HW 1

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1. Suppose you want to approximate the function

$$f(t)=\begin{cases}-1 & \text{if } -1\le t<0,\\ 0 & \text{if } t=0,\\ 1 & \text{if } 0<t\le 1\end{cases}$$

by a constant function $\varphi(x)=c$:

(a) on [−1,1] in the continuous L_1 norm,
(b) on $\{t_1,t_2,\dots,t_N\}$ in the discrete L_1 norm,
(c) on [−1,1] in the continuous L_2 norm,
(d) on $\{t_1,t_2,\dots,t_N\}$ in the discrete L_2 norm,
(e) on [−1,1] in the ∞-norm,
(f) on $\{t_1,t_2,\dots,t_N\}$ in the discrete ∞-norm.

The weighting in all norms is uniform (i.e., $w(t)\equiv 1$, $w_i=1$) and $t_i=-1+\frac{2(i-1)}{N-1}$, $i=1,2,\dots,N$. Determine the best constant c (or constants c, if there is nonuniqueness) and the minimum error.

a) $E[\varphi]=\|f(t)-\varphi\|_1=\int_{-1}^{1}|f(t)-c|\,dt$

Then note we can set:

$f(t)=c \rightarrow f^{-1}(c)=t.$

Then we have:

$$E[\varphi]=\int_{-1}^{f^{-1}(c)}(c-f(t))\,dt+\int_{f^{-1}(c)}^{1}(f(t)-c)\,dt$$

$$=\int_{-1}^{f^{-1}(c)}c\,dt-\int_{-1}^{f^{-1}(c)}f(t)\,dt+\int_{f^{-1}(c)}^{1}f(t)\,dt-\int_{f^{-1}(c)}^{1}c\,dt$$

$$=cf^{-1}(c)+c-c+cf^{-1}(c)-\int_{-1}^{f^{-1}(c)}f(t)\,dt+\int_{f^{-1}(c)}^{1}f(t)\,dt$$

$$=2cf^{-1}(c)-\int_{-1}^{f^{-1}(c)}f(t)\,dt+\int_{f^{-1}(c)}^{1}f(t)\,dt$$

$$\frac{d}{dc}[E[c]]=2f^{-1}(c)+2c\frac{d}{dc}[f^{-1}(c)]+2\left[f(f^{-1}(c))\frac{d}{dc}[f^{-1}(c)]\right]$$

$$=2f^{-1}(c)=0 \rightarrow \boxed{c=0} \longrightarrow \boxed{E[\varphi]=1-(-1)=2}$$

b) $E[\varphi]=\sum_{i=1}^{N}|f(t_i)-c|$

Since the points are uniformly distributed.

Note that if N is even than we have

half points have $f(t)=1$ and other half have $f(t)=-1$

If N is odd than we have one point has $f(t)=0$

And others similar.

Thus choose $\boxed{c=0}$ will balance the points on each side.

Then $\boxed{\begin{array}{l}E[\varphi]=N \text{ if } N \text{ is even}\\ E[\varphi]=N-1 \text{ if } N \text{ is odd}\end{array}}$

c) $E[\varphi]=\left[\int_{-1}^{1}(f(t)-c)^2\,dt\right]^{\frac{1}{2}}$

$$=\left[\int_{-1}^{0}(f(t)-c)^2\,dt+\int_{0}^{1}(f(t)-c)^2\,dt\right]^{\frac{1}{2}}$$

$$=\left[(-1-c)^2t\Big|_{-1}^{0}+(1-c)^2t\Big|_{0}^{1}\right]^{\frac{1}{2}}$$

$$=\left[(-1-c)^2+(1-c)^2\right]^{\frac{1}{2}}$$

$$=\left[1+2c+c^2+1-2c+c^2\right]^{\frac{1}{2}}=\left[2c^2+2\right]^{\frac{1}{2}}$$

Then note:

Note this can't be 0.

$$\frac{d}{dc}[E[\varphi]]=\frac{1}{2}(2c^2+2)^{-\frac{1}{2}}4c=0$$

$$\Rightarrow 4c=0 \rightarrow \boxed{c=0}$$

$$\boxed{E[\varphi]=\sqrt{2}}$$

d) $E[e] = \left[\sum (f(t)-c)^2\right]^{\frac{1}{2}}$

Since the points are uniformly distributed

Note that if N is even than we have

half points have $f(t)=1$ and other half have $f(t)=-1$

If N is odd than we have one point has $f(t)=0$

And others similar.

Thus choose $\boxed{c=0}$ will balance the points on each side.

