# Automated Market Makers for Decentralized Finance (DeFi)

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#### **Abstract**

This paper compares mathematical models for automated market makers including logarithmic market scoring rule (LMSR), liquidity sensitive LMSR (LS-LMSR), constant product/mean/sum, and others. It is shown that though LMSR may not be a good model for Decentralized Finance (DeFi) applications, LS-LMSR has several advantages over constant product/mean based automated market makers. However, LS-LMSR requires complicated computation (i.e., logarithm and exponentiation) and the cost function curve is concave. In certain DeFi applications, it is preferred to have computationally efficient cost functions with convex curves to conform with the principle of supply and demand. This paper proposes and analyzes constant circle/ellipse based cost functions for automated market makers. The proposed cost functions are computationally efficient (only requires multiplication and square root calculation) and have several advantages over widely deployed constant product cost functions. For example, the proposed market makers are more robust against front-runner (slippage) attacks.

## 1 Introduction

Decentralized finance (DeFi or open finance) is implemented through smart contracts (DApps) which are stored on a public distributed ledger (such as a blockchain) and can be activated to automate execution of financial instruments and digital assets. The immutable property of blockchains guarantees that these DApps are also tamper-proof and the content could be publicly audited.

DeFi applications range from automated markets, price oracles, to financial derivatives and many others. Most DeFi applications (such as Bancor [7], Celo [8], Uniswap [15], Compound [9], etc) enable smart token transaction instantly by using price equilibrium mechanisms based on total availability supply (or called bonding curves), though still some of DeFi applications do not carry out instant transaction. For example, the Gnosis Protocol (formerly Dfusion Protocol) [3] revised the traditional continuous double auction design to a discrete double auction design to address challenges with front-running. In a blockchain system, traders submit their transactions to the entire blockchain network, a miner in the system collects these transactions, validates them, and puts them into a valid block that is eventually added to an immutable chain of blocks. These transactions are visible to all nodes. A malicious node (the miner itself could be malicious) may construct his/her own malicious transactions based on these observed transactions. These malicious transactions may take profit with minimal or zero cost. In addition to the front-running attacks, it is also common to mount attacks against DeFi price oracles. In the DeFi market, a lender (a smart contract) normally queries an oracle to determine the fair market value (FMV) of borrower's collateral. The oracle could be on-chain-centralized, on-chain-decentralized, off-chain-centralized or off-chain-decentralized. For example, Celo [8] protocol employs stable coins based on a multi-asset tiered reserve and allows a decentralized exchange in which the different local and regional currencies and the reserve currency can be traded amongst one another without a central party. When coin supply in the Celo protocol needs to expand (when the price of Celo Dollar is above the peg), the protocol creates new coins to purchase a basket of cryptographic currencies at market rates through a smart contract. When the coin supply in the Celo protocol needs to contract, the protocol uses reserve assets to buy Celo Dollars on the open market. To determine the price of Celo stable currencies, the Celo protocol uses a Schelling-point scheme amongst stakeholders. In summary, Celo employs a kind of off-chain-centralized price oracle. Compound [9] is another DeFi application with an off-chain-centralized price oracle. Compound allows users to put their cryptographic assets in a saving account to earn interest though at the same time to use these assets in the saving account as collateral to take out loans. Uniswap [15] is a protocol with on-line-decentralized price oracles. A Uniswap smart contract holds liquidity reserves of various tokens, and trades are executed directly against these reserves using the constant product cost function. Uniswap v2 allows contracts to estimate the time-weighted average price over a given interval and supports flash swaps. tl;dr [14] describes a few practical attacks against various price oracles. The examples include attacks against DDEX's on-chain-decentralized ETH/DAI and ETH/USA oracles, bZx's DeFi applications with on-chain-decentralized Kyber Networks oracles, and Uniswap's on-line-decentralized price oracles.

Yet another popular DeFi application is flash loan that is only valid within one blockchain transaction. Flash loan is a relatively new financial instrument and it could be vulnerable to many potential attacks. On February 15, 2020, a user mounted an attack against Aave's flash loans which obtained a profit of 350k USD with a transaction fee of 132.36 USD. Qin et al [13] described this attack in detail and pointed out that if the user optimized his/her attack, she/he could have made 829.5k USD indeed.

The structure of the paper is as follows. Section 2 gives an introduction to prediction markets and introduces/proposes/analyzes various models for automated market makers: logarithmic market scoring rules (LMSR), liquidity sensitive LMSR (LS-LMSR), constant product/mean/sum markets, and constant circle/ellipse cost functions. Section 3 compares various cost functions from aspects of the principle of supply and demand, coin liquidity, and token price fluctuation. Section 4 compares price amplitude for various cost functions and Section 5 concludes the paper.

## 2 Prediction market and combinatorial market makers

### 2.1 Prediction market and combinatorial market makers

It is commonly believed that combined information/knowledge of all traders are incorporated into stock prices immediately (Fama [2] includes this as one of his "efficient market hypothesis"). For example, these information may be used by traders to hedge risks in financial markets such as stock and commodities future markets. With aggregated information from all sources, speculators who seek to "buy low and sell high" can take profit by predicting future prices from current prices and aggregated information. Inspired by these research, the concept of "information market" was introduced to investigate the common principles in information aggregation. Among various approaches to information market, a *prediction market* is an exchange-traded market for the purpose of eliciting aggregating beliefs over an unknown future outcome of a given event. As an example, in a horse race with n horses, one may purchase a security of the form "horse n beats horse n bea

For prediction markets with a huge outcome space, the continuous double-sided auction (where the market maker keeps an order book that tracks bids and asks) may fall victim of the *thin-market* problem. Firstly, in order to trade, traders need to coordinate on what or when they will trade. If there are significantly less participants than the size of the outcome space, the traders may only expect substantial trading activities in a small set of assets and many assets could not find trades at all. Thus the market has a low to poor liquidity. Secondly, if a single participant knows something about an event while others know nothing about this information, this person may choose not to release this information at all or only release this information gradually. This could be justified as follows. If any release of this information (e.g., a trade based on this information) is a signal to other participants that results in belief revision discouraging trade, the person may choose not to release the information (e.g., not to make the trade at all). On the other hand, this person may also choose to leak the information into the market gradually over time to obtain a greater profit. The second challenge for the standard information market is due to the *irrational participation* problem where a rational participant may choose not to make any speculative trades with others (thus not to reveal his private information) after hedging his risks derived from his private information.

