dfrac{du}{dx}\right) = f(x)$, $u(0) = u(1) = 0$, models the displacement $u(x)$ of a nonuniform elastic bar with stiffness $c(x) = \dfrac{1}{1+x^2}$ for $0 \le x \le 1$. (a) Find the displacement when the bar is subjected to a constant external force, $f \equiv 1$. (b) Find the Green's function for the boundary value problem. (c) Use the resulting superposition formula to check your solution to part (a). (d) Which point $0 < \xi < 1$ on the bar is the "weakest", i.e., the bar experiences the largest displacement under a unit impulse concentrated at that point?

a) First note plug in $f \equiv 1$ we have:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = 1$$

$$\Rightarrow -\frac{1}{1+x^2}\frac{du}{dx} = x + C_1$$

$$\Rightarrow \frac{du}{dx} = -(1+x^2)(x+C_1)$$

$$\Rightarrow u(x) = -\int (1+x^2)(x+C_1)\,dx + C_2$$

$$= -\frac{x^2}{2} - \frac{x^4}{4} - C_1\left(x + \frac{x^3}{3}\right) + C_2$$

They apply $u(0)=0$ and $u(1)=0$

① $u(0) = C_2 = 0$

② $u(1) = -\frac{3}{4} - \frac{4}{3}C_1 = 0 \Rightarrow C_1 = -\frac{9}{16}$

$$\boxed{u(x) = \frac{9}{16}x - \frac{x^2}{2} + \frac{3}{16}x^3 - \frac{x^4}{4}}$$

b) ① $\frac{d}{dx}\left((1+x^2)^{-1}\frac{dG}{dx}\right) = 0$

$$\Rightarrow (1+x^2)^{-1}\frac{dG}{dx} = C$$

$$\Rightarrow G(x) = C\left(x + \frac{x^3}{3}\right) + D$$

② $G(0,\xi)=0$ and $G(1,\xi)=0$

i) $x < \xi \Rightarrow G(x) = C_1\left(x + \frac{x^3}{3}\right)$

ii) $x > \xi \Rightarrow G(x) = C_2\left(x + \frac{x^3}{3}\right) + D_2$

③ $x = \xi \Rightarrow C_1\left(\xi + \frac{\xi^3}{3}\right) = C_2\left(\xi + \frac{\xi^3}{3}\right) + D_2$

④ $C_2 - C_1 = -\frac{1}{(1+\xi^2)^2}$

$$\Rightarrow \boxed{G(x;\xi) = \begin{cases} \left(1 - \frac{3}{4}\xi - \frac{1}{4}\xi^3\right)\left(x + \frac{1}{3}x^3\right), & x \le \xi \\ \left(1 - \frac{3}{4}x - \frac{1}{4}x^3\right)\left(\xi + \frac{1}{3}\xi^3\right), & x \ge \xi \end{cases}}$$

c) $u(x) = \int_0^1 G(x;\xi)\,d\xi$

$$= \int_0^1 \left(1 - \frac{3}{4}x - \frac{x^3}{4}\right)\left(\xi + \frac{\xi^3}{3}\right)d\xi + \int_x^1 \left(1 - \frac{3}{4}\xi - \frac{\xi^3}{4}\right)\left(x + \frac{x^3}{3}\right)d\xi$$

$$= \frac{9}{16}x - \frac{x^2}{2} + \frac{3}{16}x^3 - \frac{x^4}{4}$$

✓