HW5

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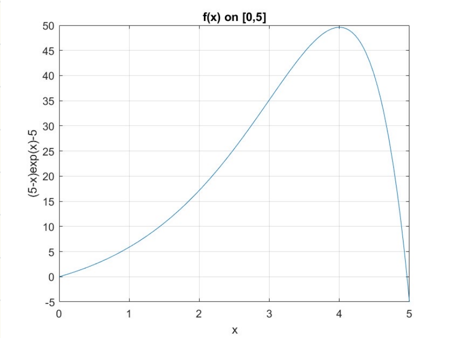
1. Use MATLAB (or another programming language) to plot the function $f(x) = (5-x)\exp(x) - 5$, for x between 0 and 5. (This function is associated with the *Wien radiation law*, which gives a method to estimate the surface temperature of a star.)

   (a) Write a bisection routine to find a root of $f(x)$ in the interval $[4,5]$, accurate to six decimal places. At each step, print out the endpoints of the smallest interval known to contain a root. Without running the code further, answer the following: How many steps would be required to reduce the size of this interval to 10^{-12}? Explain your answer.

   (b) Write a routine to use Newton's method to find a root of $f(x)$, using initial guess $x_0 = 5$. Print out your approximate solution x_k and the value of $f(x_k)$ at each step and run until $|f(x_k)| \le 10^{-8}$. Without running the code further, but perhaps using information from your code about the rate at which $|f(x_k)|$ is reduced, can you estimate how many more steps would be required to make $|f(x_k)| \le 10^{-16}$ (assuming that your machine carried enough decimal places to do this)? Explain your answer.

   (c) Modify your routine for doing Newton's method to run the secant method. Repeat the run of part (b), using, say, $x_0 = 4$ and $x_1 = 5$, and again predict (without running the code further) how many steps would be required to reduce $|f(x_k)|$ below 10^{-16} (assuming that your machine carried enough decimal places to do this) using the secant method. Explain your answer.



a)

t_b =

20×3 table

Step	an	bn
1	4	5
2	4	4.5
3	4	4.25
4	4	4.125
5	4	4.0625
6	4	4.03125
7	4	4.015625
8	4	4.0078125
9	4	4.00390625
10	4	4.001953125
11	4	4.0009765625
12	4	4.00048828125
13	4	4.000244140625
14	4	4.0001220703125
15	4	4.00006103515625
16	4	4.00003051757812
17	4	4.00001525878906
18	4	4.00000762939453
19	4	4.00000381469727
20	4	4.00000190734863

sol_b =

4.000000953674316

Then note we have:

$tol = 10^{-12}$

Then by Theorem 4.33 we have:

$$n = \left\lceil \frac{\log \frac{b-a}{tol}}{\log 2} \right\rceil = \left\lceil \frac{\log 10^{12}}{\log 2} \right\rceil$$

$$= \left\lceil \frac{12}{\log 2} \right\rceil = \boxed{40}$$

b) First note we have:

$$f'(x) = (4-x)e^x$$

t_n =

3×3 table

Step	xk	f(xk)
1	5	-5
2	4.96631026500457	-0.165642777610917
3	4.96511568630146	-0.000201201806098616

sol_n =

4.965114231746430

Note we are reaching the condition $|f(x)| \le 10^{-8}$ in 3 iterations.

Given that Newton's method converges quadratically.

$\Rightarrow$ steps for 10^{-8} $\Rightarrow$ $2^3 = 8$ $\Rightarrow$ 3 steps

Thus we have for $|f(x)| \le 10^{-16}$ is about $\boxed{4 \text{ steps}}$ since $2^4 = 16$

c) t_s =

5×3 table

Step	xk	f(xk)
1	5	-5
2	4.90842180555633	7.40202440793818
3	4.9630793363118	0.280894219802861
4	4.96523531212635	-0.0167504990409331
5	4.9651139806579	3.47314942690247e-05

sol_s =

4.965114231713327

Note that for secant method the order of convergence is greater than 1 and smaller that 2.

