AMATH 503 HW4

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◇ 3.4.6. Write down formulas for the Fourier series of both even and odd functions on $[-\ell, \ell]$.

First note we have the Fourier series formula:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos(ky)\,dy$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin(ky)\,dy$$

> for $x = \frac{\ell}{\pi} y$

① Even function

$$a_0 = \boxed{\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx} = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_k = \boxed{\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{k\pi x}{\ell}\right)dx} = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{k\pi x}{\ell}\right)dx$$

Note even function only have cosines. $\Rightarrow b_k = 0$

$$\boxed{\tilde{f}(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right)}$$

② Odd function

Note for odd function the Fourier series only contain sines:

$\Rightarrow a_0 = 0$, $a_k = 0$

$$b_k = \boxed{\frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx} = -\frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx$$

$$\boxed{\tilde{f}(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right)}$$

4.1.1. Suppose the ends of a bar of length 1 and thermal diffusivity $\gamma = 1$ are held fixed at respective temperatures $0°$ and $10°$. (a) Determine the equilibrium temperature profile. (b) Determine the rate at which the equilibrium temperature profile is approached. (c) What does the temperature profile look like as it nears equilibrium?

a) Note we have:

$$\frac{d^2u^*}{dx^2} = 0 \implies u^*(x) = ax + b$$

$$\implies u^*(0) = 0 \implies b = 0$$

$$\implies u^*(1) = 10 \implies a = 10$$

Hence we have:

$$\boxed{u^*(x) = 10x}$$

b) Note we have to solve:

$$\gamma \frac{d^2v}{dx^2} + \lambda v = 0 \quad \text{for } v(0) = v(1) = 0$$

$$\implies v(x) = a\cos\omega x + b\sin\omega x \quad \text{for } \omega = \sqrt{\lambda/\gamma}$$

$$\implies \omega = \frac{\pi}{\ell}, \frac{2\pi}{\ell}, \frac{3\pi}{\ell} \dots$$

$$\implies \lambda_k = \gamma\left(\frac{k\pi}{\ell}\right)^2 = \pi^2$$

$\implies$ The rate is $\boxed{e^{-\pi^2 t}}$

c) for $t \gg 0$ we have:

$$u(t,x) = \boxed{10x + ce^{-\pi^2 t}\sin(\pi x), \text{ for } c \neq 0}$$

4.1.4. Find a series solution to the initial-boundary value problem for the heat equation $u_t = u_{xx}$ for $0 < x < 1$ when one the end of the bar is held at $0°$ and the other is insulated. Discuss the asymptotic behavior of the solution as $t \to \infty$.

First note we have to solve:

$\frac{d^2v}{dx^2} + \lambda v = 0$ for $v(0) = 0$ and $v'(1) = 0$

for $\lambda > 0$ we have:

$\Rightarrow v(x) = \alpha \cos \omega x + \beta \sin \omega x$ for $\omega^2 = \lambda^2$

$v'(x) = -\omega\alpha \sin(\omega x) + \omega\beta \cos(\omega x)$

① $v(0) = 0$

$\Rightarrow \alpha = 0$

② $v'(1) = 0$

$\Rightarrow \omega\beta \cos(\omega) = 0$

$\Rightarrow \omega = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$= (k+\frac{1}{2})\pi$

$\Rightarrow \lambda = (k+\frac{1}{2})^2\pi^2$

Hence we have:

$$\boxed{\begin{aligned} u(t,x) &= \sum_{k=0}^{\infty} c\, e^{-(k+\frac{1}{2})^2\pi^2 t} \sin((k+\tfrac{1}{2})\pi x) \\ c &= 2\int_0^1 f(x)\sin((k+\tfrac{1}{2})\pi x)\,dx \quad \text{where } u(0,x) = f(x) \end{aligned}}$$

Note that it will decay to zero exponentially

For initial conditions for $c_1 \neq 0$, the decay rate is $e^{-\frac{\pi^2 t}{4}}$.

The solution will mostly look like $\sin(\frac{\pi}{2}x)$.

