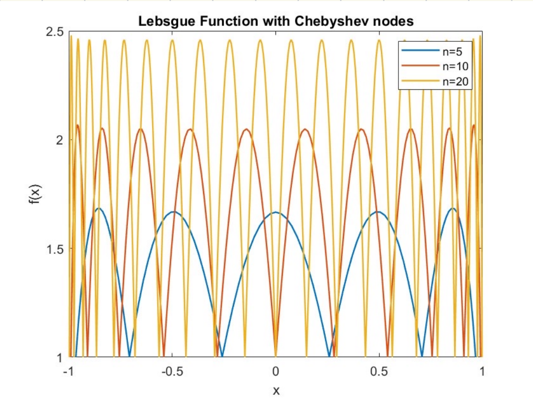
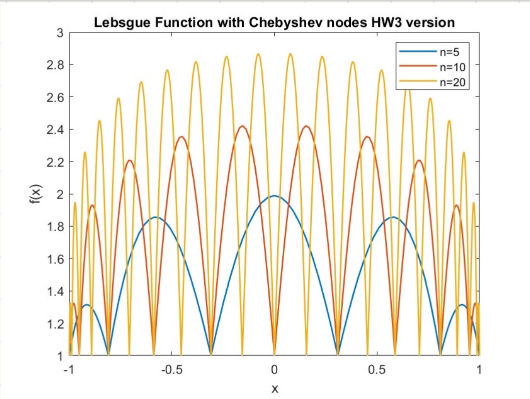
HW 3

AMATH 585 Tianbo Zhang

1. In the previous hw assignment, you plotted the Lebesgue function for equally spaced points in $[-1, 1]$ and for the points $\cos\left(\frac{2i+1}{2n+2}\pi\right)$, $i = 0, 1, \ldots, n$ (problem 27 on p. 125 of Gautschi). Now plot the Lebesgue function for the points $x_i = \cos(i\pi/n)$, $i = 0, 1, \ldots, n$. Comment on how the Lebesgue function for these points compares with that for the points $\cos\left(\frac{2i+1}{2n+2}\pi\right)$.





For $n=20$ the Lebsegue constant of using $\cos(i\pi/n)$ is about 2.87 where we have 2.48 for using $\cos(\frac{2i+1}{2n+2}\pi)$. And both case the constant increases as n increases.

I noticed that compare to the Chebyshev node we used in HW2, the one we use shows a different end behavior where the endbehavior is much lower than the center.

Both graph shows a similar concentration density.

2. Consider the following data $(x, f(x))$: $(1, 2)$, $(3/2, 6)$, $(0, 0)$, $(2, 14)$.

   (a) Recall the barycentric interpolation formula:

$$p(x) = \left(\sum_{i=0}^{n} f(x_i)\frac{w_i}{x - x_i}\right) \Big/ \left(\sum_{i=0}^{n} \frac{w_i}{x - x_i}\right),$$

   where $w_i = 1/\prod_{j \neq i}(x_i - x_j)$. Determine the weights w_i and write down the cubic polynomial that goes through the given data in barycentric form.

   (b) Write down a divided difference table and the Newton form of the interpolating polynomial for the given data.

   (c) Check that the polynomials in parts (a) and (b) are the same.

a) Note by formula from lecture we have:

$$w_i = \frac{1}{\prod_{j \neq i}(x_i - x_j)}$$

Then apply this formula we have:

$$w_0 = \frac{1}{(-1)\cdot(-3/2)\cdot(-2)} = -\frac{1}{3}$$

$$w_1 = \frac{1}{1\cdot(-1/2)\cdot(-1)} = 2$$

$$w_2 = \frac{1}{3/2\cdot 1/2\cdot(-1/2)} = -\frac{8}{3}$$

$$w_3 = \frac{1}{2\cdot 1\cdot 1/2} = 1$$

Then note by the formula we have:

$$P(x) = \left[f(x_0)\frac{w_0}{x-x_0} + f(x_1)\frac{w_1}{x-x_1} + f(x_2)\frac{w_2}{x-x_2} + f(x_3)\frac{w_3}{x-x_3}\right] \left[\frac{w_0}{x-x_0} + \frac{w_1}{x-x_1} + \frac{w_2}{x-x_2} + \frac{w_3}{x-x_3}\right]^{-1}$$

