PROBABILITY AND STATISTICS FOR ENGINEERS

LESSON INSTRUCTIONS

The lecture notes are divided into chapters. Long chapters are logically split into numbered subchapters.

Study Time

Estimated time to study and fully grasp the subject of a chapter. The time is approximate add should only be treated as a guide.

Learning Objectives

These are aims that you need to achieve at the end of each chapter. They are based on knowledge and skills required.

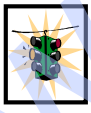
Explanation

Explanation expands on the studied material. New terms are introduced and explained in more detail. Examples are given.

Summary

Key ideas are summarized in conclusion of each chapter. If they are not clear enough at this point, it is recommended that you go back and study the chapter again.

Additional Clues



Example and Solution

Quiz

To make sure that you thoroughly understand the discussed subject, you are going to be asked several theoretical questions. You will find the answers in the brackets or at the end of the textbook in the SOLUTION KEYS section.

Practical Exercises

At the end of each long chapter practical application of the theory is presented in exercises.

1 EXPLORATORY DATA ANALYSIS

Study Time: 70 minutes

Learning Objectives

- General Concepts of Exploratory (Preliminary) Statistics
- Data Variable Types
- Statistical Characteristics and Graphical Methods of Presenting Qualitative Variables
- Statistical Characteristics and Graphical Methods of Presenting Quantitative Variables

Explanation

Original goal of statistics was to collect data about population based on population samples. By population we mean a group of all existing components available for observation during statistical research. For example:

> *If a statistical research is performed about physical hight of 15-year old girls, the population will be all girls currently aged 15.*

Considering the fact that the number of population members is usually high, the research will be based on the so-called **sample examination** where only part of the population is used. The examined part of the population is called a **sample**. What's really important is to make a definite selection that is as representative of the whole group as possible.

There are several ways to achieve it. To avoid of omitting some elements of the population the so-called **random sample** is used in which each element of population has the same chance of being selected.

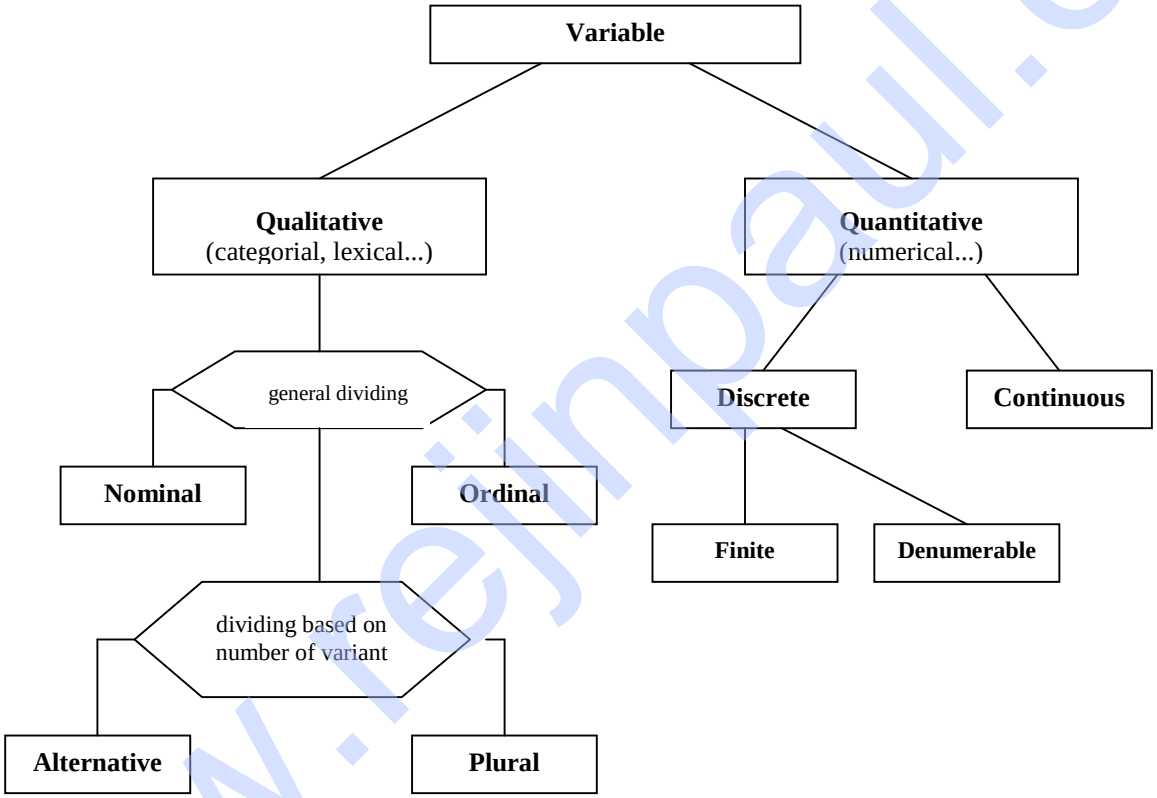
It goes without saying that sample examination can never be as accurate as examining the whole population. Why do we do prefer it then?

1. To save time and minimize costs (especially for large populations).

2. To avoid damaging samples in destructive testing (some tests like examining cholesterol in blood etc., lead to the permanent damage of examined elements).
3. Because the whole population is not available.

Now that you know that statistics can describe the whole population based on information gathered from a population sample we will move on to Exploratory Data Analysis (EDA). Data we observe will be called **the variables** and their values **variable variants**. **EDA** is often the first step in revealing information hidden in a large amount of variables and their variants.

Because the way of processing variables depends most on their type, we will now explore how variables are devided into different categories. The variables division is shown in the following diagram.



- **Qualitative variable** – its variants are expressed verbally and they split into two general subgroups according to what relation is between their values:
    - **Nominal variable** – has equivalent variants: it is impossible to either compare them or sort them (for example: sex, nationality, etc.)

- **Ordinal variable** – forms a transition between qualitative and quantitative variables: individual variant can be sorted and it is possible to compare one another (for example: cloth sizes S, M, L, and XL)

The second way of dividing them is based on number of variants:

- **Alternative variable** – has only two possible options (e.g. sex – male or female, etc.)
- **Plural variable** – has more than two possible options (e.g. education, name, eye color, etc.)

- **Quantitative variable –** is expressed numerically and it's divided into:
  - **Discrete variable** – it has finite or denumerable number of variants
    - **Discrete finite variable –** it has finite number of variants (e.g. math grades - 1,2,3,4,5)
    - **Discrete denumerable variable –** it has denumerable number of variants (e.g. age (year), height (cm), weight (kg), etc.)
  - **Continuous variable** - it has any value from $\Re$ or from some $\Re$ subset (e.g. distance between cities, etc.)

Additional clues

Imagine that you have a large statistical group and you face a question of how to best describe it. Number representations of values are used to "replace" the group elements and they become the basic attributes of the group. This is what we call statistical characteristics.

In the next chapters we are going to learn how to set up statistical characteristics for various types of variables and how to represent larger statistical groups.

1.1 Statistical Characteristics of Qualitative Variables

We know that a qualitative variable has two basic types - nominal and ordinal.

1.1.1 Nominal Variables

Nominal variable has different but equivalent variants in one group. The number of these variants is usually low and that's why the first statistical characteristics we use to describe it will be its frequency.

- **Frequency n_i** (absolute frequency)

  - is defined as the number of a variant occurrences of the qualitative variable

In case that a qualitative variable has *k* different variants (we describe their frequency as n_1, n_2… n_k) - in a statistical group (of n values) it must be true that:

$$n_1 + n_2 + ... + n_k = \sum_{i=1}^{k} n_i = n$$

If you want to express the proportion of the variant frequency on the total number of occurrences, we use relative frequency to describe the variable.

- **Relative frequency p_i**

  - is defined as:

$$p_i = \frac{n_i}{n}$$

  alternatively:

$$p_i = \frac{n_i}{n} \cdot 100 \quad [\%]$$

  (We use the second formula to express the relative frequency in percentage points).
  For relative frequency it must be true that:

$$p_1 + p_2 + \ldots + p_k = \sum_{i=1}^{k} p_i = 1$$

When qualitative variables are processed, it is good to arrange frequency and relative frequency in the so-called **frequency table**:

FREQUENCY TABLE		
Values x_i	**Absolute frequency**	**Relative frequency**
	n_i	**p_i**
x_1	n_1	p_1
x_2	n_2	p_2
⋮	⋮	⋮
x_k	n_k	p_k
Total	$\sum_{i=1}^{k} n_i = n$	$\sum_{i=1}^{k} p_i = 1$

The last characteristic of nominal variable is the mode.

- **Mode**

- is defined as a variant that occurs most frequently

The mode represents a typical element of the group. Mode cannot be determined if there are more values with maximum frequency in the statistical group.

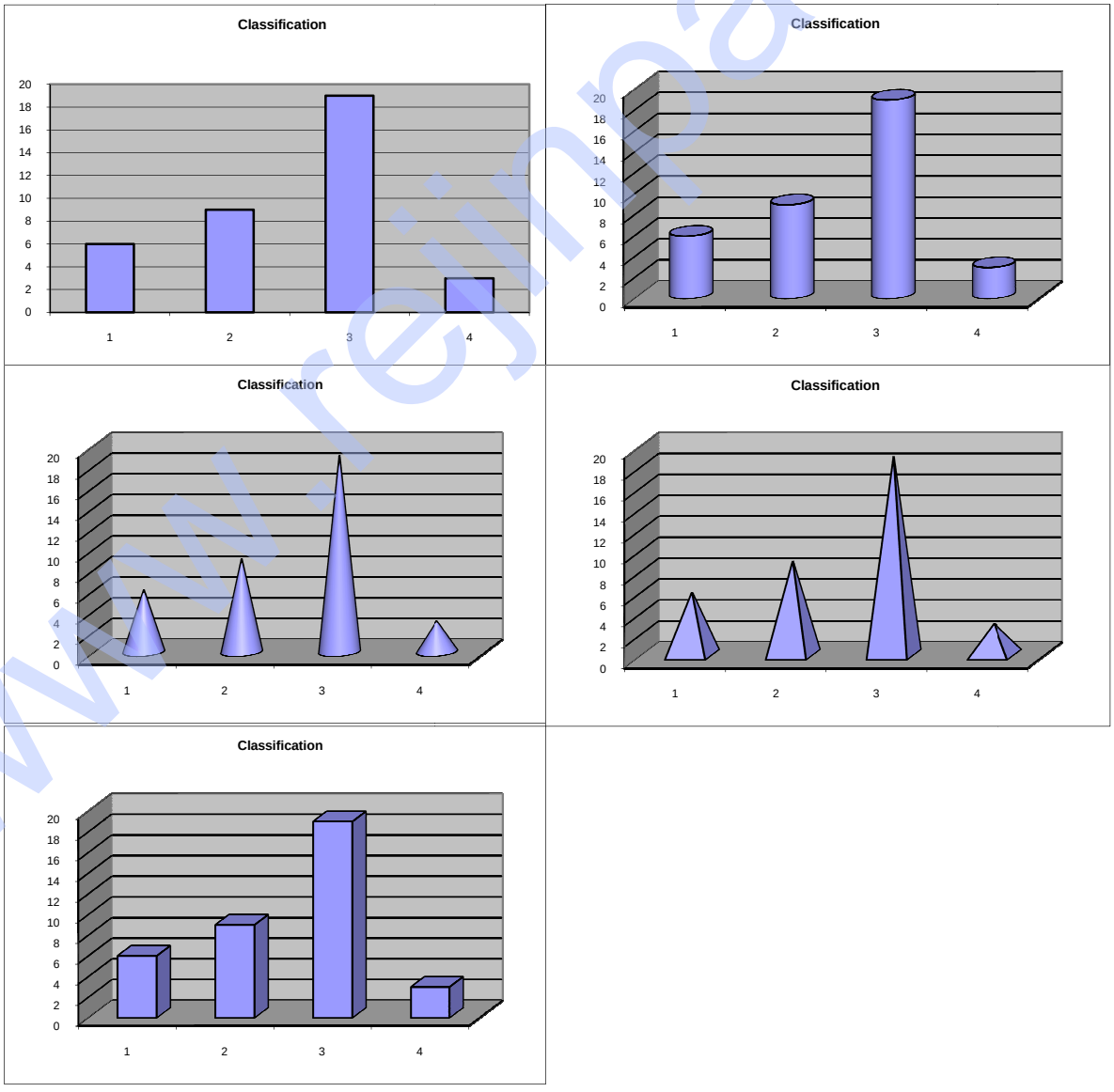
1.1.2 Graphical Methods of Presenting Qualitative Variables

The statistics often uses **graphs** for better analysis of variables. There are two types of graphs for analyzing nominal variable:

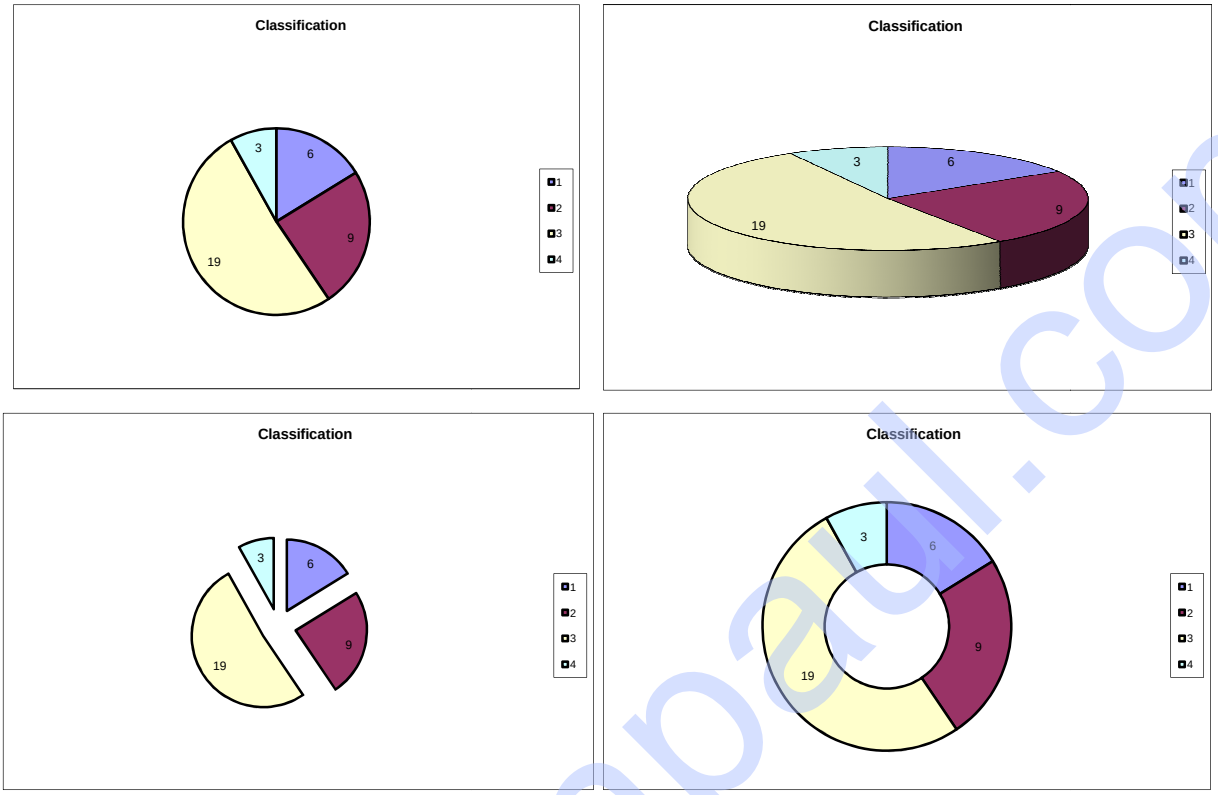
- **Histogram** (bar chart)
- **Pie chart**

Histogram is a standard graph where variants of the variable are represented on one axis and variable frequencies on the other exis. Individual values of the frequency are then displayed as bars (boxes, vectors, squared logs, cones, etc.)

Examples:

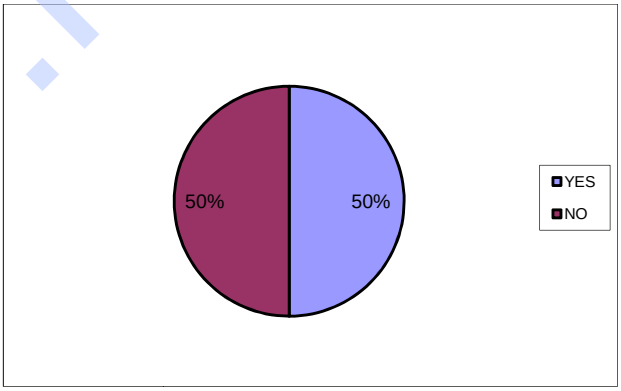


Pie chart represents relative frequencies of individual variants of a variable. Frequencies are presented as proportions in a sector of a circle. When we change the angle of the circle, we can get elliptical, three-dimensional effect.

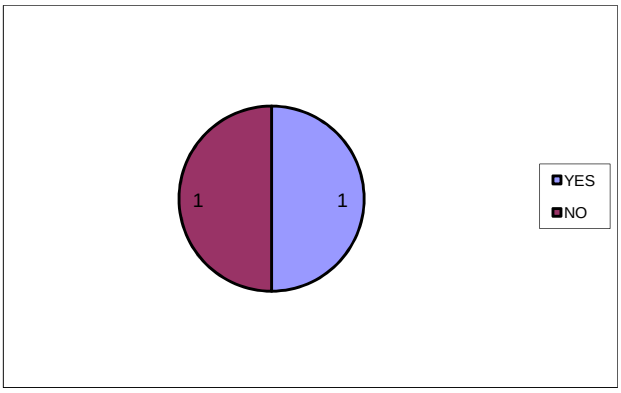


REMEMBER! Describing the pie chart is necessary. Marking individual sectors by relative frequencies only without adding their absolute values is not sufficient.

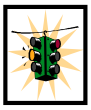
Example: An opinion poll has been carried out about launching high school fees. Its results are shown on the following chart:



Aren't the results interesting? No matter how true they may be, it is recommended that the chart be modified as follows:



What is the difference? From the second chart it is obvious that only two people were asked - the first one said YES and the second one said NO. What can be learned from that? Make charts in such a way that their interpretation is absolutely clear. If you are presented with a pie chart without absolute frequencies marked on it, you can ask yourselves whether it is because of the author's ignorance or it is a deliberate bias.



Example and Solution

An observational study has been undertaken on the use of an intersection. The collected data are in the table below. The data is made up of colours of cars that pass through the intersection. Analyze the data and interpret the results in a graphical form.

red	blue	red	Green
blue	red	red	White
green	green	blue	Red

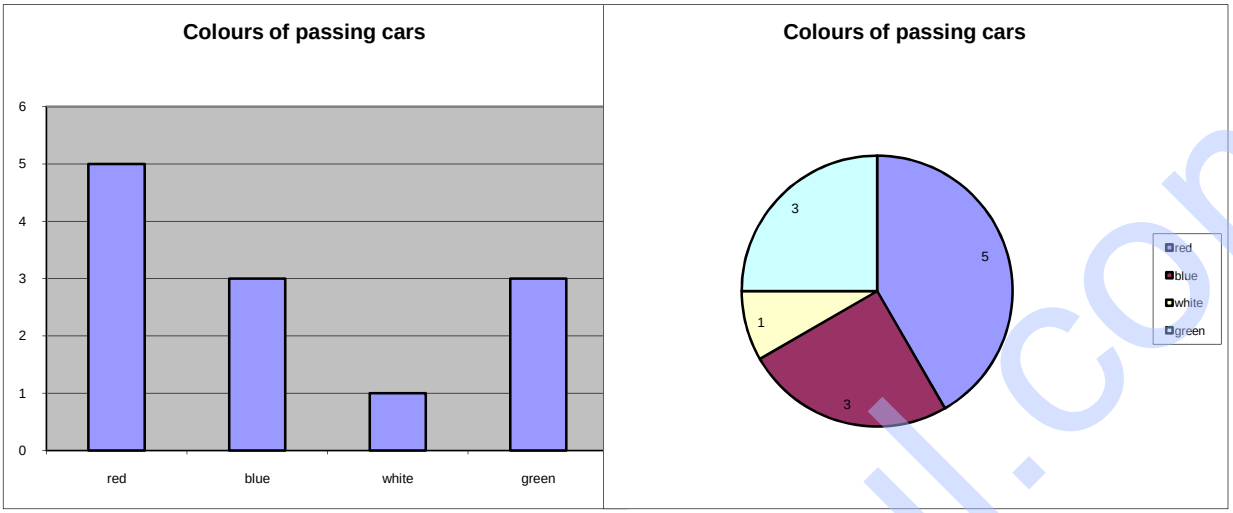
Solution:
From the table it is obvious that the collected colours are qualitative (lexical) variables, and because there is no order or comparison between them, we can say they are nominal variables.

For better description we create a frequency table and we determine the mode. We are going to present the colours of the passing vehicles by a histogram and a pie chart.

FREQUENCY TABLE		
Colors of passing cars	**Absolute frequency**	**Relative frequency**
	$\mathbf{n_i}$	$\mathbf{p_i}$
red	5	5/12 = 0.42
blue	3	3/12 = 0.25
white	1	1/12 = 0.08
green	3	3/12 = 0.25
Total	12	1.00

We observed 12 cars total.

Mode = red (i.e. in our sample most cars were red)



1.1.3 Ordinal Variable

Now we are going to have a look at describing ordinal variables. The ordinal variable (just like the nominal variable) has various verbal variants in the group but these variants can be sorted i.e. we can tell which one is "smaller" and which one is "bigger"

For describing ordinal variables we use the same statistical characteristics and graphs as for nominal variables (frequency, relative frequency, mode viewed by histogram or pie chart) plus two others characteristics (cumulative frequency and cumulative relative frequency) thus including information about how they are sorted.

- **Cumulative frequency of the i-th variant m_i**

  - is a number of values of a variable showing the frequency of variants less or equal the i-th variant

  E.g. we have a variable called "grade from Statistics" that has the following variants: "1", "2", "3" or "4" (where 1 is the best and 4 the worst grade). Then, for example, the cumulative frequency for variant "3", will be equal number of students who get grade "3" or better.

  If variants are sorted by their "size" ("$x_1 < x_2 < \ldots < x_k$") then the following must be true:

  $$m_i = \sum_{j=1}^{i} n_j$$

  So it is self-evident that cumulative frequency k-th ("the highest") variant is equal to the variable n.

  $$m_k = n$$

The second special characteristic for ordinal variable is cumulative relative frequency.

- **Cumulative relative frequency of i-th variant F_i**

  - a part of the group are the values with the i-th and lower variants. They are expressed by the following formula:

$$F_i = \sum_{j=1}^{i} p_j$$

  This is nothing else then relative expression of the cumulative frequency:

$$F_i = \frac{m_i}{n}$$

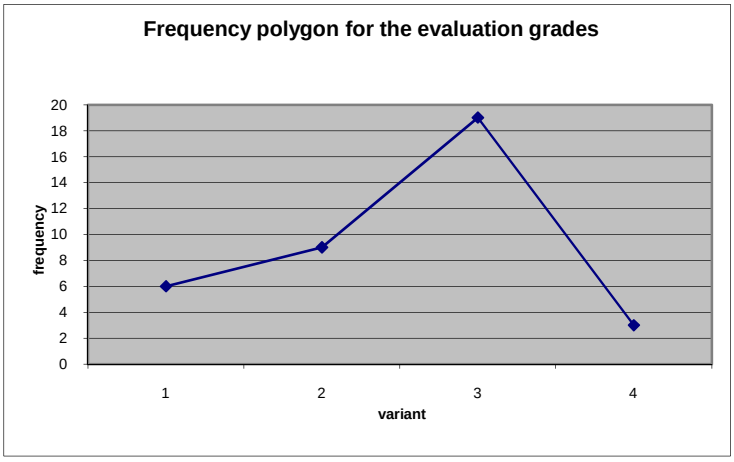
Just as in the case of nominal variables we can present statistical characteristics using frequency table for ordinal variables. In comparison to the frequency table of nominal variables it also contains values of cumulative and cumulative relative frequencies.

FREQUENCY TABLE				
Values x_i	**Absolute frequency**	**Cumulative frequency**	**Relative frequency**	**Relative cumulative frequency**
	n_i	**m_i**	**p_i**	**F_i**
x_1	n_1	$m_1 = n_1$	p_1	$F_1 = p_1$
x_2	n_2	$m_2 = n_1 + n_2 = m_1 + n_2$	p_2	$F_2 = p_1 + p_2 = F_1 + p_2$
⋮	⋮	⋮	⋮	⋮
x_k	n_k	$m_k = n_{k-1} + n_k = n$	p_k	$F_k = F_{k-1} + p_k = 1$
Total	$\sum_{i=1}^{k} n_i = n$	-----	$\sum_{i=1}^{k} p_i = 1$	-----

1.1.4 Graphical Presentation of Ordinal Variables

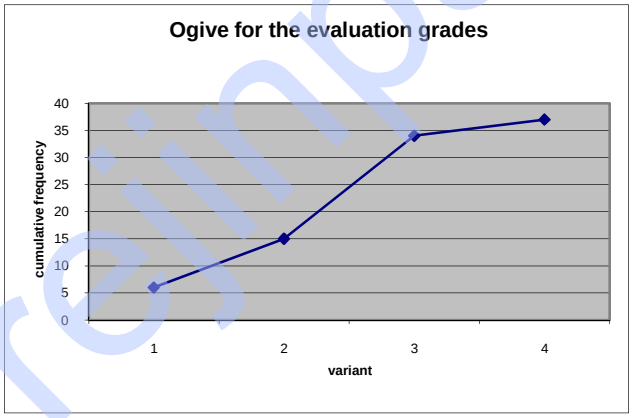
We briefly mentioned histogram and the pie chart as good ways of presenting the ordinal variable. But these graphs don't reflect variants' sorting. To achieve that, we need to use polygon (also known as Ogive) and Pareto graph.

Frequency Polygon
- is a line chart. The frequency is placed along the vertical axis and the individual variants of the variable are placed along the horizontal axis (sorted in ascending order from the "lowest" to the "highest"). The values are attached to the lines.



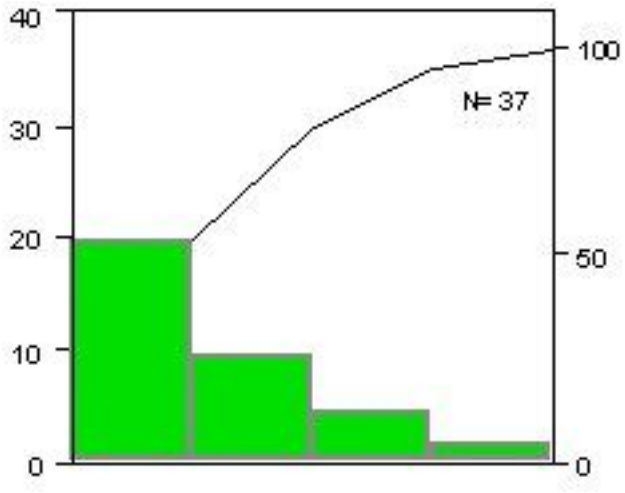
Ogive (Cumulative Frequency Polygon)

- is a frequency polygon of the cumulative frequency or the relative cumulative frequency. The vertical axis is the cumulative frequency or relative cumulative frequency. The horizontal axis represents variants. The graph always starts at zero, at the lowest variant, and ends up at the total frequency (for a cumulative frequency) or 1.00 (for a relative cumulative frequency).

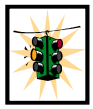


Pareto Graph

- is a bar chart for qualitative variable with the bars arranged by frequency
- variants are on horizontal axis and are sorted from the “highest” importance to the “lowest”



Notice the decline of cumulative frequency. It drops as the frequency of variables decreases.



Example and Solution

Following data represent t-shirts sizes that a cloths retailer offers on sale:

S, M, L, S, M, L, XL, XL, M, XL, XL, L, M, S, M, L, L, XL, XL, XL, L, M

a) Analyze the data and interpret results in a graphical form.
b) Determine what percentage of people bought t-shirts of L size maximum.

Solution:

a) The variable is qualitative (lexical) and t-shirt sizes can be sorted, therefore it is an ordinal variable. For its description you use frequency table for the ordinal variable and you determine the mode.

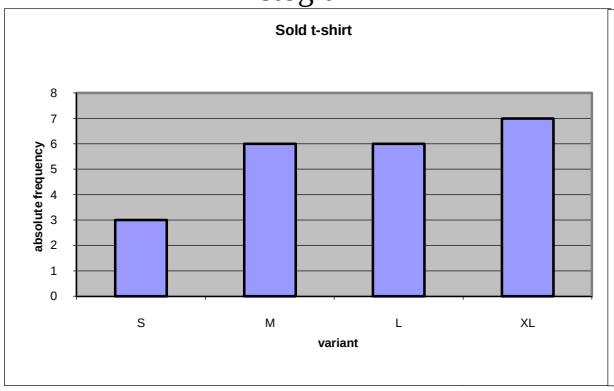
FREQUENCY TABLE		
Colors of passing cars	**Absolute frequency**	**Relative frequency**
	n_i	**p_i**
red	5	5/12=0.42
blue	3	3/12=0.25
white	1	1/12=0.08
green	3	3/12=0.25
Total	12	1.00

Mode = XL (the most people bought t-shirts with XL value)

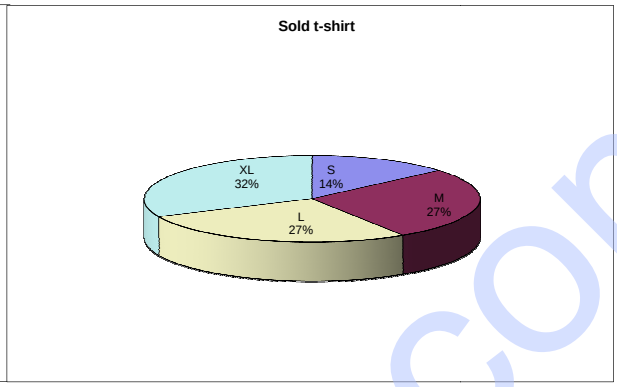
For graphical representation use histogram, pie graph and cumulative frequency polygon (you don't create Pareto graph because it is mostly used for technical data).