## 2.2 Logarithmic market scoring rules (LMSR)

Market scoring rules are commonly used to overcome the thin market and the irrational participation problems discussed in the preceding section. Market scoring rule based automated market makers implicitly/explicitly maintain prices for all assets at certain prices and are willing to trade on every assets. In recent years, Hanson's logarithmic market scoring rules (LMSR) automated market maker [5, 6] has become the de facto automated market maker mechanisms for prediction markets.

Let X be an independent random variable with a finite outcome space  $\Omega$ . Let  $\mathbf{p}$  be a reported probability estimate for the random variable X. That is,  $\sum_{\omega \in \Omega} \mathbf{p}(\omega) = 1$ . In order to study rational behavior (decision) with fair fees,

Good [4] defined a reward function with the logarithmic market scoring rule (LMSR) as follows:

$$\{s_{\omega}(\mathbf{p}) = b \ln(2 \cdot \mathbf{p}(\omega))\} \tag{1}$$

where b>0 is a constant. A participant in the market may choose to change the current probability estimate  $\mathbf{p}_1$  to a new estimate  $\mathbf{p}_2$ . This participant will be rewarded  $s_{\omega}(\mathbf{p}_2)-s_{\omega}(\mathbf{p}_1)$  if the outcome  $\omega$  happens. Thus the participant would like to maximize his expected value (profit)

$$S(\mathbf{p}_1, \mathbf{p}_2) = \sum_{\omega \in \Omega} \mathbf{p}_2(\omega) \left( s_{\omega}(\mathbf{p}_2) - s_{\omega}(\mathbf{p}_1) \right) = b \sum_{\omega \in \Omega} \mathbf{p}_2(\omega) \ln \frac{\mathbf{p}_2(\omega)}{\mathbf{p}_1(\omega)} = bD(\mathbf{p}_2 || \mathbf{p}_1)$$
(2)

by honestly reporting his believed probability estimate, where  $D(\mathbf{p}_2||\mathbf{p}_1)$  is the relative entropy or Kullback Leibler distance between the two probabilities  $\mathbf{p}_2$  and  $\mathbf{p}_1$ . An LMSR market can be considered as a sequence of logarithmic scoring rules where the market maker (that is, the patron) pays the last participant and receives payment from the first participant.

Equivalently, an LMSR market can be interpreted as a market maker offering  $|\Omega|$  securities where each security corresponds to an outcome and pays \$1 if the outcome is realized [5]. In particular, changing the market probability of  $\omega \in \Omega$  to a value  $\mathbf{p}(\omega)$  is equivalent to buying the security for  $\omega$  until the market price of the security reaches  $\mathbf{p}(\omega)$ . As an example for the decentralized financial (DeFi) automated market maker on blockchains, assume that the market maker offers n categories of tokens. Let  $\mathbf{q}=(q_1,\cdots,q_n)$  where  $q_i$  represents the number of outstanding tokens for the token category i. The market maker keeps track of the cost function

$$C(\mathbf{q}) = b \ln \sum_{i=1}^{n} e^{q_i/b} \tag{3}$$

and a price function for each token

$$P_i(\mathbf{q}) = \frac{\partial C(\mathbf{q})}{\partial q_i} = \frac{e^{q_i/b}}{\sum_{j=1}^n e^{q_j/b}}$$
(4)

It should be noted that the equation (4) is a generalized inverse of the scoring rule function (1). The cost function captures the amount of total assets wagered in the market where  $C(\mathbf{q}_0)$  is the market maker's maximum subsidy to the market. The price function  $P_i(\mathbf{q})$  gives the current cost of buying an infinitely small quantity of the category i token. If a trader wants to change the number of outstanding shares from  $\mathbf{q}_1$  to  $\mathbf{q}_2$ , the trader needs to pay the cost difference  $C(\mathbf{q}_2) - C(\mathbf{q}_2)$ .

Chen et al [1] showed that it is #P-hard (in the variable n) to compute price function  $P_i(\mathbf{q})$  for subset betting and pair betting. However, in DeFi applications, a patron may only offer automated markets with a small n (e.g., a pair of tokens). Thus the results in [1] do not incur challenges for implementing LMSR based DeFi applications. LMSR automated market makers have been implemented in Augur [12] and Gnosis [3].