The the estimate of the steps is about $\boxed{8 \text{ or } 9 \text{ steps}}$

11. The equation $x^2 - a = 0$ (for the square root $\alpha = \sqrt{a}$) can be written equivalently in the form

$$x = \varphi(x)$$

in many different ways, for example,

$$\varphi(x) = \frac{1}{2}\left(x + \frac{a}{x}\right), \quad \varphi(x) = \frac{a}{x}, \quad \varphi(x) = 2x - \frac{a}{x}.$$

Discuss the convergence (or nonconvergence) behavior of the iteration $x_{n+1} = \varphi(x_n)$, $n = 0, 1, 2, \ldots$, for each of these three iteration functions. In case of convergence, determine the order of convergence.

Case 1: $\varphi(x) = \frac{1}{2}(x + \frac{a}{x}) = \frac{1}{2}x + \frac{a}{2}x^{-1}$

Then we have:

$$\varphi'(\alpha) = \frac{1}{2} - \frac{a}{2}x^{-2}\Big|_{x=\alpha=\sqrt{a}} = \frac{1}{2} - \frac{1}{2} = 0$$

$$\varphi''(\alpha) = ax^{-3}\Big|_{x=\alpha=\sqrt{a}} = a^{-\frac{1}{2}} \neq 0$$

$\Rightarrow$ The fixed point $\boxed{\text{converges quadratically.}}$

Case 2: $\varphi(x) = \frac{a}{x} = ax^{-1}$

Then we have:

$$\varphi'(\alpha) = -ax^{-2}\Big|_{x=\alpha=\sqrt{a}} = -a/a = -1$$

Then note for $x_0 \neq 0$ we have:

$x_1 = ax_0^{-1}$, $a_2 = ax_1^{-1} = x_0$

Thus the $\boxed{\text{iteration cycles.}}$

Case 3: $\varphi(x) = 2x - \frac{a}{x} = 2x - ax^{-1}$

Then we have:

$$\varphi'(\alpha) = 2 + ax^{-2}\Big|_{x=\alpha=\sqrt{a}} = 2 + 1 = 3$$

Then the $\boxed{\text{iteration diverges unless: } x_0 = \sqrt{a} = \alpha}$

3. If you enter a number into a handheld calculator and repeatedly press the cosine button, what number (approximately) will eventually appear? Provide a proof. [Note: Set your calculator to interpret numbers as radians rather than degrees; otherwise you will get a different answer.]

The number that will eventually appear is 0.7309851

Note we have the following sequence:

$x_0 = x_0$

$x_1 = \cos(x_0)$

$x_2 = \cos(x_1)$

$\vdots$

Then note as $n \to \infty$ we have:

$x_n = \cos(x_n)$

Then we can solve the equation using Newton's method:

$f(x) = x - \cos(x)$

$f'(x) = 1 + \sin(x)$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\vdots$

$x = 0.7309851$

4. The equation of motion of a pendulum of length L meters is

$$\theta''(t) = -(g/L)\sin(\theta(t)),$$

where θ is the angle of the pendulum with the downward vertical axis and $g \approx 9.8m/s^2$ is the gravitational constant. For simplicity, assume that $g/L = 1$. Suppose we wish to set the pendulum in motion, swinging from some given initial position $\theta(0) = \alpha$ with some unknown angular velocity $\theta'(0)$ in such a way that the pendulum ends up at a desired position β at time $t = T$. Then we must solve the following 2-point boundary value problem:

$$\theta''(t) = -\sin(\theta(t)), \quad 0 < t < T,$$
$$\theta(0) = \alpha, \quad \theta(T) = \beta.$$

We saw in a previous homework how to approximate the second derivative with a finite difference quotient:

$$\theta''(t) \approx \frac{\theta(t+h) + \theta(t-h) - 2\theta(t)}{h^2}.$$

If we divide the interval $[0, T]$ into n subintervals, each of width $h = T/n$, and let θ_i denote the approximate value of θ at $t = ih$, $i = 1, \ldots, n-1$, then we end up with a system of $n-1$ equations in $n-1$ unknowns:

$$\frac{1}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + \sin(\theta_i) = 0, \quad i = 1, \ldots, n-1,$$

where $\theta_0 = \alpha$ and $\theta_n = \beta$.

Taking $T = 2\pi$, $\alpha = \beta = 0.7$, write down the Jacobian matrix corresponding to this nonlinear system of equations and the Newton iteration that you would use to solve this system. Write a code to solve this system using $\theta_i^{(0)} = 0.7\cos(t_i) + 0.5\sin(t_i)$, $i = 1, \ldots, n-1$ as an initial guess. Plot the approximate solution at each step of Newton's method. Try some different initial guesses and see if you can converge to different solutions, again plotting the approximations from Newton's method at each step. Turn in your plots and discuss your results.

