HW 2

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3.2. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

Let λ be an eigenvalue of A with the correspondance eigenvector x.

Then by definition we will have:

$Ax = \lambda x$

$\Longrightarrow \|Ax\| = \|\lambda x\|$

Then by definition of norm we have:

$\|\lambda x\| = |\lambda| \|x\|$

Then by triangle inequality we have:

$\|Ax\| \leq \|A\| \|x\|$

Then combining the two results we have:

$\|A\| \|x\| \geqslant |\lambda| \|x\|$

$\Longrightarrow \|A\| \geqslant |\lambda|$

Since $\rho(A)$ is the largest $|\lambda|$ of A we have:

$\|A\| \geqslant \rho(A)$. $\square$

4.4. Two matrices $A, B \in \mathbb{C}^{m\times m}$ are *unitarily equivalent* if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

Suppose A and B are unitarily equivalent.

Then we have:

$A = QBQ^*$ where $Q \in \mathbb{C}^{m\times m}$ is unitary

Then suppose B's SVD is:

$B = U\Sigma V^*$

Then we have:

$A = QU\Sigma V^* Q^* = (QV)\Sigma(V^*Q^*)$

Then by socks and shoes theorem we have:

$V^*Q^* = (QU)^*$

Then we have:

$A = (QV)\Sigma(QV)^*$

Thus B and A shares same singular values.

$\Longrightarrow$ Suppose A and B have the same singular values:

Let A and B's SVD be:

$A = U_1\Sigma V_1^*$

$B = U_2\Sigma V_2^*$

Then since U_2 and V_2 are both unitary we have:

$\Sigma = U_2^* B V_2$

Then plug in the new expression into A we have:

$A = U_1U_2^* B V_2 V_1^*$

However, note $U_1U_2^* \neq V_1V_2^*$

Thus the converse is not true.

$\boxed{\text{Hence the statement is false}}$. □

5.3. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Find A^{-1} not directly, but via the SVD.

(e) Find the eigenvalues λ_1, λ_2 of A.

(f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

a). $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$ $A^* = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}$

① $AA^* = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 4+121 & 20+55 \\ 20+55 & 100+25 \end{bmatrix} = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix}$

② $\det(AA^* - \lambda I) = \begin{vmatrix} 125-\lambda & 75 \\ 75 & 125-\lambda \end{vmatrix} = (125-\lambda)^2 - 75^2 = 15625 - 250\lambda + \lambda^2 - 5625$

$= \lambda^2 - 250\lambda + 10000$

$= (\lambda - 50)(\lambda - 200)$

$\lambda_1 = 50$, $\lambda_2 = 200$

③ $\lambda_1 = 50$

$(AA^* - \lambda_1 I)\vec{x} = \begin{bmatrix} 125-50 & 75 \\ 75 & 125-50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 75 & 75 \\ 75 & 75 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{1+1}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 200$

$(AA^* - \lambda_2 I)\vec{y} = \begin{bmatrix} 125-200 & 75 \\ 75 & 125-200 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -75 & 75 \\ 75 & -75 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \frac{1}{\sqrt{1+1}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then note we have: $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{200} & 0 \\ 0 & \sqrt{50} \end{bmatrix} = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$

④ Then note $V_i = \frac{1}{\sigma_i} A^* u_i$

$V_1 = \frac{1}{10\sqrt{2}}\begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{5\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{11}{10\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{10} - \frac{1}{2} \\ \frac{11}{20} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

$V_2 = \frac{1}{5\sqrt{2}}\begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5\sqrt{2}} & -\frac{2}{\sqrt{2}} \\ \frac{11}{5\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} -\frac{2}{10} + 1 \\ \frac{11}{10} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

Then we have:

$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ $\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$ $V = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