4.1.7. A metal bar of length $\ell = 1$ and thermal diffusivity $\gamma = 1$ is fully insulated, including its ends. Suppose the initial temperature distribution is $u(0,x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1. \end{cases}$

(a) Use Fourier series to write down the temperature distribution at time $t > 0$.
(b) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$?
(c) How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss.

a) First note we have to solve:

$$\frac{d^2v}{dx^2} + \lambda v = 0 \quad \text{for } v(0)=0 \quad \text{and } v'(1)=0$$

for $\lambda > 0$ we have:

$$V(x) = \alpha\cos(wx) + \beta\sin(wx) \quad \text{for } w^2 = \lambda$$

$$V'(x) = -w\alpha\sin(wx) + w\beta\cos(wx)$$

① $V'(0) = 0$

$$\Rightarrow w\beta = 0 \Rightarrow \beta = 0$$

② $V'(L) = 0$

$$\Rightarrow -w\alpha\sin(Lw) = 0$$

$$\Rightarrow w = \frac{k\pi}{L}$$

$$\Rightarrow u_k(t,x) = A_k e^{-\gamma(\frac{k\pi}{L})^2 t}\cos\left(\frac{n\pi}{L}x\right)$$

Then we have:

$$u(t,x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k e^{-\gamma(\frac{k\pi}{L})^2 t}\cos\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow A_0 = \int_0^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^{1} 1-x\,dx = \frac{1}{2}$$

$$A_k = 2\int_0^1 f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$\Rightarrow \frac{A_k}{2} = \frac{2\cos(\frac{k\pi}{2}) - 1 - (-1)^k}{k^2\pi^2}$$

$$\boxed{u(x,t) = \frac{1}{4} + 2\sum_{k=1}^{\infty} \frac{2\cos(\frac{k\pi}{2}) - 1 - (-1)^k}{k^2\pi^2}\cos(k\pi x)e^{-\gamma k^2\pi^2 t}}$$

b) For $t \gg 0$ note:

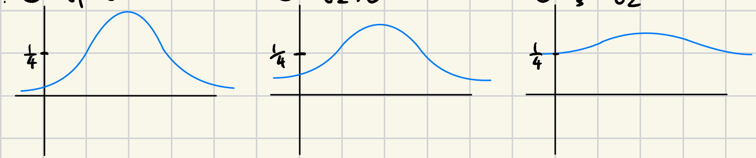
$$\lim_{t\to\infty} e^{-\gamma k^2\pi^2 t} = 0$$

$$\Rightarrow \lim_{t\to\infty} u(x,t) = \boxed{\frac{1}{4}}$$

c) The solution is approached at exponential speed.

$$\Rightarrow e^{-\gamma n^2\pi^2 t} \ll 1.$$

d). ① $t_1 = 0$ ② $t_2 > 0$ ① $t_3 > t_2$



The graph shows eventually the temperature will go towards 0.25 and also become smooth.

4.1.10. For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature as $t \to \infty$: (a) $\cos x$, (b) $\sin^3 x$, (c) $|x|$, (d) $\begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$

a) i) Note $\cos(x)$ is already a Fourier cosine series:

$$u(x,t) = \boxed{\cos(x)\, e^{-t}}$$

ii) $u(x,\infty) = \frac{1-1}{2} = \boxed{0}$

b) i) Note $\sin^3(x) = \frac{3\sin x - \sin 3x}{4}$

$$u(x,t) = \boxed{\frac{3}{4}\sin(x) e^{-t} - \frac{1}{4}\sin(3x)\cdot e^{-9t}}$$

ii) $u(x,\infty) = \boxed{0}$

c) i) Note $|x|$ is even hence we only consider cosine.