Next we use an example to show how to design automated market makers using LMSR. Assume that b=1 and the patron sets up an automated market marker  $\mathbf{q}_0=(1000,1000)$  by depositing 1000 coins of token A and 1000 coins of token B. The initial market cost is  $C(\mathbf{q}_0)=\ln\left(e^{1000}+e^{1000}\right)=1000.693147$ . The instantaneous prices for a coin of tokens are

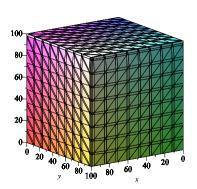
$$P_A(\mathbf{q}_0) = \frac{e^{1000}}{e^{1000} + e^{1000}} = 0.5$$
 and  $P_B(\mathbf{q}_0) = \frac{e^{1000}}{e^{1000} + e^{1000}} = 0.5$ 

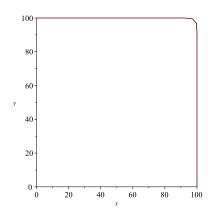
If this automated market maker is used as a price oracle, then one coin of token A equals  $\frac{P_A(\mathbf{q}_0)}{P_B(\mathbf{q}_0)}=1$  coin of token B. If a trader uses 0.689772 coins of token B to buy 5 coins of token A from market  $\mathbf{q}_0$ , then the market moves to a state  $\mathbf{q}_1=(995,1000.689772)$  with a total market cost  $C(\mathbf{q}_1)=1000.693147=C(\mathbf{q}_0)$ . The instantaneous prices for a coin of tokens in  $\mathbf{q}_1$  are  $P_A(\mathbf{q}_1)=0.003368975243$  and  $P_B(\mathbf{q}_1)=295.8261646$ . Now a trader can use 0.0033698 coins of token B to purchase 995 coins of token A from the automated market maker  $\mathbf{q}_1$  with a resulting market maker state  $\mathbf{q}_2=(0,1000.693147)$  and a total market cost  $C(\mathbf{q}_2)=1000.693147=C(\mathbf{q}_0)$ . The instantaneous prices for a coin of tokens in market maker  $\mathbf{q}_2$  are  $P_A(\mathbf{q}_2)=2.537979907\times 10^{-435}$  and  $P_B(\mathbf{q}_2)=1$ .

The above example shows that LMSR based automated market maker works well only when the outstanding shares of the tokens are evenly distributed (that is, close to 50/50). When the outstanding shares of the tokens are not evenly distributed, a trader can purchase all coins of the token with lesser outstanding shares and let the price ratio  $\frac{P_A(\mathbf{q})}{P_B(\mathbf{q})}$ 

changes to an arbitrary value with a negligible cost. This observation is further justified by the LMSR cost function curves in Figure 1. The first plot is for the cost function C(x, y, z) = 100 with three tokens and the second plot is for the cost function C(x,y) = 100 with two tokens. The second plot shows that the price for each token fluctuates smoothly only in a tiny part (the upper-right corner) of the curve with evenly distributed token shares. Outside of this part, the tangent line becomes vertical or horizontal. That is, one can use a tiny amount of one token to purchase all outstanding coins of the other token in the market maker. In a conclusion, LMSR based automated market makers may not be a good solution for DeFi applications.

Figure 1: LMSR market maker cost function curves for C(x, y, z) = 100 and C(x, y) = 100





## Liquidity-sensitive automated market maker LS-LMSR

In the traditional prediction market, the three desired properties for a pricing rule to have include: path independence, translation invariance, and liquidity sensitivity. Path independence means that if the market moves from one state to another state, the payment/cost is independent of the paths that it moves. For example, this means that a trader cannot place a series of transactions and profit without assuming some risk. Translation invariance requires that the cost of buying a guaranteed payout of x always costs x. Liquid sensitivity means that a fixed-size investment moves prices less in thick (liquid) markets than in thin (illiquid) markets.

**Definition 2.1** (see, e.g., Othman et al [11]) For a pricing rule P,

- 1. P is path independent if the value of line integral (cost) between any two quantity vectors depends only on those quantity vectors, and not on the path between them.
- 2. P is translation invariant if  $\sum_i P_i(\mathbf{q}) = 1$  for all valid market state  $\mathbf{q}$ .
- 3. P is liquidity insensitive if  $P_i(\mathbf{q} + (\alpha, \dots, \alpha)) = P_i(\mathbf{q})$  for all valid market state  $\mathbf{q}$  and  $\alpha$ .

Othman et al [11] showed that no market maker can satisfy all three of the desired properties at the same time. Furthermore, Othman et al [11] showed that LMSR satisfies translation invariance and path independence though not liquidity sensitivity. Then, by relaxing the translation invariance, Othman et al [11] proposed the Liquidity-Sensitive LMSR market. In particular, LS-LMSR changes the constant b in the LMSR formulas to  $b(\mathbf{q}) = \alpha \sum_i q_i$  where  $\alpha$  is a constant and requiring the cost function to always move forward in obligation space. Specifically, for  $\mathbf{q}=(q_1,\cdots,q_n)$ , the market maker keeps track of the cost function

$$C(\mathbf{q}) = b(\mathbf{q}) \ln \sum_{i=1}^{n} e^{q_i/b(\mathbf{q})}$$
(5)

and a price function for each token

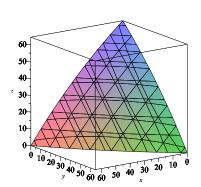
$$P_{i}(\mathbf{q}) = \alpha \ln \left( \sum_{j=1}^{n} e^{q_{j}/b(\mathbf{q})} \right) + \frac{e^{q_{i}/b(\mathbf{q})} \sum_{j=1}^{n} q_{j} - \sum_{j=1}^{n} q_{j} e^{q_{j}/b(\mathbf{q})}}{\sum_{j=1}^{n} q_{j} \sum_{j=1}^{n} e^{q_{j}/b(\mathbf{q})}}$$
(6)