Then $\boxed{\begin{array}{l} E[e]=\sqrt{N} \text{ if } N \text{ is even} \\ E[e]=\sqrt{N-1} \text{ if } N \text{ is odd} \end{array}}$

e) $E[e] = \max\limits_{-1\le t\le 1} |f(t)-c|$

Note the only values for $f(t)$ is: $-1, 0, 1$

Then the constant that minimizes the maximum absolute difference

is the one halfway the extreme values.

$\Rightarrow \boxed{c=0}$

$\boxed{E[e]=1}$

f). $E[e] = \max\limits_{1\le t\le N} |f(t)-c|$

Since the points are uniformly distributed

Note that if N is even than we have

half points have $f(t)=1$ and other half have $f(t)=-1$

If N is odd than we have one point has $f(t)=0$

And others similar.

Thus choose $\boxed{c=0}$ will balance the points on each side.

$\boxed{E[e]=1}$

2. Consider the data

$$f(t_i) = 1, \; i = 1, 2, \ldots, N-1; \quad f(t_N) = y \gg 1.$$

(a) Determine the discrete L_∞ approximant to f by means of a constant c (polynomial of degree zero).
(b) Do the same for discrete (equally weighted) least square approximation.
(c) Compare and discuss the results, especially as $N \to \infty$.

a) Set $\varphi(x) = c$

Then we have:

$$E[\varphi] = \|f(t) - c\|_\infty = \max_{1 \le t \le N} |f(t) - c|$$

Note since $f(t_N) = y \gg 1$

Then note the value c that will minimize the maximum difference should be the halfway between two extreme values: 1 and y

Thus $\boxed{c = \frac{1+y}{2}}$

b) Set $\varphi(x) = c$

Then we have:

$$E[\varphi] = \|f(t) - c\|_2 = \left[\sum_{i=1}^{N} (f(t_i) - c)^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{N-1} (1-c)^2 + (y-c)^2\right]^{\frac{1}{2}}$$

$$= \left[(N-1)(1-c)^2 + (y-c)^2\right]^{\frac{1}{2}}$$

$$\frac{d}{dc}\left[E[\varphi]^2\right] = 2(N-1)(1-c)\cdot(-1) + 2(y-c)\cdot(-1)$$

$$= -2N + 2 + 2Nc - \cancel{2c} - 2y + \cancel{2c}$$

$$= 2Nc - 2N - 2y + 2 = 0$$

$$2Nc = 2y + 2N - 2$$

$$\boxed{c = \frac{y+N-1}{N}}$$

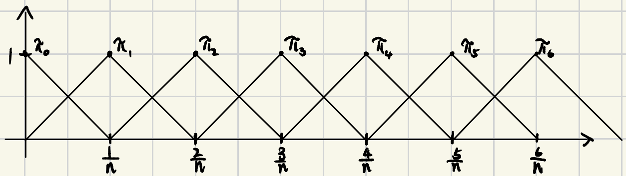
c). Note for L_∞ as N goes to infinity.
c will stays the same.
Since L_∞ only cares about the maximum deviation.
Thus it only cares about the extreme scenario.
However this is less accurate when $N \to \infty$.

Then for L_2 as N goes to infinity
c will converge to 1.
Which makes more sense since there will be only one outlier $f(t_N) = y$.
Thus in general its more accurate.

13. Given an integer $n \geq 1$, consider the subdivision Δ_n of the interval $[0, 1]$ into n equal subintervals of length $1/n$. Let $\pi_j(t)$, $j = 0, 1, \ldots, n$, be the function having the value 1 at $t = j/n$, decreasing on either side linearly to zero at the neighboring subdivision points (if any), and being zero elsewhere.

(a) Draw a picture of these functions. Describe in words the meaning of a linear combination $\pi(t) = \sum_{j=0}^{n} c_j \pi_j(t)$.
(b) Determine $\pi_j(k/n)$ for $j, k = 0, 1, \ldots, n$.
(c) Show that the system $\{\pi_j(t)\}_{j=0}^{n}$ is linearly independent on the interval $0 \leq t \leq 1$. Is it also linearly independent on the set of subdivision points $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$ of Δ_n? Explain.
(d) Compute the matrix of the normal equations for $\{\pi_j\}$, assuming $d\lambda(t) = dt$ on [0,1]. That is, compute the $(n+1) \times (n+1)$ matrix $A = [a_{ij}]$, where $a_{ij} = \int_0^1 \pi_i(t)\pi_j(t)dt$.

a)