Graphical output:

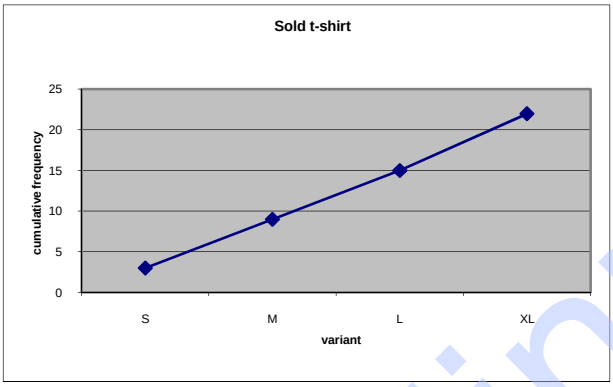
Histogram



Pie Chart



Cumulative Frequency Polygon



Total sales were 22 t-shirts.

b) You get the answer from the value of the relative cumulative frequency for variant L. You see that 68% of people bought t-shirts of L size and smaller.

1.2 Statistical Characteristics of Quantitative Variables

To describe quantitative variable, most of the statistical characteristics for ordinal variable description can be used (frequency, relative frequency, cumulative frequency and cumulative relative frequency). Apart from those, there are two additional ones:

- **Measures of location** – those indicate a typical distribution of the variable values

and

- **Measures of variability** – those indicate a variability (variance) of the values around their typical position

1.2.1 Measures of Location and Variability

The most common measure of position is the variable mean. The mean represents average or typical value of the sample population. The most famous mean of quantitative variable is:

- **Arithmetical mean** $\bar{x}$

  It is defined by the following formula:

$$\bar{x} = \frac{\sum_{i=!}^{n} x_i}{n}$$

  where: x_i ... are values of the variable
  n ... size of the sample population (number of the values of the variable)

  Properties of the arithmetical mean:

1. $\sum_{i=!}^{n} (x_i - \bar{x}) = 0$

   - sum of all diversions of variable values from their arithmetical mean is equal to zero which means that arithmetical mean compensates mistakes caused by random errors.

2. $\forall (a \in \Re): \left(\bar{x} = \frac{\sum_{i=!}^{n} x_i}{n} \Rightarrow \frac{\sum_{i=!}^{n} (a + x_i)}{n} = a + \bar{x} \right)$

   - if the same number is added to all the values of the variable, the arithmetical mean increases by the same number

3. $\forall (b \in \Re): \left(\bar{x} = \frac{\sum_{i=!}^{n} x_i}{n} \Rightarrow \frac{\sum_{i=!}^{n} (bx_i)}{n} = b\bar{x} \right)$

   - if all the variable values are multiplied by the same number the arithmetical Mean increases accordingly

Arithmetical mean is not always the best way to calculate the mean of the sample population. For example, if we work with a variable representing relative changes (cost indexes, etc.) we use the so-called geometrical mean. To calculate mean when the variable has a form of a unit, harmonical mean is often used.

Considering that the mean uses the whole variable values data set, it carries maximum information about the sample population. On the other hand, it's very sensitive to the so-called **outlying observations (outliers).** Outliers are values that are substantially different from the rest of the values in a group and they can distort the mean to such a degree that it no longer represents the sample population. We are going to have a closer look at the Outliers later.

Measures of location that are less dependent on the outlying observations are:

- **Mode** $\hat{x}$

  In the case of mode we will differentiate between discrete and continuous quantitative variable. **For discrete variable** we define **mode** as the most frequent value of the variable (similarly as with the qualitative variable).

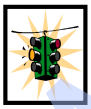
  But in the case of **continuous variable** we think of the mode as the value around which most variable values are concentrated.

  For assessment of this value we use **shorth**. Shorth is the shortest interval with at least 50% of variable values. In case of a sample as large as $n = 2k \ (k \in \mathrm{N})$ (with even number of values) k values lie within shorth - which is n/2 (50%) variable values. In the case of a sample as large as $n = 2k+1 (k \in \mathrm{N})$ (with odd number of values) $k+1$ values lies within short - which is about 1/2 plus 50% variable values (n/2+1/2).

  Then, the **mode** $\hat{x}$ can be defined as the centre of the shorth.

  From what has been said so far it is clear that the shorth length (top boundary - bottom boundary) is unique but its location is not.

  If the mode can be determined unambiguously we talk about **unimode variable**. When a variable has two modes we call it **bimode**. When there are two or more modes in a sample, it usually indicates a heterogenity of variable values. This heterogenity can be removed by dividing the sample into more subsamples (for example bimode mark for person's height can be divided by sex into two unimode marks - women's height and men's height).



Example and Solution

The following data shows ages of musicians who performed at a concert. Age is a continuous variable. Calculate Mean, Shorth and Mode for the variable.

22 82 27 43 19 47 41 34 34 42 35

Solution:

a) Mean:

In this case we use arithmetical mean:

$$\bar{x} = \frac{\sum_{i=!}^{n} x_i}{n} = \frac{22+82+27+43+19+47+41+34+34+42+35}{11} = 38.7\,years$$

The musicians' average age is 38.7 years.

b) Shorth:

Our sample population has 11 values. 11 is an odd number. 50% of 11 is 5.5 and the nearest higher natural number is 6 - otherwise: n/2+1/2 = 11/2+1/2 = 12/2 = 6. That means that 6 values will lie in the Shorth.

And what are the next steps?

- You need to sort the variable
- You determine the size of all the intervals (having 6 elements) where $x_i < x_{i+1} < \ldots < x_{i+5}$
- The shortest of these intervals will be the shorth (size of the interval = $x_{i+5} - x_i$)

Original data	Sorting data	Size of intervals (having 6 elements)
22	19	16 (= 35 – 19)
82	22	19 (= 41 – 22)
27	27	15 (= 42 – 27)
43	***34***	***9*** (= 43 – 34)
19	***34***	13 (= 47 – 34)
47	***35***	47 (= 82 – 35)
41	***41***	
34	***42***	
34	***43***	
42	47	
35	82	

From the table you can see that the shortest interval has the value of 9. There is only one interval that corresponds to this size and that is: $\langle 34;43 \rangle$.

Shorth = $\langle 34;43 \rangle$ and that means that half of the musicians are between 34 and 43 years of age.

c) Mode:

Mode is defined as the center of shorth:

$$\hat{x} = \frac{34+43}{2} = 38.5$$

Mode = 38.5 years which means that the typical age of the musicians who performed at the concert was 38.5 years.

Among other characteristics describing quantitative variables are **quantiles**. Those are used for more detailed illustration of the distribution of the variable values within the scope of the population.

- **Quantiles**
  Quantiles describe location of individual values (within the variable scope) and are resistant to outlying observations similarly like the mode. Generally the quantile is defined as a value that divides the sample into two parts. The first one contains values that are smaller than given quantile and the second one with values larger or equal than the given quantile. The data must be sorted ascendingly from the lowest to the highest value.

  Quantile of variable x that separates 100% smaller values from the rest of the samples (i.e. from 100(1-p)% values) will be called **100p % quantile** and marked $\mathbf{x_p}$.

  In real life you most often come across the following quantiles:

- **Quartiles**

In case of the four-part division the values of the variate corresponding to 25%, 50%, and 75% of the total distribution are called quartiles.

Lower quartile $\mathbf{x_{0,25}}$ = 25% quantile - divides a sample of data in a way that 25% of the values are smaller than the quartile, i.e. 75% are bigger (or equal)

Median $\mathbf{x_{0,5}}$ = 50% quantile - divides a sample of data in a way that 50% of the values are smaller than the median and 50% of values are bigger (or equal)

Upper quartile $\mathbf{x_{0,75}}$ = 75% quantile - divides a sample of data in a way that 75% of values are smaller than the quartile, i.e. 25% are bigger (or equal)

Example:

Data	6 47 49 15 43 41 7 39 43 41 36
Data in ascending order	6 7 15 36 39 41 41 43 43 47 49
Median	41
Upper quartile	43
Lower quartile	15

The difference between the 1st and 3rd quartile is called the **Inter-Quartile Range (IQR)**.

$$IQR = x_{0.75} - x_{0.25}$$

Example:

Data	2 3 4 5 6 6 6 7 7 8 9
Upper quartile	7
Lower quartile	4
IQR	7 - 4 = 3

- **Deciles** – $x_{0.1;}\, x_{0.2;} \ldots ;\, x_{0.9}$

  The deciles divide the data into 10 equal regions.

- **Percentiles –** $x_{0.01};\, x_{0.02};\, \ldots;\, x_{0.99}$

  The percentiles divide the data into 100 equal regions.

  For example, the 80th percentile is the number that has 80% of values below it and 20% above it. Rather than counting 80% from the bottom, count 20% from the top.

  Note: The 50th percentile is the median.

- **Minimum x_{min} and Maximum x_{max}**

  $x_{\min} = x_0$, i.e. 0% of values are less than minimum
  $x_{\max} = x_1$, i.e. 100% of values are less than maximum

There is the following process to determine quantiles:

1. The sample population needs to be ordered by size
2. The individual values are sequenced so that the smallest value is at the first place and the highest value is at n-th place (n is the total number of values)
3. 100p% quantile is equal to a variable value with the sequence z_p where: $z_p = n \cdot p + 0.5$
   z_p has to be rounded to integer!!!

REMEMBER!!!
In case of a data set with an even number of values the median is not uniquely defined. *Any number between two middle values (including these values) can be accepted as the median. Most often it is the middle value.*

We are now going to discuss the **relation** between **quantiles** and **the cumulative relative frequency**. The value p denotes cumulative relative frequency of quantile x_p

i.e. relative frequency of those variable values that are smaller than quantile x_p. Quantile and cumulative relative frequency are inverse concepts.

Graphical or tabular representation of the ordered variable and appropriate cumulative frequencies is known as **distribution function of the cumulative frequency** or **empirical distribution function**.

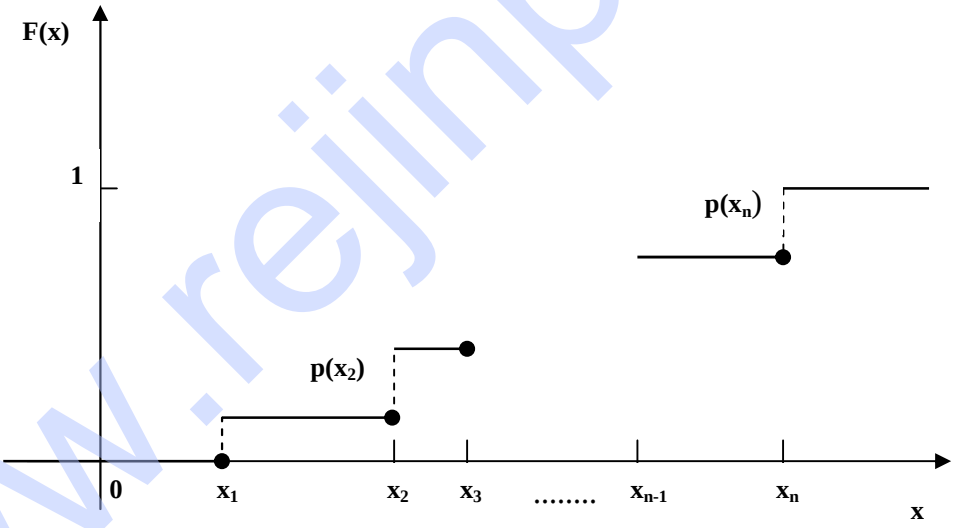
- **Empirical Distribution Function F(x) for the Quantitative Variable**

  We put the sample population in ascending order ($x_1<x_2<\ \ldots\ <x_n$) and we denote $p(x_i)$ as relative frequency of the value x_i. For empirical distribution function F(x) it must then be true that:

$$F(x)=\begin{cases} 0 & for\ x \le x_1 \\ \sum_{i=1}^{j} p(x_i) & for\ x_j < x \le x_{j+1}, 1 \le j \le n-1 \\ 1 & for\ x_n < x \end{cases}$$

  The empirical distribution function is a monotonous, increasing function and it runs from the left.

$$p(x_i)=\lim_{x \to x_{i+}} F(x)-F(x_i)$$



- **MAD**

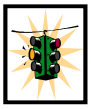
  MAD is a short for **Median Absolute Deviation** from the median.

  MAD is determined as follows:

  1. Order the sample population by size

  2. Determine the median of the sample population

3. For each value determine absolute value of its deviation from the median
4. Put absolute deviations from the median in ascending order by size
5. Determine the median of the absolute deviations from the median i.e. MAD



Example and Solution

There is the following data set: 22, 82, 27, 43, 19, 47, 41, 34, 34, 42, 35 (the data from the previous example).
Determine:

a) All quartiles
b) Inter-Quartile Range
c) MAD
d) Draw the Empirical Distribution Function

Solution:

a) You need to determine Lower Quartile $x_{0,25}$; Median $x_{0,5}$ and Upper Quartile $x_{0,75}$. First, you order the data by size and assign a sequence number to each value.

Original data	Ordered data	Sequence
22	19	1
82	22	2
27	**27**	**3**
43	34	4
19	34	5
47	**35**	**6**
41	41	7
34	42	8
34	**43**	**9**
42	47	10
35	82	11

Now you can divide the data set into quartiles and mark their variable values accordingly:

Lower Quartile $x_{0,25}$: $p = 0.25;\ n = 11 \Rightarrow z_p = 11 \text{ x } 0.25 + 0.5 = 3.25 \cong 3 \Rightarrow x_{0.25} = 27$
i.e. 25% of musicians are under 27 (75% of them are 27 years old or older).

Median $x_{0,5}$: $p = 0.5;\ n = 11 \Rightarrow z_p = 11 \text{ x } 0.5 + 0.5 = 6 \Rightarrow x_{0.5} = 35$
i.e. a half of the musician are under 35 (50% of them are 35 years old or older).

Upper Quartile $x_{0,75}$: $p = 0.75;\ n = 11 \Rightarrow z_p = 11 \text{ x } 0.75 + 0.5 = 8.75 \cong 9 \Rightarrow x_{0.75} = 43$

i.e. 75% musicians are under 43 (25% of them are 43 years old or older).

b) **Inter-Quartile Range IQR:**

IQR = $x_{0.75} - x_{0.25} = 43 - 27 = 16$

c) **MAD**

If you want to determine this characteristic you must follow its definition (the median of absolute deviations from the median).

$x_{0.5} = 35$

Original data x_i	**Ordered data y_i**	Absolute values of deviations of the ordered data from their median $\lvert y_i - x_{0.5} \rvert$	Ordered absolute values **M_i**
22	19	$16 = \lvert 19 - 35 \rvert$	0
82	22	$13 = \lvert 22 - 35 \rvert$	1
27	27	$8 = \lvert 27 - 35 \rvert$	1
43	34	$1 = \lvert 34 - 35 \rvert$	6
19	34	$1 = \lvert 34 - 35 \rvert$	7
47	35	$0 = \lvert 35 - 35 \rvert$	**8**
41	41	$6 = \lvert 41 - 35 \rvert$	8
34	42	$7 = \lvert 42 - 35 \rvert$	12
34	43	$8 = \lvert 43 - 35 \rvert$	13
42	47	$12 = \lvert 47 - 35 \rvert$	16
35	82	$47 = \lvert 82 - 35 \rvert$	47

MAD = $M_{0.5}$
$p = 0.5; n = 11 \Rightarrow z_p = 11 \text{ x } 0.5 + 0.5 = 6 \Rightarrow x_{0.5} = 8$

(MAD is a median absolute deviation from the median i.e. 6th value of ordered absolute deviations from the median)

MAD = 8.

d) The last task was to draw the Empirical Distribution Function. Here is its definition:

$$F(x) = \begin{cases} 0 & for\, x \le x_1 \\ \sum_{i=1}^{j} p(x_i) & for\, x_j < x \le x_{j+1}, 1 \le j \le n-1 \\ 1 & for\, x_n < x \end{cases}$$

- Arrange the variable values as well as their frequencies and relative frequencies in ascending order and write them down in the table. Then derive the empirical distribution function from them:

Original data x_i	Ordered data a_i	Absolute frequencies of the ordered values $\mathbf{n_i}$	Relative frequencies of the ordered values $\mathbf{p_i}$	Empirical distribution function $F(a_i)$
22	19	1	1/11	0
82	22	1	1/11	1/11
27	27	1	1/11	2/11
43	34	2	2/11	3/11
19	35	1	1/11	5/11
47	41	1	1/11	6/11
41	42	1	1/11	7/11
34	43	1	1/11	8/11
34	47	1	1/11	9/11
42	82	1	1/11	10/11
35				

As by its definition - the empirical distribution function F(x) - equals 0 for each x <19; F(x) equals 1/11 for all 22≥x>19; F(x) equals 1/11 + 1/11 for all 27≥x> 22; and so it goes on.

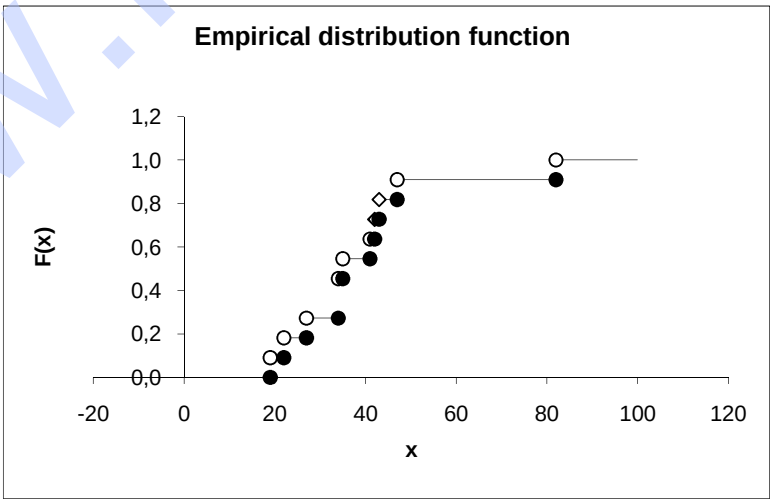
X	$\langle -\infty;19\rangle$	$\langle 19;22\rangle$	$\langle 22;27\rangle$	$\langle 27;34\rangle$	$\langle 34;35\rangle$
F(x)	0	1/11	2/11	3/11	5/11

X	$\langle 35;41\rangle$	$\langle 41;42\rangle$	$\langle 42;43\rangle$	$\langle 43;47\rangle$	$\langle 47;82\rangle$	$\langle 82;\infty)$
F(x)	6/11	7/11	8/11	9/11	10/11	11/11

Means, mode and median (i.e. measures of location) represent imaginary centre of the variable. However, we are also interested in the distribution of the individual values of the variable around the centre (i.e. measures of variability).

The following three statistical characteristics allow description of the sample population variability. Shorth and Inter-Quartile Range are classified as measures of variability.

- **Sample Variance s^2**



- is the most common measure of variability

The sample variance is given by:

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

- Sample Variance is the sum of all squared deviations from their mean divided by one less than the sample size

General **properties of the sample** variance are for example:

- The sample variance of a constant number is zero

  In other words: if all variable values are the same, the sampling has zero diffuseness

- $$\forall a \in \Re : \left[\left(s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \right) \wedge \left(y_i = a + x_i \right) \right] \Rightarrow \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} = s^2$$

  In other words: if you add the same constant number to all variable values, the sample variance doesn't change

- $$\forall b \in \Re : \left[\left(s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} \right) \wedge \left(y_i = bx_i \right) \right] \Rightarrow \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} = b^2 s^2$$

  In other words: if you multiply all variable values by an arbitrary constant number (b) the sample variance increases by square of this constant number (b^2)

Disadvantage of using the sample variance as a measure of variablility is that it employs squared values of the variable. For example: if the variable represents cash denominated in EUR, then the sample variation of this variable will be in EUR^2. That is why we use another measure of variability called standard deviation.

- **Standard Deviation s**

  - is calculated by the square root of the variance

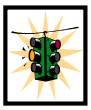
$$s = \sqrt{s^2} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

Another disadvantage of using the sample variation and the standard deviation is that variability of the variable can't be compared in different units. Which variable has bigger variability - height or weight of an adult? To answer that, Coefficient of Variation has to be used.

- **Coefficient of Variation V_x**

  - it represents relative measure of variability of the variable x and it is often expressed as a percentage
  - it is the ratio of the sample standard deviation to the sample mean:

$$V_x = \frac{s}{\bar{x}}$$



Example and Solution

A table glass manufacturer has developed less expensive technology for improving the fire-resistant glass. 10 glass table sheets were selected for testing. Half of them were treated by the new technology while the other half was used for comparison.
Both lots were tested by fire until they cracked. These are the results:

Critical temperature (glass cracked) [°C]	
Old technology x_i	**New technology y_i**
475	485
436	390
495	520
483	460
426	488

Compare both technologies by means of basic characteristics of the exploratory analysis (mean, variation, etc.).

Solution:

- First you compare both technologies by the mean:

Mean for the old technology:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{475+436+\ldots+426}{5} = 463.0 \ [°C]$$

Mean for the new technology:

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{485+390+\ldots+488}{5} = 468.6 \ [°C]$$

Based on the calculated means the new technology could be recommended because the temperature it can withstand is 6°C higher.

- now you determine the measures of variability

The old technology:

Sample Variance:

$$s_x^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} = \frac{(475-463.0)^2+(436-463.0)^2+\ldots+(426-463.0)^2}{5-1} = 916.3 \quad [°C^2]$$

Standard Deviation:

$$s_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} = \sqrt{s_x^2} = \sqrt{916.3} = 30.3 \quad [°C]$$

New technology:

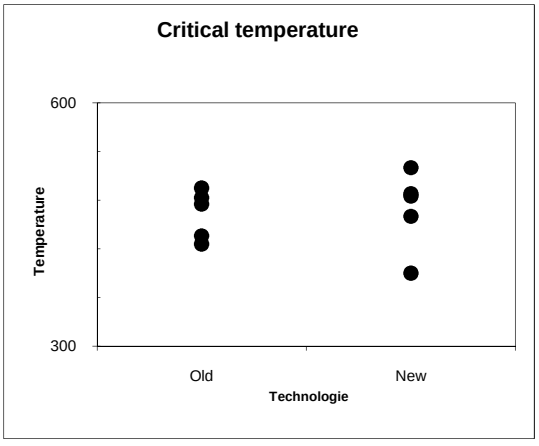
Sample variance:

$$s_y^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1} = \frac{(485-468.6)^2+(390-468.6)^2+\ldots+(488-468.6)^2}{5-1} = 2384.4 \quad [°C^2]$$

Standard deviation:

$$s_y = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}} = \sqrt{s_y^2} = \sqrt{2384.4} = 48.8 \quad [°C]$$

Sample variance (standard deviation) for the new technology is significantly larger. What is the possible reason? Look at the graphical representation of the collected data. Critical temperatures are much more spread out which means this technology is not fully under control and its use can't guarantee higher production quality. In this case the critical temperature can either be much higher or much lower. For that reason it is recommended that it should be subjected to additional research. These conclusions are based only on exploratory analysis. Statistics provides us with more exact methods for analysing similar problems (hypothesis testing).



Now we are going to return to exploratory analysis as such. We mentioned outliers. So far we know that outliers are variable values that are substentially different from the rest of the values and this impacts on mean. How can these values be identified?

- **Identification of the Outliers**

  In the statistical practice we are going to come across a few methods that are capable of identifying outliers. We'll mention three and go through them one by one.

  1. The outlier can be every value x_i that by far exceeds 1,5 IQR lower (or upper) quantile.

     $$\left[\left(x_i < x_{0.25} - 1.5IQR\right) \vee \left(x_i < x_{0.75} + 1.5IQR\right)\right] \Rightarrow x_i \, is an outlier$$

  2. The outlier can be every value x_i of which the absolute value of the **z-score** is greater then 3.

     $$z - score_i = \frac{x_i - \bar{x}}{s}$$

     $$\left(\left|z - score_i\right| > 3\right) \Rightarrow x_i \, is an outlier$$

  3. The outlier can be every value x_i of which the absolute value of the **median-score** is greater then 3.

     $$median - score_i = \frac{x_i - x_{0.5}}{1.483.MAD}$$

     $$\left(\left|median - score_i\right| > 3\right) \Rightarrow x_i \text{ is an outlier}$$

  Any of the three rules can be used to identify outliers in real-life problems. The Z-axis is "less strict" than the median-axis to outliers. It's because establishing the z-axis is based on mean and standard deviation and they are strongly influenced by outlying values. Meanwhile, establishing the median-axis is based on median and MAD and they are immune to outliers.

  When you identify a value as an outlier you need to decide its type unless it is caused by:

  - mistakes, typing errors, human error, technology whims, etc.
  - faults, results of wrong measurements, etc.

  If you know the outlier cause and make sure that it will not occur again, it can be cleared from the process. In other cases you must consider carefully if by getting rid of an outlier you won't lose important information about events with low frequencies.

The others characteristics describing qualitative variable are **skewness** and **kurtosis**. Their formulas are rather complex therefore specialized software is used for the calculation.

- **Skewness**

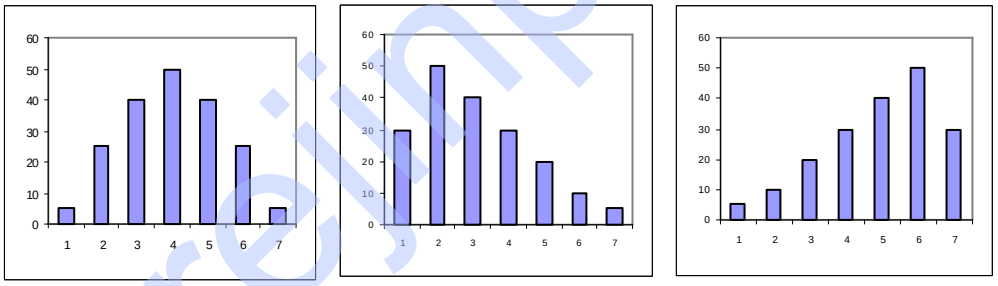
  - Skewness is defined as asymmetry in the distribution of the variable values. Values on one side of the distribution tend to be further away from the "middle" than values on the other side.

  - The following formula is used:

$$\alpha = \frac{n}{(n-1)(n-2)} \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^3}{s^3}$$

Skewness interpretation:

$\alpha = 0$	...	variable values are distributed symmetrically around the mean
$\alpha > 0$	...	values smaller than the mean are predominant
$\alpha < 0$	...	values larger than the mean are predominant



α=0 α>0 α<0

- **Kurtosis**

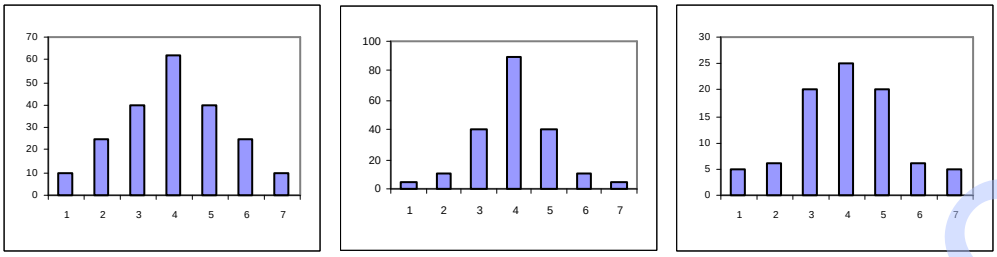
  - Kurtosis represents concentration of variable values around their mean.

The following formula is used to get its value:

$$\beta = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \cdot \frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{s^4} - 3\frac{(n-1)^2}{(n-2)(n-3)}$$

Kurtosis interpretation:

$\beta = 0$ ... Kurtosis corresponds to normal distribution
$\beta > 0$ ... "peaked" distribution of the variable
$\beta < 0$ ... "flat" distribution of the variable



β=0 β>0 β<0

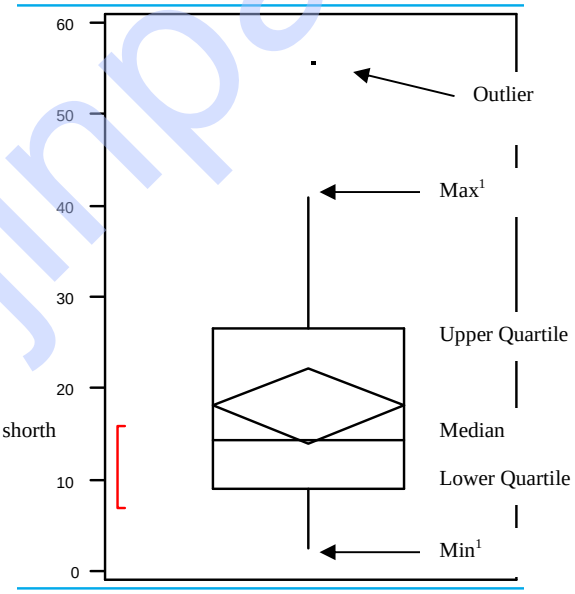
We have now defined all numerical characteristics of the quantitative variable Next we are going to take a look at how they can be interpreted graphically.

1.2.2 Graphical Methods of Presenting Quantitative Variables

Box plot

A box plot is a way of summarizing a data set on an interval scale. It is often used in exploratory data analysis. It is a graph that shows the shape of the distribution, its centre point, and variability. The resulting picture consists of the most extreme values in the data set (maximum and minimum), the lower and upper quartiles, and the median.

A box plot is especially helpful for indicating whether a distribution is skewed and whether there are any unusual observations (outliers) in the data set.



Notice: *A box plot construction begins by marking outliers and then other characteristics (min[1], max[1], quartiles and shorth).*

Stem and Leaf Plot

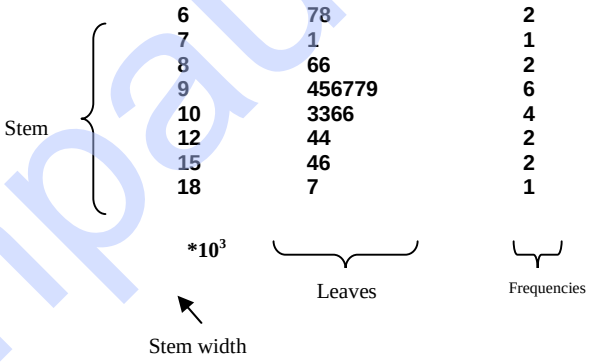
As we saw, simplicity is an advantage of the box plot. However, information about specific values of the variable is missing. The missing numeric values would have to be specifically marked down onto the graph. The Stem and leaf plot will make up for that limitation.

