HW 1

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1\.

1.1. Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices.
(b) Write it again as a product ABC (same B) of three matrices.

a) Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$

$A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $A_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ $A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Then we have $\boxed{A_5 A_3 A_2 B A_1 A_4 A_6 A_7}$

b) $A = A_5 A_3 A_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$C = A_1 A_4 A_6 A_7 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Results from Matlab:

```
Set up B:

B =

    0.8909    0.1493    0.8143    0.1966
    0.9593    0.2575    0.2435    0.2511
    0.5472    0.8407    0.9293    0.6160
    0.1386    0.2543    0.3500    0.4733

Double column 1:

B =

    1.7818    0.1493    0.8143    0.1966
    1.9186    0.2575    0.2435    0.2511
    1.0944    0.8407    0.9293    0.6160
    0.2772    0.2543    0.3500    0.4733

Halve row 3:

B =

    1.7818    0.1493    0.8143    0.1966
    1.9186    0.2575    0.2435    0.2511
    0.5472    0.4204    0.4646    0.3080
    0.2772    0.2543    0.3500    0.4733

Add row 3 to row 1:

B =

    2.3290    0.5697    1.2789    0.5046
    1.9186    0.2575    0.2435    0.2511
    0.5472    0.4204    0.4646    0.3080
    0.2772    0.2543    0.3500    0.4733
```

```
Interchange columns 1 and 4:

B =

    0.5046    0.5697    1.2789    2.3290
    0.2511    0.2575    0.2435    1.9186
    0.3080    0.4204    0.4646    0.5472
    0.4733    0.2543    0.3500    0.2772

Subtract row 2 from each of the other row:

B =

    0.2535    0.3121    1.0354    0.4104
    0.2511    0.2575    0.2435    1.9186
    0.3080    0.4204    0.4646    0.5472
    0.4733    0.2543    0.3500    0.2772


B =

    0.2535    0.3121    1.0354    0.4104
    0.2511    0.2575    0.2435    1.9186
    0.0569    0.1629    0.2211   -1.3714
    0.4733    0.2543    0.3500    0.2772


B =

    0.2535    0.3121    1.0354    0.4104
    0.2511    0.2575    0.2435    1.9186
    0.0569    0.1629    0.2211   -1.3714
    0.2222   -0.0032    0.1065   -1.6413

Replace column 4 by column 3:

B =

    0.2535    0.3121    1.0354    1.0354
    0.2511    0.2575    0.2435    0.2435
    0.0569    0.1629    0.2211    0.2211
    0.2222   -0.0032    0.1065    0.1065
```

```
Delete column 1:

B =

    0.3121    1.0354    1.0354
    0.2575    0.2435    0.2435
    0.1629    0.2211    0.2211
   -0.0032    0.1065    0.1065
```

```
Compute matrix A:

A =

    1.0000   -1.0000    0.5000         0
         0    1.0000         0         0
         0   -1.0000    0.5000         0
         0   -1.0000         0    1.0000

Compute matrix C:

C =

     0     0     0
     1     0     0
     0     1     1
     0     0     0

Result for ABC:

ans =

    0.3121    1.0354    1.0354
    0.2575    0.2435    0.2435
    0.1629    0.2211    0.2211
   -0.0032    0.1065    0.1065
```

3. **2.1.** Show that if a matrix A is both triangular and unitary, then it is diagonal.

Suppose A is triangular and unitary

Corollary: We claim if A is a nonsingular $m \times m$ triangular matrix then A^{-1} is also a triangular matrix of same kind.

proof. Suppose A is upper-triangular

Note one way to find inverse of a triangular matrix is by back s back substitution.

$$\Rightarrow \left[\begin{array}{ccccc|ccccc} a_{11} & a_{12} & \cdots & \cdots & a_{1m} & 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & & & \vdots & 0 & 1 & 0 & & \vdots \\ \vdots & 0 & \ddots & & \vdots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & a_{mm} & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

Where we use row $m-1$ minus (row m / a_{mm}) $\cdot \alpha$ for the first operation

Than we use row $m-2$ minus (row $m-1 / a_{m-1,m-1}$) $\cdot \beta$ for the second

Than we keep doing similar operation.

Note the upper triangular structure is reserved by there operation.

Thus we conclude A^{-1} is also upper triangular.