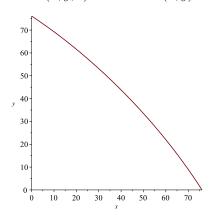
Furthermore, in order to always move forward in obligation space, we need to revise the cost that a trader should pay. In the proposed "no selling" approach, assume that the market is at state  $\mathbf{q}_1$  and the trader tries to impose an obligation  $\mathbf{q}_{\delta} = (q'_1, \cdots, q'_n)$  to the market with  $\bar{q}_{\delta} = \min_i q'_i < 0$ . That is, the trader puts  $q'_i$  coins of token i to the market if  $q'_i \geq 0$  and receives  $-q'_i$  coins of token i from the market if  $q'_i < 0$ . Let  $\bar{\mathbf{q}}_{\delta} = (-\bar{q}_{\delta}, \cdots, -\bar{q}_{\delta})$ . Then the trader should pay

$$C(\mathbf{q} + \mathbf{q}_{\delta} + \bar{\mathbf{q}}_{\delta}) + \bar{q}_{\delta} - C(\mathbf{q}) \tag{7}$$

and the market moves to the new state  $\mathbf{q} + \mathbf{q}_{\delta} + \bar{\mathbf{q}}_{\delta}$ . In the proposed "covered short selling approach", the market moves in the same way as LMSR market except that if the resulting market  $\mathbf{q}'$  contains a negative component, then the market  $\mathbf{q}'$  automatically adds a constant vector to itself so that all components are non-negative. In either of the above proposed approach, if  $\mathbf{q} + \mathbf{q}_{\delta}$  contains negative components, extra shares are automatically mined and added to the market to avoid negative outstanding shares. This should be avoided in DeFi applications. In DeFi applications, one should require that  $\mathbf{q}_{\delta}$  could be imposed to a market  $\mathbf{q}_{0}$  only if there is no negative component in  $\mathbf{q} + \mathbf{q}_{\delta}$  and the resulting market state is  $\mathbf{q} + \mathbf{q}_{\delta}$ . LS-LMSR is obviously path independent since it has a cost function. Othman et al [11] showed that LS-LMSR has the desired liquidity sensitive property. Figure 2 displays the curve of the cost function C(x, y, z) = 100 for LS-LMSR market maker with three tokens and the curve of the cost function C(x, y) = 100 for LS-LMSR market maker with two tokens. It is clear that these two curves are concave.

Figure 2: LS-LMSR market maker cost function curves for C(x, y, z) = 100 and C(x, y) = 100





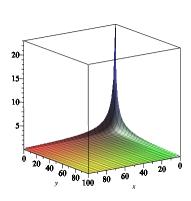
### 2.4 Constant product/sum/mean automated market makers

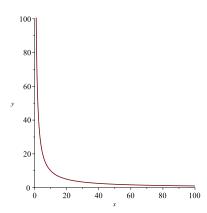
Constant product market makers have been used in DeFi applications (e.g., Uniswap [15]) to enable on-chain exchanges of digital assets and on-chain-decentralized price oracles. In this market, one keeps track of the cost function  $C(\mathbf{q}) = \prod_{i=1}^{n} q_i$  as a constant. For this market, the price function for each token is defined as

$$P_i(\mathbf{q}) = \frac{\partial C(\mathbf{q})}{\partial q_i} = \prod_{j \neq i} q_j.$$

Figure 3 shows the curve of the constant product cost function xyz = 100 with three tokens and the curve of the constant product cost function xy = 100 with two tokens. It is clear that the constant product cost function is convex which conforms to the principle of supply and demand.

Figure 3: Constant product cost function curves for xyz = 100 and xy = 100



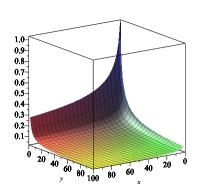


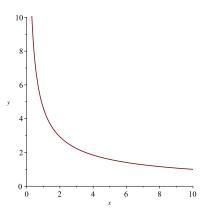
The cost function  $C(\mathbf{q}) = \prod_{i=1}^n q_i^{w_i}$  has been used to design constant mean automated market makers [10] where  $w_i$  are positive real numbers. In the constant mean market, the price function for each token is

$$P_i(\mathbf{q}) = \frac{\partial C(\mathbf{q})}{\partial q_i} = w_i q_i^{w_i - 1} \prod_{j \neq i} q_j.$$

Figure 4 shows the curve of the constant mean cost function  $xy^2z^3=100$  with three tokens and the curve of the constant mean cost function  $x^2y^3=100$  with two tokens. It is clear that the constant mean cost function is a convex function.

Figure 4: Constant mean cost function curves for  $xy^2z^3 = 100$  and  $x^2y^3 = 100$ 

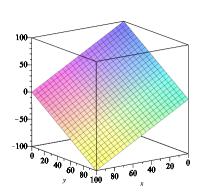


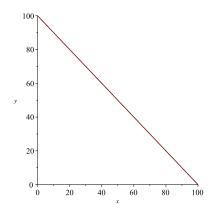


One may also use the cost function  $C(\mathbf{q}) = \sum_{i=1}^{n} q_i$  to design constant sum market makers. In this market, the price for each token is always 1. That is, one coin of a given token can always be used to trade for one coin of another token at any time when supply lasts. Figure 5 shows the curve of the constant sum cost function x + y + z = 100 with three tokens and the curve of the constant sum cost function x + y = 100 with two tokens.

It is straightforward to check that constant product/mean/sum automated market makers achieve path independence but not translation invariance. Furthermore, constant product/mean automated market makers are liquidity sensitive

Figure 5: Constant sum market maker cost function curves for x + y + z = 100 and x + y = 100





and constant sum automated market maker is liquidity insensitive.

## 2.5 Constant ellipse/circle automated market makers

Section 3 compares the advantages and disadvantages of LMSR, LS-LMSR, and constant product/mean/sum automated market makers. The analysis shows that none of them is ideal for DeFi applications. In this section, we propose automated market makers based on constant ellipse/circle cost functions. That is, the automated market maker's cost function is defined by

$$C(\mathbf{q}) = \sum_{i=1}^{n} (q_i - a)^2 + b \sum_{i \neq j} q_i q_j$$
 (8)

where a, b are constants. In constant ellipse/circle automated market makers, the price function for each token is

$$P_i(\mathbf{q}) = \frac{\partial C(\mathbf{q})}{\partial q_i} = 2(q_i - a) + b \sum_{i \neq i} q_i.$$

For automated market makers, we only use the first quadrant of the coordinate plane. By adjusting the parameters a, b in the equation (8), one may keep the cost function to be concave (that is, using the upper-left part of the ellipse/circle) or to be convex (that is, using the lower-left part of the ellipse/circle). By adjusting the absolute value of a, one may obtain various price amplitude and price fluctuation rates based on the principle of supply and demand for tokens. It is observed that constant ellipse/circle automated market maker price functions are liquidity sensitive and path independent but not translation invariance.