We have a variable representing average month salary of bank employees in the Czech Republic.

Average month pay [CZK]									
10,654	9,765	8,675	12,435	9,675	10,343	18,786	15,420	8,675	7,132
6,732	6,878	15,657	9,754	9,543	9,435	10,647	12,453	9,987	10,342

Average month pay [CZK] – data in ascending order									
6,732	6,878	7,132	8,675	8,675	9,435	9,543	9,675	9,754	9,765
9,987	10,342	10,343	10,647	10,654	12,435	12,453	15,420	15,657	18,786

How do we bring the data onto the graph? The place values that are regarded as "unimportant" are ignored and data on the higher places are put in order.
We are especially interested in the values from the third (hundreds') place. The values on the fourth (thousands') place are written down in ascending order thus creating a **stem**. Under the graph we append a **stem width** that will also act as a coefficient used to multiply values in the graph.

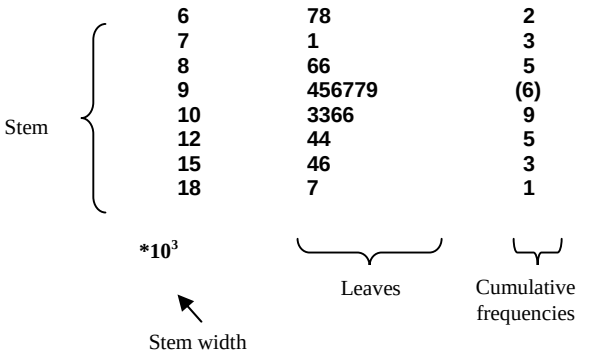


The second column of the graph - known as **leaves** – are the numbers representing "important" place values. They are written down in corresponding rows.

The third column is **absolute frequency** for particular rows.

For example: the first row in the graph represents two values - (6.7 and 6.8)*10^3 CZK i.e. 6,700 CZK and 6,800 CZK, the sixth row represents two values too - (12.4 and 12.4)*10^3 CZK, i.e. two employees have the average month pay of 12,400 CZK, etc.

There are various modifications of this graph. For example the third column could store cumulative frequencies and in the median row the absolute frequency is shown in parentheses. From this row the absolute frequencies either cumulate from the smallest values or diminish from the highest values – as seen on the picture.



Finally, you need to keep in mind that there are different ways of constructing a stem and leaf plot and you need to be aware of one particular problem. Nowhere is it said which place values of the variable are important and which ones are not. This is left to the observer. However there is a tip to follow. A long stem with short leaves and a short stem with long leaves indicate incorrect choice of scale. Look at the picture.

0	**66788999999**	**11**
1	**000022558**	**9**

***10^4**

Quiz

1. What is exploratory statistics concerned with?

2. Characterize the basic types of variables.

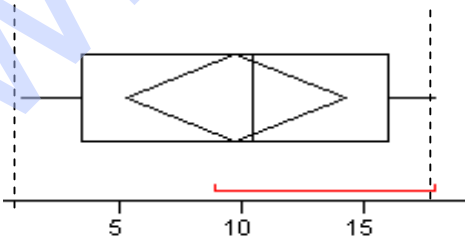
3. Which statistical characteristics can be contained in frequency table (for what type of variable)?

4. What are the outliers and how do you define them?

5. Which characteristics are sensitive to outliers?
   a) Median
   b) Arithmetical Mean
   c) Upper Quartile

6. How do you depict the qualitative (quantitative) variables?

7. The following box plot represents students' earnings during holiday.



Mark statements that do not correspond to the displayed reality:
a) A student earned 19 thousand CZK maximum.
b) Inter-quartile range is approximately 10 thousands CZK.
c) Half of the students earned less than 11 thousands CZK.
d) Shorth is rougly an interval of (5;15) thousand CZK

Practical Exercises

Exercise 1: The following data represents car manufacturers' countries of origin. Analyze the data (frequency, relative frequency, cumulative frequency and cumulative relative frequency, mode) and interpret it in the graphical form (histogram, pie chart).

USA	USA	Germany	Czech Rep.
Germany	Germany	Germany	Czech Rep.
Czech Rep.	Czech Rep.	USA	Germany

Exercise 2: The following data represents customers' waiting time (min) when dealing with the customer service. Draw box plot and stem and leaf plot.

120	80	100	90
150	5	140	130
100	70	110	100

Exercise 3: A traffic survey was carried out to establish a vehicle count at an intersection. A student data collector recorded the numbers of cars waiting in queue each time the green light jumped on. These are his/her outcomes:

3 1 5 3 2 3 5 7 1 2 8 8 1 6 1 8 5 5 8 5 4 7 2 5 6 3 4 2 8 4 4 5 5 4 3 3 4 9 6 2 1
5 2 3 5 3 5 7 2 5 8 2 4 2 4 3 5 6 4 6 9 3 2 1 2 6 3 5 3 5 3 7 6 3 7 5 6

Draw box plot, empirical distribution function and calculate the mean, standard deviation, shorth, mode and inter-quartile range.

2. PROBABILITY THEORY

Study Time: 70 minutes

Learning Objectives - you will be able to

- Characterize probability theory
- Explain general notions of probability theory
- Explain and use general relations between events
- Explain a notion of probability
- Define probability by basic axioms
- Define properties of probability function
- Use a conditional probability
- Explain theorem of total probability and Bayes theorem

2.1. Introduction to Basic Concepts

Study time: 20 minutes

Learning Objectives

- Characterize probability theory
- Explain general notions of probability theory

Explanation

Probability theory is the deductive part of statistics. Its purpose is to give a precise mathematical definition or structure to what has so far been an intuitive concept of randomness. Defining randomness will allow us to make exact probability statements. For example when discussing association, we could only make rough statements in terms of tendencies.
Mathematically, probability is a set function which means it is premised on sets. Therefore, we are begining this discussion by explaining the fundamental nature of sets and the basic operations performed on sets and elements as the main ideas behind the probability function.

- **General Notions of the Probability Theory**

Definition of a Set - set A is a collection of elements. Elements are basic intuitive mathematically undefined entities. To define a set, it is necessary to be able to determine whether any element is included or not included in the set. The notion of inclusion is also an intuitive undefined concept.

Definition of Elementary Events - In probability theory, the probability assumptions are based on elements that are part of a set. The sets' elements are called elementary events. In practice, these elementary events may be measurable by units, cases, samples, points, etc.

Example:

{heads or tails} – when tossing a coin

{1,2,3,4,5,6} – when throwing a dice

The set of all results will be denoted by Ω and called **sample space** (of the elementary events)**.** The elementary event {ω} is a subset of the Ω set which contains one element ω from Ω set, $\omega \in \Omega$.
Then the event A will be an arbitrary subset of Ω, $A \subset \Omega$.

From statistical data we can easily establish that share of boys born in particular years with respect to all born children is moving around 51.5%. Despite the fact that in individual cases we can't predict sex of a child we can make a relatively accurate guess about how many boys there are among 10 000 children.

As the example suggests, relative frequencies of some events are stabilized with increased repetition of certain values. We shall call this phenomenon *Stability of the Relative Frequencies* and it is is an empirical fundamental principle of the probability theory. **Relative frequency** is number n(A)/n where n is a total number of random observations and n(A) is a number of observations with an A result.

Summary

Probability theory is a mathematical branch using axiomatic logical structure.
Mathematical statistics is a science concerned with data mining, data analysis, and formulating results
Random observation is every finite process where the result is not set by conditions under which it is run.
Sample space Ω is a set of all possible outcomes of an a random observation.
Relative frequencies of some events with increased repetition indicate some level of **stability**.

2.2. Operations with the elementary events

Study time: 20 minutes

Learning Objectives

- Types of elementary events
- General relations between events

Explanation

What are the types of elementary events?

If an elementary event $\omega \in \Omega$ ($\omega \in$ A) occurs then you can say that an even A has occured.

This result is denoted by $\omega \in$ A and is **favorable to the event A.**

Certain event
- is the event which occurs with each random experiment. It is equivalent to the Ω set.
The certain event for example occurs when you *throw a dice and you end up with one of the six numbers: 1,2,3,4,5,6*

Impossible event
- is the event which never occurs in an experiment. It will be denoted by $\varnothing$.
The impossible event for example would be *throwing number 8* (with the same dice).

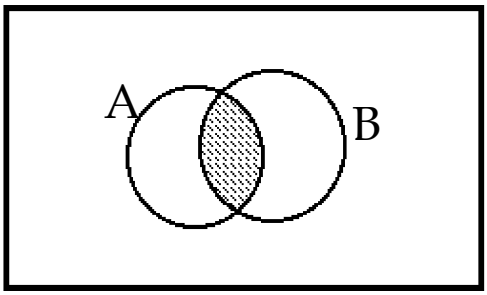
What are relations between events?

Operations on Sets - The operations of union, intersection, complementation (negation), subtraction, the concept of subset, and the null set and universal set or sample space are the algebra of sets.

Intersection A ∩ B
- Is the set of all elements that are both in A and in B.

Graphical example:

$$A \cap B = \{\omega \mid \omega \in A \wedge \omega \in \mathrm{B}\}$$

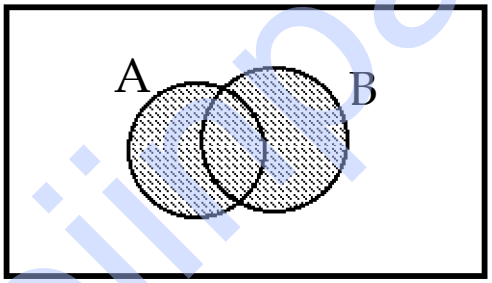


Example for throwing a dice: Numbers 2, 3 or 4 are thrown as event A and an even number is thrown as event B. It is obvious that $A \cap B = \{2,4\}$.

Union $A \cup B$
It is the set of all elements that are either in A or in B.

Graphical example:

$$A \cup B = \{\omega \mid \omega \in A \vee \omega \in B\}$$



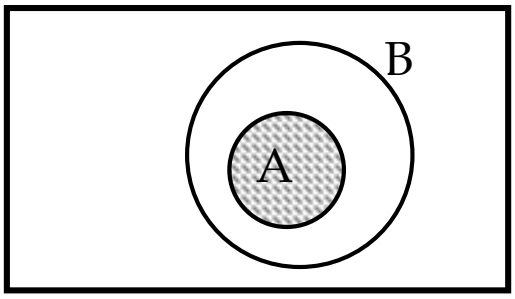
Example – throwing a dice: Event A = {1,3,4} and event B is an even number. It's obvious that $A \cup B = \{1,2,3,4,6\}$.

Disjoint events $A \cap B = \varnothing$
Two events A and B can't occur together. They have no common result.
Example for throwing a dice: You throw an even number as event A and an odd number as event B. These events never have the same result. If event A occurs then event B can't happen.

Subsets (Subevent) $A \subset B$
A is a subset of B if each element of A is also an element of B. It means that if event A occurs than event B occurs as well.

Graphical example:

$$A \subset B \Leftrightarrow \{\omega \in A \Rightarrow \omega \in B\}$$



Example for throwing a dice: You throw number 2 as event A and you throw an even number as event B. The event A is subevent of event B.

Events A and B are equivalent **A** = B if $A \subset B$ and at the same time $B \subset A$.

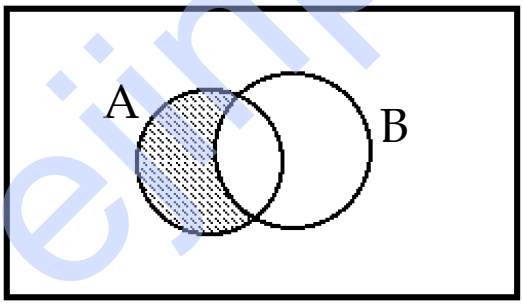
Example for throwing a dice: You throw an even number as event A and you throw a number that is dividable by number 2 as event B. These events are equivalent.

Subtraction A-B
The set of all elements that are in A but not in B

$$A - B = A \cap \overline{B}$$

$$A - B = \{\omega \mid \omega \in A \wedge \omega \notin B\}$$

Graphical example:

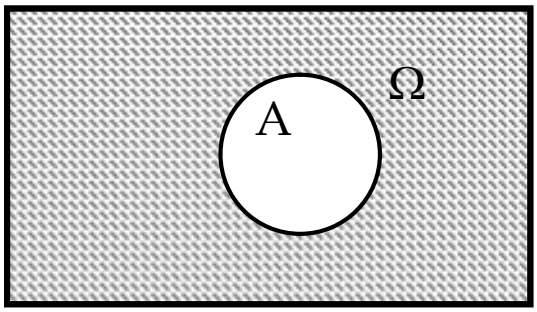


Example for throwing a dice: You throw a number greater than 2 as event A and you throw an even number as event B. Subtraction of the two events A – B = {3,5}.

Complement of event A (opposite event)
The set of all elements that are not in A.

$$\overline{A} = \{\omega \mid \omega \notin A\}$$

Graphical example:



Example for throwing a dice: You throw an even number as event A and you throw an odd number as event $\overline{A}$

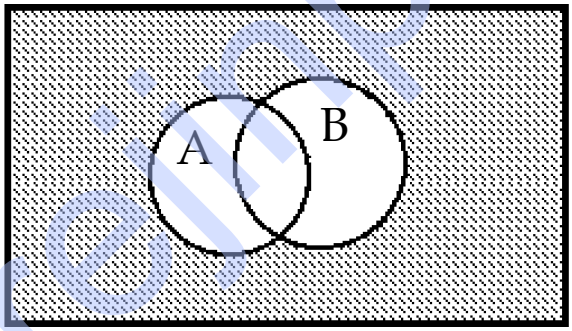
DeMorgan's Laws

- DeMorgan's Laws are logical conclusions of the fundamental concepts and basic operations of the set theory.

Law no. 1

The set of all elements that are neither in A nor in B.

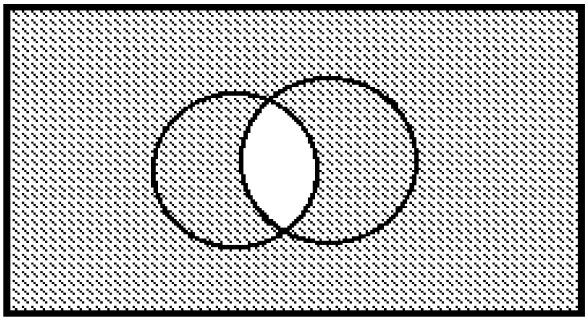
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$



Law no. 2

The set of all elements that are either not in A or not in B (that are not in the intersection of A and B).

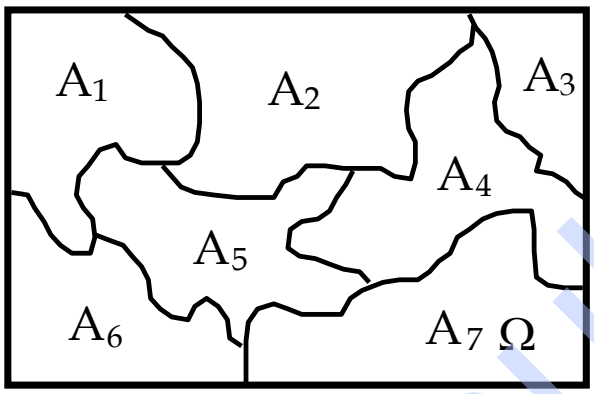
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$



Mutually disjoint sets and partitioning the sample space

The collection of sets $\{A_1, A_2, A_3, \ldots\}$ partition the sample space Ω: $A_i \bigcap A_j = \varnothing$ for $i \neq j$

$$\Omega = \bigcup_{i=1}^{n} A_i$$



2.3. Probability Theory

Study time: 30 minutes

Learning Objectives

- Notion of probability
- Basic theorems and axioms of probability
- Types of probability
- Conditional probability
- Theorem of total probability and Bayes' theorem

Explanation

Concept of Probability, Classical Definition

Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes $A_1, A_2, , . . . A_N$ **. We are interested in the event A which occurs if anyone of M elementary outcomes occurs,** $A_1, A_2, , . . . A_M$**, i.e.**

$$A = \bigcup_{i=1}^{M} A_i$$

Since the events are mutually exclusive and equally probable, we introduce the probability of the event A, P(A) as follows:

$$P(A) = \frac{k}{n}$$

where
***k* is number of outcomes of interest**
***n* is total number of possible outcomes**

This result is very important because it allows computing the probability with the methods of combinatorial calculus; its applicability is however limited to the case in which the event of interest can be decomposed in a finite number of mutually exclusive

and equally probable outcomes. Furthermore, the classical definition of probability entails the possibility of performing repeated trials; it requires that the number of outcomes be finite and that they be equally probable, i.e. it defines probability resorting to a concept of frequency.

We are also going to introduce *Axiomatic Probability Definition*.

Axiomatic Probability Definition

Probability space is a triad (Ω, *S*, P) where

(i) Ω is sample space (elements of Ω are elementary events)

(ii) *S* is a set of subsets of Ω where the following is true:

a) $\Omega \in S$;

b) if $A \in S$ then $\overline{A} = \Omega - A \in S$;

c) if $A_1, A_2, A_3, \ldots \in S$ then $\bigcup_{i=1}^{\infty} A_i \in S$

Elements of S are called **events**.

(iii) P is a function derived from S where the following is true:

a) $P(\Omega) = 1$
Probabilities are scaled to lie in the interval [0,1];

b) $P(\overline{A}) = 1 - P(A)$ for every $A \in S$;

c) For a collection of mutually disjoint sets, the probability of their union is equal to the sum of their probabilities.

$$\text{If } A \cap B = \varnothing, \text{ then}$$
$$P\{A \cup B\} = P\{A\} + P\{B\}$$

In general,

$$A_i \cap A_j = \varnothing, \ \forall 1 \le i, j \le \infty; i \ne j,$$
$$P\{\bigcup_{i=1}^{\infty} A_i\} = \sum_{i=1}^{\infty} P\{A_i\}$$

Function P is called **probability measure** or simply **probability**.

Example for throwing a dice:
Ω = {1,2,3,4,5,6,},
S is a set of subsets of Ω (sometimes we denote *S* by exp Ω) and probability is defined by $P(A) = \frac{\text{card}A}{6}$ where cardA is number or A elements in the set.

Basic Theorems of Probability

The following theorems are logical conclusions of the three basic probability axioms postulated so far.

1. For disjoint events A and B the following is true:
   $A \cap B = \varnothing$ then

$P\{A \cup B\} = P\{A\} + P\{B\}$

2. If for two events A,B:
   $B \subset A$ then $P\{B\} \leq P\{A\}$

Note that A is partitioned by B and its complement, and hence P{A} is sum of these two parts

3. For every event A the following is true: $P\{\overline{A}\} = 1 - P\{A\}$
   The union of the two sets is the sample space, the intersection is the null sets.

4. It holds that: $P\{\varnothing\} = 0$

5. It holds that: $P\{B - A\} = P\{B\} - P\{B \cap A\}$
   Note that B-A and B intersection A are two disjoint sets whose union is B

6. Particularly if $A \subset B$ then $P\{B - A\} = P\{B\} - P\{A\}$

7. For arbitrary events A,B it holds that:
   $P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$

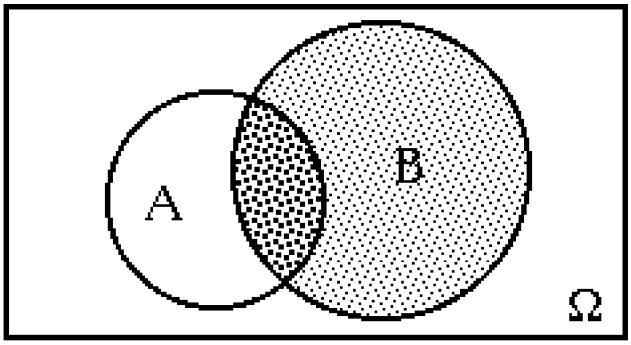
8. In accordance with the de Morgan's laws:
   $P\{A \cup B\} = 1 - P\{\overline{A \cup B}\}$
   $= 1 - P\{\overline{A} \cap \overline{B}\}$

Definition of Conditional Probability

The definition of conditional probability determines how probabilities adjust to changing conditions. When we say that the condition B applies, we mean that the set B is known to have occurred and therefore the rest of the sample space in the complement of B has zero probability. Under these new circumstances, the revised probability of any other event, A, can be determined from the following definition of conditional probability:

$$P\{A|B\} = \frac{P\{A \cap B\}}{P\{B\}}$$

By this formula, the probability of that part of the event A which is in B or intersects with B is revised upwards to reflect the condition that B has occurred and becomes the new probability of A. It is assumed that the probability of B is not zero.



$P\{A|B\}$ - *probability of the event A conditioned by the event B*

Conditional Probability Definition of Independence

If the condition that B has occurred does not affect the probability of A, then we say that A is independent of B.

$$P\{A|B\} = P\{A\}$$

From the definition of conditional probability, this implies

$$P\{A\} = \frac{P\{A \cap B\}}{P\{B\}}$$

and hence,

$$P\{A \cap B\} = P\{A\} \cdot P\{B\}$$

It is clear from this demonstration that if A is independent of B, then B is also independent of A.

Example for throwing a dice:
If for events A - you throw 1 in the first throw, and for event B - you throw 1 in the second throw, and for event C = A ∩ B - you throw 1 in both throws, then the following is true:

$$P\{C\} = P\{A \cap B\} = P\{A\} \text{ x } P\{B\} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

Theorem of Total Probability

If a collection of sets $\{B_1, B_2, B_3, \ldots, B_n\}$ partitions the sample space Ω, that is,

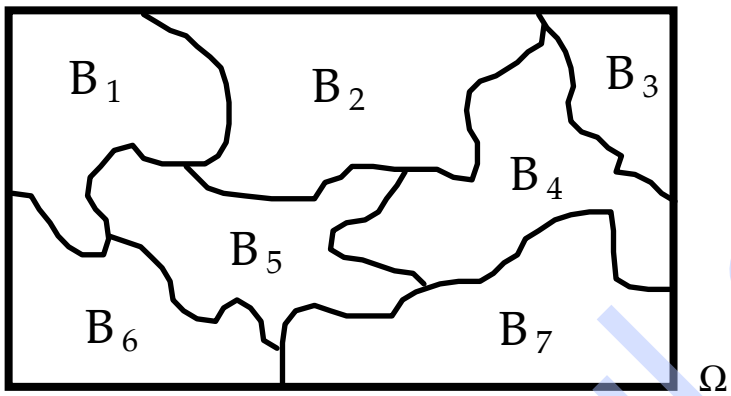
$$B_i \cap B_j = \varnothing; \forall \; i \neq j$$

$$\bigcup_{i=1}^{n} B_i = \Omega$$

then for any set A ($P\{A\}\neq 0$) in the sample space Ω,

$$P\{A\} = \sum_{i=1}^{n} P\{A|B_i\} \cdot P\{B_i\}$$

n=7



Proof: Since the collection of sets $\{B_1, B_2, B_3, \ldots, B_n\}$ partitions the sample space Ω,

$$P\{A\} = \sum_{i=1}^{n} P\{A \cap B_i\}$$

From the definition of conditional probability

$$P\{A \cap B_i\} = P\{A|B_i\}\, P\{B_i\}$$

Bayes' Theorem

If the collection of sets $\{B_1, B_2, B_3, \ldots, B_n\}$ partitions the sample space Ω, then

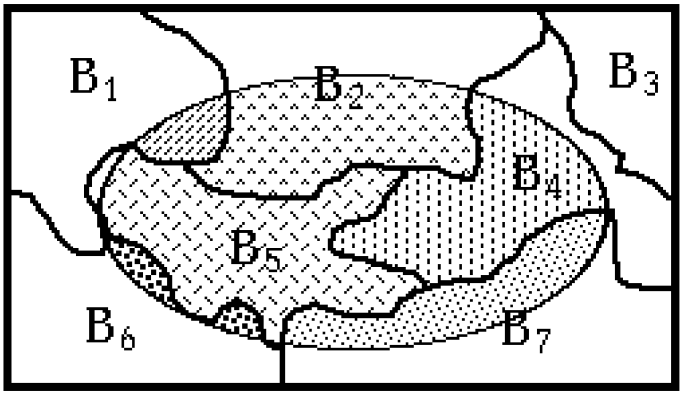
$$P\{B_k|A\} = \frac{P\{A|B_k\}\, P\{B_k\}}{\sum_{i=1}^{n} P\{A|B_i\}\, P\{B_i\}}$$

Proof: From the definition of conditional probability,

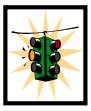
$$P\{B_k|A\} = \frac{P\{B_k \cap A\}}{P\{A\}} = \frac{P\{A\}P\{B_k\}}{P\{A\}}$$

The proof comes from substituting $P\{A\}$ as defined by the Theorem of Total Probability.

Graphical representation of Bayes' theorem (the marked area represents event A):



PROBABILITY THEORY – EXAMPLES AND SOLUTIONS



Example and Solution

Probability of the extinguishing system failure is 20%. Probability of the alarm system failure is 10% and probability that both systems fail is 4%. What is the probability that:
a) at least one of the systems will stay working?
b) both systems will stay working?

Solution:

H ... extinguishing system works
S ... alarming system works

It is known that: $P(\overline{H}) = 0{,}20$
$P(\overline{S}) = 0{,}10$
$P(\overline{H} \cap \overline{S}) = 0{,}04$

You must find:

ada) $P(H \cup S)$

There are two possible solutions:

By definition: Events H and S are not disjoint events and hence:

$P(H \cup S) = P(H) + P(S) - P(H \cap S)$,

but it would be a problem to determine $P(H \cap S)$

By the opposite events from the de Morgan's laws you can say**:**

$P(H \cup S) = 1 - P(\overline{H \cup S}) = 1 - P(\overline{H} \cap \overline{S})$,

$\underline{\underline{P(H \cup S) = 1 - 0{,}04 = 0{,}96}}$

The probability (that at least one system will be working) is 96%.

adb) $P(H \cap S)$

We can't solve it by the definition**:**

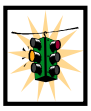
$(P(H \cap S) = P(H|S) \cdot P(S) = P(S|H) \cdot P(H))$,

because there is little information about how dependent the failures are on individual systems. Hence we try to use the opposite event:

$$P(H \cap S) = 1 - P(\overline{H \cap S}) = 1 - P(\overline{H} \cup \overline{S}) = 1 - [P(\overline{H}) + P(\overline{S}) - P(\overline{H} \cap \overline{S})],$$

$$\underline{\underline{P(H \cap S) = 1 - [P(\overline{H}) + P(\overline{S}) - P(\overline{H} \cap \overline{S})] = 1 - [0{,}20 + 0{,}10 - 0{,}04] = 0{,}74}}$$

The probability (that both systems will be working) is 74%.



Example and Solution

120 students passed mathematics and physics exams. 30 of them failed both exams. 8 of them failed only the math exam and 5 of them failed ~~to pass~~ only the physics exam. What is the probability that a random student:

a) passed the math exam if you know that he had failed the physics exam
b) passed the physics exam if you know that he had failed the math exam
c) passed the math exam if you know that he had passed the physics exam

Solution:

M ... he passed the math exam
F... he passed the physics exam

You know that:

$$P(\overline{M} \cap \overline{F}) = \frac{30}{120}$$
$$P(\overline{M} \cap F) = \frac{8}{120}$$
$$P(M \cap \overline{F}) = \frac{5}{120}$$

You must find:

ada) $P(M|\overline{F})$

by the definition of conditional probability:

$$P(M|\overline{F}) = \frac{P(M \cap \overline{F})}{P(\overline{F})} = \frac{P(M \cap \overline{F})}{P(M \cap \overline{F}) + P(\overline{M} \cap \overline{F})},$$

$$\underline{\underline{P(M|\overline{F}) = \frac{P(M \cap \overline{F})}{P(M \cap \overline{F}) + P(\overline{M} \cap \overline{F})} = \frac{\frac{5}{120}}{\frac{5}{120} + \frac{30}{120}} = \frac{5}{35} = \frac{1}{7} \cong 0{,}14}}$$

The probability (that he passed the math exam if you know that he had failed the physics exam) is 14%.

adb) $P(F|\overline{M})$

the same way as ada):

$$P(F|\overline{M}) = \frac{P(F \cap \overline{M})}{P(\overline{M})} = \frac{P(F \cap \overline{M})}{P(F \cap \overline{M}) + P(\overline{F} \cap \overline{M})},$$

$$\underline{\underline{P(F|\overline{M}) = \frac{P(F \cap \overline{M})}{P(F \cap \overline{M}) + P(\overline{F} \cap \overline{M})} = \frac{\frac{8}{120}}{\frac{8}{120} + \frac{30}{120}} = \frac{8}{38} = \frac{4}{19} \cong 0.21}}$$

The probability (that he passed the physics exam if you know that he had failed the math exam) is 21%.

adc) $P(M|F)$

from the definition:

$$P(M|F) = \frac{P(M \cap F)}{P(F)},$$

there are two possibilities:

1)

$$\underline{\underline{P(M|F) = \frac{P(M \cap F)}{P(F)} = \frac{1 - P(\overline{M \cap F})}{1 - P(\overline{F})} = \frac{1 - P(\overline{M} \cup \overline{F})}{1 - \left[P(\overline{F} \cap M) + P(\overline{F} \cap \overline{M})\right]} = \frac{1 - \left[P(\overline{F}) + P(\overline{M}) - P(\overline{F} \cap \overline{M})\right]}{1 - \left[P(\overline{F} \cap M) + P(\overline{F} \cap \overline{M})\right]} =}}$$

$$= \frac{1 - \left[\left[P(\overline{F} \cap M) + P(\overline{F} \cap \overline{M})\right] + \left[P(F \cap \overline{M}) + P(\overline{F} \cap \overline{M})\right] - P(\overline{F} \cap \overline{M})\right]}{1 - \left[P(\overline{F} \cap M) + P(\overline{F} \cap \overline{M})\right]} =$$

$$= \frac{1 - \left[P(\overline{F} \cap M) + P(F \cap \overline{M}) + P(\overline{F} \cap \overline{M})\right]}{1 - \left[P(\overline{F} \cap M) + P(\overline{F} \cap \overline{M})\right]} = \frac{1 - \left[\frac{5}{120} + \frac{8}{120} + \frac{30}{120}\right]}{1 - \left[\frac{5}{120} + \frac{30}{120}\right]} = \frac{\frac{77}{120}}{\frac{85}{120}} = \frac{77}{85} \cong \underline{\underline{0.91}}$$

2)
You record the data into a table:

	They passed the math exam	**They failed the math exam**	**Total**
They passed the physics exam		8	
They failed the physics exam	5	30	35
Total		38	**120**

and you calculate and fill in the remaining data:

How many students passed the physics exam? It is the total number of students (120) minus the number of students who failed the physics exam (35) and that is 85. Analogously for the number of students who passed the math exam: 120 – 38 = 82. And for the number of students who passed both exams: 82 – 5 = 77.