The linear combination of $\pi(t)$ combines all the function $\pi_j(t)$. Where each function's contribution is determined by the coefficient c_j. If c_j varies than it will create a bunch of triangles along $[0,1]$ with different height.

b) $\pi_j(k/n) = \begin{cases} 1, & \text{for } k=j \\ 0, & \text{otherwise} \end{cases}$

c) First note that $\{\pi_j(t)\}_{j=0}^{n}$ is linearly independent if:

$\sum_{j=0}^{n} c_j \pi_j(t) = 0$ for $t \in [0,1] \longrightarrow c_j = 0$ for $j \in [0,n]$

Then note $\pi_j(\frac{k}{n}) = 1$ for $j=k$ and $\pi_j(\frac{k}{n}) = 0$ for $j \neq k$

Then also note we have $n+1$ sums:

$S_0 = \sum_{j=0}^{n} c_j \pi_j(0) = \pi_0 = 0$

$S_{\frac{1}{n}} = \sum_{j=0}^{n} c_j \pi_j(\frac{1}{n}) = \pi_1 = 0$

$S_{\frac{2}{n}} = \sum_{j=0}^{n} c_j \pi_j(\frac{2}{n}) = \pi_2 = 0$

$\vdots$

$S_n = \sum_{j=0}^{n} c_j \pi_j(n) = \pi_n = 0$

Thus $\pi_0, ---, \pi_n = 0$ if $\sum_{j=0}^{n} c_j \pi_j(t) = 0$ for $t \in [0,1]$

Thus it is lin. indep. on $t \in [0,1]$ □

Note it's not linearly independent on the subintervals.

If we let k be a fix value

Then $\sum_{j=0}^{n} c_j \pi_j(\frac{k}{n})$ would only require $c_k = 0$

And $c_1, ---, c_{k-1}, c_{k+1}, ---, c_n$ can be any value

Since $\pi_j(\frac{k}{n}) = 0$.

Thus not lin. indep. □

d)

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \; 1 \end{bmatrix} = I^{n+1}$$

4. Using an orthogonal polynomial basis, find the best least squares polynomial approximation of degree 2 to the function $f(t) = e^t$ on $[0, 1]$; i.e., find $p_2(t)$ that minimizes

$$\int_0^1 (e^t - p_2(t))^2 \, dt.$$

If the monomial basis is used, will the resulting polynomial be the same as the one obtained with the orthogonal basis? Explain.

First note a common choice for the orthogonal polynomial basis is the Legendre polynomial:

$P_0^*(t) = 1$

$P_1^*(t) = t$

$P_2^*(t) = \frac{1}{2}(3t^2-1)$

Then since we are focused on $[0,1]$ we have: (plug in $2t-1$)

$P_0^*(t) = 1$

$P_1^*(t) = 2t-1$

$P_2^*(t) = 6t^2-6t+1$

Then set:

$P_2(t) = C_1 + C_2(2t-1) + C_3(6t^2-6t+1)$

Then we have:

$$E[P_2(t)] = \int_0^1 (e^t - P_2(t))^2 \, dt$$

$$\frac{\partial}{\partial C_1}\left[E[P_2(t)]\right] = \int_0^1 2(e^t - P_2(t)) \, dt$$

$$= 2\left(e^t\Big|_0^1 - C_1 t\Big|_0^1 - C_2 t^2\Big|_0^1 + C_2 t\Big|_0^1 - 2C_3 t^3\Big|_0^1 + 3C_3 t^2\Big|_0^1 - C_3 t\Big|_0^1\right)$$

$$= 2(e - 1 - C_1 - \cancel{C_2} + \cancel{C_2} - \cancel{2C_3} + \cancel{3C_3} - \cancel{C_3})$$

$$= 2e - 2 - 2C_1 = 0 \longrightarrow \boxed{C_1 = e-1}$$

$$\frac{\partial}{\partial C_2}\left[E[P_2(t)]\right] = \int_0^1 2(e^t - P_2(t))(2t-1) \, dt$$

$$= 2\left(3 - \frac{C_2 + 3e}{3}\right) = 0 \longrightarrow \boxed{C_2 = 9-3e}$$

$$\frac{\partial}{\partial C_3}\left[E[P_2(t)]\right] = \int_0^1 2(e^t - p_2(t))(6t^2-6t+1) \, dt$$

$$= 2\left(-\frac{C_3 - 35e}{5} - 19\right) = 0$$

$$\Rightarrow \boxed{C_3 = -95+35e}$$

Note by definition we know.

Any monimal basis can be written as a linear combination of orthogonal polynomial basis.

Thus if we use monomial basis, the result will be the same.