Figure 6 shows the curve of the constant ellipse cost function

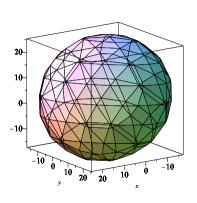
$$(x-10)^2 + (y-10)^2 + (z-10)^2 + 1.5(xy + xz + yz) = 350$$

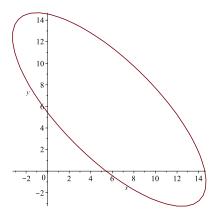
with three tokens and the curve of the the constant ellipse cost function

$$(x-10)^2 + (y-10)^2 + 1.5xy = 121$$

with two tokens. As mentioned in the preceding paragraphs, one may use convex or concave part of the ellipse for the cost function. For example, in the second plot of Figure 6, one may use the lower-left part in the first quadrant as a convex cost function or use the upper-right part in the first quadrant as a concave cost function.

Figure 6: Constant ellipse/circle cost function curves for three and two tokens





## 3 Supply and demand, coin liquidity, and token price fluctuation

Without loss of generality, this section considers automated market makers consisting of two tokens: a USDT token where each USDT coin costs one US dollar and an imagined spade suit token . The current market price of a token coin could have different values such as half a USDT coin, one USDT coin, two USDT coins, or others. In Decentralized Finance (DeFi) applications, the patron needs to provide liquidity by depositing coins of both tokens in the automated market maker. Without loss of generality, we assume that, at the time when the automated market maker is incorporated, the market price for a coin of spade suit token is equivalent to one USDT coin. For general cases that the market price for one . coin is not equivalent to one USDT coin at the time when the market maker is incorporated, we can create virtual shares in the automated market maker by dividing or merging actual coins. That is, each share of USDT (respectively ) in the automated market maker consists of a multiple or a portion of USDT (respectively ) coins. One may find some examples in Section 4.

To simplify our notations, we will use  $\mathbf{q}=(x,y)$  instead of  $\mathbf{q}=(q_1,q_2)$  to represent the market state. In this section, we will only study the price fluctuation of the first token based on the principle of supply and demand and the trend of the price ratio  $\frac{P_y(\mathbf{q})}{P_y(\mathbf{q})}$ . By symmetry of the cost functions, the price fluctuation of the second token and the ratio  $\frac{P_y(\mathbf{q})}{P_x(\mathbf{q})}$  have the same property. In the following, we analyze the token price fluctuation for various automated market maker models with the initial market state  $\mathbf{q}_0=(1000,1000)$ . That is, the patron creates the automated market maker by depositing 1000 USDT coins and 1000 spade suit coins in the market. The analysis results are summarized in Table 1.

Table 1: Token price comparison

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AMM type	market cost	$P_x(\mathbf{q})/P_y(\mathbf{q})$	tangent line slope $\frac{\partial y}{\partial x}$
LS-LMSR	2386.294362	(0.6481, 1.5430)	(-1.5430,-0.6481)
constant product	1000000	$(0,\infty)$	$(-\infty,0)$
constant sum	2000	1	-1
constant circle	50000000	(0.6236, 1.6036)	(-1.6036, -0.6236)

### 3.1 LS-LMSR

For the LS-LMSR based automated market maker, the market cost is

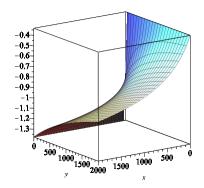
$$C(\mathbf{q}_0) = 2000 \cdot \ln\left(e^{1000/2000} + e^{1000/2000}\right) = 2386.294362.$$

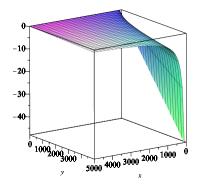
At market state  $\mathbf{q}_0$ , the instantaneous prices for a coin of tokens are  $P_x(\mathbf{q}_0) = P_y(\mathbf{q}_0) = 1.193147181$ . A trader may use 817.07452949 spade suit coins to purchase 1000 USDT coins with a resulting market state  $\mathbf{q}_1 = (0, 1817.07452949)$  and a resulting market cost  $C(\mathbf{q}_1) = 2386.294362$ . At market state  $\mathbf{q}_1$ , the instantaneous prices for a coin of tokens are  $P_x(\mathbf{q}_1) = 0.8511445298$  and  $P_y(\mathbf{q}_1) = 1.313261687$ . Thus we have  $P_x(\mathbf{q}_1)/P_y(\mathbf{q}_1) = 0.6481149479$ . The tangent line slope of the cost function curve indicates the token price fluctuation stability in the automated market. The tangent line slope for the LS-LMSR cost function curve at the market state  $\mathbf{q} = (x, y)$  is

$$\frac{\partial y}{\partial x} = -\frac{P_x(\mathbf{q})}{P_y(\mathbf{q})} = -\frac{(x+y)\left(e^{\frac{x}{x+y}} + e^{\frac{y}{x+y}}\right)\ln\left(e^{\frac{x}{x+y}} + e^{\frac{y}{x+y}}\right) + y\left(e^{\frac{x}{x+y}} - e^{\frac{y}{x+y}}\right)}{(x+y)\left(e^{\frac{x}{x+y}} + e^{\frac{y}{x+y}}\right)\ln\left(e^{\frac{x}{x+y}} + e^{\frac{y}{x+y}}\right) + x\left(e^{\frac{y}{x+y}} - e^{\frac{x}{x+y}}\right)}.$$