	They passed math exam	**They failed to pass math exam**	**Total**
They passed physics exam	77	8	85
They failed to pass physics exam	5	30	35
Total	82	38	**120**

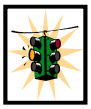
The probabilities are:

$$P(M \cap F) = \frac{77}{120}; \quad P(F) = \frac{85}{120},$$

which implies that:

$$P(M|F) = \frac{P(M \cap F)}{P(F)} = \frac{\frac{77}{120}}{\frac{85}{120}} = \frac{77}{85} \cong 0{,}91$$

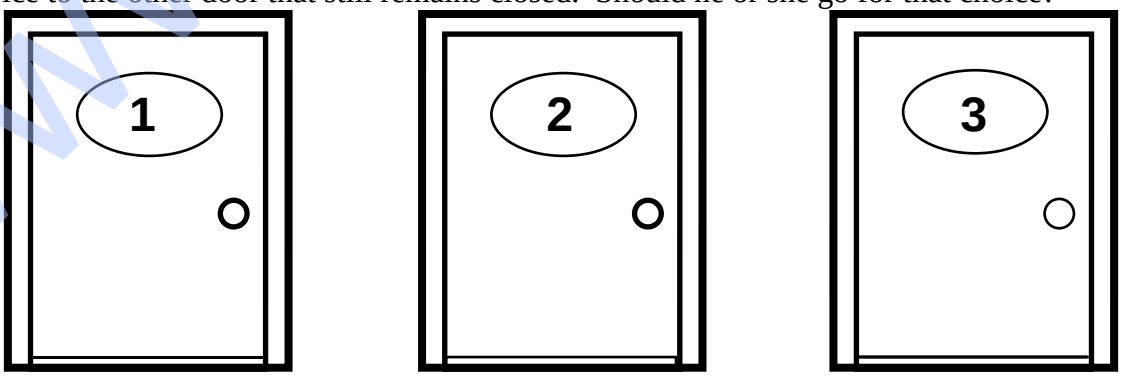
The probability (that he passed the math exam if you know that he had passed the physics exam) is 91%.



Example and Solution (Application of Bayes's Theorem)

In a famous television show, the winner of the preliminary round is given the opportunity to increase the winnings. The contestant is presented with three closed doors and told that behind one of the doors there is a new car while behind the other two doors there are goats. If the contestant correctly selects the door to the car he or she will win the car.

The host asks the contestant to make a selection and then opens one of the other two doors to see if there is a goat. The contestant is then given the option of switching his/her choice to the other door that still remains closed. Should he or she go for that choice?



Solution:

The sample space consists of three possible arrangements {AGG, GAG, GGA}.

Assume that each of the three arrangements has the following probabilities:

$p_1 = P\{AGG\}$ $p_2 = P\{GAG\}$ $p_3 = P\{GGA\}$

where $p_1 + p_2 + p_3 = 1$.

Assume that the contestant's first choice is Door #1 and the host opens Door #3 to reveal a goat. Based on this information we must revise our probability assessments. It is clear that the host cannot open Door #3 if the car is behing it.

$$P\{\text{Door } \#3 \mid GGA\} = 0$$

Also, the host must open Door #3 if Door #2 leads to the car since he cannot open Door #1, the contestant's choice.

$$P\{\text{Door } \#3 \mid GAG\} = 1$$

Finally if the car is behind the contestant's first choice, Door #1, the host can choose to open either Door #2 or Door #3. Suppose he chooses to open Door #3 with some probability q.

$$P\{\text{Door } \#3 \mid AGG\} = q$$

Then according to Bayes' Theorem, we can compute the revised probability that the car is behind Door #2 as follows:

$$P\{GAG \mid \text{Door } \#3\} = \frac{P\{\text{Door } \#3 \mid GAG\} \cdot P\{GAG\}}{P\{\text{Door } \#3\}}$$

Substituting the known values into this equation we obtain,

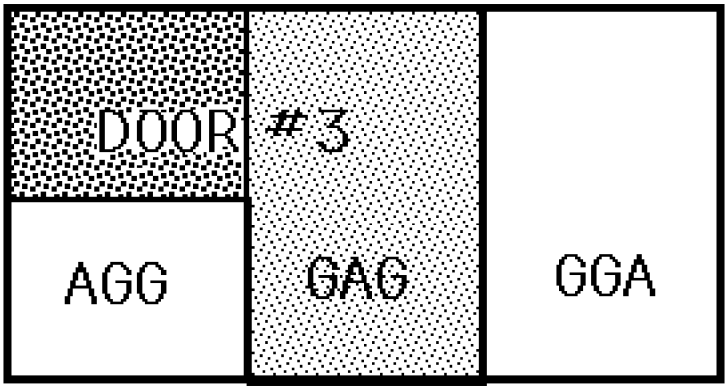
$$P\{GAG \mid \text{Door } \#3\} = \frac{1 \times p_2}{(q \times p_1) + (1 \times p_2) + (0 \times p_3)} = \frac{p_2}{qp_1 + p_2}$$

Thus the probability that the car is behind Door #2 after the host has opened Door #3 is greater than 50% if,

$$qp_1 < p_2 .$$

In this case, the contestant should make another choice. Under normal circumstances, where the original probabilities of the three arrangements, p_i, are equal, and the host chooses randomly between Door #2 and Door #3, the revised probability of Door #2 leading to the car will be greater 50%. Therefore, unless the contestant has a strong a priori belief that Door #1

conceals the car, and/or believes that the host will prefer to open Door #3 before Door #2, he should switch his/her choice.



As the above diagram illustrates, if the original probabilities of all three arrangements are equal and the host chooses randomly which door to open, then of the one half of the sample space covered by opening Door #3, two thirds falls in the region occupied by arrangement GAG. Therefore, if the host opens Door #3, Door #2 becomes twice as likely as Door #1 to conceal the automobile.

Summary

Random experiment is every finite process whose result is not determined in advance by conditions it runs under and which is at least theoretically infinitely repeatable.
Possible results of the random experiment are called **elementary events.**
A set of all elementary events are called **sample space**.
Probability measure is real function defined upon a subset system of the sample space which is non-negative, normed and σ-aditive.
Conditional probability is a probability of an event under condition that some other (not impossible) event has happened.
A and B events are **independent** if their intersection probability is equal to a product of individual event probabilities.
Total probability theorem gives us a way to determine probability of some event A while presuming that complete set of mutual disjoint events is given.
Bayes's theorem allows us to determine conditional probabilities of individual events in this complete set while presuming that A event has happened.

Quiz

1. How do you determine the probability of two events' union?
2. How do you determine the probability of two events' intersection?
3. When are two events independent?

Practical Exercises

Exercise 1: Suppose that there is a man and a woman, each having a pack of 52 playing cards. Each one of them draws a card from his/her pack. Find the probability that they each draw the ace of clubs.

{*Answer:* independent events - 0.00037}

Exercise 2: A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of choosing a red marble? a green marble? a blue marble? a yellow marble?

{*Answer:* P(red)=3/11, P(green)=5/22, P(blue)=4/11, P(yellow)=3/22}

Exercise 3: Suppose that there are two bowls full of cookies. Bowl #1 has 10 chocolate chip cookies and 30 plain cookies, while bowl #2 has 20 of each. Fred picks a bowl at random, and then picks a cookie at random. You may assume there is no reason to believe Fred treats one bowl differently from another, likewise for the cookies. The cookie turns out to be a plain one. How probable is it that Fred picked it out of the bowl #1?

{*Answer*: Conditional probability - 0.6}

Exercise 4: Suppose a certain drug test is 99% accurate, that is, the test will correctly identify a drug user as testing positive 99% of the time, and will correctly identify a non-user as testing negative 99% of the time. This would seem to be a relatively accurate test, but Bayes's theorem will reveal a potential flaw. Let's assume a corporation decides to test its employees for opium use, and 0.5% of the employees use the drug. You want to know the probability that, given a positive drug test, an employee is actually a drug user.

{*Answer:* Bayes's theorem - 0.3322}

3. RANDOM VARIABLES

Study time: 80 minutes

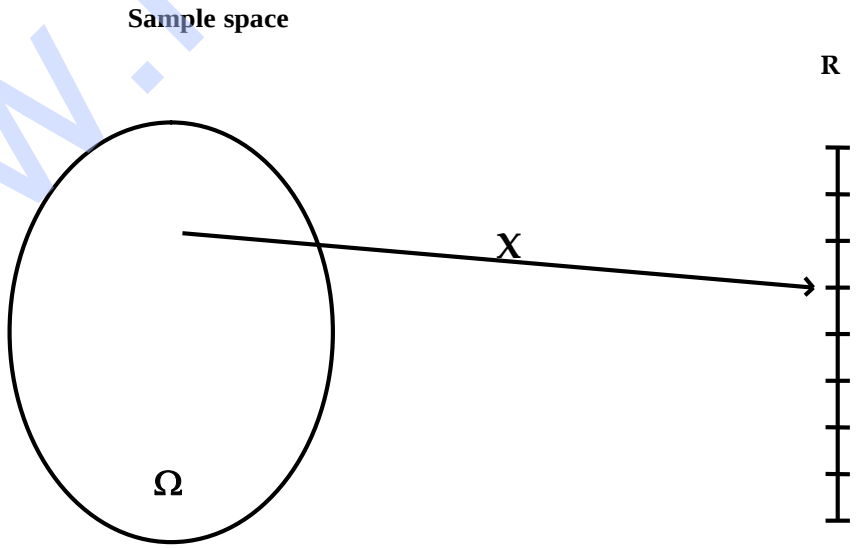
Learning objectives - you will be able to

- Describe the random variable by the distribution function
- Characterize a discrete and a continuous random variable
- Understand the hazard rate function
- Determine the numerical characteristics of the random variable
- Transform the random variable

Explanation

3.1. Definition of a Random Variable

Let us consider a probability space (Ω, *S*, P). A **random variable X** (RV) on a sample space Ω is a real function X(ω) where for each real x∈ R the set is $\{\omega \in \Omega \mid X(\omega < x)\} \in S$, i.e. it is a random event. Therefore, the random variable is a function X: Ω → R where for each x∈ R holds: $X^{-1}((-\infty, x)) = \{\omega \in \Omega \mid X(\omega < x)\} \in S$. The definition implies that we can determine the probability of $X(\omega) < x$ for any x∈R.



A group of all values $\{x = X(\omega),\ \omega \in \Omega\}$ is called **sample space**.

3.2. Distribution Function

Definition: The distribution function of a random variable X is $F(t)$ and for each $t \in R$ it has the value:

$$F(t) = P\{X \in (-\infty, t)\} = P(X \leq t).$$

Properties of the probability distribution function:

1. $0 \leq F(x) \leq 1$ for $-\infty < x < +\infty$

2. the distribution function is a monotonic increasing function of x, i.e. $\forall x_1, x_2 \in R: x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

3. the distribution function F(x) is left-continuous

4. $\lim_{x \to +\infty} F(x) = 1;\ \lim_{x \to -\infty} F(x) = 0$

5. $\forall$ a, b $\in$ R; a < b : $P(a \leq X < b) = F(b) - F(a)$

6. $P(x = x_0) = \lim_{x \to x_0 +} F(x) - F(x_0)$

If the range of the random variable function is discrete, then the random variable is called a discrete random variable. Otherwise, if the range includes a complete interval on the real line, the random variable is continuous.

3.3. Discrete Random Variable

You can speak about discrete random variable if a random variable is from some finite and enumerable set. The most often it is an integer random variable e.g. a number of students that entered the main building of VSB TUO before midday (0,1,2,...), a number of house occupants (1,2,3,...), a number of car accidents on the Prague - Brno highway in one day (0,1,2,...), etc..

Definition
You can say that a random **variable X has a discrete probability distribution** when:

$\exists$ finite or enumerable set of real numbers M={ $x_1, \ldots, x_{n, \ldots}$} that

$P(X = x_i) > 0 \qquad i = 1, 2, \ldots$

$$\sum_i P(X = x_i) = 1$$

Function $P(X = x_i) \Leftrightarrow P(x_i)$ is called **probability function of random variable** X.
A distribution function of such a distribution is a step function with steps in $x_1, \ldots, x_{n, \ldots}$

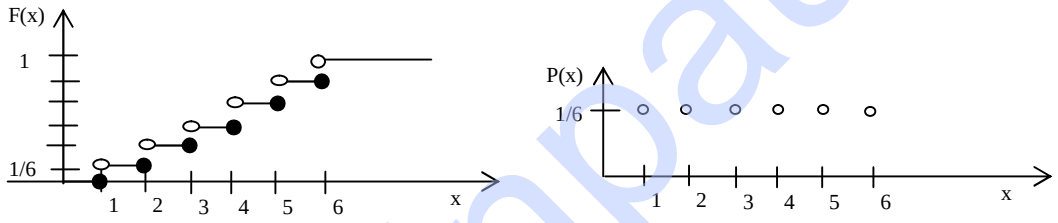
For a distribution function of a discrete random variable the following is true:

$$F(x) = \sum_{x_i < x} P(X = x_i)$$

Example

Throwing a dice, X … a number of dots obtained

x_i	$P(X = x_i)$	$F(x_i)$
1	1/6	0
2	1/6	1/6
3	1/6	2/6
4	1/6	3/6
5	1/6	4/6
6	1/6	5/6



3.4. Continuous Random Variable

If a random variable has any value from a certain interval it is a random variable with continuous distribution. Product life expectancy (0, ∞) or the length an object are examples. In this case, a density function as well as distribution function can be used to describe a distribution of random variable.

Definition
Random variable has a **continuous probability distribution** when a function f(x) exists that

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad \text{for } -\infty < x < \infty$$

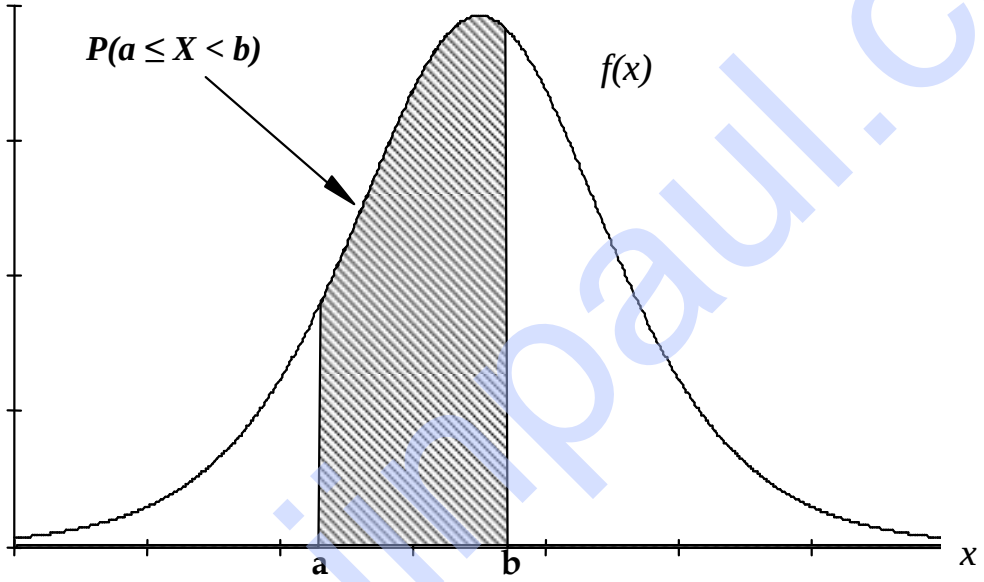
Function *f(x)* is called a **probability density function** of continuous random variable X. It is a non-negative real function.
It is evident that in all points where a derivation of distribution function exists the following is true:

$$f(x) = \frac{dF(x)}{dx}$$

If you know the distribution function you can easily determine the probability density function and vice versa.

The area below the *f(x)* curve for $x \in < a;b\,);\ (a,b \in R\,)$ in any interval is the probability that X will get a value within this interval. It also fully corresponds with the density definition.

$$P(a \le X < b\,) = F(\,b\,) - F(\,a\,) = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt = \int_{a}^{b} f(t)\,dt$$



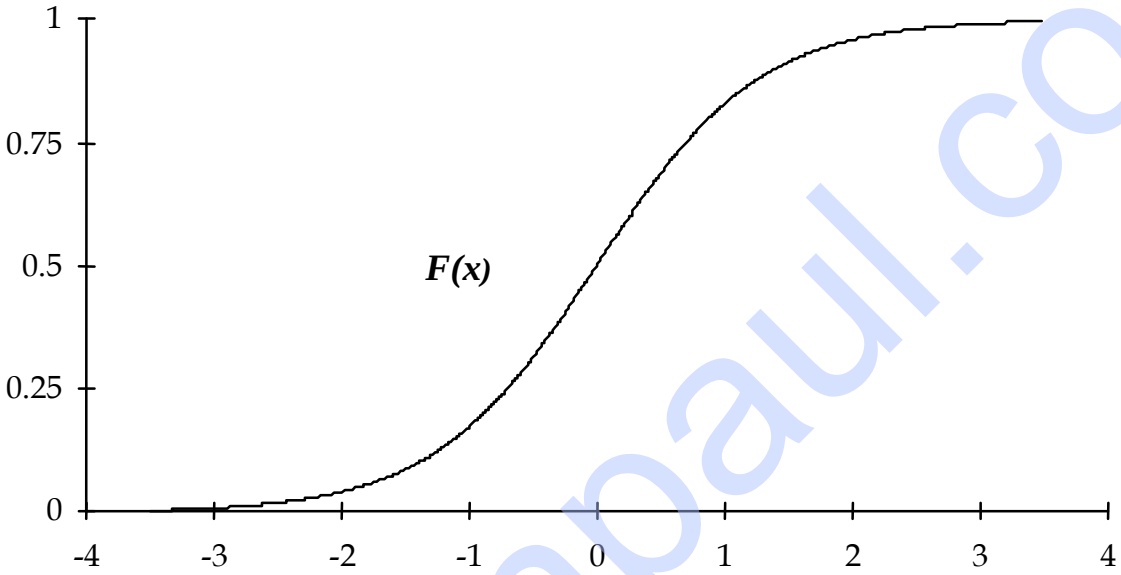
One of the attributes of probability density is the fact that the total area under the curve equals one. It is analogical to a discrete random variable where the sum of probabilities for all possible results also equals one. The following equation describes the attribute:

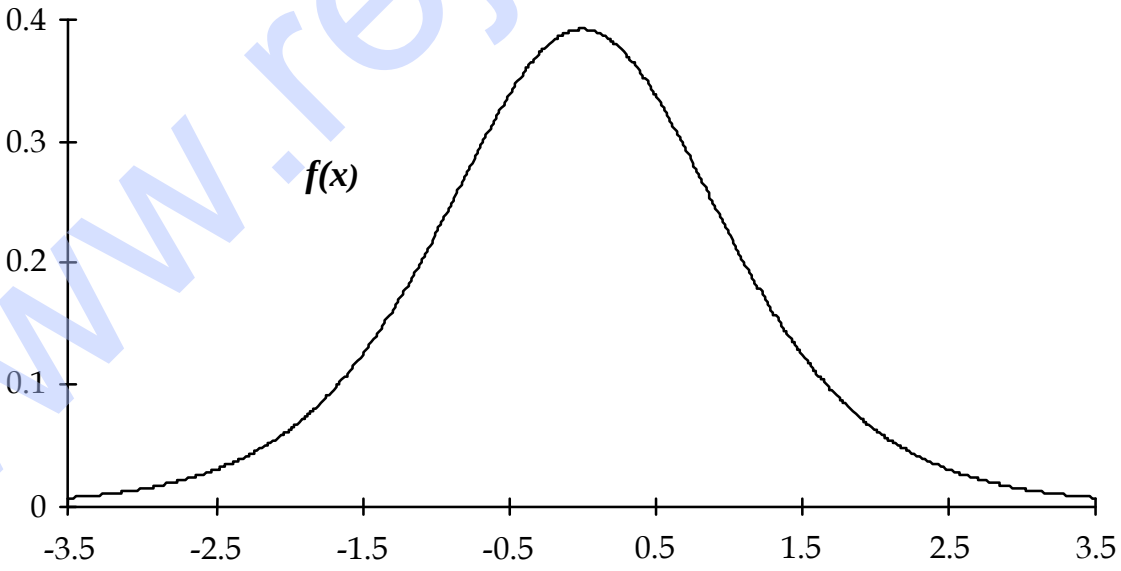
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Example

Logistic probability distribution has the following distribution function *F(x)* and probability density *f(x):*

$$F(x) = \frac{1}{1+e^{-(\beta_0+\beta_1 x)}} \qquad f(x) = \frac{\beta_1 e^{-(\beta_0+\beta_1 x)}}{(1+e^{-(\beta_0+\beta_1 x)})^2}$$





3.5. Failure Rate

Let X be a non-negative random variable with continuous distribution. Then, the failure rate for F(t) < 1 is defined as:

$$\lambda(t) = \frac{f(t)}{1 - F(t)}.$$

The following formula can easily be derived:

$$\lambda(t) = \lim_{\Delta t \to 0+} \frac{P(t < X \le t + \Delta t | X > t)}{\Delta t} = \frac{f(t)}{1 - F(t)}$$

Let X be a **mean time to failure** of any system. Then, the failure rate means that if in the t-time there was no failure, the probability of failure in a small subsequent interval Δt is approximately $\lambda(t).\Delta t$:

$$P(t < X \le t + \Delta t | X > t) \approx \frac{f(t)}{1 - F(t)} \Delta t = \lambda(t).\Delta t$$

The failure rate characterizes the probability distribution of non-negative random variable.

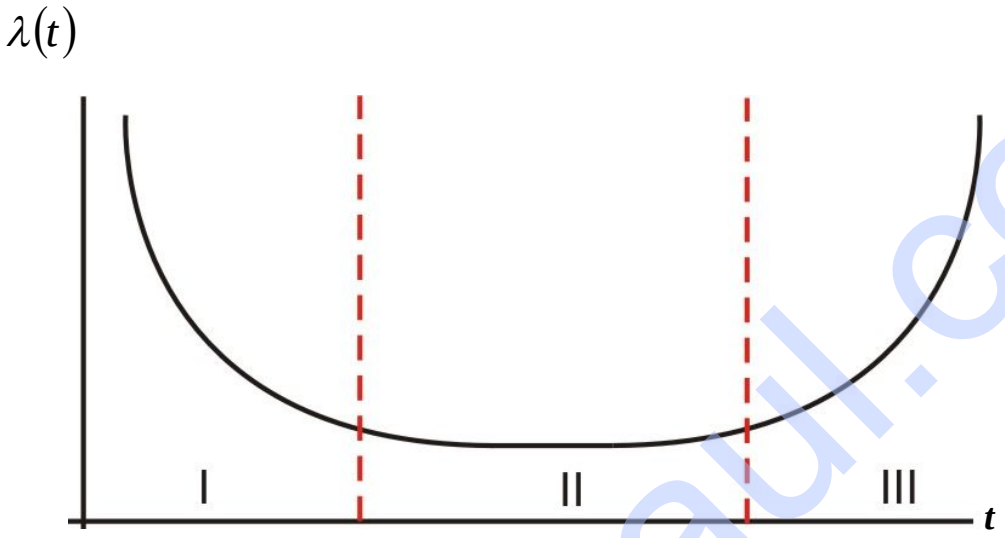
Table 1 shows the mutual conversions between $f(t), F(t), \lambda(t)$:

	$F(t)$	$f(t)$	$\lambda(t)$
$F(t)$	$F(t)$	$\int_0^t f(x)dx$	$1 - \exp\left[-\int_0^t \lambda(x)dx\right]$
$f(t)$	$\frac{dF(t)}{dt}$	$f(t)$	$\lambda(t) \cdot \exp\left[-\int_0^t \lambda(x)dx\right]$
$\lambda(t)$	$\frac{\frac{dF(t)}{dt}}{1 - F(t)}$	$\frac{f(t)}{1 - \int_0^t f(x)dx}$	$\lambda(t)$

Table 1

- **The Most Commonly Used Graphical Interpretation of Failure Rate**

Let a random variable *X* be a **mean time to failure** of any system. Then, a typical form of failure rate is shown in the following figure. The curve in this figure is called the **bathtub curve.**



I … The first part is a decreasing failure rate, known as early failures or **infant mortality.**

II … The second part is a constant failure rate, known as random failures.

III … The third part is an increasing failure rate, known as wear-out failures.

3.6. Numerical Characteristics of Random Variable

The probability distribution of each random variable X is fully described by its distribution function F(x). In many cases we can summarize the total information by several numbers. These numbers are called the **numerical characteristics of the random variable X.**

1. Moments

R-th General Moment is denoted $\mu_r' = EX^r$ r = 0,1,2, …

discrete RV: $\mu_r' = \sum_i x_i^r \, . \, P(x_i)$

continuous RV: $\mu_r' = \int_{-\infty}^{\infty} x^r . f(x)\, dx$ r = 0,1,2, …

if stated progression or integral converge absolutely.

R-th Central Moment is denoted $\mu_r = E[X - EX]^r$ r = 0,1,2, …

discrete RV: $\mu_r = \sum_i [\, x_i - EX \,]^r \,.\, P(x_i)$

continuous RV: $\mu_r = \int_{-\infty}^{\infty} (x - EX)^r \,.\, f(x)\, dx$

if stated progression or integral converge absolutely.

2. Expected Value (Mean) $EX = \mu_1'$

discrete RV: $EX = \sum_i x_i \,.\, P(X = x_i)$

continuous RV: $EX = \int_{-\infty}^{\infty} x.\, f(x)\, dx$

Properties:

1. $E(aX + b) = a.\, EX + b \quad a,b \in R$
2. $E(X_1 + X_2) = EX_1 + EX_2$
3. X_1, X_2 ... independent RV $\Rightarrow E(X_1.\, X_2) = EX_1 \,.\, EX_2$
4. $Y = g(X)$; g(X) is a continuous function: $EY = E(g(X))$

   Y is a continuous RV: $EY = \int_{-\infty}^{\infty} g(x).\, f(x)\, dx$

   Y is a discrete RV: $EY = \sum_i g(x_i) \,.\, P(X = x_i)$

3. Variance $DX = \mu_2 = E(X - EX)^2 = EX^2 - (EX)^2$

discrete RV: $DX = \sum_i x_i^2 . P(x_i) - (\sum_i x_i.\, P(x_i))^2$

continuous RV: $DX = \int_{-\infty}^{\infty} x^2.\, f(x)\, dx - (\int_{-\infty}^{\infty} x.\, f(x)\, dx)^2$

Properties::

1. $D(aX + b) = a^2.\, DX$
2. X_1, X_2 ... independent $\Rightarrow D(X_1 + X_2) = DX_1 + DX_2$

4. **Standard Deviation** $\sigma_x = \sqrt{DX}$

5. **Skewness** $a_3 = \mu_3 / \sigma_x^3$

Is a level of symmetry for the given probability distribution:

$a_3 = 0$ … symmetrical distribution

$a_3 < 0$ ... negatively skewed set

$a_3 > 0$ ... positively skewed set

6. **Kurtosis** $a_4 = \mu_4 / \sigma_x^4$

Is a level of kurtosis (flatness):

$a_4 = 3$ ... normal kurtosis (i.e. kurtosis of normal distribution)

$a_4 < 3$ ... lower kurtosis than kurtosis of normal distribution (flatter)

$a_4 > 3$ ... greater kurtosis than kurtosis of normal distribution (sharper)

7. Quantiles

$p \in (0,1)$

x_p ... 100p% quantile $\quad x_p = \sup\{x \mid F(x) \leq p\}$

continuous RV: $\quad F(x_p) = p$

Special types of quantiles:

$x_{0.5}$… 50% quantile is called the median

$x_{0.25}$ and $x_{0.75}$… 25% quantile is called the lower quartile and 75% quantile is called the upper quartile

$x_{k/10}$ … k =1,2,… 9 the k-th decile

$x_{k/100}$ … k =1,2,… 99 the k-th percentile

8. Mode

The mode $\hat{x}$ of a discrete RV X is a value that holds:

$$P(X = \hat{x}) \geq P(X = x_i) \qquad i = 1, 2, ...$$

It means that the mode is a value in which the discrete RV comes with the highest probability.

The mode $\hat{x}$ of a continuous RV X is a value that meets the following:

$$f(\hat{x}) \geq f(x) \qquad \text{pro } -\infty < x < \infty$$

It is a point where the probability density has the maximum value.