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$$

$$a_k = \frac{2}{\pi}\int_0^\pi x\cos(kx)dx \quad , u = x, dv = \cos(kx)$$

$$= \frac{2}{\pi}\left[\frac{1}{k^2}\cos(kx)\Big|_0^\pi\right] = \frac{2}{\pi}\frac{\cos(kx)}{k^2} \text{ for } k = 1, 3, 5, \ldots$$

$$= \frac{2}{\pi}\frac{\cos((2j-1)x)}{(2j-1)^2}$$

$$\Rightarrow u(x,t) = \boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j-1)x)}{(2j-1)^2} e^{-(2j-1)^2 t}}$$

ii) $u(x,\infty) = \boxed{\frac{\pi}{2}}$

d) i) $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^0 1\,dx + \int_0^\pi 0\,dx\right] = 1$

$$a_k = \frac{1}{\pi}\int_{-\pi}^0 \cos(kx)dx = \frac{1}{\pi}\left[\frac{\sin(kx)}{k}\Big|_{-\pi}^0\right] = 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^0 \sin(kx)dx = \frac{1}{\pi}\left[\frac{\cos(kx)}{k}\Big|_{-\pi}^0\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{k} - \frac{\cos(k\pi)}{k}\right]$$

$$= \frac{1}{\pi}\frac{\cos((2j-1)\pi)}{(2j-1)} \text{ for } j = 1, 2, \ldots$$

$$u(x,t) = \boxed{\frac{1}{2} - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j-1)\pi)}{(2j-1)} e^{-(2j-1)^2 t}}$$

ii) $u(x,\infty) = \boxed{\frac{1}{2}}$

◇ 4.1.17. The *convection-diffusion equation* $u_t + cu_x = \gamma u_{xx}$ is a simple model for the diffusion of a pollutant in a fluid flow moving with constant speed c. Show that $v(t, x) = u(t, x + ct)$ solves the heat equation. What is the physical interpretation of this change of variables?

First let $v(t,x) = u(t, x+ct)$

Then we have:

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

And also note

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

Then plug in the expressions we have:

$$v_t = \gamma u_{xx} = \gamma v_{xx}$$

Hence $v(t,x)$ solves the equation. $\square$

Interpretation:

The process of change of variable can be viewed as a shift. We are moving along the fluid flow with velocity of c.

In the shifted frame we are viewing the diffusion as there is no velocity. Which simplifies the problem.

◇ 4.1.19. Let $\gamma > 0$ and $\lambda \le 0$. (a) Find all solutions to the differential equation $\gamma\, v'' + \lambda\, v = 0$. (b) Prove that the only solution that satisfies the boundary conditions $v(0) = 0$, $v(\ell) = 0$, is the zero solution $v(x) \equiv 0$.

a) Note we have to solve:

$$v'' + \frac{\lambda}{\gamma} v = 0$$

① $\lambda = 0$

$\Rightarrow v'' = 0$

$\Rightarrow \boxed{v(x) = Ax + B}$

② $\lambda < 0$

$\Rightarrow -w^2 = \frac{\lambda}{\gamma}$

$\boxed{v(x) = C_1 e^{wx} + C_2 e^{-wx}}$

b) We have $v(0) = 0$, $v(\ell) = 0$

① $\lambda = 0$

$v(0) = B = 0$

$v(\ell) = A\ell = 0 \Rightarrow A = 0$

$v(x) \equiv 0$ ✓

② $\lambda < 0$

$v(0) = C_1 + C_2 = 0$

$v(\ell) = C_1 e^{w\ell} + C_2 e^{-w\ell} = 0$

Note $w = \sqrt{\frac{\lambda}{\gamma}} \neq 0$

$\Rightarrow C_1 = C_2 = 0$

$\Rightarrow v(x) \equiv 0$ ✓

4.2.2. How much longer would a piano string have to be to make the same sound when it is pulled twice as tight?

Note we have the frequency:

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

L: length

T: tension of the string

μ: linear mass density of the string

Then set the frequencies to be equal we have:

$$\frac{1}{2L_1}\sqrt{\frac{T}{\mu}} = \frac{1}{2L_2}\sqrt{\frac{2T}{\mu}}$$

$$\Rightarrow \frac{L_2}{L_1} = \sqrt{2}$$

$$\Rightarrow L_2 = \sqrt{2}\, L_1$$

Hence the length have to be $\boxed{\sqrt{2} \text{ of the original length}}$