For the LS-LMSR automated market maker with an initial state  $\mathbf{q}_0=(1000,1000)$ , the tangent line slope (see Figure 7) changes smoothly and stays between -1.542936177 and -0.6481149479. Thus the token price fluctuation is quite smooth. By the principle of supply and demand, it is expected that when the token supply increases, the token price decreases. That is, the cost function curve should be convex. However, the cost function curve for LS-LMSR market is concave. This can be considered as a disadvantage of LS-LMSR markets for certain DeFi applications. Though LS-LMSR does not satisfy the translation invariance property, it is shown in [11] that the sum of prices are bounded by  $1 + \alpha n \ln n$ . For the two token market with  $\alpha = 1$ , the sum of prices are bounded by  $1 + 2 \ln 2 = 2.386294362$  and this value is achieved when x = y.

Figure 7: Tangent line slopes for LS-LMSR cost function (first) and constant product cost function (second)





As an additional example of LS-LMSR automated market makers, a trader may spend 10 USDT coins to purchase 10.020996 coins of spade suit token at market state  $\mathbf{q}_0$  or spend 500 USDT coins to purchase 559.926783 coins of spade suit from the market state  $\mathbf{q}_0$  with a resulting market state (1500, 440.073217). Furthermore, in the market state (1500, 440.073217), the value of one USDT coin is equivalent to the value of 1.260346709 coins of spade suit token.

#### 3.2 Constant product, constant mean, and constant sum

For the constant product automated market maker, the market cost is  $C(\mathbf{q}_0) = 1000000$  and the constant product cost function is  $x \cdot y = 1000000$ . At market state  $\mathbf{q}_0$ , the instantaneous token prices are  $P_x(\mathbf{q}_0) = P_y(\mathbf{q}_0) = 1000$ . Thus we have  $\frac{P_x(\mathbf{q})}{P_y(\mathbf{q})} = 1$ . A trader may use one USDT coin to buy approximately one coin of spade suit token and vice

versa at the market state  $\mathbf{q}_0$ . However, as market state moves on, the prices could change dramatically based on token supply in the market and the pool of a specific coin will never run out. Specifically, at market state  $\mathbf{q}_0$ , a trader may spend 10 USDT coins to purchase 9.900990099 spade suit coins. On the other hand, a user may spend 500 USDT coins to purchase only 333.3333333 coins of spade suit token from the market state  $\mathbf{q}_0$  with a resulting market state  $\mathbf{q}_1 = (1500, 666.6666667)$ . Note that in the example of LS-LMSR market example, at market state  $\mathbf{q}_0$ , a trader can spend 500 USDT coins to purchase 559.926783 coins of spade suit. Furthermore, in the market state  $\mathbf{q}_1$ , one USDT coin could purchase 0.4444444445 coins of spade suit token. The tangent line slope of the cost function curve at the market state  $\mathbf{q} = (x, y)$  is

$$\frac{\partial y}{\partial x} = -\frac{P_x(\mathbf{q})}{P_y(\mathbf{q})} = -\frac{y}{x}.$$

That is, the tangent line slope for the cost function curve (see Figure 7) can go from  $-\infty$  to 0 and the token price fluctuation could be very sharp. Specifically, if the total cost of the initial market  $\mathbf{q}_0$  is "small" (compared against attacker's capability), then a trader/attacker could easily control and manipulate the market price of each coins in the automated market maker. In other words, this kind of market maker may not serve as a reliable price oracle. A good aspect of the constant product cost function is that the curve is convex. Thus when the token supply increases, the token price decreases. On the other hand, the sum of prices  $P_x(\mathbf{q}) + P_y(\mathbf{q}) = x + y$  in constant product market is unbounded. Thus constant production cost function could not be used in prediction markets since it leaves a chance for a market maker to derive unlimited profit from transacting with traders.

For constant mean automated market makers, Figure 4 displays an instantiated constant mean cost function curve. The curve in Figure 4 is very similar to the curve in Figure 3 for the constant product cost function. Thus constant mean automated market maker has similar properties as that for constant product automated market maker and we will not go into details.

For constant sum automated market makers, the market cost is  $C(\mathbf{q}_0)=2000$  and the constant sum cost function is x+y=2000. A trader can always use one USDT coin to buy one spade suit token coin in the market and vice versa. This price is fixed and will not change as long as token supply lasts in the market. For example, a trader may spend 1000 USDT coins to purchase 1000 spade suit coins with a resulting market state  $\mathbf{q}_1=(2000,0)$ . At the market state  $\mathbf{q}_1$ , no one can purchase spade suit coins any more until someone spends some spade suit coins to purchase USDT coins from this market. Due to the fact that coin prices are fixed in this constant sum market maker, this kind of market may only be useful for stable coins whose relative prices do not change over time.

## 3.3 Constant circle and ellipse

As we have mentioned in the preceding Sections, one may use the upper-right part of the curve for a concave cost function or use the lower-left part of the curve for a convex cost function. In order to conform to the principle of supply and demand, we analyze the convex cost functions based on constant circle/ellipse. Constant circle and constant ellipse share many similar properties though they have different characteristics. By adjusting corresponding parameters, one may obtain different cost function curves with different properties (e.g., different price fluctuation range, different tangent line slope range, etc). The approaches for analyzing these cost function curves are similar. Our following analysis uses the low-left convex part of the circle  $(x - 6000)^2 + (y - 6000)^2 = 2 \times 5000^2$  as the constant cost function.