4. RANDOM VECTOR

Study time: 60 minutes

Learning Objectives - you will be able to

- Describe a random vector and its joint distribution
- Explain the concepts of marginal and conditional probability distribution
- Explain a stochastic independence of random variables

Explanation

4.1. Random Vector

For continuous random variables, the joint distribution can be represented either in the form of a distribution function or a probability density function.

$$F(x_1, x_2, ..., x_n) = P(X_1 < x_1, X_2 < x_2, ..., X_n < x_n),\ F: R^n \to R$$

$$f(x_1, x_2, ..., x_n) = \frac{\partial^n F(x_1, x_2, ..., x_n)}{\partial x_1 \partial x_2 ... \partial x_n}$$

Both forms are equivalent. In terms of the joint probability density function, the joint distribution function of $X_1,...,X_n$ is

$$F(x_1, x_2, ..., x_n) = \int_{-\infty}^{x_n} ... \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(t_1, t_2, ..., t_n)\ dt_1\ dt_2\ ...dt_n.$$

Although in theory the joint distribution of a discrete variable with a continuous variable does exist, there is no practical algebraic formulation of such a distribution. Those distributions are only represented in conditional form.

4.2. Marginal Distribution

Definition

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be a random vector. The random vector $\boldsymbol{Y} = (X_{i_1}, X_{i_2}, \ldots, X_{i_k})$, where $k < n$, $i_j \in \{1,2,\ldots,n\}$, $i_u \neq i_v$ *pro* $u \neq v$, is called the **marginal random vector.** Especially, X_i is the **marginal random variable** for every $i=1,2,\ldots,n$. The probability distribution of $\boldsymbol{Y}$ is called the **marginal probability distribution**.

Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random variable with given distribution function $F(x_1, x_2)$.

$$F_1(x_1) = \lim_{x_2 \to +\infty} F(x_1, x_2) = F(x_1, +\infty) \ldots \text{marginal distribution function of random variable } X_1$$

$$F_2(x_2) = \lim_{x_1 \to +\infty} F(x_1, x_2) = F(+\infty, x_2) \ldots \text{marginal distribution function of random variable } X_2$$

For continuous random variables, the marginal probability density of one jointly distributed variable is established by integrating the joint density function with respect the other variable.

$$f_1(x_1) = \int_{x_2} f(x_1, x_2) dx_2 \quad \text{for } X_1$$

$$f_2(x_2) = \int_{x_1} f(x_1, x_2) dx_1 \quad \text{for } X_2 .$$

For discrete random variables, the marginal distributions are given by:

$P_1(x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2) \ldots$ marginal distribution function of X_1

$P_2(x_2) = \sum_{x_1} P(X_1 = x_1, X_2 = x_2) \ldots$ marginal distribution function of X_2

4.3. Conditional Distribution

The conditional distribution is the distribution of one variable at a fixed value of the other jointly distributed random variable. For two discrete variables, the conditional distribution is given by the ratio of the joint probabilities to the corresponding marginal probability.

$$p(x_1 \mid x_2) = P(X_1 = x_1 \mid X_2 = x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P_2(x_2)} = \frac{p(x_1, x_2)}{P_2(x_2)} .$$

For continuous random variables, the conditional densities are given analogously by the ratio of the joint density to the corresponding marginal density.

$$f(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} .$$

$f_2(x_2) = \int_{-\infty}^{+\infty} f(x_1, x_2)dx_1$ is the corresponding marginal density of X_2.

4.4. Independence of Random Variables

Definition

$X_1 \ldots X_n$ are **mutually independent** $\Leftrightarrow$ random events $\{X_i < x_i\}$, $(i=1,2,\ldots,n$, where $x_i \in R$) are mutually independent.

Therefore, $X_1 \ldots X_n$ are mutually independent $\Leftrightarrow$ $F(x_1,\ldots,x_n) = F_1(x_1) \ldots F_n(x_n)$.

The above is true because:
$F(x_1,\ldots,x_n) = P(X_1<x_1, \ldots , X_n<x_n) = P(X_1< x_1) \, . \, P(X_2<x_2) \ldots P(X_n<x_n) = F_1(x_1) \, . \, F_2(x_2) \ldots F_n(x_n)$.

It implies the following rule:
$X_1 \ldots X_n$ are mutually independent $\Leftrightarrow$ $f(x_1,\ldots,x_n) = f_1(x_1) \, . \ldots f_n(x_n)$.

Example:
X_1, X_2 are mutually independent. Determine the variance $X_1 + X_2$.

In general, if X_1 and X_2 are not independent, the variance of their sum is given by

$$D(X_1 + X_2) = D(X_1) + D(X_2) + 2\,Cov(X_1, X_2)$$

where the covariance of X_1 and X_2 is defined by

$$Cov(X_1, X_2) = E[(X_1 - E(X_1))(X_2 - E(X_2))]$$

When X_1 and X_2 are independent, the covariance is zero.
An alternate expression for the covariance similar to that of the variance and simpler for computation is

$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2).$$

Correlation Coefficient

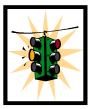
The correlation coefficient measures the strength of the relation between two random variables, X_1 and X_2. The correlation coefficient is defined by

$$\rho_{X_1X_2} = \frac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}}.$$

The correlation coefficient properties are:
1. $-1 \le \rho \le 1$

2. $\rho(X,Y) = \rho(Y,X)$

The correlation assumes values between -1 and 1. A value close to 1 implies a strong positive relationship, a value close to -1 implies a strong negative relationship, and a value close to zero implies little or no relationship.



Example and Solution

Imagine that you are going to repeat a test in a form of a trial and you are going to repeat it three times (you known the success probability, similarly as for throwing a coin)

Write down all possible combinations: (S - success, F – failure):

{ FFF; SFS; SSF; FSS; FSF; FFS; SFF; SSS }

Now you can specify the following random variables:

Y ... a number of attempts until the first success
Z ... a number of the subsequent successes

a) determine the probability function P(Y), P(Z)
b) set the joint probability function Y, Z
c) determine the marginal distribution function and P(Y | Z), P(Z | Y)

Solution:

ada) Y and Z are the discrete RV and that is why Y and Z can claim the values: 0, 1, 2, 3
All element events of the sample space need to be given their names:

A1 ... FFF	$P(A1) = (1 - p)^3$
A2 ... SFS	$P(A2) = p^2.(1 - p)$
A3 ... SSF	$P(A3) = p^2.(1 - p)$
A4 ... FSS	$P(A4) = p^2.(1 - p)$
A5 ... FSF	$P(A5) = p.(1 - p)^2$
A6 ... FFS	$P(A6) = p.(1 - p)^2$
A7 ... SFF	$P(A7) = p.(1 - p)^2$
A8 ... SSS	$P(A8) = p^3$

The fact that the F and S variables are independent needs to be taken into consideration for the further calculation:

Y ... a number of attempts until the first success			
0	1	2	3
SFS, SSF, SFF, SSS	FSS, FSF	FFS	FFF

Z ... a number of subsequent successes			
0	1	2	3
FFF	SFS, FSF, FFS, SFF	SSF,FSS	SSS

Since A1, ... , A8 events are disjoint we can simply determine the probability function (p=0.5).

Y ... a number of attempts until the first success			
P(Y= 0)	P(Y= 1)	P(Y= 2)	P(Y= 3)
0.5	0.25	0.125	0.125

Z ... a number of subsequent successes			
P(Z = 0)	P(Z = 1)	P(Z = 2)	P(Z = 3)
0.125	0.5	0.25	0.125

adb) you proceed in the same way like in establishing the probability function

	Z			
Y	0	1	2	3
0	-	SFS, SFF	SSF	SSS
1	-	FSF	FSS	-
2	-	FFS	-	-
3	FFF	-	-	-

	Z			
Y	0	1	2	3
0	0	0.25	0.125	0.125
1	0	0.125	0.125	0
2	0	0.125	0	0
3	0.125	0	0	0

adc) marginal probability functions - P(Y), P(Z)

		Z			
Y	0	1	2	3	P(Y)
0	0	0.25	0.125	0.125	0.5
1	0	0.125	0.125	0	0.25
2	0	0.125	0	0	0.125
3	0.125	0	0	0	0.125
P(Z)	0.125	0.5	0.25	0.125	1

$P(Y \mid Z) = P(Y \wedge Z) / P(Z)$

	Z			
Y	0	1	2	3
0	0	0.5	0.5	1
1	0	0.25	0.5	0
2	0	0.25	0	0
3	1	0	0	0

$P(Z \mid Y) = P(Y \wedge Z) / P(Y)$

	Z			
Y	0	1	2	3
0	0	0.5	0.25	0.25
1	0	0.5	0.5	0
2	0	1	0	0
3	1	0	0	0

Summary

Random variable X is a real function which can be characterized by a **distribution function** *F(t).*

Distribution function is a function that assigns to each real number a probability that the random variable will be less then this real number. Distribution function has some general properties like $\forall$ a, b $\in$ R; a < b : $P(a \leq X < b) = F(b) - F(a)$.

Depending on the type of values the random variable is getting, there is continuous and discrete variable.

The discrete random variable is characterized by a **probability function**, the continuous variable by a **density function**.

Quite often it is useful to describe the whole information about random variable by several numbers that characterize its properties while allowing the comparison of different random variables. These numbers are called the **numerical characteristics** of random variable.

A **random vector** is a vector consisting of random variables $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ that is characterized by **joint distribution function.**

From the joint distribution function of the random vector you can easily determine the **marginal probability distribution** of particular random variables the vector is composed of.

Quiz

1. What is the mutual relationship between distribution function and probability function of a discrete random variable?
2. What is the mutual relationship between distribution function and probability density function of a continuous random variable?
3. What are the median and the mode?
4. Explain the term Conditional Probability Distribution.
5. Explain the term Stochastic Independence of Random Variables.
6. What does a value of correlation coefficient tell us?

Practical Exercises

Exercise 1: Let Y be a continuous variable defined by a probability density function:

$$f(y) = \begin{cases} c.(1+y).(1-y); & -1 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find a constant c, a distribution function, an expected value and a variance of this variable.

{*Answer:* c=0.75; $F(y) = 0.25\,(3y - y^3+2)$; EY = 0; DY = 0.2}

Exercise 2: Let a random variable W be defined as a linear transformation of random variable Y, defined in previous example.

$$W = 5Y + 6$$

Find a probability density function, a distribution function, an expected value and a variance of random variable W.

{*Answer:* $f(w) = -\frac{3}{500}\,(w^2-12w+11)$; $F(w) = 0.25\,[3(\frac{w-6}{5}) - (\frac{w-6}{5})^3+2]$; EW = 6; DW = 5}

Exercise 3: Let a random variable Z be defined as:

$$f(z) = 1 / [\,(1+e^z).(1+e^{-z})\,]; \qquad -\infty < z < \infty$$

Find a distribution function of the random variable Z.

{*Answer:* $F(z) = \frac{e^z}{1+e^z}$ }

5. SOME IMPORTANT PROBABILITY DISTRIBUTIONS

5.1. Discrete Probability Distributions

Study time: 50 minutes

Learning Objectives - you will be able to

- Characterize Bernoulli trials and types of discrete distributions
- Characterize Poisson process and Poisson distribution
- Describe contexture in between discrete distributions

Explanation

A lot of discrete random variables exist and now we will summarize basic information about the most common discrete variables.

Bernoulli Trials:

- a sequence of Bernoulli trials is defined as a sequence of random events which are mutually independent and which have only two possible outcomes (e.g. success-nonsuccess, 1-0)
- probability of event occurrence (a success) p is constant in any trial

$$P\{\text{Trial 'i'} = \text{"Success"}\} = p$$

Binomial Random Variable:

The most natural random variable to define on the sample space of Bernoulli trials is the number of successes. Such a random variable is called a binomial random variable. If X is the number of successes in n Bernoulli trials where the probability of success at each trial is p, then we represent the distribution of X by the short-hand notation:

$$X \rightarrow B(n, p)$$

where B indicates that X has a binomial distribution and n and p are the parameters determining which particular distribution from the binomial family applies to X.

The probability distribution of a binomial random variable can be expressed algebraically as:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k};\ 0 \le k \le n$$

$$EX = \sum_{k=0}^{n} k \cdot P(X = k) = \sum_{k=0}^{n} k \cdot \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k} =$$

$$n \cdot p \sum_{k=0}^{n} \frac{(n-1)!}{(n-k)!\cdot(k-1)!} \cdot p^{k-1} \cdot (1-p)^{n-k} = n \cdot p$$

$$DX = EX^2 - (EX)^2$$

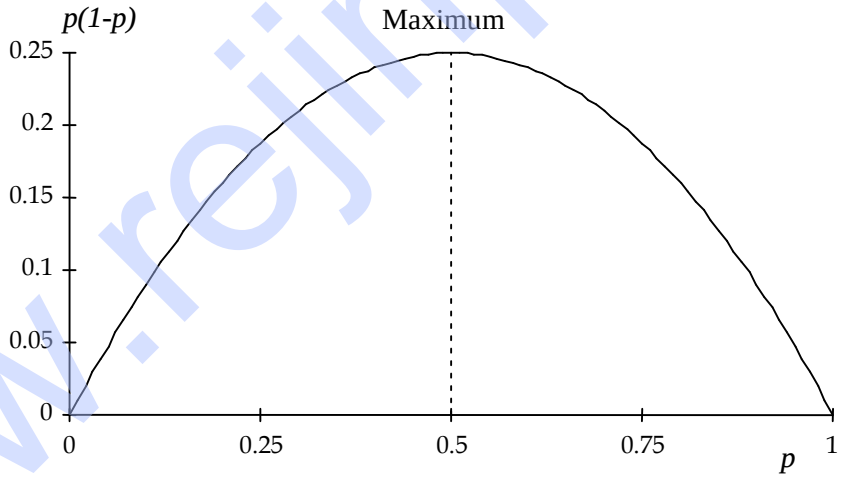
$$EX^2 = \sum_{k=1}^{n} k^2 \cdot P(X = k) = \sum_{k=1}^{n} k \cdot (k-1) \cdot \frac{n!}{(n-k)!\cdot k!} \cdot p^k \cdot (1-p)^{n-k} + EX =$$

$$= n \cdot (n-1) \cdot p^2 \cdot \sum_{k=2}^{n} \frac{(n-2)!}{(n-k)!\cdot(k-2)!} \cdot p^{k-2} \cdot (1-p)^{n-k} + EX =$$

$$= n \cdot (n-1) \cdot p^2 + n \cdot p = (np)^2 - np^2 + np$$

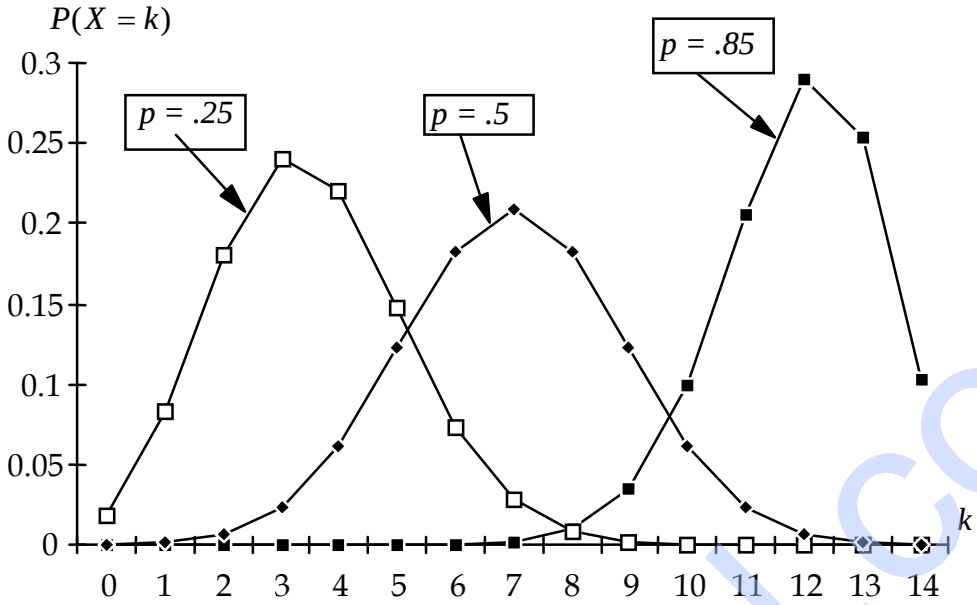
$$DX = EX^2 - (EX)^2 = n.p.(1 - p)$$

Notice that the variance of the binomial distribution is maximum when $p = 0.5$.



Example:

Some examples of binomial distributions for $n = 14$ trials are illustrated below. Notice that as p, the probability of success at each trial increases, the location of the distribution shifts to higher values of the random variable. Also notice that when p = 0.5, the distribution is symmetric around 7.5.



Geometric Distribution:

This distribution has a single parameter, p, and we denote the family of geometric distributions by

$$X \rightarrow G(p)$$

G(p) … the geometric random variable is defined as the number of trials until a success occurs or until the first success

The probability distribution for a geometric random variable is:

$$P(X = k) = p(1-p)^{k-1};\ 1 \le k < \infty$$

The expression for the mean of the geometric distribution is

$$EX = \sum_{k=1}^{\infty} k \cdot P(X = k) = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1} = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = p \cdot \frac{\partial}{\partial(1-p)} \cdot \left(\sum_{k=1}^{\infty} (1-p)^{k} \right) = \frac{1}{p}$$

Note: By first evaluating the series and then taking its derivative the result is obtained.

The mean number of Bernoulli trials until the first success is the inverse of the success probability at each trial, again an entirely intuitive result. That is, if 10% of the trials are successful, on average it will take ten trials to obtain a success.

To find the variance, we first evaluate the expected value of X^2 and then modify the expression using the same technique as for the binomial case. We note that the expression now has the form of the second derivative of the same geometric series we evaluated for the mean. Taking derivatives of this evaluated expression, we obtain

$$E(X^2) = \sum_{k=1}^{\infty} k^2\ p(1-p)^{k-1}$$

$$= p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} + p\sum_{k=2}^{\infty} k(1-p)^{k-1}$$

$$= p(1-p)\frac{\partial^2 \sum_{k=1}^{\infty}(1-p)^k}{\partial^2(1-p)} + p\frac{\partial \sum_{k=1}^{\infty}(1-p)^k}{\partial(1-p)}$$

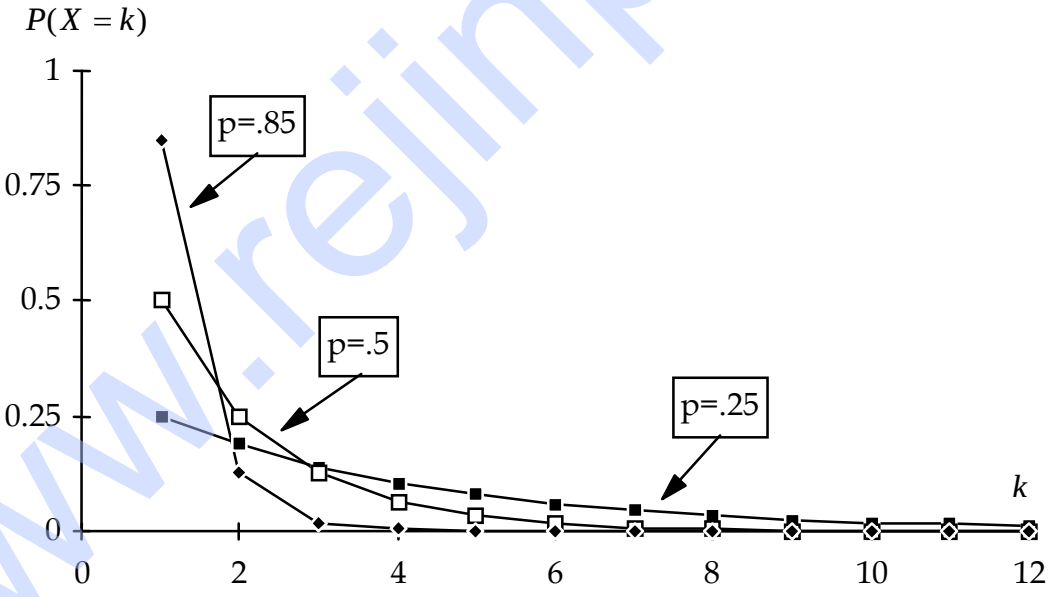
$$= p(1-p)\frac{\partial^2\left(\frac{1-p}{p}\right)}{\partial^2(1-p)} + p\frac{\partial\left(\frac{1-p}{p}\right)}{\partial(1-p)} = \frac{2(1-p)}{p^2} + \frac{1}{p}$$

From the mean and expected value of X^2 we can derive the variance.

$$DX = EX^2 - (EX)^2 = \frac{1-p}{p^2}$$

Example:

Some examples of geometric distributions are illustrated below. Not surprisingly, the probability of long sequences without success decreases rapidly as the success probability, *p*, increases.



Negative Binomial Random Variable

The negative binomial distribution has two parameters, k and p and is denoted by

$$X \rightarrow NB(k, p)$$

The negative binomial is the number of Bernoulli trials until the kth success.

The negative binomial distribution is:

$$P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}; \; k \le n < \infty$$

The mean and variance of the negative binomial distribution can be computed easily by noting that a negative binomial random variable with parameters k and p is just the sum of k independent geometric random variables with parameter p. Independence the geometric random variables follow from the independence assumption of sequences of Bernoulli trials. Thus if

$$W_i \rightarrow G(p); 1 \le i \le k$$

then

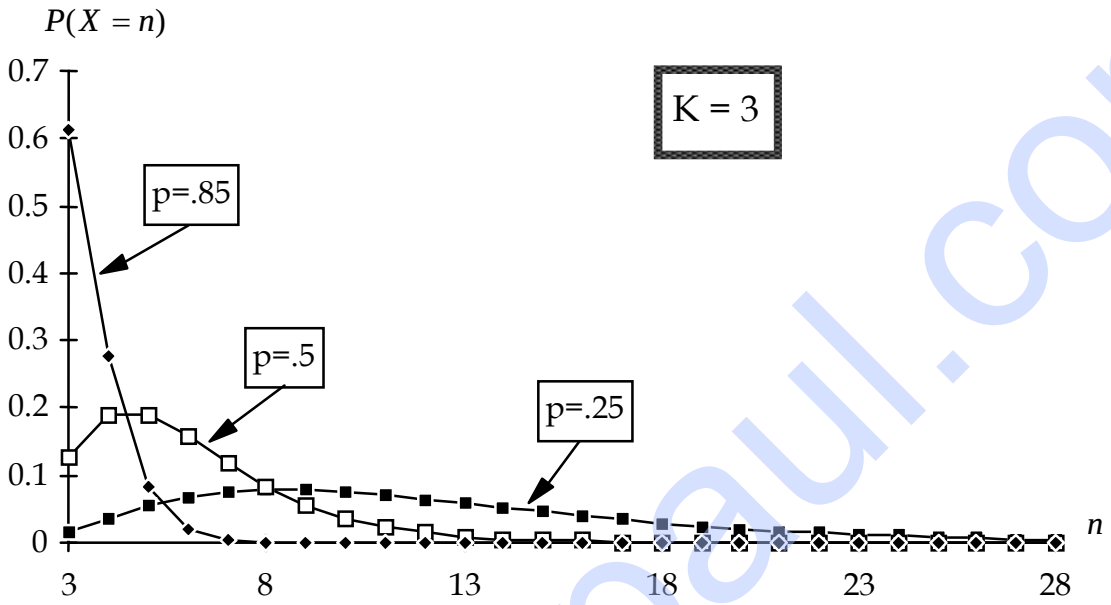
$$X = \sum_{i=1}^{k} W_i$$

$$E(X) = \sum_{i=1}^{k} E(W_i) = \frac{k}{p}$$

$$D(X) = \sum_{i=1}^{k} D(W_i) = \frac{k(1-p)}{p^2}$$

Example:

The following chart illustrates some examples of negative binomial distributions for k=3. Notice that for values of *p* near 0.5, the distribution has a single mode near 5. This mode moves away from the origin and diminishes in magnitude as *p* decreases indicating an increase in variance for small *p*. The negative binomial distribution has a shape similar to the geometric distribution for large values of *p*.



***Note*:**

Comparison of Binomial and Negative Binomial Distributions

It is interesting to compare the distributions of binomial and negative binomial random variables. Notice that except for the combinatorial term at the beginning of each distributional expression, the portion contributed by the probability of single sample space elements is identically $p^k (1 - p)^{n-k}$.

Binomial distribution

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}; 0 \le k \le n$$

For the binomial, the number of trials (*n*) is fixed and the number of successes (*k*) is variable.

Negative binomial distribution

$$P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}; k \le n < \infty$$

For the negative binomial, the number of successes (*k*) is fixed and the number of trials (*n*) is variable.

Poisson Process

The Poisson process is a second general type of sample space model which is widely applied in practice. The Poisson process may be viewed as the continuous time generalization of a sequence of Bernoulli trials, sometimes called the Bernoulli process. The Poisson process describes the sample space of randomly occurring events in some time interval. The Poisson process assumes that the **rate at which events occur is constant** throughout the interval or region of observation and those events occur **independently** of each other.

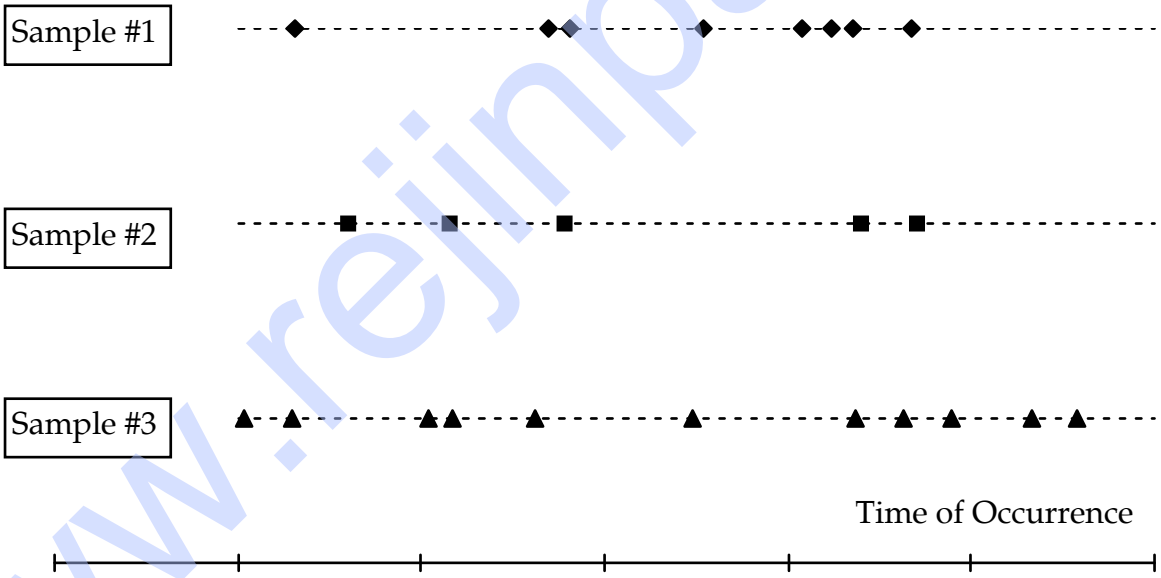
λ … lambda $\rightarrow$ the rate at events occur

However, events occurring over some region can also be modeled by a Poisson process. The appearance of defects in some product, or mold on the surface of a leaf under certain conditions could follow a Poisson process.

Examples:

- Customers arriving at a bank to transact business.
- Patients arriving at a clinic for treatment
- Telephone inquiries received by a government office, *etc.*

Some examples of elements of the sample space for a Poisson process are illustrated below. This is a complex difficult to characterize sample space. The number of elements in the sample space is uncountable, infinite and in this sense continuous. Probabilities cannot be assigned to individual elements of this sample space, only to subsets.



The Poisson process describes events which occur randomly over some time interval or spatial region.

Poisson Distribution

The Poisson distribution has a single parameter and therefore we denote this random variable by the symbolic notation,

$$X \rightarrow P(\lambda t)$$

Consider a Poisson process that is observed for a time period *t*. Suppose the rate of occurrence of events is λ during the time period. Then the total rate of occurrence over the entire observation interval is λt. Now divide the interval *t* into *n* subintervals of equal length *t/n*. Occurrence of events in each of these intervals will be mutually independent at constant rate $\lambda t/n$. If *n* becomes large enough, the interval lengths, *t/n*, will become small enough that the probability of two events in one interval is effectively zero and the probability of one event is proportional to $\lambda t/n$. Then the distribution of the number of events in the total interval *t* can be approximated by the distribution of a binomial random variable with parameters *n* and $\lambda t/n$. Thus,

$$P(X = k) = \binom{n}{k}\left(\frac{\lambda t}{n}\right)^k\left(1-\frac{\lambda t}{n}\right)^{n-k}$$

Taking the limit as *n* goes to infinity, this expression becomes

$$\lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda t}{n}\right)^k\left(1-\frac{\lambda t}{n}\right)^{n-k} = \frac{(\lambda t)^k}{k!}\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k}\left(1-\frac{\lambda t}{n}\right)^{n-k} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad ,$$

We can express the distribution of a Poisson random variable as:

$$P(X = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!};\ 0 \le k < \infty$$

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty}\frac{(\lambda t)^k e^{-\lambda t}}{k!} = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!} = 1$$

Calculation of mean:

$$E(X) = \sum_{k=0}^{\infty} kP(X = k) = \sum_{k=0}^{\infty} k\frac{(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t\sum_{k=1}^{\infty}\frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} = \lambda t$$

The expected value of X^2 is found using the Taylor series expansion of the exponential function as well as the same algebraic manipulation as was invoked for Bernoulli random variables.

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X = k) = \sum_{k=2}^{\infty} k(k-1)\frac{(\lambda t)^k e^{-\lambda t}}{k!} + \mathrm{E}(\mathrm{X}) =$$

$$= (\lambda t)^2 \sum_{k=2}^{\infty} \frac{(\lambda t)^{k-2} e^{-\lambda t}}{(k-2)!} + \lambda t = (\lambda t)^2 + \lambda t$$

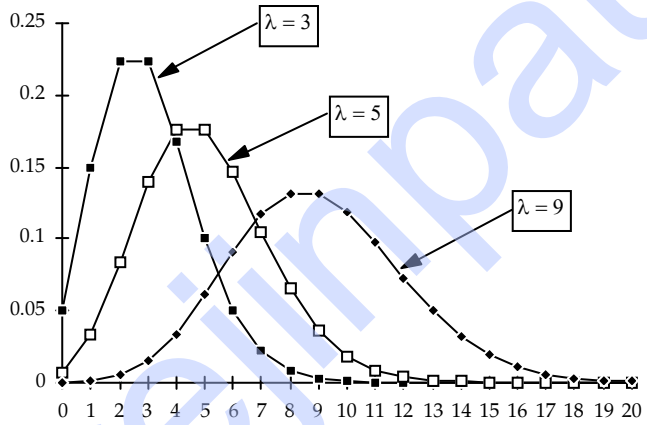
From this result, the variance follows directly.

$$D(X) = E[(X - E(X))^2] = \mathrm{E}(\mathrm{X}^2) - (\mathrm{E}(\mathrm{X}))^2 = \lambda t$$

We notice that the Poisson distribution has the remarkable property that the variance is equal to the mean and by implication that the variance of the Poisson random variable will increase as the rate λ increases.