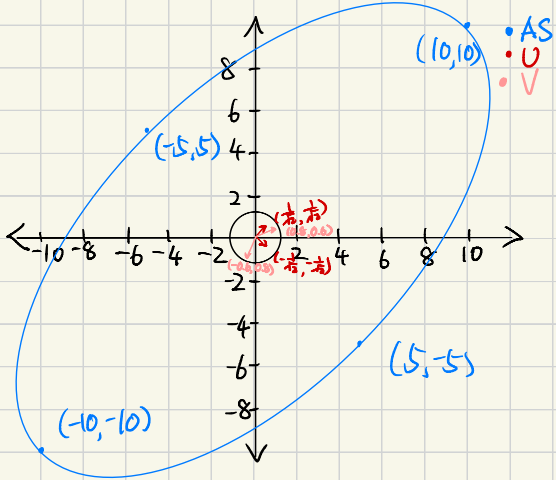
b) Singular values: $10\sqrt{2},\ 5\sqrt{2}$

Left singular vectors: $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Right singular vectors: $\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

$$\sigma_1 u_1 = 10\sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\sigma_2 u_2 = 5\sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$



c) $\|A\|_1 = 16$

$\|A\|_2 = 10\sqrt{2}$

$\|A\|_\infty = 15$

$\|A\|_F = 250$

d) $A^{-1} = V\Sigma^{-1}U^T$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{50\sqrt{2}} & \frac{4}{25\sqrt{2}} \\ \frac{4}{50\sqrt{2}} & \frac{3}{25\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{100} + \frac{4}{50} & -\frac{3}{100} - \frac{4}{50} \\ \frac{4}{100} + \frac{3}{50} & \frac{4}{100} - \frac{3}{50} \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & -\frac{11}{100} \\ \frac{11}{100} & -\frac{1}{50} \end{bmatrix}$$

e) $$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{vmatrix} = (-2-\lambda)(5-\lambda)+110$$
$$= -10-5\lambda+2\lambda+\lambda^2+110$$
$$= \lambda^2-3\lambda+100$$

$$\lambda = \frac{3 \pm \sqrt{9-400}}{2} = 1.5 \pm \frac{\sqrt{-391}}{2}$$

$$\lambda_1 = \boxed{1.5 + \frac{\sqrt{-391}}{2}},\quad \lambda_2 = \boxed{1.5 - \frac{\sqrt{-391}}{2}}$$

f) $\det(A) = -2\cdot 5 + 10\cdot 11 = -10 + 110 = 100$

$$\lambda_1\lambda_2 = \left(1.5+\frac{\sqrt{-391}}{2}\right)\left(1.5-\frac{\sqrt{-391}}{2}\right) = 1.5^2 - \left(\frac{\sqrt{-391}}{2}\right)^2 = 2.25 - \left(\frac{-391}{4}\right) = 100$$

$\Rightarrow |A| = \lambda_1\lambda_2$

$\sigma_1\sigma_2 = 10\sqrt{2}\cdot 5\sqrt{2} = 50\cdot 2 = 100 = |\det A|$ □

g) $x = \sqrt{10^2+10^2} = \sqrt{200}$ $\quad y = \sqrt{5^2+5^2} = \sqrt{50}$

$A = \pi\sqrt{200}\sqrt{50} = \boxed{100\pi}$

4. Flag Compressions

a)

Matrix	Rank	Low Rank
A	1	✓
B	2	✓
C	2	✓
D	3	✓
E	5	✗

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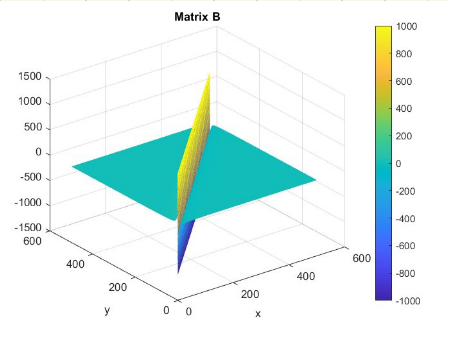
b) For A: $a = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$ $b = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$

c) For B: $a = \begin{bmatrix}0\\0\\1\\1\\0\\0\end{bmatrix}$ $b = \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$ $c = \begin{bmatrix}1\\1\\0\\0\\1\\1\end{bmatrix}$ $d = \begin{bmatrix}0\\0\\1\\1\\0\\0\end{bmatrix}$