For automated market makers based on this cost function  $C(\mathbf{q})=(x-6000)^2+(y-6000)^2$ , the market cost is  $C(\mathbf{q}_0)=50000000$ . At market state  $\mathbf{q}_0$ , the instantaneous prices for a coin of tokens are  $P_x(\mathbf{q}_0)=P_y(\mathbf{q}_0)=-10000$ . A trader may use 1258.342613 spade suit coins to purchase 1000 USDT coins with a resulting market state  $\mathbf{q}_1=(0,2258.342613)$  and a resulting market cost  $C(\mathbf{q}_1)=C(\mathbf{q}_0)$ . At market state  $\mathbf{q}_1$ , the instantaneous prices for a coin of tokens are  $P_x(\mathbf{q}_1)=12000$  and  $P_y(\mathbf{q}_1)=7483.314774$ . Thus we have  $\frac{P_x(\mathbf{q}_1)}{P_y(\mathbf{q}_1)}=1.603567451$ . The tangent line slope of the cost function curve at the market state  $\mathbf{q}=(x,y)$  is

$$\frac{\partial y}{\partial x} = -\frac{P_x(\mathbf{q})}{P_y(\mathbf{q})} = -\frac{x - 6000}{y - 6000}.$$

This tangent line slope function (see Figure 8) changes very smoothly and stays in the interval [-1.603567451, -0.6236095645]. Thus the token price fluctuation is quite smooth. Furthermore, this cost function has a convex curve which conforms to the principle of supply and demand. That is, token price increases when token supply decreases. For constant circle

cost function market, the sum of prices are bounded by  $P_x(\mathbf{q}) + P_y(\mathbf{q}) = 2(x+y) - 4a$ . Similar bounds hold for constant ellipse cost function market. Thus, when it is used for prediction market, there is a limit on the profit that a market maker can derive from transacting with traders.

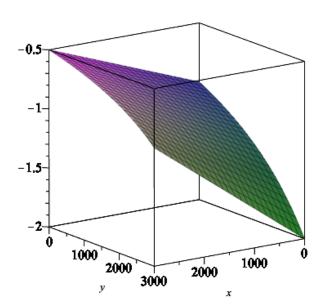


Figure 8: The tangent line slope for constant circle automated market maker

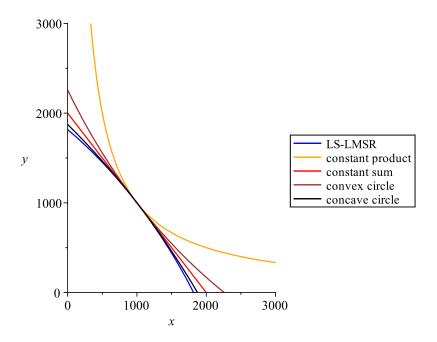
Figure 9 compares the cost function curves for different automated market makers that we have discussed. These curves shows that constant circle/ellipse cost function is among the best ones for DeFi applications.

### 3.4 Front running attacks

Front running attacks, which is closely related to slippage, have been well known for automated market makers and can be launched against all automated market makers with non-constant tangent line slopes. In the following, we compare this attack on a constant product automated market maker and on a constant circle automated market maker.

- 1. For a constant product market maker with an initial market state  $\mathbf{q}_0 = (1000, 1000)$ , assume that Alice submits 50 coins of USDT to purchase coins of spade suit token. The front-runner (e.g., a miner) intercepts this request and uses 200 coins of USDT token to get 166.6666667 coins of spade suit token leaving the market state at  $\mathbf{q}_1 = (1200, 833.3333333)$ . The front-runner submits Alice's order to the market  $\mathbf{q}_1$  which returns 33.33333333 coins of spade suit token to Alice with a resulting market state  $\mathbf{q}_2 = (1250, 800)$ . Now the front-runner uses 152.3809524 coins of spade suit token to get 200 USDT coins from  $\mathbf{q}_2$  with a resulting state  $\mathbf{q}_3 = (1050, 952.3809524)$ . Through this process, the front-runner obtained 166.6666667 152.3809524 = 14.2857143 coins of spade suit token for free.
- 2. For a constant circle market maker with cost function  $C(\mathbf{q}) = (x-6000)^2 + (y-6000)^2$  and an initial market state  $\mathbf{q}_0 = (1000, 1000)$ , assume that Alice submits 50 coins of USDT to purchase coins of spade suit token. The front-runner intercepts this request and uses 200 coins of USDT token to get 192.301994 coins of spade suit token leaving the market state at  $\mathbf{q}_1 = (1200, 807.698006)$ . The front-runner submits Alice's order to the market  $\mathbf{q}_1$  which returns 45.779716 coins of spade suit token to Alice with a resulting market state  $\mathbf{q}_2 = (1250, 761.918290)$ . Now the front-runner uses 188.576785 coins of spade suit token to get 200 USDT

Figure 9: Cost functions (order from bottom up):  $(x+y) \ln \left(e^{\frac{x}{x+y}} + e^{\frac{y}{x+y}}\right) = 2000 \cdot \ln \left(2e^{1/2}\right), (x+6000)^2 + (y+6000)^2 = 2 \times 7000^2, x+y = 2000, (x-6000)^2 + (y-6000)^2 = 2 \times 5000^2, \text{ and } xy = 1000000$ 



coins from  $\mathbf{q}_2$  with a resulting state  $\mathbf{q}_3 = (1050, 950.495075)$ . Through this process, the front-runner obtained 192.301994 - 188.576785 = 3.725209 coins of spade suit token for free.

The above two examples show that if front-runner's budget is 200 USDT token coins, then the front-runner could make 3.725209 spade suit coins of profit in constant circle market and 14.2857143 spade suit coins of profit in constant product market. In other words, the slippage for Alice is 14.2857143 spade suit coins in constant product market and is 3.725209 spade suit coins in constant circle product market.