Example:

Some examples of Poisson distributions are illustrated below. Notice that at the value λ = 9, the distribution becomes almost symmetric.



5.2. Continuous Probability Distributions

Study time: 50 minutes

Learning Objectives - you should be able to

- Characterize types of continuous distributions : exponential, Gamma and Weibull
- Describe contexture in between continuous distributions

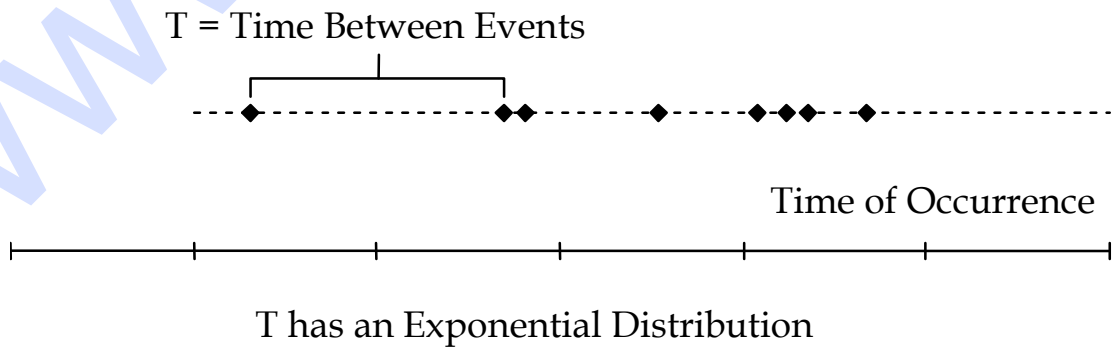
Explanation

- **How does a basic description for exponential, gamma and Weibull distributions look like?**

Exponential Distribution

The exponential random variable is a second very natural variable which can be defined on the sample space generated by a Poisson process. If a continuous time process satisfies the assumptions of a Poisson process, the time between events, or equivalently because of the assumption of independence, the time until the next event will have an exponential distribution. The exponential random variable is analogous to the geometric random variable defined for a Bernoulli process.

The range of possible values for the exponential random variable is the set of non-negative numbers.



T has an Exponential Distribution

Strictly speaking, the sample space for an exponential random variable consists of intervals of varying length terminated by a single event, just as the sample space of the geometric random variable consists of a sequence of failures terminated by a success.

The probability density function and distribution function of an exponential distribution have the following simple form.

$$f(t) = \lambda e^{-\lambda t}; t \geq 0$$

$$F(t)=P(T < t) = P(N_t \geq 1) = 1 - P(N_t < 1)$$
$$= 1 - e^{-\lambda t}$$

where λ is the rate at which events occur. The family of exponential random variables is identified by the single parameter, λ, the same parameter which defines the Poisson distribution.

$$T \rightarrow E(\lambda)$$

The mean of the exponential distribution is the reciprocal of the rate parameter. The result can be obtained through integrating the expected value integral by parts.

$$E(T) = \int_{t=0}^{\infty} \lambda t\, e^{-\lambda t}\, dt = \frac{1}{\lambda}$$

The variance of the exponential distribution is obtained from evaluating the following integral again through integration by parts.

$$DT = ET^2 - (ET)^2 = \ldots = \frac{1}{\lambda^2}$$

The variance equals the square of the mean and therefore the mean equals the standard deviation for an exponential distribution.

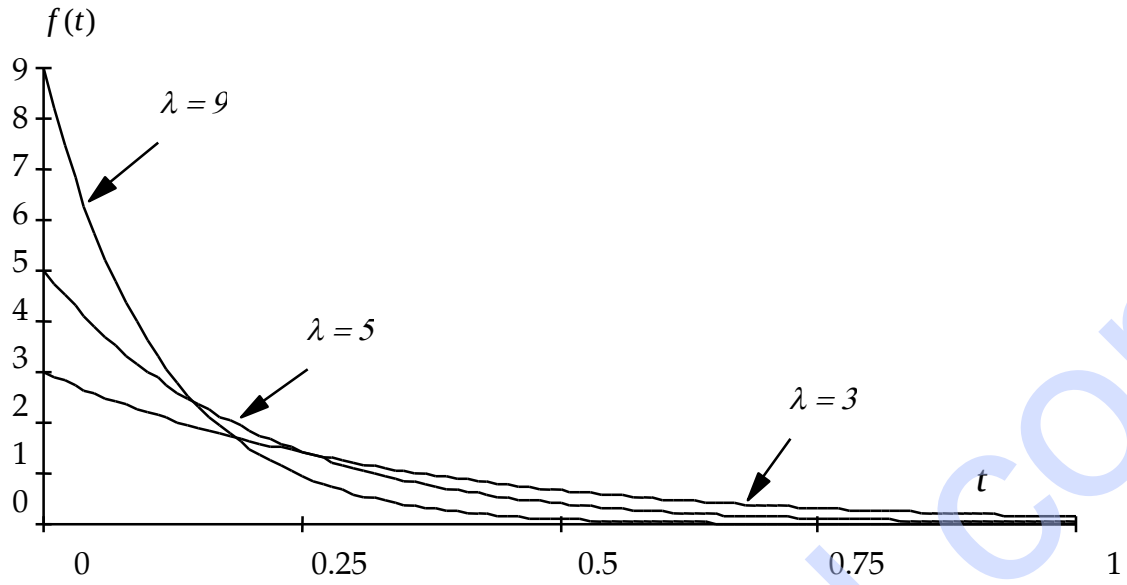
The hazard function is given by:

$$h(t) = \frac{f(t)}{1 - F(t)} \quad \text{if } F(t) < 1$$

$h(t) = \lambda = const. \Rightarrow$ the "no memory" property of the exponential distribution

Example:

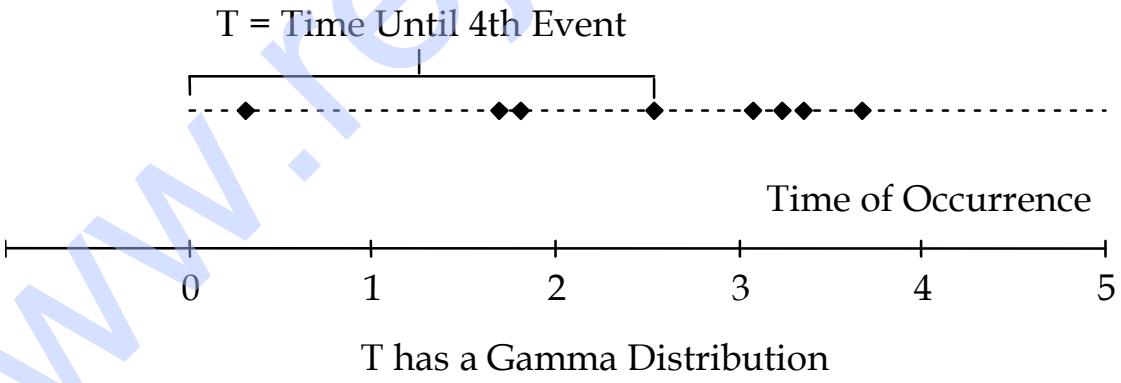
The following graph illustrates some examples of the probability density functions of exponential random variables. Notice that the shape of the exponential density is similar to the shape of the geometric probability distribution. The exponential distribution of time to next event is the continuous time equivalent of the geometric distribution which is the number of trials to next event where event may be considered a "successful" trial.



Gamma Distribution

$T \rightarrow Ga(k, \lambda)$

The sample space generated by the Poisson process gives rise to a second random variable closely associated with the exponential random variable. The total time until some specified number of events, say *k*, occur is called a Gamma random variable and arises as a sum of *k* identical independent exponential random variables. If the exponential distribution is the continuous time equivalent of the geometric, then it follows that the Gamma distribution is the continuous time equivalent of the negative binomial.



The sample of a gamma random variable arising as the sum of 4 independent exponential random variables, that is as the time until the fourth event in a Poisson process will consist of intervals of varying length, all having three events and terminated by a fourth event.

The gamma distribution function for any integer value of *k* can be derived by the following argument. Since the gamma arises as the sum of *k* independent, identically distributed exponential random variables, the distribution function of the gamma is the probability that

the sum of *k* exponentials is less than or equal to some value *t*. This implies that there have been at least *k* occurrences of a Poisson process within time *t*, the probability of which is given by the cumulative distribution of a Poisson random variable with rate parameter λt, where λ is the rate of the underlying Poisson process.

$$T_k = X_1 + X_2 + X_3 + \ldots + X_k$$

$$X_i \rightarrow E(\lambda)$$

$$F(t) = P(T_k < t) = P\left(\sum_{i=1}^{k} X_i < t\right) = P(N_t \geq k) = 1 - P(N_t < k) =$$

$$1 - \sum_{j=0}^{k-1} e^{-\lambda t} \cdot \frac{(\lambda t)^j}{j!} = 1 - e^{-\lambda t}\left[\sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}\right]$$

and the probability density function is

$$f(t) = \lambda e^{-\lambda t}\left[\sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}\right] - e^{-\lambda t}\left[\sum_{j=1}^{k-1} \frac{\lambda(\lambda t)^{j-1}}{(j-1)!}\right]$$

$$= \lambda^k e^{-\lambda t}\left[\frac{t^{k-1}}{(k-1)!}\right]; \quad t > 0$$

Since the gamma random variable is the sum of *k* identical independent exponential random variables, the mean and variance will be k times the mean and variance of an exponential random variable. This same argument was used to derive the mean and variance of the negative binomial from the moments of the geometric distribution.

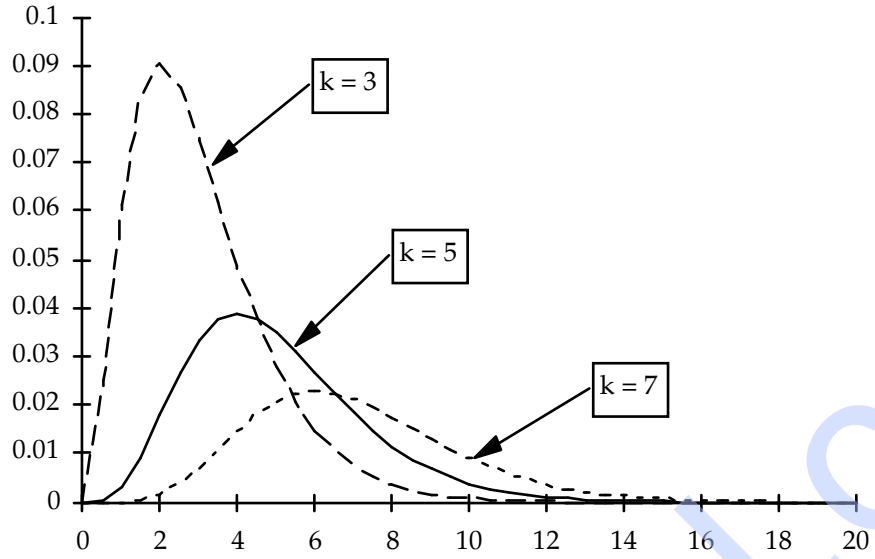
$$ET_k = EX_1 + EX_2 + \ldots + EX_k = \frac{1}{\lambda} + \ldots + \frac{1}{\lambda} = \frac{k}{\lambda}$$

$$DT_k = \ldots. = \frac{k}{\lambda^2}$$

The form of the gamma distribution presented here where the parameter *k* is restricted to be a positive integer is actually a special case of the more general family of gamma distributions where *k* is a shape parameter which need only be a positive real number. The special case we have discussed is sometimes called the **Erlang Distribution**.

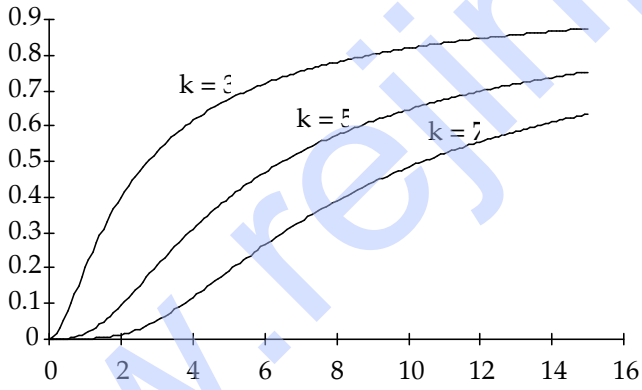
Example:

Examples of gamma probability density functions for $\lambda = 1$ are illustrated in the following chart. Notice that the gamma density has a single mode which moves away from the origin as *k* increases. Also the dispersion increases and the distribution become more symmetric and *k* increases.



The hazard function is given by:

$$h(x) = \frac{\lambda}{(k-1)!\sum_{j=0}^{k-1} \frac{1}{(k-1-j)!(\lambda x)^j}}$$



The hazard function of Gamma distribution, λ=1

h(x) is a sharply increasing function for k > 1 ⇒ this distribution is suitable for modeling of ageing and wear processes

Weibull Distribution

The distribution function is:

$F(x) = 1 - e^{-\left(\frac{x}{\Theta}\right)^{\beta}}$, $\Theta > 0, \beta > 0, x > 0$ β ... a shape parameter, Θ ... a scale parameter.

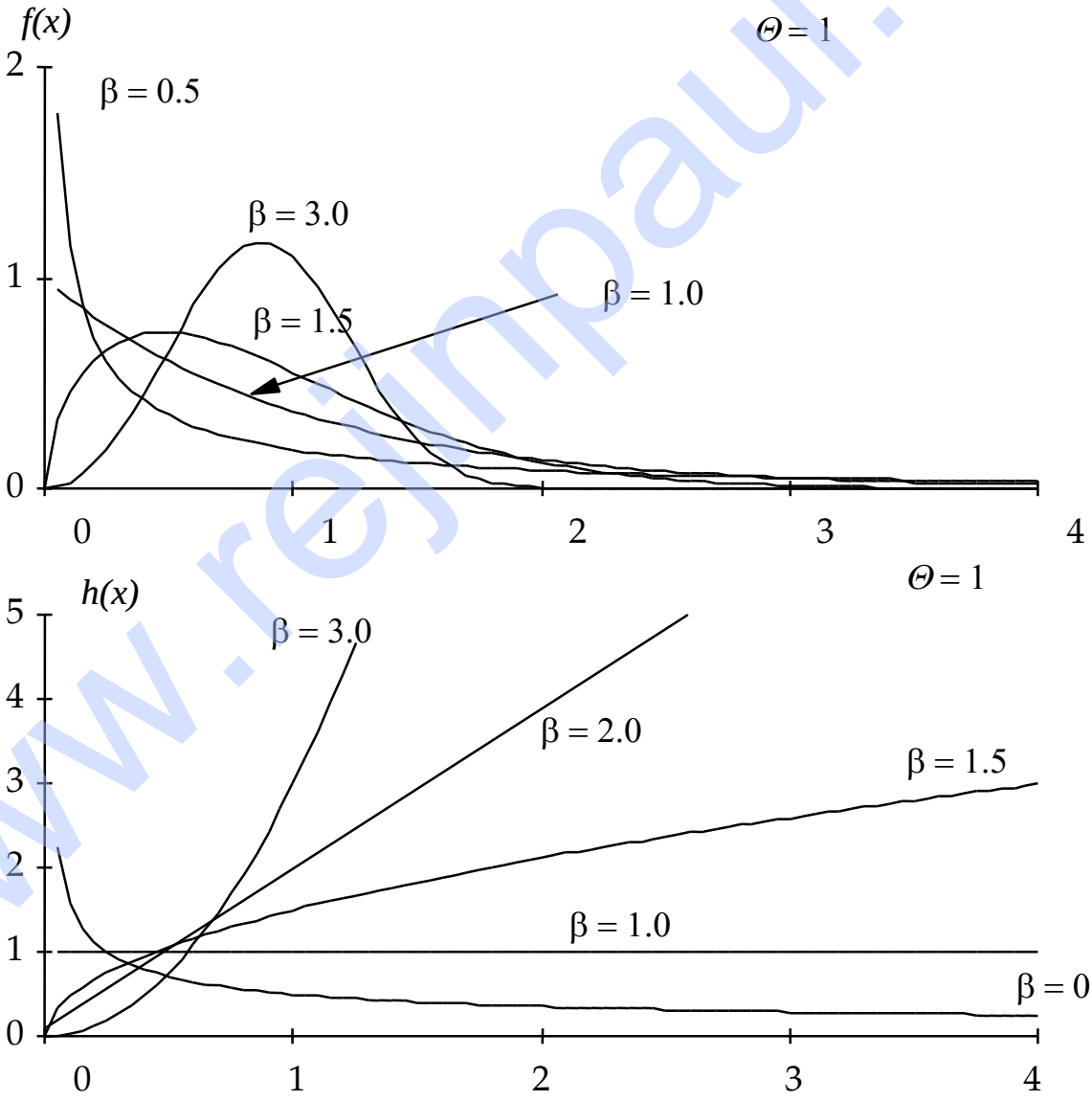
The probability density function for the Weibull is:

$$f(x) = \frac{\beta}{\Theta}\left(\frac{x}{\Theta}\right)^{\beta-1} e^{-\left(\frac{x}{\Theta}\right)^{\beta}}$$

And the hazard function for the Weibull is:

$$h(x) = \frac{\beta}{\Theta}.\left(\frac{x}{\Theta}\right)^{\beta-1}$$

Some examples of the Weibull density and the Weibull hazard function are illustrated below.



The Weibull distribution is very flexible and we use it in Reliability theory for modeling of the random variable "time to failure".

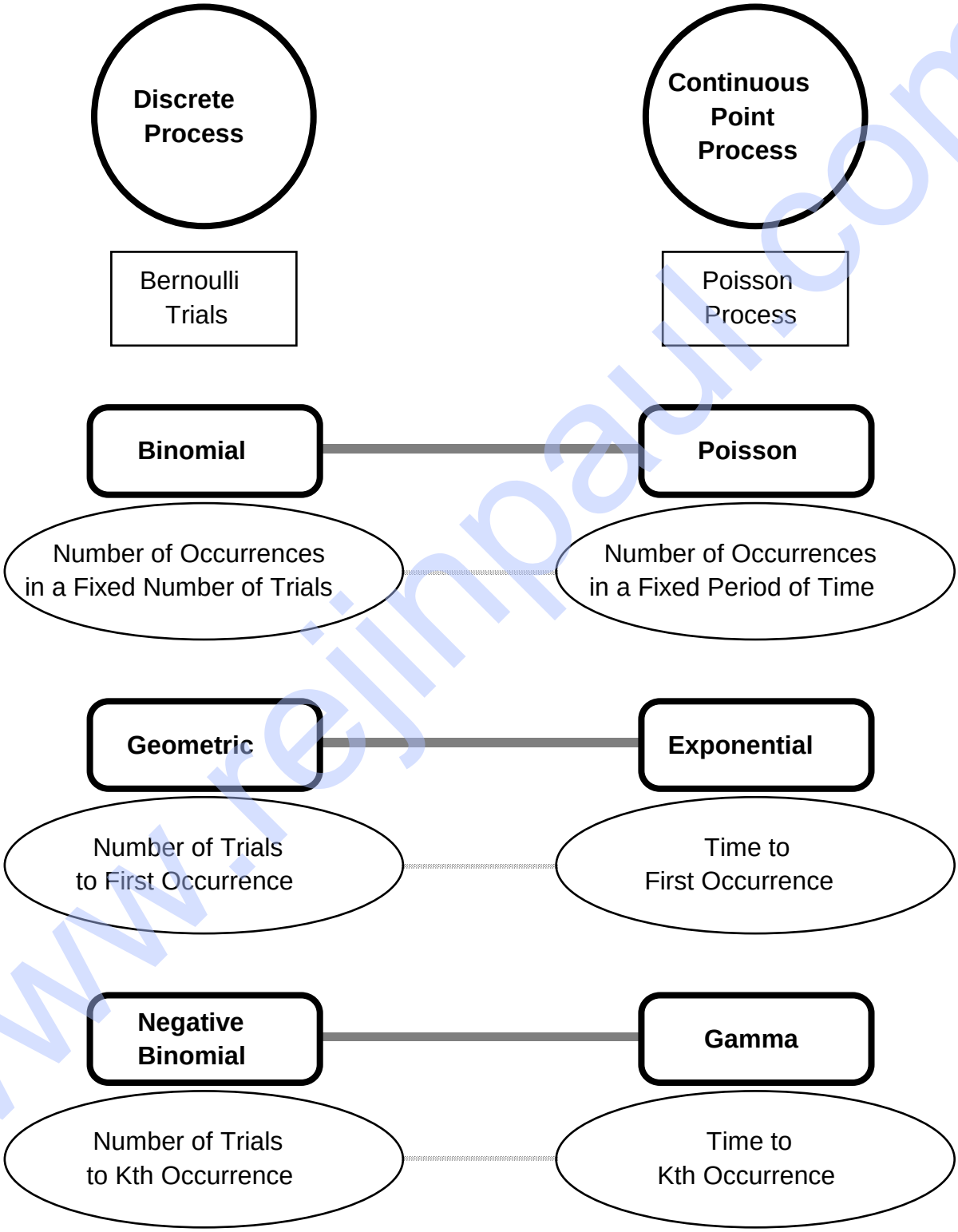
Summary

A sequence of **Bernoulli trials** is defined as a sequence of random events which are mutually independent and which have only two possible outcomes (e.g. success-nonsuccess, 1-0) and the probability of event occur (a success) p is constant in any trial.
On the basis of these trials expectations we can define the following random variables: **binomial**, **geometric** and **negative binomial**.
A number of events occurrences on any deterministic interval from 0 to t can be describe (at certain expectations) by a **Poisson distribution**.
If a continuous time process satisfies the assumptions of a Poisson process, the time between events, or equivalently because of the assumption of independence, the time until the next event will have an **Exponential distribution**.
A Gamma distribution describes a time to k-th event occurrence.
A Weibull distribution is generalization of the exponential distribution and it is very flexible.

Quiz

1. Which discrete and continuous distributions do you know?
2. Characterize the Bernoulli trials and individual types of the discrete distributions. Determine the mean of the binomial random variable.
3. What is Gamma distribution used for? How is it related to exponential distribution?
4. For what β of Weibull distribution is the hazard function linearly increasing?

Diagram: A Contexture Among Distributions



Practical Exercises

Exercise 1*:* Suppose that a coin with probability of heads $p = 0.4$ is thrown 5 times. Let X denote the number of heads.

a) Compute the density function of X explicitly.
b) Identify the mode.
c) Find $P(X > 3)$.

{*Answer:* Let $f(k) = P(X = k) = \binom{5}{k}(0.4)^k (0.6)^{5-k}$ for $k = 0, 1, 2, 3, 4, 5$.

a) $f(0) = 0.0778$, $f(1) = 0.2592$, $f(2) = 0.3456$, $f(3) = 0.2304$, $f(4) = 0.0768$, $f(5) = 0.0102$.
b) mode: $k = 2$
c) $P(X > 3) = 0.9870$. }

Exercise 2*:* Suppose that the number of misprints N on a web page has the Poisson distribution with parameter 2.5.

a) Find the mode.
b) Find $P(N > 4)$.

{*Answer:* a) mode: $n = 2$, b)$P(N > 4) = 0.1088$}

Exercise 3: Message arrive at a computer at an average rate of the 15 messages/second. The number of messages that arrive in 1 second is known to be a Poisson random variable.

a) Find the probability that no messages arrive in 1 second.
b) Find the probability that more than 10 messages arrive in a 1-second period.

{*Answer:* a) $3.06(10^{-7})$, b) 0.8815 }

Exercise 4: If there are 500 customers per eight-hour day in a check-out lane, what is the probability that there will be exactly 3 in line during any five-minute period?

{*Answer:* Poisson - 0.1288 }

6. THE NORMAL DISTRIBUTION AND THE LIMIT THEOREMS

Study time: 60 minutes

Learning Objectives - you will be able to

- Characterize the normal and the standard normal distribution
- Formulate and use the limit theorems
- Describe special distributions

Explanation

6.1. Normal Distribution

Astronomers were responsible for one of the earliest attempts to formally model the random variation inherent in the measurement process. The probability density function which was adopted at that time has been alternatively called the error function, the Gaussian curve, and today most commonly, the normal distribution. The normal distribution is the most widely used model of random variation. Its popularity is partly based on its intuitive appeal as a simple mathematical model of our instinctual notions of what constitutes random variation. However, there is also sound theoretical support for the belief that the normal distribution frequently occurs in practice.

The form of the normal density model is a simple symmetric bell-shaped curve with a single mode. This shape is achieved by the use of a negative exponential function whose argument is the square of the distance from the mode. Since squared distance makes values near zero smaller, the normal curve has a smooth rounded shape in the region of its mode. The normal density has two parameters, μ, its mode and the point about which the density is symmetric, and σ, a scale or dispersion parameter which determines the concentration of the density about the mode and the rate of decrease of the density towards the tails of the distribution. The family of normally distributed random variables is denoted by

$$X \rightarrow N(\mu, \sigma^2)$$

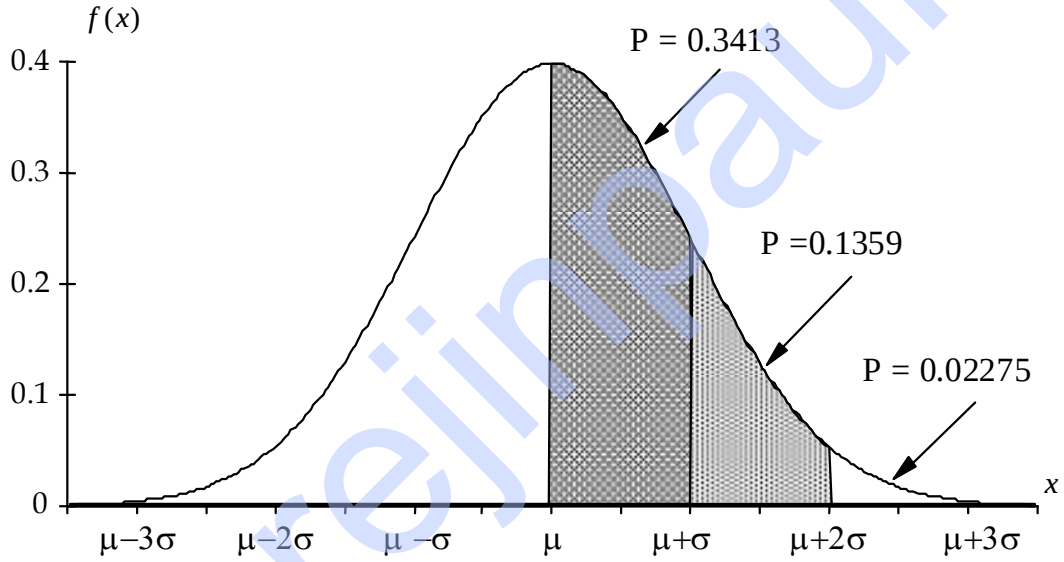
The probability density of the random variable X with the normal distribution: $X \rightarrow N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty < x < +\infty$$

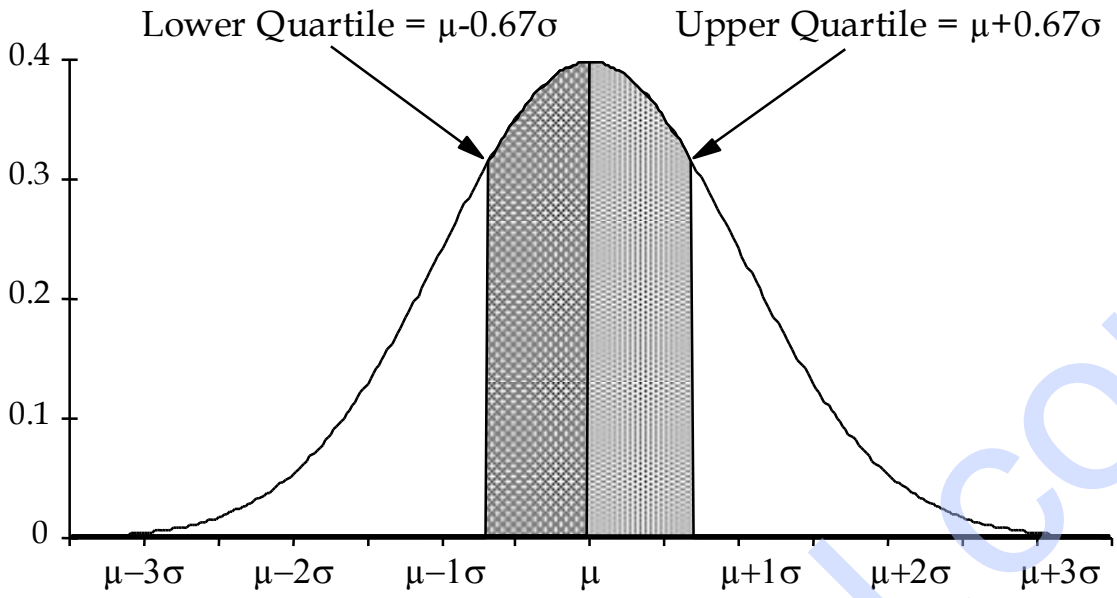
pro - $\infty < \mu < \infty$, $\sigma^2 > 0$

This density is symmetric about μ and therefore the mean, median, and mode are all equal to μ. Also due to the symmetric bell shape of this density, the interquartile range equals the Shorth which is twice the MAD.

