HW 4 Tianbo Zhang 1938501

1. **10.1.** Determine the (a) eigenvalues, (b) determinant, and (c) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof.

a) Eigenvalues

First note the householder reflector is:

$$F = I - \frac{2vv^*}{v^*v}$$

Then suppose x is an eigenvector of F with eigenvalue λ.

Then we have:

$$Fx = \lambda x$$

$$\left(I - \frac{2vv^*}{v^*v}\right)x = \lambda x$$

$$x - \frac{2vv^*}{v^*v}x = \lambda x$$

$$(1-\lambda)x = \frac{2vv^*}{v^*v}x$$

Note for the reflections there are two possibilities.

① x is orthogonal to v.

Then we have $v^*x = 0$

$\Rightarrow (1-\lambda)x = 0$

$$\boxed{\lambda = 1}$$

② x is in the direction of v.

Then we have $v = \alpha x$ for α is the scalar factor.

$\Rightarrow (1-\lambda)\alpha v = \frac{2vv^*\alpha v}{v^*v} = 2\alpha v \cancel{v^*v} / \cancel{v^*v} = 2\alpha v$

$\Rightarrow (1-\lambda) = 2$

$$\boxed{\lambda = -1}$$

Geometric Interpretation for eigenvalue:

Note the Householder reflector reflcts vectors over a hyperplane.

Then there are two options:

① Fx is the orthogonal projection of x.

Note x remain unchanged after reflection.

Then it's scaled by the factor of 1

② Fx is in the direction of x.

Then it gets negated.

Then it's scaled by the factor of -1.

b) Determinant

Note the determinant of F is the product of eigenvalues.

Then note all eigenvalues are -1 or 1.

$\Rightarrow \det(F) = -1$

c) Note the singular values are the eigenvalues of F^*F

Note since F is a reflection matrix we have:

$F^*F = I$

Note eigenvalue of I is 1.

Thus the singular value is $\boxed{1}$.

2、a)

```
function y = circmatvec(c,v)
%computes y = Cv, where C is the circulant matrix with column 1 = c.
d = fft(c);
y = ifft(d.*(fft(v)));
end
```

d=fft(c)

Note fft() will caculate the calculate the Fast Fourier Transform.

Thus fft will return the diagonal elements of $\Lambda = \text{diag}(Fc)$

Then note:

fft(v) computes Fv

Then d.* fft(v) will have:

$$FCF^{-1}Fv = FCv$$

Then ifft(FCv) will return:

$$F^{-1}FCv = Cv = y$$

proof. First note we have:

$$FCF^{-1} = \Lambda$$

$$F^{-1}FCv = y$$

Then note:

$$Cv = y_{true}$$

$$\Rightarrow FCv = Fy_{true}$$

$$\Rightarrow F^{-1}FCv = y_{true}$$

$$\Rightarrow \quad y = y_{true} \quad \square$$

(b) **Fast Toeplitz matvec.** An $m \times m$ Toeplitz matrix T that is not circulant can be embedded in a $2m \times 2m$ circulant matrix,

$$C_T = \begin{bmatrix} T & B \\ B & T \end{bmatrix}.$$

Figure out what B should be, and then use this fact to write a function `y = toepmatvec(ct,rt, v)` that computes $y = Tv$ in only $\mathcal{O}(m \log m)$ operations. Include a description of B in your writeup, and a pseudocode description of your algorithm.

Note If we want to make the matrix T in a circulant way, than we would wrap around such that the new elements on the right is a continuation of diagonals of T.

Take the followin:

$$T = \begin{bmatrix} t_0 & t_1 & t_2 \\ t_{-1} & t_0 & t_1 \\ t_{-2} & t_{-1} & t_0 \end{bmatrix} \longrightarrow B = \begin{bmatrix} t_0 & t_{-2} & t_{-1} \\ t_2 & t_0 & t_{-2} \\ t_1 & t_2 & t_0 \end{bmatrix}$$

Note since T has t_0 as a constant on the diagonal. We can keep t_0 in B

Then note for the last column of B what we did is Take the first column of T, the remove first element Then add t_0 to the end.

Then for the 2nd last column we delete the first entry of last column and append first element of last column of T.

And repeat this process.

So we have $B = \begin{bmatrix} t_0 & t_{-m+1} & \cdots & t_{-1} \\ t_{m-1} & t_0 & & t_{-2} \\ \vdots & & \ddots & \vdots \\ & & t_0 & t_{-m+1} \\ t_1 & & & t_0 \end{bmatrix}$

Pseudocode $\boxed{\text{if we want to compute } B}$:

1. Put first element of C_t to last element than store in last column of B
2. for: $i = 1$, $i \leq m-1$
   holder = B(last i column)
   delete first element of holder
   place $r_t(m-i)$ as last element in holder
   Save holder to last ith column of B

C_T = Combine T and B.

d = fft(first column of C_T)

y = ifft(d.*(fft(v)));

(c) **Hermitian transpose.** Verify that the Hermitian transpose of a Toeplitz matrix is Toeplitz.

Let $T = \begin{bmatrix} a_0 & a_1 & \cdots & a_{m-1} \\ a_{-1} & a_0 & \ddots & \vdots \\ \vdots & & \ddots & a_1 \\ a_{-m+1} & \cdots & a_{-1} & a_0 \end{bmatrix}$

Then $T^* = \begin{bmatrix} \bar{a}_0 & \bar{a}_{-1} & \cdots & \bar{a}_{-m+1} \\ \bar{a}_1 & \bar{a}_0 & \ddots & \vdots \\ \vdots & & \ddots & \bar{a}_{-1} \\ \bar{a}_{m-1} & & & \bar{a}_0 \end{bmatrix}$

Note T^* is still Toeplitz.

Another approach:

Note we can express element of T in following way:

$T_{ij} = f(\underline{i-j})$ f assign value for $i-j$

$\Rightarrow T^*_{ij} = \overline{f(j-i)}$

for $i-j = k \rightarrow j-i = -k$

Then $\underbrace{\text{diag}}(T) = \underbrace{\text{diag}}(T^*)$

Then the structure is preserved.

d) My implementation is to use toepmatvec to calculate Y and B.

for i in 1 to k

Y(i th column) = toepmatvec(ct, rt, G(i th column)).

for i in 1 to k

B(i th column) = toepmatvec(ct, rt, G(i th column)).