For C: $a = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}$ $b = \begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$ $c = \begin{bmatrix}0\\1\\0\\1\\0\\1\end{bmatrix}$ $d = \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$

d) Low rank flags have a lot repeated patterns and there are a lot of redundancy. Full rank flags tend to have more complex patterns. Note that identity matrix have full rank which means that each column can't be expressed by a combination of other columns. Thus a full rank flag will have different pattern.

e)

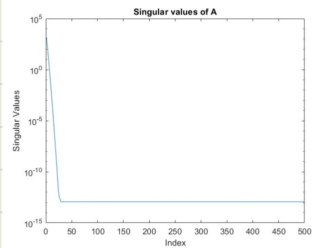


I think the matrix is low rank. After looking at the image in 2D, I can see there is a pattern in the graph.

Coding:

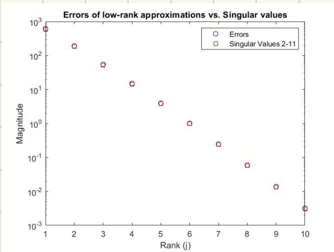
```
A = [1 1 1 1 1 1;
     1 1 1 1 1 1;
     1 1 1 1 1 1;
     1 1 1 1 1 1;
     1 1 1 1 1 1];
[u_a, s_a, v_a] = svd(A);

B = [0 0 1 1 0 0;
     0 0 1 1 0 0;
     1 1 1 1 1 1;
     1 1 1 1 1 1;
     0 0 1 1 0 0;
     0 0 1 1 0 0];
[u_b, s_b, v_b] = svd(B);

C = [1 1 1 0 0 0;
     1 1 1 1 1 1;
     0 0 0 0 0 0;
     1 1 1 1 1 1;
     0 0 0 0 0 0;
     1 1 1 1 1 1];
[u_c, s_c, v_c] = svd(C);

D = [1 0 0 0 0 1;
     0 1 0 0 1 0;
     0 0 1 1 0 0;
     0 0 1 1 0 0;
     0 1 0 0 1 0;
     1 0 0 0 0 1];
[u_d, s_d, v_d] = svd(D);

E = [1 1 0 0 0 0;
     1 1 1 0 0 0;
     0 1 1 1 0 0;
     0 0 1 1 1 0;
     0 0 0 1 1 1;
     0 0 0 0 1 1];
[u_e, s_e, v_e] = svd(E);

sv1 = diag(s_a);
fprintf('Diagonal of matrix s of A:\n');
disp(sv1);

sv2 = diag(s_b);
fprintf('Diagonal of matrix s of B:\n');
disp(sv2);

sv3 = diag(s_c);
fprintf('Diagonal of matrix s of C:\n');
disp(sv3);
sv4 = diag(s_d);
fprintf('Diagonal of matrix s of D:\n');
disp(sv4);

sv5 = diag(s_e);
fprintf('Diagonal of matrix s of E:\n');
disp(sv5);
```

5. a)



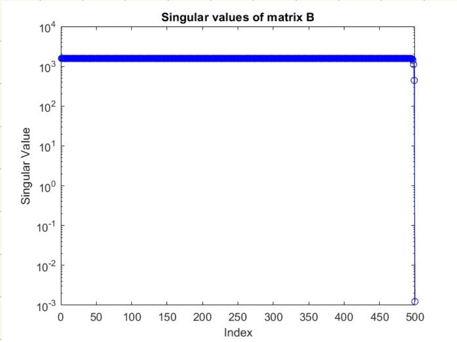
I noticed that the singular kept decreasing

b)



I noticed that the error $E(j)$ is almost identical to the singular value
Just like what Eckart-Mirsky-Young Theorem stated.

c)



By Eckart-Young-Mirsky Theorem, the error in the 2-norm for rank 10 approximation is the 11th singular value.
Then the lower bound for 2-norm is:
1.5708×10^{-3} (which is also the 11th singular value).
The lower bound for Frobenius norm is:
$\|E\|_F = \sqrt{S_{12}^2 + \cdots + S_{500}^2}$

There isn't any drop after the 10th singular value.
Thus I don't think a rank 10 matrix exists that effectively approximate the matrix.