The following charts illustrate the distribution of probabilities for a normal random variable. The first chart shows that the probability of being between 0 and 1 standard deviation (σ) above the mean (μ) is 0.3413 or approximately one-third. Since the distribution is symmetric, the probability of being between 0 and 1 standard deviation (σ) below the mean (μ) is also approximately one third. Therefore the probability of being more than one standard deviation from the mean in either direction is again one third.



Conversely, the upper and lower quartiles are two thirds of a standard deviation above and below the mean. Thus μ+/- 0.67 σ divides the probability of the distribution into four equal parts of 25%.



The mean and variance of a normal random variable are equal to its location parameter μ, and the square of its scale parameter σ^2, respectively.

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \mu$$

$$V(X) = E[(X - E(X))^2] = \int_{-\infty}^{+\infty} \frac{(x-\mu)^2}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$$

The distribution function of the normal distribution is:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}dt} dt$$

Standard Normal Distribution

A normal random variable with location parameter 0 and scale parameter 1 is called a standard normal random variable. Because of the form of the normal density, it is possible to determine probabilities for any normal random variable from the distribution function of the standard normal variable. Consequently, the standard normal random variable has been given the special symbolic destination, Z, from which the z-score derives. The standard normal distribution function is given the special symbol, Φ.

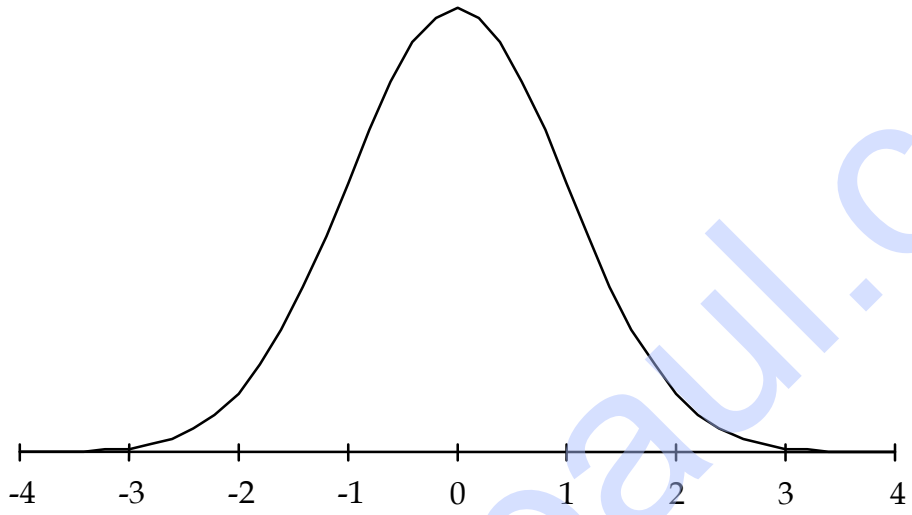
Then,

$$\Phi(z) = P(Z < z) = \int_{-\infty}^{z} \varphi(u)du = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

where φ(z) is the standard normal density

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} ; -\infty < z < +\infty$$

Standard Normal Distribution



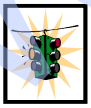
Standardization - relation between normal and standard normal distribution

Therefore if X is any normal random variable $N(\mu, \sigma^2)$ we can define a related standard normal random variable $Z = \frac{X - \mu}{\sigma}$ and it has the standard normal distribution.

$$X \dots N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \quad , Z \dots \Phi(0, 1)$$

The distribution function of X can therefore be computed from the derived random variable Z which has a standard normal distribution:

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \Phi\left(\frac{x-\mu}{\sigma}\right)$$



EXAMPLE AND SOLUTION

X ... N(2, 25), determine P(2 < X < 8)

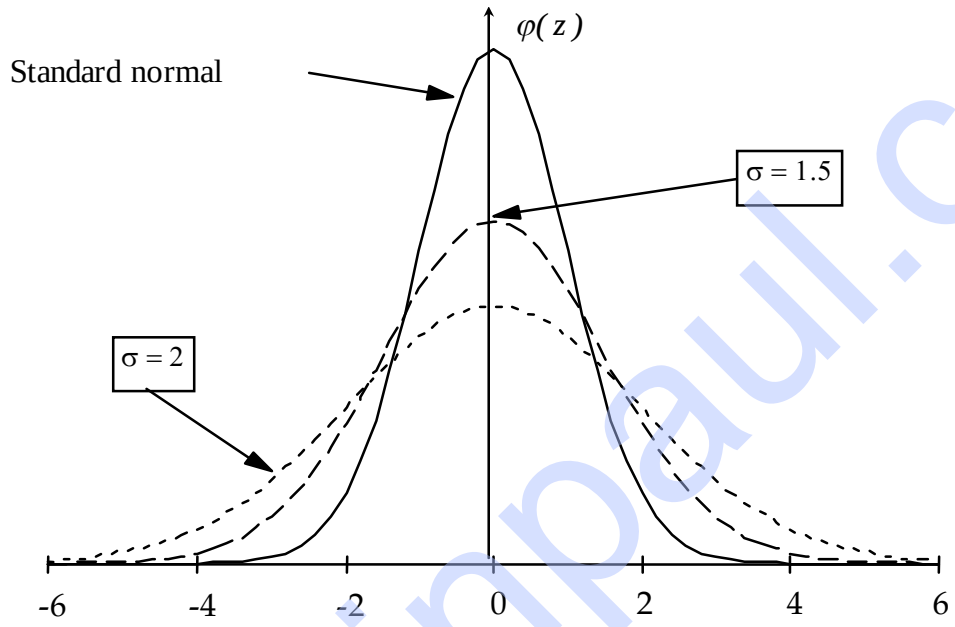
Solution:

$$P(2 < X < 8) = F(8) - F(2) = \Phi\left(\frac{8-2}{\sqrt{25}}\right) - \Phi\left(\frac{2-2}{\sqrt{25}}\right) = \Phi(1.2) - \Phi(0) = 0.885 - 0.5 = 0.385$$

In the tables or by suitable software we can find: $\Phi(1.2) = 0.885$, $\Phi(0) = 0.5$

Example:

The following chart illustrates the normal density with zero mean for selected values of σ. It is clear that the mean, median, and mode of a normal random variable are all equal, and the two parameters of the normal distribution are the embodiment of our intuitive notions about the general distributional characteristics of location and scale.



6.2. Limit Theorems

Basic definitions

Convergence in probability:

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1 \quad \Rightarrow X_n \xrightarrow{p} X;$$ *i.e. a sequence of random variables* $\{X_n\}$ *converges in probability to random variable X*

Convergence in distribution:
$\{F_n(x)\}$ ... is a sequence of distribution functions corresponding to random variables $\{X_n\}$

The sequence $\{X_n\}$ *converges towards X in distribution,* if:

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for every real number x at which F is continuous.

Consequence:

The sequence $\{X_n\}$ converges in distribution to distribution $N(\mu, \sigma^2)$, i.e. the random variable X_n has *asymptotical normal distribution* if $\lim_{n\to\infty} F_n(x) = \Phi(\frac{x-\mu}{\sigma})$.

Limit Theorems

Chebyshev's Inequality:

X ... is an arbitrary random variable with mean EX and variance DX.
Then

$$P(\,|X - EX\,| \geq \varepsilon) \leq \frac{DX}{\varepsilon^2}, \qquad \varepsilon > 0$$

This relation results from the variance definition:

$$D(X) = \int_{-\infty}^{+\infty} (x\text{-}E(X))^2 f(x)dx = \left\{ \int_{|x-E(X)|<\varepsilon} (x\text{-}E(X))^2 f(x)dx + \int_{|x-E(X)|\geq\varepsilon} (x\text{-}E(X))^2 f(x)dx \right\}$$
$$\geq \int_{|x-E(X)|\geq\varepsilon} (x\text{-}E(X))^2 f(x)dx \geq \varepsilon^2 P(|X - E(X)| \geq \varepsilon)$$

Using the Chebyshev's inequality (for calculating probabilities):

$$P(|X - E(X)| < k\sigma) > 1 - \frac{1}{k^2}$$

e.g. we can apply it to $X = \overline{X}$ with respect to following limit theorems:

$$P(|\overline{X} - \mu| < k\frac{\sigma}{\sqrt{n}}) > 1 - \frac{1}{k^2}$$
$$P(\frac{|\overline{X} - \mu|}{\sigma} < \frac{k}{\sqrt{n}}) > 1 - \frac{1}{k^2}$$

Law of Large Numbers:

$\{X_n\}$ ... is a sequence of independent random variables, each having a mean $EX_n = \mu$ and a variance $DX_n = \sigma^2$.

Define a new variable

$$\overline{X_n} = \frac{1}{n}.\sum_{j=1}^{n} X_j, \qquad n \in N$$

The sequence $\{\overline{X_n}\}$ converges in probability to μ: $\overline{X_n} \xrightarrow{p} \mu$.

Notion: This affirmation results from the Chebyshev's inequality.

Bernoulli theorem: $\{X_n\}$ ... is a sequence of the binomial independent random variables with parameters $n=1$ a $p \in (0,1)$ (so-called alternative random variable, let $X_n = 1$ in case the event will be at one trial and $X_n = 0$ in case the event won't be; $P(X_n = 1) = p$ a $P(X_n = 0) = 1\text{-}p$). Then we know that

$$\overline{X_n} = \frac{1}{n}.\sum_{j=1}^{n} X_j \xrightarrow{p} p$$

The expression on the left side represents a relative frequency of the event occurrence in the sequence n trials. That is why we can estimate a probability ingoing any occurrence by relative frequency of this event occurrence in the sequence n trials when we have a great number of the trials.

Central Limit Theorem

Lindeberg's Theorem:
Let $X_1, X_2, \ldots, X_n \ldots$ be a sequence of independent random variables, $n \to \infty$.
X_i ... has the same probability distribution, $EX_i = \mu$, $DX_i = \sigma^2$.
Then

$Y_n = \dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}$ has an asymptotic normal distribution $N(0,1) \Rightarrow \lim_{n\to\infty} P(Y_n < u) = \Phi(u)$

for $-\infty < u < \infty$, it means that Y_n converges in distribution to distribution $N(0,1)$.

For sufficiently large numbers n the following is true:

1. $X_n = \sum_{i=1}^{n} X_i \Rightarrow EX = n\mu$, $DX = n\sigma^2$, we can approximate the distribution X_n by the distribution $N(n\mu, n\sigma^2)$, i.e. X_n *has the asymptotic normal distribution,* $X_n = \sum_{i=1}^{n} X_i \to N(n\mu, n\sigma^2)$.
2. Analogously for $\overline{X}$:

$\overline{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ *has the asymptotic normal distribution with parameters* $E\overline{X} = \mu$, $D\overline{X} = \dfrac{\sigma^2}{n}$,

$$\overline{X} \to N(\mu, \frac{\sigma^2}{n})$$

A special case of the above theorem is the Moivre-Laplace Theorem:

Let S_n ... Bi(n, p); $ES_n = np$; $DS_n = np(1\text{-}p)$

[$S_n = \sum_{i=1}^{n} X_i$, X_i ... has the alternative distribution thus binomial Bi(1, p)]

then for large n it holds that $U_n = \frac{S_n - np}{\sqrt{np(1-p)}} \to N(0,1)$.

Applications of the Central Limit Theorem – Normal Approximations to the Binomial and Poisson distributions

Taking n observations of a Bernoulli distribution and computing the sample average $\hat{p}$ is equivalent to defining the sample proportion random variable

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{k}{n}$$ ... proportion of "successes" in n Bernoulli trials

The sample proportion will have a Binomial distribution will the values re-scaled from 0 to 1. That is, if X has a binomial distribution with parameters n and p.

$$\mathrm{P}\left[\hat{p} = \frac{k}{n}\right] = \mathrm{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Furthermore:

$$E(\hat{p}) = p; \qquad D(\hat{p}) = \frac{p(1-p)}{n}$$

Therefore by the Central Limit Theorem, we can approximate the binomial distribution by the normal distribution for large n.

$$\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \to N(0,1)$$

$$\frac{X - np}{\sqrt{np(1-p)}} \to N(0,1)$$

Probabilities of value ranges for these variables can then be calculated as

$$P(p_1 < \hat{p} < p_2) = \Phi\left(\frac{p_2 - p}{\sqrt{\frac{p(1-p)}{n}}}\right) - \Phi\left(\frac{p_1 - p}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$P(k_1 < S_n < k_2) = \Phi\left(\frac{k_2 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1 - np}{\sqrt{np(1-p)}}\right)$$

For smaller sample sizes a so-called continuity correction is often employed to improve the accuracy of the approximation. Thus we would compute the preceding probability as

$$P(k_1 < S_n < k_2) = \Phi\left(\frac{k_2 + 0{,}5 - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k_1 - 0{,}5 - np}{\sqrt{np(1-p)}}\right)$$

The Central Limit Theorem applies broadly to most distributions. In particular, the normal distribution can be used to approximate the Poisson distribution when the interval of observation, *t*, and hence the expected number of events, λt, is large.

$$\hat{\lambda} = \frac{X}{t} = \frac{count\ \ per}{unit\ \ time} = \frac{rate\ \ of}{occurrence}$$

We know that the mean and variance of the number of events during an interval *t* is λt, and therefore the mean and variance of the rate at which events occur is

$$E(\frac{X}{t}) = \lambda; \qquad D(\frac{X}{t}) = \frac{\lambda}{t}$$

Probabilities concerning Poisson counts or rates can then be calculated as

$$P(k_1 < X < k_2) = \Phi\left(\frac{k_2 - \lambda t}{\sqrt{\lambda t}}\right) - \Phi\left(\frac{k_1 - \lambda t}{\sqrt{\lambda t}}\right)$$

$$P(g_1 < \frac{X}{t} < g_2) = \Phi\left(\frac{g_2 - \lambda}{\sqrt{\frac{\lambda}{t}}}\right) - \Phi\left(\frac{g_1 - \lambda}{\sqrt{\frac{\lambda}{t}}}\right)$$

where,

$$g_1 = \frac{k_1}{t}; \qquad g_2 = \frac{k_2}{t}$$

Applying the continuity correction, we would calculate the probability as,

$$P(k_1 < X < k_2) = \Phi\left(\frac{k_2 + 0{,}5 - \lambda t}{\sqrt{\lambda t}}\right) - \Phi\left(\frac{k_1 - 0{,}5 - \lambda t}{\sqrt{\lambda t}}\right)$$

6.3. Special Sampling Distribution

Chi-Square Distribution

The chi-squared random variable arises as the sum of squared standard normal random variables. The distribution has a single parameter *n*, the number of squared normal random variables in the sum. This parameter is called the degrees of freedom of the chi-squared distribution.

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2$$

A χ-squared random variable with one degree of freedom χ_1^2 is simply a squared standard normal random variable. The distribution function of this random variable is

$$\begin{aligned} F_{\chi_1^2}(y) &= P(Z^2 < y) = P(-\sqrt{y} < Z < \sqrt{y}) \\ &= \Phi\left(\sqrt{y}\right) - \Phi\left(-\sqrt{y}\right) \qquad y > 0 \end{aligned}$$

The probability density function of a χ_1^2 random variable can be found from the derivative of its distribution function.

$$f_{\chi_1^2}(y) = \frac{\partial F_{\chi_1^2}(y)}{\partial y} = \left(\frac{\partial \Phi(\sqrt{y})}{\partial \sqrt{y}} + \frac{\partial \Phi(-\sqrt{y})}{\partial \sqrt{y}} \right) \frac{\partial \sqrt{y}}{\partial y}$$

$$= \frac{e^{-\frac{y}{2}}}{\sqrt{2\pi\, y}} \qquad y > 0$$

The general density for a χ_n^2 random variable is

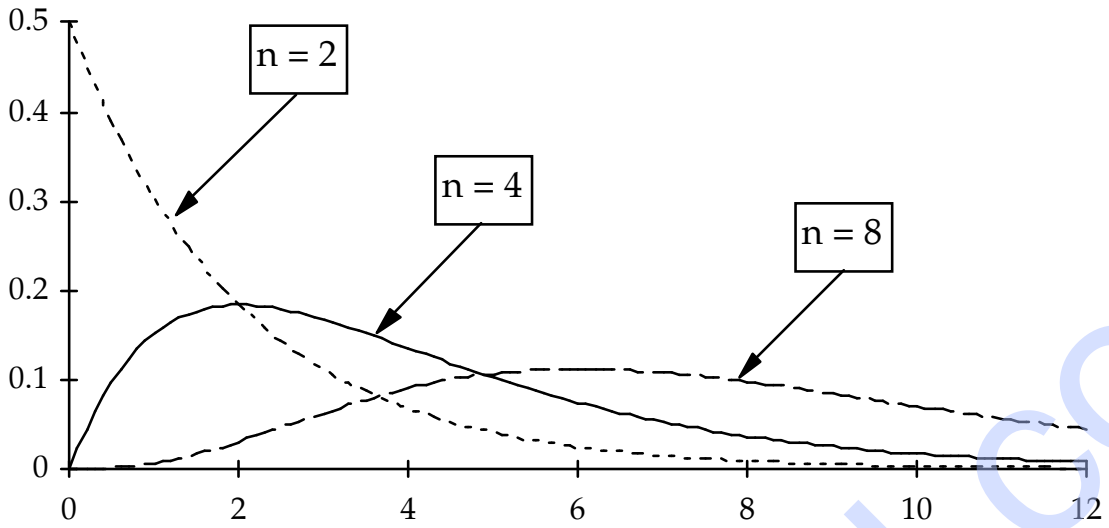
$$f_\chi(x) = \frac{x^{\left(\frac{n}{2}-1\right)} e^{-\frac{x}{2}}}{2^{\frac{n}{2}}\, \Gamma\left(\frac{n}{2}\right)}$$

where $\Gamma(t)$ is a gamma function.

The mean and variance of the chi-squared distribution are

$$\begin{aligned} E(\chi_n^2) &= n \\ D(\chi_n^2) &= 2n \end{aligned}$$

The density function for various values of the parameter *n* is:



The χ^2_n is the sampling distribution of the sample variance

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

If the X_i is a sample from a normal population with mean μ and standard deviation σ, then the sample variance has a distribution

$$s^2 \rightarrow \chi^2_{n-1}\frac{\sigma^2}{n-1}$$

To see this, consider the sum of squared standardized observations from a normal population with mean μ and standard deviation σ.

$$\sum_{i=1}^{n}\left(\frac{x-\mu}{\sigma}\right)^2 \rightarrow \chi^2_n$$

This expression clearly has a χ^2_n distribution. Now re-express the numerator by the added and subtracting the sample mean from the squared terms.

$$\frac{\sum_{i=1}^{n}(x-\bar{x}+\bar{x}-\mu)^2}{\sigma^2} \rightarrow \chi^2_n$$

Expanding and simplifying, we obtain

$$\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{\sigma^2}+\frac{(\bar{x}-\mu)^2}{\sigma^2/n}\rightarrow\chi_n^2$$

The second term is simply the squared standardized sample mean and therefore has a χ_1^2 distribution. Since the χ_n^2 is the sum of *n* squared normal, the first term must be the remaining *n-1* squared normal. Therefore,

$$\frac{(n-1)\,s^2}{\sigma^2}=\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{\sigma^2}\rightarrow\chi_{n-1}^2$$

Note that this argument is only heuristic and it is not a formal proof. The independence of the sample mean from the deviations about the sample mean has not been established.

This fact is important at the statistical hypothesis testing.
1. We use this distribution for verification of the random variables independence.
2. We can use chi-square distribution to test that the random variables follow certain distribution. This test is known as "Goodness-of-Fit Test".

Student's Distribution (*t*-distribution)

The Student's t-distribution is the sampling distribution of the standardized sample mean when the sample variance is used to estimate the true population variance. The origin of its name has an interesting history. An Irish statistician, W. S. Gosset first published this distribution anonymously under the pseudonym "Student" because his employer, Guiness Breweries of Dublin, prohibited its employees from publishing under their own names for fear that its competitors would discover the secret of their excellent beer. In his original paper, Gosset used the designation "t" for his statistic. Hence the name is Student's.

The Student's *t*-distribution with *n* degrees of freedom is the ratio of a standard normal random variable to the square root of a chi-squared random variable divided by its degrees of freedom. The *t*-distribution has a single parameter, *n* the degrees of freedom of the chi-squared random variable in the denominator.

$$t_n=\frac{Z}{\sqrt{\frac{\chi_n^2}{n}}}$$

The probability density function of this random variable is

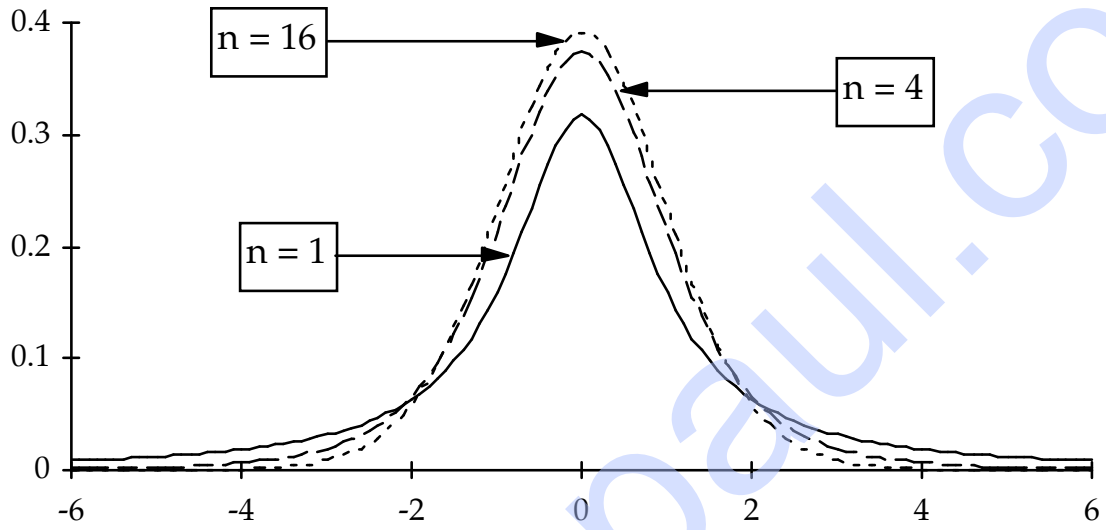
$$f(t)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\;\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

The mean and the variance of the t-distribution are

$$E(t_n) = 0$$

$$D(t_n) = \frac{n}{n-2}$$

The following figure shows the density function for different values of the degrees of freedom:



If random variables X_1, X_2,..., X_n have the normal distribution $N(\mu,\sigma^2)$ and they are **independent** then

$$\overline{X} \rightarrow N(\mu, \sigma^2/n)$$

and

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \rightarrow N(0,1)$$

then

$$\frac{\overline{X}-\mu}{S}\sqrt{n} = \frac{\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)}{\sqrt{\frac{\frac{S^2}{\sigma^2}(n-1)}{n-1}}} \rightarrow t_{n-1}$$

Student's t-distribution has a wide usage

Fisher-Snedecor‘s Distribution - F-distribution

Snedecor's F-distribution arises as the ratio of two chi-squared distributions divided by their respective degrees of freedom. The F-distribution has two parameters, n - the degrees of freedom of the chi-squared random variable in the numerator and m - the degrees of freedom of the chi-squared random variable in the denominator.

$$F_{n,m} = \frac{\frac{\chi_n^2}{n}}{\frac{\chi_m^2}{m}}$$

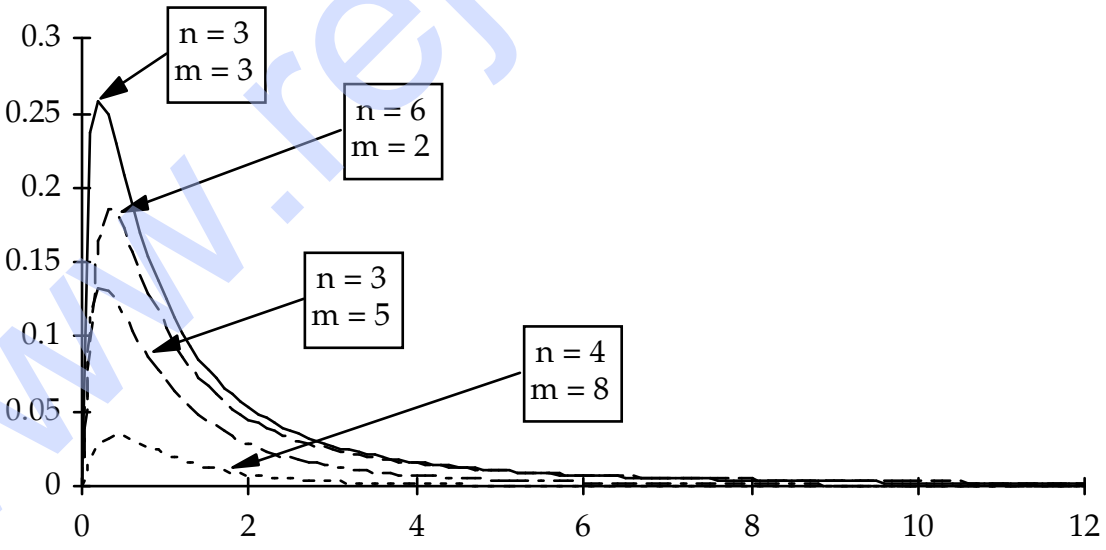
The probability density function of this random variable is

$$f(x) = \frac{\Gamma\left(\frac{n+m}{2}\right) n^{\frac{n}{2}} m^{\frac{m}{2}} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)(m+nx)^{\frac{n+m}{2}}}$$

The mean and variance of the t distribution are

$$E(F_{n,m}) = \frac{m}{m-2} \qquad D(F_{n,m}) = \frac{2m^2\left(1+\frac{m-2}{n}\right)}{(m-2)^2(m-4)}$$

The following figure shows the density function for different values of m a n:



Clearly this distribution would arise as the sampling distribution of the ratio of the sample variances of two independent populations with the same standard deviation σ. / The degrees

of freedom represent one less than the samples sizes of the numerator and denominator sample variances respectively.

$\overline{X_1}; \overline{X_2}$

$X_{1j} \rightarrow N(\mu_1, \sigma); \quad j = 1, n_1$

$X_{2j} \rightarrow N(\mu_2, \sigma); \quad j = 1, n_2$

$$S_i^2 = \frac{\sum_{j=1}^{n_i} \left(X_{ij} - \overline{X}_i\right)^2}{n_i - 1}; \; i = 1, 2$$

$$\frac{S_1^2}{S_2^2} \rightarrow F_{n_1, n_2}$$

We use this distribution for evaluation of statistical analysis results.

Summary

One of the most important continuous distributions is the **normal distribution**. It is a distribution with two parameters – the mean and the variance. The standard normal distribution is achievable with selected parameters (when the mean equals 0 and the variance equals 1).

Chebyshev's inequality puts an upper boundary on the probability that an observation should be far from its mean.

Chí-square distribution is a distribution derived from the sum of squared standard normal random variables.

Central limit theorem describes asymptotic statistical behavior of the mean. We can use it for substituting the binomial (Poisson) distribution by the normal distribution.

Student's t-distribution with *n* degrees of freedom is the ratio of the standard normal random variable to the square root of the chi-squared random variable divided by its degrees of freedom.

F-distribution is the ratio of two chi-squared distributions divided by their respective degrees of freedom.

Quiz

1. Define the relationship between the normal and the standard normal distributions.
2. What is the Chebyshev's inequality?
3. Explain the law of large numbers.

4. Describe the chi-square distribution.

Practical Exercises

Exercise 1: If the mean (μ) height of a group of students equals 170cm with a standard deviation (σ) of 10 cm, calculate the probability that a student is between 160cm and 180cm.

{*Answer:* 0,6828}

Exercise 2: Let X be the "height of a randomly selected male", and suppose that X is normally distributed with μ = 176cm and $\sigma^2 = 25\ cm^2$,i.e. X has N(176,25):

(i) calculate the probability that the height of a randomly selected male is less than or equal to 182cm
(ii) calculate the probability that the height of a randomly selected male is less than or equal to 170cm
(iii) calculate the probability that the height of a randomly selected male is not greater than 176cm
(iv) calculate the probability that the height of a randomly selected male is between 170 and 182 cm
(v) calculate the probability that the height of a randomly selected male is not less than 160cm

{*Answer:* (i) 0.885, (ii) 0.115, (iii) 0.5, (iv) 0.7698. (v) 0.9993}

Exercise 3: You take the heights of 9 males and you assume that the heights are N(176,25) as in the previous exercise. What is the probability that the sample mean height is between 174cm and 178cm?

{*Answer:* 0.7698}

Exercise 4: Premature babies are those born more than 3 weeks before the due date. A local newspaper reports that 10% of the live births in a country are premature. Suppose that 250 live births are randomly selected and the number Y of "preemies" is determined.

(i) What is the probability that X lies in between 15 and 30 (both included)?
(ii) Find the proportion of the event that fewer than 20 births are premature?

{*Answer:* (i) 0.86, (ii) 0.12}

7. INTRODUCTION TO STATISTICAL INFERENCE

Study time: 50 minutes

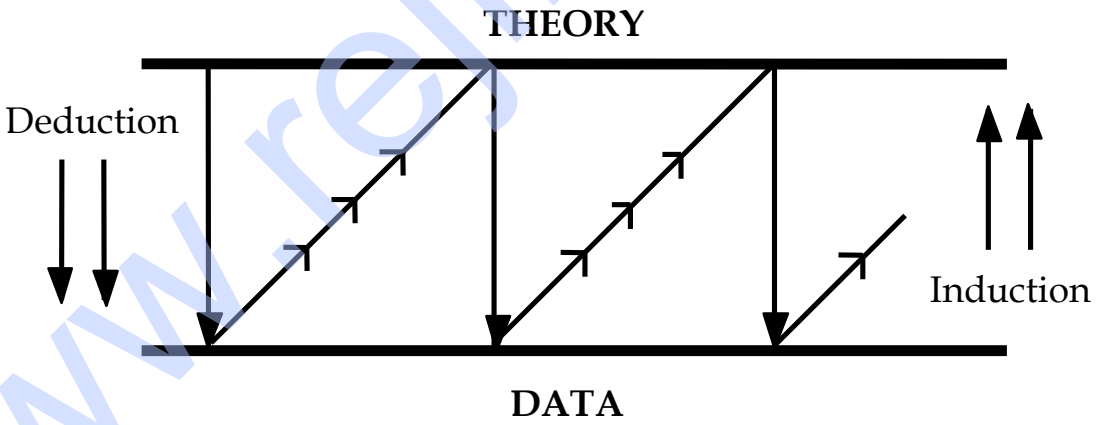
Learning Objectives - you will be able to

- Understand the random sampling term
- Use the sampling distribution and their properties

Explanation

7.1. Introduction – The Scientific Method

Science is a process of systematic learning which proceeds by alternating between inductive and deductive methods of investigation.