① $\dfrac{f(x_0)w_0(x-x_1)(x-x_2)(x-x_3) + f(x_1)w_1(x-x_0) - - - - -}{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}$

② $\dfrac{(x-x_0)\cdot(x-x_1)\cdot(x-x_2)\cdot(x-x_3)}{w_0(x-x_1)\cdots(x-x_3) + w_1(x-x_0)\cdots(x-x_3) + - - - -}$

① $(x-x_1)(x-x_2)(x-x_3) = x^3 - \frac{9}{2}x^2 + \frac{13}{2}x - 3$

② $(x-x_0)(x-x_2)(x-x_3) = x^3 - \frac{7}{2}x^2 + 3x$

③ $(x-x_0)(x-x_1)(x-x_3) = x^3 - 3x^2 + 2x$

④ $(x-x_0)(x-x_1)(x-x_2) = x^3 - \frac{5x^2}{2} + \frac{3x}{2}$

Then we have the denominator for right be:

$$-\frac{1}{\cancel{3}}\cancel{x^3} + 2\cancel{x^3} - \frac{8}{\cancel{3}}\cancel{x^3} + \cancel{x^3} + \frac{3}{\cancel{2}}\cancel{x^2} - \cancel{7}x^2 + \cancel{8}x^2 - \frac{\cancel{5x^2}}{\cancel{2}} - \frac{13}{\cancel{6}}x + \cancel{6}x - \frac{16}{\cancel{3}}x + \frac{3}{\cancel{2}}x + 1$$

Then note the denominator on the left cancel with neumerator on the right.

Thus we have:

$$P(x) = x\left[4(x-3/2)(x-2) - 16(x-1)(x-2) + 14(x-1)(x-3/2)\right]$$

$$= \boxed{2x^3 - x^2 + x}$$

b)

x	$f(x)$	$F(x_i, x_j)$	$F(x_i, x_j, x_k)$	$F(x_i, x_j, x_k, x_l)$
0	0			
		2		
1	2		4	
		8		2
3/2	6		8	
		16		
2	14			

Then by Newton's divided difference interpolation formula we have:

$$P(x) = 0 + 2\cdot x + 4\cdot(x-1)\cdot x + 2\cdot(x-3/2)\cdot(x-1)\cdot x$$

$$= 2x + \cancel{4x^2} - 4x + 2x(x^2 - \cancel{\tfrac{3}{2}x} - \cancel{x} + \tfrac{3}{2})$$

$$\boxed{= 2x^3 - x^2 + x}$$

c) They are the same.

3. Let f be a given function satisfying $f(0) = 1$, $f(1) = 2$, and $f(2) = 0$. A *quadratic spline* interpolant $r(x)$ is defined as a piecewise quadratic that interpolates f at the nodes ($x_0 = 0$, $x_1 = 1$, and $x_2 = 2$) and whose first derivative is continuous throughout the interval. Find the quadratic spline interpolant of f that also satisfies $r'(0) = 0$.

Note for the interval $[0, 1]$ we have:

$r(x) = a_1 + b_1 x + c_1 x(x-1)$

Then plug in: $r(0)=1$, $r(1)=2$ we have:

$a_1 = 1$

$b_1 + 1 = 2 \rightarrow b_1 = 1$ $\quad x^2 c_1 - c_1 x$

Then note: $r'(x) = 1 + 2c_1 x - c_1$

$\Rightarrow r'(0) = 1 - c_1 = 0 \rightarrow c_1 = 1$

We have $\boxed{r(x) = x^2 + 1 \text{ for } x \in [0, 1]}$

Then for interval $[1, 2]$ we have:

$r(x) = a_2 + b_2(x-1) + c_2(x-1)(x-2)$

plug in $r(1)=2$, $r(2)=0$ we have:

