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6.1. If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Note since P is an orthogonal projector we have the following two properties:

i) $P^2 = P$

ii) $P = P^*$

Then note:

$$(I-2P)(I-2P)^* = (I-2P)(I^*-2P^*)$$
$$= (I-2P)(I-2P)$$
$$= I^2 - 4P + 4P^2$$

Note $P^2 = P$, thus we have:

$$(I-2P)(I-2P)^* = I^2 - 4P + 4P = I^2 = I$$

Similarly $(I-2P)^*(I-2P) = I$

Thus $I-2P$ is unitary $\square$

Geometric Interpretation:

Note $I-P$ will reflects a vector over the orthogonal complement of the subspace.

Then $I-2P$ will reflects a vector across the subspace where it also scales by a factor of 2.

Double the magnitude

6.2. Let E be the $m \times m$ matrix that extracts the "even part" of an m-vector: $Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, \ldots, x_m)^*$ to $(x_m, \ldots, x_1)^*$. Is E an orthogonal projector, an oblique projector, or not a projector at all? What are its entries?

First note since F flips $(x_1, \cdots, x_m)^*$ to $(x_m, \cdots, x_1)^*$
Then we have:

$$F = \begin{bmatrix} 0 & \cdots & & 1 \\ \vdots & & ⋰ & \vdots \\ 1 & \cdots & & 0 \end{bmatrix}$$

Then take x as I using linearity we have:

$$E = \frac{I+F}{2}$$

$$\Rightarrow E^2 = \frac{I + 2F + F^2}{4}$$

Then note F^2 flips the vector twice where turns the vector back to original:

$$\Rightarrow F^2 = I$$

Then plug this back in we have:

$$E^2 = \frac{I + 2F + I}{4} = \frac{2I + 2F}{4} = \frac{I+F}{2} = E$$

Thus E is a projector.

Then note:

$$E^* = \frac{I^* + F^*}{2} = \frac{I+F}{2} = E$$

Thus E is an orthogonal projector where:

$$E = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}$$

where $\frac{1}{2}$ on main and off diagonal and zero elsewhere.

7.5. Let A be an $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

(b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. What does this imply about the rank of A? Exactly k? At least k? At most k? Give a precise answer, and prove it.

a) Suppose A has rank n

$\Rightarrow$ A has linearly independent columns

Then note by the definition of reduced QR factorization we have:

$Q: m \times n$

$R: n \times n$

Note that Q forms an orthonormal basis for $\mathrm{col}(A)$

Then we have R must have linearly independent columns.

$\Rightarrow$ This make the diagonal of R nonzero.

Conversely, suppose R have non-zero diagonals

$\Rightarrow$ R has linearly independent columns.

Then since Q has orthonormal columns

We have $A = QR$ and A also has linearly independent columns.

Thus A has rank n.

b) Suppose R has k nonzero diagonal entries

$\Rightarrow$ R has k linearly independent columns.

Then note there are $n-k$ columns that are linear combination of other columns.

Then we have:

$A = QR$

Since Q is orthonormal

We have each columns of A is a linear combination of columns of Q with column of R as weights.

Since $n-k$ columns of R are combination of k lin. indep. columns.

We have $\mathrm{Rank}(A) = \boxed{k}$

$\square$

3. **Even skinnier QR.** Let $A \in \mathbb{C}^{m \times n}$, $m \geq n$, and suppose $\operatorname{rank}(A) = k < n$. Prove that there exists a factorization $A = Q_k R_k P^T$, where Q_k is an $m \times k$ matrix with orthonormal columns, R_k is an upper triangular $k \times n$ matrix (think of this as "upper trapezoidal"), and P is a permutation matrix. *Note: Algorithms that compute this factorization are called pivoted QR decomposition algorithms.*

Let $\operatorname{rank}(A) = k$

Then we know that there are k linearly independent column in A

Then we can reorder A so that the linearly independent columns comes first.

We let p be the permutation matrix for this then we have:

AP

Then if we perform the G-R decomposition for the first k columns of AP we will have:

Q_k which by the GR process will have orthonormal columns with size $m \times k$.

Then note by QR decomposition we will have:

$A_k = Q_k R_k$ where R_k is upper triangular with the size of $k \times k$

first k columns of AP

Then the remaining columns can be expressed as:

$A_{n-k} = Q_k R_{n-k}$ where R_{n-k} is the coefficients of the linear combination with size $k \times (n-k)$

Combine R_k and R_{n-k} and possibly some rearranging we will result in:

$R_k \to k \times n$ "trapezoidal" matrix

$\Rightarrow A = Q_k R_k P^T$

$\square$

4. a) pseudocode for LRsvd:

i) Compute the QR decomposition of $X \longrightarrow Q_X, R_X$

ii) Compute the QR decomposition of $Y \longrightarrow Q_Y, R_Y$

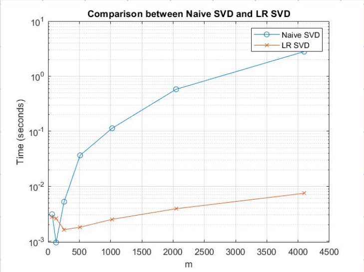
iii) Let $B = R_X \cdot R_Y^*$

iv) Then calculate the SVD on B: $[U_b, \Sigma, V_b] = \text{svd}(B)$

v) $U = Q_x \cdot U_b$

vi) $V = Q_y \cdot V_b$

vii) Return U, Σ, V

c) 

From the accuracy test we can see that the error is really small.
Which means that LRSVD algorithm is pretty accurate.
And from the timing test we can see that LRSVD is faster than the normal svd algorithm.
As m become bigger the difference become more obvious.
Since LRSVD run in $O(mk^2+k^3)$ but naive algorithm runs in $O(m^2n)$.
LRSVD will be faster as m become bigger