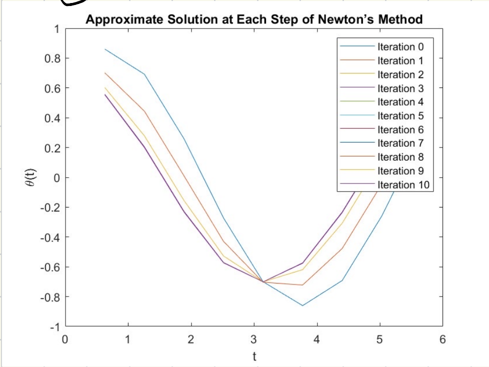
To compute the Jacobian matrix we have:

① $J_{i,i} = \frac{\partial}{\partial\theta_i}\left(\frac{1}{h^2}\theta_{i-1} - \frac{2}{h^2}\theta_i + \frac{1}{h^2}\theta_{i+1} + \sin\theta_i\right)$
$= -\frac{2}{h^2} + \cos(\theta_i)$

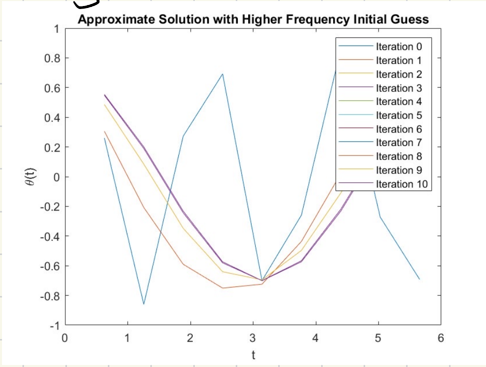
② $J_{i,i-1} = J_{i-1,i} = \frac{\partial}{\partial\theta_{i-1}}\left(\frac{1}{h^2}\theta_{i-1} - \frac{2}{h^2}\theta_i + \frac{1}{h^2}\theta_{i+1} + \sin\theta_i\right)$
$= \frac{\partial}{\partial\theta_{i+1}}\left(\frac{1}{h^2}\theta_{i-1} - \frac{2}{h^2}\theta_i + \frac{1}{h^2}\theta_{i+1} + \sin\theta_i\right)$
$= \frac{1}{h^2}$

③ All other elements are 0.

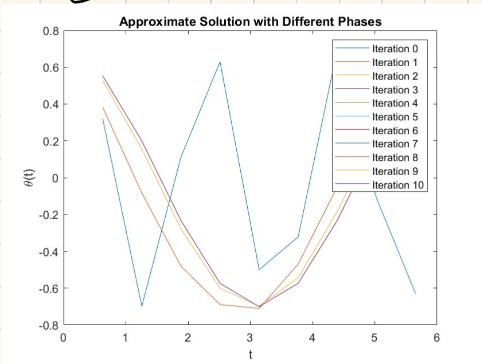
① Using $\theta_i^{(0)} = 0.7\cos(t_i) + 0.5\sin(t_i)$



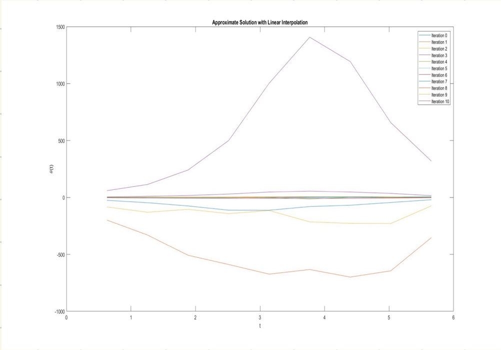
② Using $\theta_i^{(0)} = 0.7\cos(3\cdot t_i) + 0.5\sin(3\cdot t_i)$



③ Using $\theta_i^{(0)} = 0.5\cos(3\cdot t_i) + 0.5\sin(3\cdot t_i)$



④ Using $\theta = t_i$



Summary: Note that from the graph with different initial guess we can see that the outcome is not high sensitive to the initial guess as long as we use trig functions. And the system converge to the same solution. However if use initial guess that is not based on trig functions, the result may be inaccurate.