$a_2 = 2$

$2 + b_2 = 0 \rightarrow b_2 = -2$

Then note $r'(x) = -2 + 2x c_2 - 3c_2$

$\Rightarrow r'(1) = 1 + 2 - 1 = 2$

$= -2 - c_2 \rightarrow c_2 = -4$

We have: $\boxed{r(x) = -4x^2 + 10x - 4 \text{ for } x \in [1, 2]}$

4. Determine all the values of a, b, c, d, e, and f for which the following function is a cubic spline:

$$s(x) = \begin{cases} ax^2 + b(x-1)^3 & x \in (-\infty, 1] \\ cx^2 + d & x \in [1, 2] \\ ex^2 + f(x-2)^3 & x \in [2, \infty) \end{cases}$$

In order to have continuous function for $s(x)$

Note we have:

$s(1) = a = c+d$

$s(2) = 4c+d = 4e$

Then note:

$$s'(x) = \begin{cases} 2ax + 3b(x-1)^2 & x \in (-\infty, 1] \\ 2cx & x \in [1, 2] \\ 2ex + 3f(x-2)^2 & x \in [2, -\infty) \end{cases}$$

$\Rightarrow s'(1) = 2a = 2c \rightarrow a = c$

$s'(2) = 4c = 4e \rightarrow c = e$

we have $a = c = e$ and $a = c+d$

$\Rightarrow d = 0$

Then note:

$$s''(x) = \begin{cases} 2a + 6b(x-1) \\ 2a \\ 2a + 6f(x-2) \end{cases}$$

We don't have any information since it's already continuous.

Finally we have:

$$\boxed{S(x) = \begin{cases} ax^2 + b(x-1)^3 & x \in (-\infty, 1] \\ ax^2 & x \in [1, 2] \\ ax^2 + f(x-2)^3 & x \in [2, \infty) \end{cases} \quad a, b, f \text{ can be any number}}$$

5. Let $f(x) = x^4$ on $[0, 2]$. Find the natural cubic spline interpolant of f using the two subintervals $[0, 1]$ and $[1, 2]$.

First to set up the equation we have the following:

$$S(x) = \begin{cases} a + bx + cx^2 + dx^3, & x \in [0,1] \\ e + f(x-1) + g(x-1)^2 + h(x-1)^3, & x \in [1,2] \end{cases}$$

$\Rightarrow S(1) = a+b+c+d = e$

$$S'(x) = \begin{cases} b + 2cx + 3dx^2, & x \in [0,1] \\ f + 2g(x-1) + 3h(x-1)^2, & x \in [1,2] \end{cases}$$

$S'(1) = b+2c+3d = f$

$$S''(x) = \begin{cases} 2c + 6dx, & x \in [0,1] \\ 2g + 6h(x-1), & x \in [1,2] \end{cases}$$

$S''(1) = 2c+6d = 2g \rightarrow g = c+3d$

Then note another condition is: $S''(2)=0$, $S''(0)=0$

$\Rightarrow g+3h=0$, $\boxed{c=0}$

Then note $f(0)=0 \rightarrow \boxed{a=0}$

$f(1)=1 \rightarrow \boxed{e=1}$

Then we have: ① $b+d=1$

② $b+3d=f$

③ $3d=g$

④ $g+3h=0$

Also note:

$f(2)=16 \rightarrow 1+f+g+h=16$

use ②-① we have:

$2d = f-1 \rightarrow 2d+1=f$

Use ③ and upper 2 equation we have:

$1+2d+1+3d+h=16$

$5d+h=14$

plug ③ into ④ we have

$h=-d$

$\Rightarrow 4d=14 \rightarrow \boxed{d=\frac{7}{2}} \rightarrow \boxed{h=-\frac{7}{2}}$

$\Rightarrow b=-\frac{5}{2}$, $f=8$

We have:

$$\boxed{S(x) = \begin{cases} -\frac{5}{2}x + \frac{7}{2}x^3, & x \in [0,1] \\ 1 + 8(x-1) + \frac{21}{2}(x-1)^2 - \frac{7}{2}(x-1)^3, & x \in [1,2] \end{cases}}$$