(Noter lower-triangular use similar arithmatic).

Since A is alse unitary we have:

$A^* = A^{-1} \longrightarrow A^*$ is also triangular with the same kind

Since the transpose is also triangular we have: A is diagonal.

$\square$

2.2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.

(b) Show that this computation also establishes the general case, by induction.

a) Claim: For a set of 2 orthogonal vectors: $\{x_1, x_2\}$

$\|x_1+x_2\|^2 = \|x_1\| + \|x_2\|$

Note $\|x_1+x_2\|^2 = (x_1+x_2)^*(x_1+x_2)$

Note inner product is bilinear so we have:

$(x_1^* + x_2^*)(x_1+x_2) = x_1^*x_1 + x_2^*x_1 + x_1^*x_2 + x_2^*x_2$

Since x_1 and x_2 are orthogonal we have:

$x_2^*x_1 = x_1^*x_2 = 0$

Thus $(x_1^*+x_2^*)(x_1+x_2) = x_1^*x_1 + x_2^*x_2$

$= \|x_1\|^2 + \|x_2\|^2$

Thus the theorem is true in the case of $n=2$.

$\square$

b) Base case: $n=2$

We have already proven $n=2$ case in part a).

Induction Step:

Suppose theorem is true for $n=k$.

$$\Rightarrow \left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \left\|\sum_{i=1}^{k} x_i + x_{k+1}\right\|^2$$

$$= \left(\sum_{i=1}^{k} x_i + x_{k+1}\right)^*\left(\sum_{i=1}^{k} x_i + x_{k+1}\right)$$

Note inner product is bilinear, we have:

$$\left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \left(\sum_{i=1}^{k} x_i\right)^*\left(\sum_{i=1}^{k} x_i\right) + x_{k+1}^*\left(\sum_{i=1}^{k} x_i\right) + \left(\sum_{i=1}^{k} x_i\right)^* x_{k+1} + x_{k+1}^* x_{k+1}$$

Note since all the vectors are orthogonal

$$\Rightarrow x_{k+1}^*\left(\sum_{i=1}^{k} x_i\right) = \left(\sum_{i=1}^{k} x_i\right)^* x_{k+1} = 0$$

$$\Rightarrow \left\|\sum_{i=1}^{k+1} x_i\right\|^2 = \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2$$

$$= \sum_{i=1}^{k+1} \|x_i\|^2$$

We have proven the statement by induction

$\square$

2.6. If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)?

① Suppose u and v are non-zero m-vectors
Let A be nonsingular and $A = I + uv^*$
Then note:

$$(I+\alpha uv^*)(I+uv^*) = I + \alpha uv^* + uv^* + \alpha uv^*uv^*$$
$$= I + (\alpha + 1 + \alpha v^*u)\, uv^*$$

matrix associativity and the fact v^*u is scalar

Then note:

$$u \cdot v^* = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} [v_1 \cdots v_m] = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_m \\ \vdots & & & \vdots \\ u_mv_1 & \cdots & \cdots & u_mv_m \end{bmatrix}$$

$$v^*u = [v_1 \cdots v_m] \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = v_1u_1 + v_2u_2 + \cdots v_mu_m$$

Then note $u_iv_j + 1$ is the coefficients when we calculate the determinant of A.
Since A is nonsingular we have $1 + v^*u \neq 0$
Thus for $\alpha + 1 + \alpha v^*u$ we can choose α s.t.

$$\alpha + 1 + \alpha v^*u = 0 \implies \alpha = -\frac{1}{1+v^*u}$$

Then we have $(I + \alpha uv^*) \cdot A = I \implies A^{-1} = I + \alpha uv^*$
where $\boxed{\alpha = -\frac{1}{1+v^*u}}$

② A is singular if $\boxed{1 + v^*u = 0}$
We have proven this statement from part ①.

③ If A is singular then: $1 + v^*u = 0$
Let $w \in \text{null}(A)$

$$\implies Aw = (I + uv^*)w = 0$$
$$= w + uv^*w$$
$$= w + (v^*w)u$$

matrix associativity and the fact v^*w is scalar

$$\implies w + (v^*w)u = 0$$
$$w = -(v^*w)u$$

Note v^*w is a scalar

$$\implies w \in \text{span}(u)$$
$$\implies \boxed{\text{Null}(A) = \text{span}(u)}$$