Front-runner (slippage) attacks can always be launched if the tangent line slope for the cost function curve is not a constant. The more the tangent line slope fluctuates around the current market state, the more profit the front-runner can make. The analysis in preceding sections show that tangent line slopes for LS-LMSR and constant circle/ellipse cost functions fluctuate smoothly and tangent line slopes for constant product/mean cost functions fluctuate sharply. Thus LS-LMSR and constant circle/ellipse cost function automated markets are more robust against front running attacks (as shown in the two examples of the preceding paragraph). In practice, when a trader submits a transaction buying coins of token A with coins of token B (or vice versa), the trader may submit the order at the limit. But the front runner can always try to profit by letting the trader's order be executed at the limit price.

# 4 Price amplitude

For constant product/mean automated market makers, the relative price  $\frac{P_1(\mathbf{q})}{P_2(\mathbf{q})}$  of the two tokens ranges from 0 (not inclusive) to  $\infty$ . At the time when a tiny portion of one token coin is equivalent to all coins of the other token in the market maker, no trade is essentially feasible. Thus the claimed advantage that no one can take out all shares of one token from the constant product/mean market seems to have limited value. For a given LS-LMSR (or constant circle/ellipse) automated market with an initial state  $\mathbf{q}_0$ , the relative price  $P_1(\mathbf{q})/P_2(\mathbf{q})$  can take values only from a fixed interval. If the market changes and this relative price interval no long reflects the market price of the two tokens,

one may need to add tokens to the market to adjust this price interval. On the other hand, it may be more efficient to just cancel this automated market maker and create a new automated market maker when this situation happens.

In the following example, we show how to add liquidity to an existing LS-LMSR automated market maker to adjust the relative price range. Assume that the market price for a coin of token A is 100 times the price for a coin of token B when the automated market maker is incorporated. The patron uses 10 coins of token A and 1000 coins of token B to create an automated market maker with the initial state  $\mathbf{q}_0 = (1000, 1000)$ . The total market cost is  $C(\mathbf{q}_0) = 2386.294362$ . Assume that after some time, the automated market maker moves to state  $\mathbf{q}_1 = (100, 1750.618429)$ . At  $\mathbf{q}_1$ , we have  $P_1(\mathbf{q}_1)/P_2(\mathbf{q}_1) = 0.6809820540$  which is close to the lowest possible value 0.6481149479. In order to adjust the automated market maker so that it still works when the value  $P_1/P_2$  in the real world goes below 0.6481149479, the patron can add some coins of token A to  $\mathbf{q}_1$  so that the resulting market state is  $\mathbf{q}_2 = (1750.618429, 1750.618429)$ . To guarantee that one coin of token B is equivalent to  $\frac{P_2(\mathbf{q}_1)}{100 \cdot P_1(\mathbf{q}_1)} = 0.01468467479$  coins of token A in  $\mathbf{q}_2$ , we need to have the following mapping from outstanding shares in  $\mathbf{q}_2$  to actual token coins (note that this mapping is different from that for  $\mathbf{q}_0$ ):

- Each outstanding share of token A corresponds to 0.01468467479 coin of token A.
- Each outstanding share of token B corresponds to one coin of token B.

Thus there are  $1750.618429 \times 0.01468467479 = 25.70726231$  coins of token A in  $\mathbf{q}_2$ . Since there are only one coin of token A in  $\mathbf{q}_1$ , the patron needs to deposit 24.70726231 coins of token A to  $\mathbf{q}_1$  to move the automated market maker to state  $\mathbf{q}_2$ . If the market owner chooses not to deposit these tokens to the market, the market maker will still run, but there is a chance that the outstanding shares of token A goes to zero at certain time.

In the above scenario, one may ask whether it is possible for the market maker to automatically adjust the market state to  $\mathbf{q}_3=(1750.618429,1750.618429)$  by re-assigning the mapping from shares to coins? If  $\mathbf{q}_2$  automatically adjusts itself to  $\mathbf{q}_3$  without external liquidity input, then a trader may use one share of token A to get one share of token B in  $\mathbf{q}_3$ . Since we only have one equivalent coin of token A but 1750.618429 outstanding shares in  $\mathbf{q}_3$ , each outstanding share of token A in  $\mathbf{q}_3$  is equivalent to 0.0005712267068 coins of token A. That is, the trader used 0.0005712267068 coins of token A to get one coin of token B (note that each outstanding share of token B corresponds to one coin of token B in  $\mathbf{q}_3$ ). By our analysis in the preceding paragraphs, at  $\mathbf{q}_3$ , one coin of token B has the same market value of 0.01468467479 coins of token A. In other words, the trader used 0.0005712267068 coins of token A to get equivalent 0.01468467479 coins of token A. Thus it is impossible for the automated market to adjust its relative price range without an external liquidity input.

## 5 Conclusion

The analysis in the paper shows that constant circle/ellipse cost functions are a better choice for building automated market makers in Decentralized Finance (DeFi) applications. One may argue that constant circle/ellipse cost function based markets have less flexibility after the market is launched since the price amplitude is fixed. We have mentioned that, though the token price could range from 0 to  $\infty$  in the constant product cost model, when the price for one token is close to infinity, any meaningful trade in the market is infeasible. Thus the old market needs to be stopped and a new market should be incorporated. Indeed, it is an advantage for an automated market maker to have a fixed price amplitude when it is used as a price oracle for other DeFi applications. For the constant product cost market, if the patron incorporates the automated market maker by deposing a small amount of liquidity, an attacker with a small budget can manipulate the token price significantly in the automated market maker and take profit from other DeFi applications that use this automated market maker as a price oracle. For constant circle/ellipse based automated market makers, the patron can use a small amount of liquidity to set up the automated market and the attacker can only manipulate the token price within the fixed price amplitude.

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