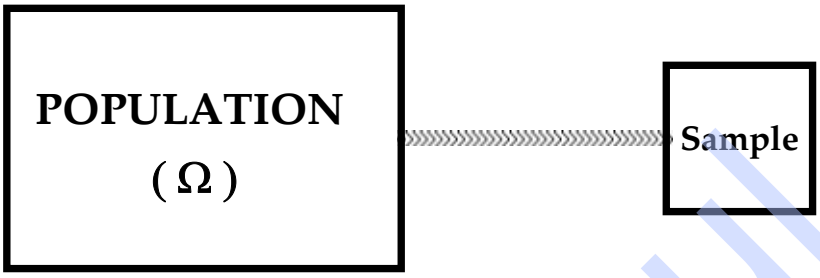
The methods of induction and deduction are the connections between data and theory of a science. The deductive method proceeds in a logically consistent fashion to project what data should result from a particular theory. Induction is an informal process which tries to postulate some theory to reasonably explain the observed data.

Statistics to be a complete science must embody both inductive and deductive methods. The first topic of the course, exploratory data analysis, was an attempt to understand observed distributions of data and was therefore an inductive method. Without some theory of randomness however, the ability of EDA methods to induce precise

explanations was limited. Therefore, we introduced the theory of probability and discussed its applications to various hypothetical sample spaces. Probability theory is the basis of deductive methods of statistics.

Probability theory proceeds by assuming a hypothetical sample space or population on which a probability measure is defined. Probability distributions of random variables defined on this sample space are then derived mathematically. The probability of any sample observation from the hypothetical population can then be determined.

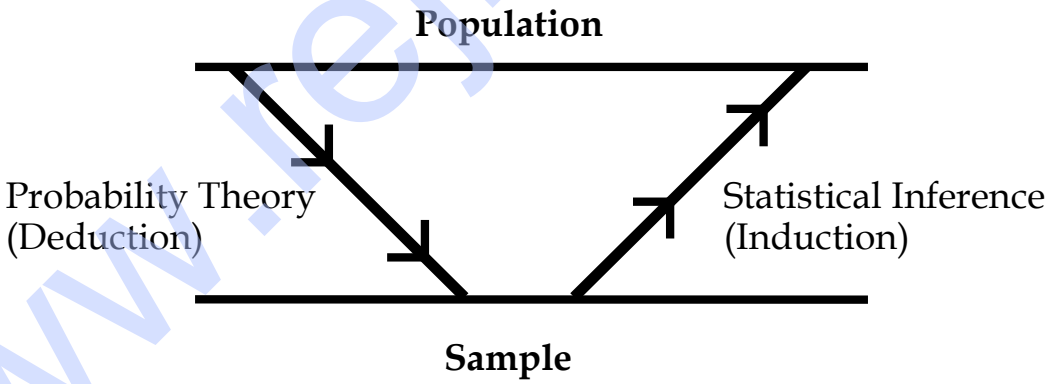
Deductive Theory of Probability



Known Probability Distribution of Population

Sampling Distributions of Sample Statistics

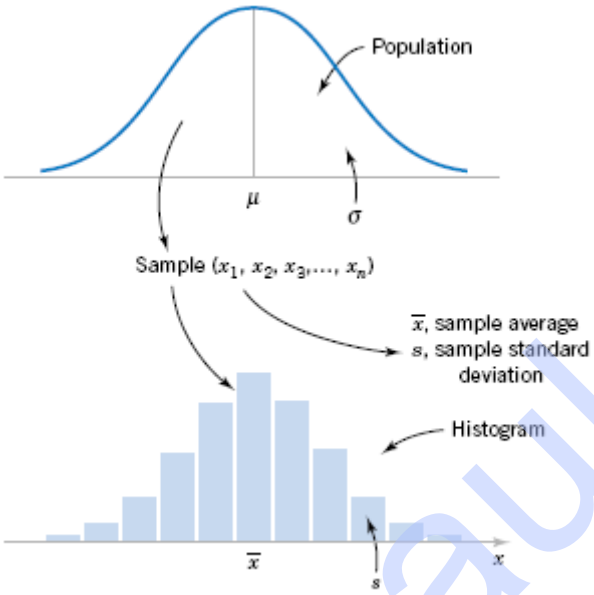
If probability theory is the deductive method of statistics, then by implication, theory in statistical science must be represented by some well-defined population with a known probability distribution and data by the sample drawn from that population. Statistical inference then becomes the inductive methods for using sample data to make inferences about the probability distribution of the population from which the sample was drawn.



Statistical inference then is the inverse of the probability theory. It is the process of making statements about an unknown population on the basis of a known sample from that population.

7.2. Random Sampling

In most statistics problems, we work with a sample of observations selected from the population you are interested in. The following figure illustrates the relationship between the population and the sample.



We have informally discussed these concepts before, however, we are now going to present formal definitions for some of those terms.

Definition:
A **population** consists of the totality of the observations with which we are concerned.

In any particular problem, the population may be small, large but finite, or infinite. The number of observations in the population is called the **size** of the population. For example, the number of undefiled bottles produced on one day by a soft-drink company is a population of finite size. The observations obtained by measuring the carbon monoxide levels every day is a population of infinite size. We often use a **probability distribution** as a **model** for a population.

For example, a structural engineer might consider the population of tensile strengths of a chassis structural element to be normally distributed with mean and variance. We could refer to this as a **normal population** or a normally distributed population. In most situations, it is impossible or impractical to observe the entire population. For example, we could not test the tensile strength of all the chassis structural elements because it would be too time-consuming and expensive. Furthermore, some (perhaps many) of these structural elements do not yet exist at the time a decision is to be made, so to a large extent, we must view the population as **conceptual.** Therefore, we depend on a subset of observations from the population to help make decisions about the population.

Definition:
A **sample** is a subset of observations selected from a population.

For statistical methods to be valid, the sample must be representative of the population. It is often tempting to select the observations that are most convenient as the sample or to exercise judgment in sample selection. These procedures can frequently introduce **bias** into the sample, and as a result the parameter of interest will be consistently underestimated (or overestimated) by such a sample. Furthermore, the behavior of a judgment sample cannot be statistically described. To avoid these difficulties, it is desirable to select a **random sample** as the result of some chance mechanism. Consequently, the selection of a sample is a random experiment and each observation in the sample is the observed value of a random variable. The observations in the population determine the probability distribution of the random variable. To define a random sample, let X be a random variable that represents the result of one selection of an observation from the population. Let $f(x)$ denotes the probability density function of X. Suppose that each observation in the sample is obtained independently, under unchanging conditions. That is, the observations for the sample are obtained by observing X independently under unchanging conditions, say, n times. Let X_i denote the random variable that represents the i-th replicate. Then, $X_1, X_2 \ldots X_n$ is a random sample and the numerical values obtained are denoted as $x_1, x_2, \ldots, x_n$. The random variables in a random sample are independent with the same probability distribution $f(x)$ because of the identical conditions under which each observation is obtained. That is, the marginal probability density function of $X_1, X_2 \ldots X_n$ is

$$f(x_1),\ f(x_2),\ldots f(x_n)$$

respectively, and by independence the joint probability density function of the random sample is

$$f_{X1X2\ldots Xn}(x_1, x_2, \ldots, x_n) = f(x_1)\, f(x_2)\ldots f(x_n).$$

Definition:

The random variables $X_1, X_2 \ldots X_n$ are a random sample of size n if (a) the X_i's are independent random variables, and (b) every X_i has the same probability distribution.

To illustrate this definition, suppose that we are investigating the effective service life of an electronic component used in a cardiac pacemaker and that component life is normally distributed. Then we would expect each of the observations on component life in a random sample of n components to be independent random variables with exactly the same normal distribution. After the data are collected, the numerical values of the observed lifetimes are denoted as $x_1, x_2, \ldots, x_n$.

The primary purpose in taking a random sample is to obtain information about the unknown population parameters. Suppose, for example, that we wish to reach a conclusion about the proportion of people in the United States who prefer a particular brand of soft drink. Let p represent the unknown value of this proportion. It is impractical to question every individual in the population to determine the true value of p. In order to make an inference regarding the true proportion p, a more reasonable procedure would be to select a random sample (of an appropriate size) and use the observed proportion $\hat{p}$ of people in this sample favoring the brand of soft drink.

The sample proportion, $\hat{p}$ is computed by dividing the number of individuals in the sample who prefer the brand of soft drink by the total sample size n. Thus, $\hat{p}$ is a function of the observed values in the random sample. Since many random samples are possible from a

population, the value of $\hat{p}$ will vary from sample to sample. That is, $\hat{p}$ is a random variable. It is called a **statistic.**

Definition:
A **statistic** is any function of the observations in a random sample.

We have encountered statistics before. For example, if $X_1, X_2 \ldots X_n$ is a random sample of size n, the **sample mean** $\overline{X}$ the **sample variance** S^2, and the **sample standard deviation** S are statistics.

Although numerical summary statistics are very useful, **graphical displays** of sample data are a very powerful and extremely useful way to visually examine the data. In the first lecture we presented a few of the techniques that are most relevant to engineering applications of probability and statistics.

7.3. Sampling Distribution

Let's assume that given random sample comes from normal distribution:

$\underline{X}=(X_1, \ldots, X_n)'$, $X_i \to N(\mu,\sigma^2)$

1. $\overline{X_n} = \dfrac{\sum_{i=1}^{n} X_i}{n} \to N\left(\mu, \dfrac{\sigma^2}{n}\right)$ … comes from central limit theorem for large number n

2. $Z_n = \dfrac{\overline{X_n}-\mu}{\sigma}\cdot\sqrt{n} \to N(0,1)$ … comes from a transformation of previous distribution

3. $\dfrac{S_n^2}{\sigma^2}\cdot(n-1) \to \chi^2(n-1)$ … was explained in χ^2 discussion

where

$$S_n^2 = \frac{1}{n-1}\cdot\sum_{i=1}^{n}(X_i-\overline{X})^2 \quad ; \frac{S_n^2}{\sigma^2}\cdot(n-1) = \sum_{i=1}^{n}\frac{(X_i-\overline{X})^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2$$

4. $\dfrac{\overline{X}_n-\mu}{s}\cdot\sqrt{n} \to t_{n-1}$ … was derived in the discussion about using of Student's distribution since:

$$\frac{\dfrac{\overline{X}_n-\mu}{\sigma}\cdot\sqrt{n}}{\sqrt{\dfrac{\dfrac{S^2}{\sigma^2}\cdot(n-1)}{n-1}}} = \frac{\overline{X}_n-\mu}{S}\cdot\sqrt{n}$$

Now assume two samples from the normal distribution
$\underline{X}=(X_1, \ldots, X_n)'$, $X_i \to N(\mu_1,\sigma_1^2)$, $\underline{Y}=(Y_1, \ldots, Y_m)'$, $Y_j \to N(\mu_2,\sigma_2^2)$. Then it holds:

5. $$\frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}} \to N(0,1) \qquad \overline{X} \sim N\left(\mu_1, \frac{\sigma_1^2}{n}\right)$$
$$\overline{Y} \sim N\left(\mu_2, \frac{\sigma_2^2}{m}\right)$$
$$\overline{X}-\overline{Y} \sim N\left(\mu_1-\mu_2, \frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}\right)$$

6. $$\frac{\frac{S_x^2}{\sigma_1^2}\cdot(n-1)}{\frac{n-1}{\frac{S_y^2}{\sigma_2^2}\cdot(m-1)}} = \frac{\frac{S_x^2}{\sigma_1^2}}{\frac{S_y^2}{\sigma_2^2}} \to F_{n-1,m-1}$$ … explained in F – distribution

Now assume that the variances are the same and are unknown: $\sigma_1^2=\sigma_2^2$. Then the following is proved to be true:

7. $$\frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{\sqrt{S_x^2(n-1)+S_y^2(m-1)}}\cdot\sqrt{\frac{n.m.(n+m-2)}{n+m}} \to t_{n+m-2}$$

Summary

The **random sample** is the special random vector whose elements are independent random variables with the same probability distribution.

If the random sample comes from the normal distribution of probability we can derive other significant statistics with known distribution from given random sample, e.g. t-statistics $\frac{\overline{X}_n-\mu}{s}\cdot\sqrt{n} \to t_{n-1}$ or two-sample t-statistics:

$$\frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{\sqrt{S_x^2(n-1)+S_y^2(m-1)}}\cdot\sqrt{\frac{n.m.(n+m-2)}{n+m}} \to t_{n+m-2}.$$

These other statistics will be later used for the construction of interval estimation or for hypothesis testing.

Quiz

1. What is the statistical induction?
2. Characterize the term: random sample.

8. HYPOTHESIS TESTING

Study time: 80 minutes

Learning Objectives - you will be able to

- Conclude by the pure significance test
- Use basic sample and two sample tests
- Conclude by paired test and tests for proportions

Explanation

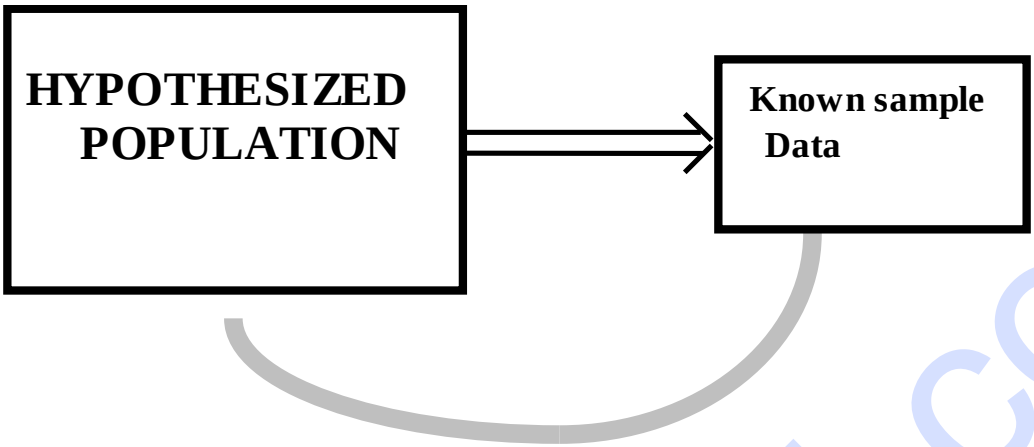
8.1. Introduction

This chapter is going to deal with constructing a test that is used as an aid in accepting or rejecting some population hypotheses.
The most common situation is when the population can be described by some probability distribution which depends on θ parameter. Based on the trial result we can accept or reject an opinion that θ has some concrete value θ_0. In other situation and for hypothesis validity we will be interested whether the given population comes from a certain distribution. Procedures leading to similar decisions are called **significance tests**.
Statistical hypothesis - the assumptions about population of which trueness can be verified by statistical significance tests
Significance tests - procedures which decide if a verified hypothesis should be accepted or rejected based on a random sample
Null hypothesis H_0 – a verified hypothesis of which rejection is decided by the significance test
Alternative hypothesis H_A – a hypothesis which is accepted when the null hypothesis is rejected

8.2. Pure Significance Tests

The pure significance test asks whether the sample result is extreme with respect to some hypothesized distribution.

If the sample data lies at an extremely high or extremely low percentile of the hypothesized distribution, then the hypothesis is in doubt.



Is data consistent with hypothesized population ?

The pure significance test consists of the following components:

1. Null Hypothesis: H_0

– The null hypothesis expresses some belief about the nature of the population. It must be specified precisely enough to define a probability measure on the population.

2. Sample Statistic: T(X)

– The sample statistic is a function of the sample data drawn from the population. The choice of sample statistic is determined by the characteristics of the population's probability distribution with which the null hypothesis is concerned.

3. Null Distribution: $F_0(x)$

$$F_0(x) = P(T(\underline{X}) < x \mid H_0)$$

- The null distribution is the probability distribution of the sample statistic when the null hypothesis is correct. The null hypothesis must be specified precisely enough to determine the null distribution.

4. To determine whether the observed sample statistic $t=x_{OBS}$ is extreme with respect to the null distribution, a statistic known as the p-value is computed. The p-value has 3 definitions depending on the context of the null hypothesis, but in all cases, the interpretation of the p-value is the same.

Definition 1: $P_{VALUE} = F_0(x_{OBS})$

This definition is used when we are concerned that the distribution of the sample statistics may be less than the null distribution.

Definition 2: $P_{VALUE} = 1 - F_0(x_{OBS})$

This definition is used when we are concerned that the distribution of the sample statistics may be greater than the null distribution. Such is the case in our first example.

Definition 3: $P_{VALUE} = 2 \min [F_0(x_{OBS}), 1 - F_0(x_{OBS})]$

This definition is used when we are concerned that the distribution of the sample statistics may be either greater or less than the null distribution. Note that this definition is only applied when the null distribution is symmetric.

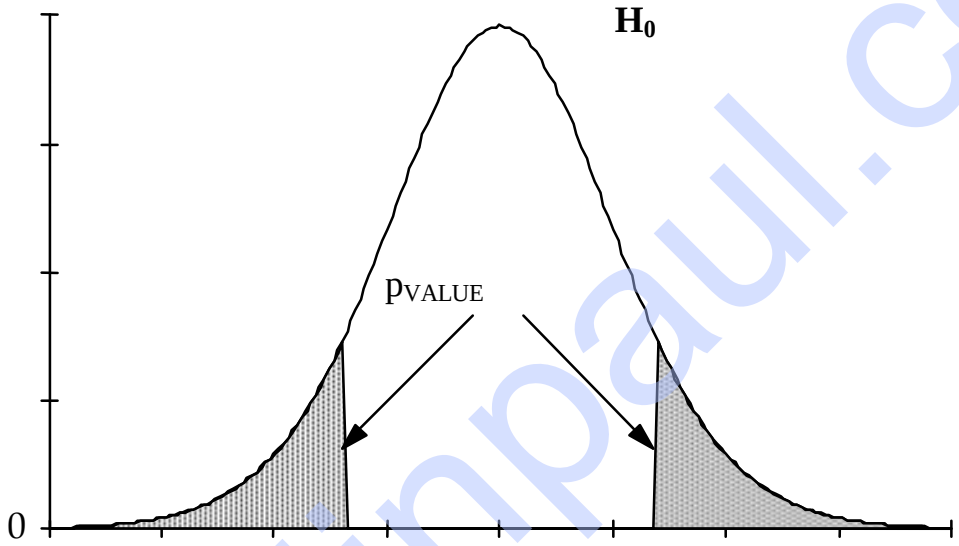


Figure: *Graphical presentation of p_{VALUE} for definition 3 by area below spline of density of the null distribution.*

When the null hypothesis is correct, the distribution of the p-value under all three definitions is uniform. That is,

$$P(p_{VALUE}(\underline{X}) < p | H_0) = p$$

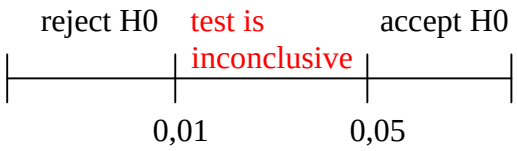
Therefore, the p-value has the same interpretation for all null hypotheses independent of the original null distribution. Clearly, smaller p-values are more extreme with respect to null distribution. Therefore, the smaller the p-value is, the stronger is the evidence of the sample statistic against the null hypothesis. But how small must the p-value be before the evidence is strong enough to reject the null hypothesis? Strictly speaking, this would again depend on the context in which the hypothesis is tested. However, since the weight of evidence against the null hypothesis increases continuously with decreasing p-value, it would be unreasonable to designate a single p-value cut-off point below which the null hypothesis is rejected and above which it is accepted. Rather we should expect an inconclusive region separating accept and reject p-values.

5. Conclusion in terms of p_{VALUE}

$p_{VALUE} < 0{,}01$ reject H_0

$0{,}01 < p_{VALUE} < 0{,}05$ test is inconclusive

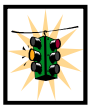
$p_{VALUE} > 0{,}05$ accept H_0



8.3. Alternate Hypothesis

From the definition of the *p-value*, it is clear that the pure significance test procedure for hypothesis testing requires not only a specific null hypothesis but also a notion of which alternative might be correct if the null hypothesis is rejected. The alternate hypothesis does not need to be specified as precisely as the null hypothesis. To select the appropriate definition of the *p-value*, it is only necessary to know the direction of the alternative with respect to the null. However, the alternate hypothesis will also influence the choice of the sample statistic. Those values of the sample statistic which have a small *p-value* under the null hypothesis should tend to have a larger *p-value* for prospective alternatives and vice versa. (Large null *p-values* should have small alternate *p-values*).

8.4. Hypothesis Tests for Mean and Median



Example and Solution 1

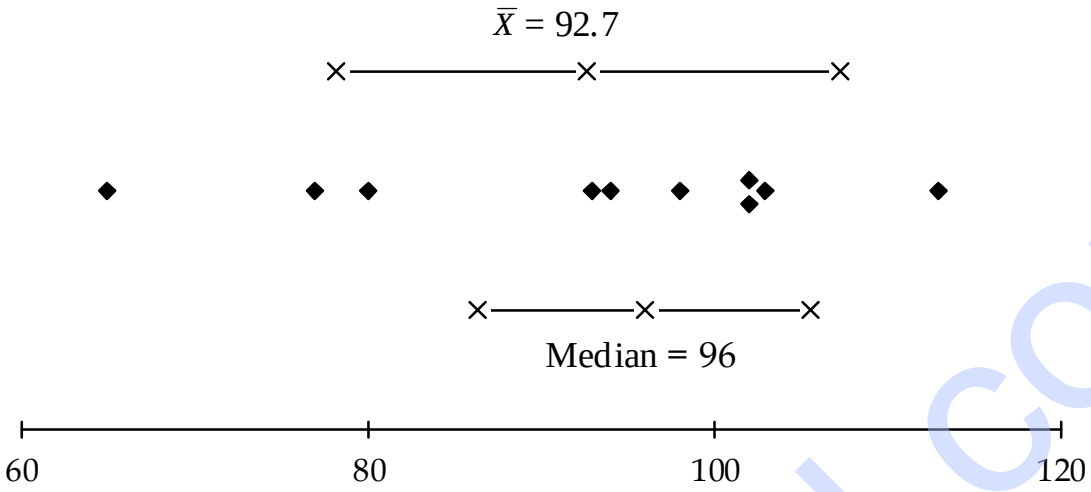
Consider the following ten IQ scores. IQ test scores are scaled to have a mean of 100 and a standard deviation of 15.

65	98	103	77	93
102	102	113	80	94

We wish to test the hypothesis that the mean is 100.

Solution:

We can illustrate this sample:



H_0: X (IQ) has N(100, 15); $\mu_0 = 100$; $\sigma = 15$

Under H_0 $\Rightarrow$ $\bar{X} \to N\left(100, \frac{15}{\sqrt{10}}\right)$

$\bar{X} = 92.7$; $s = 14.51$

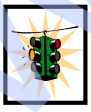
$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{92.7 - 100}{15/\sqrt{10}} = -1.54$$

$p_{VALUE} = \Phi(-1.54) = 0{,}06178$

Accept H_0: $\mu_0 = 100$

Notes: a) When the sample size n is large, the Central Limit Theorem permits the use of this test when the original population is not normally distributed.

b) If σ is not known and the original population is not normally distributed, the sample standard deviation s may be substituted when the sample size is large.



Example and Solution 2

Use the same data as in Example1

Student's test for mean of small samples

H_0: X je N(100, σ); $\mu_0 = 100$; σ is unknown

$\bar{X} = 92.7$; $s = 14.51$

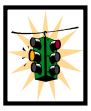
$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{92.7 - 100}{14.51/\sqrt{10}} = -1.59$$

$t \rightarrow t_{n-1} = t_9$

$p_{VALUE} = t_{n-1}(-1.59) = 0{,}073149$

Accept H_0: $\mu_0 = 100$.

Notes: a) When the sample size n becomes larger than about 30, the t-distribution becomes very similar to the normal distribution.



Example and Solution 3

Use the same data as in Example1

Sign test for median

An alternative to testing the hypothesis that the mean equals 100 is to test the median. If the median is m_0, then the probability of any observation exceeding the median is 0.5. Therefore, the number of observations in a sample of n which exceed the hypothesized median will have a binomial distribution with parameters n and 0.5.

H_0: X (IQ) has median m0 = 100

Let Y = number of observations > m_0

$$Y \rightarrow B\left(n, \frac{1}{2}\right) = B(10, \frac{1}{2})$$

$Y = 4$

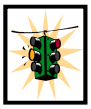
$$p_{value} = P(Y \leq 4)$$

$$= \sum_{k=0}^{4} \binom{10}{k} \frac{1}{2^{10}} = \frac{386}{1024} = 0{,}377$$

The test result shows no inconsistency between the data and the hypothesis.

Notes: a) The following test makes no assumption about the form of the original distribution and can therefore be applied to any distribution.

b) The sign test has lower power than the t test when the original distribution is normal or the z test when the Central Limit Theorem applies but is not affected by departures from these conditions and is not sensitive to outliers.



Example and Solution 4

Use the same data as in Example1

Wilcoxon signed-rank test for medians

A second alternative to tests for sample means based on the normal distribution is to replace the observed values by their ranks and calculate a test statistic from the ranks. To test whether the median is equal to some hypothesized value m_0, we first calculate the absolute difference of each observation from m_0. The absolute differences are then replaced by their ranks or the number of their position. The ranks are then signed -1 if the original observation is less than m_0 and +1 if the original observation is greater than m_0. If the hypothesis that the true median equals m_0 is true, then each rank or integer between 1 and n, the sample size has equal probability of being positive or negative. Therefore, the expected value of the mean of the signed ranks should be zero. Therefore calculating the mean and standard deviation of the signed ranks and forming the z-score as we do for the t test would produce a reasonable test statistic.

H_0: X (IQ) has median $m_0 = 100$

$y_i = |x_i - m_0|$

$r_i = \text{rank}(y_i)$

$r_i^* = \text{sgn}(x_i - m_0)\, r_i = \text{signed rank}(y_i)$

For the observations of IQ scores these results are as follows:

IQ score	Absolute difference y_i	Rank of absolute difference r_i	Signed rank r_i^*
93	7	6	-6
94	6	5	-5
77	23	9	-9
80	20	8	-8
103	3	4	4
113	13	7	7
98	2	2	-2
102	2	2	2
65	35	10	-10
102	2	2	2

For the three observations which have the same absolute difference from the hypothesized median, the average of the three ranks has been assigned.

The test statistic of the ranks is calculated as follows. First calculate the mean and standard deviation of the signed ranks.

$$\bar{r} = \frac{\sum_{i=1}^{n} r_i^*}{n} = -2.5; \quad s_r = \sqrt{\frac{\sum_{i=1}^{n} (r_i - \bar{r})^2}{n-1}} = 5.9675$$

Then calculate the z-score for the mean signed rank remembering that the expected value of the mean signed rank is zero under H_0.

$$w = \frac{\bar{r}}{s_r / \sqrt{n}} = -1.325$$

$$p_{VALUE} = W(-1.325) = \Phi(-1.325) = .09257$$

Accept H_0.

Notes: a) Like the sign test, the Wilcoxon signed rank test makes no assumption about the form of the original distribution. If the original distribution is normal, the Wilcoxon test will have less power than the t test, but will be less sensitive to departures from the assumption of normality. As a general rule for small samples it is reasonable to compute both the usual t test and the Wilcoxon test. If the two tests give very different p-values, this would act as a warning that the original distribution may be seriously non-normal.

b) Because the ranks are fixed pre-determined values, the Wilcoxon statistic will not be sensitive to outliers.

c) Computationally simpler formulas for computing the Wilcoxon test statistic which exploit the fact that ranks are fixed values are given in some books.

8.5. Errors Through Testing

When you make a decision between competing hypotheses, there are two possibilities of being correct and two possibilities of making a mistake. This can be depicted by the following table.

		True situation	
		H_0	H_A
Decision	H_0	OK	Error II
	H_A	Error I	OK

If your decision leans toward H_0 and H_0 is indeed true (true situation), then you did not make an error. If your decision leans toward H_A and H_A is true, then again you did not make an error. These are the probabilities appearing in the upper left and lower right corners.

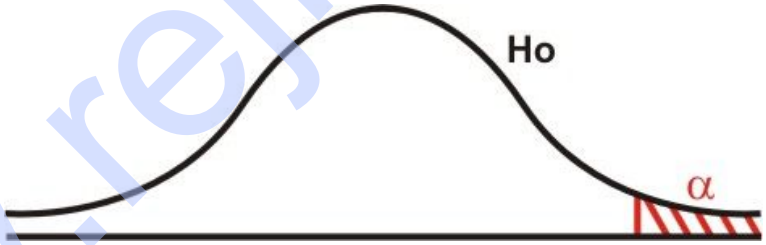
The probabilities in the lower left and upper right corners errors made by not making the correct decisions. These probabilities are designated by the Greek letters α ("alpha") and β ("beta").

α is called a Type I Error. It is the probability of falsely rejecting H_0. It is often referred to as the significance level and it represents the risk one is willing to take in rejecting falsely. The user or researcher has a complete control over α. Typical (and subjective) α -values are 0.05 and 0 .01. If the consequence of a Type I Error is something in the nature of increased risk of death for a patient or increased risk in financial loses, then one would use a level of significance no greater than 0.01.

β is called a Type II Error. It is the probability of falsely accepting H_0. Unlike a Type I Error, it is difficult to quantify β. More will be said about this later. If the consequence of H_A is extremely attractive and if the results of a Type I Error are not catastrophic, it may be advisable to increase the risk of making a Type I Error and use a level of significance that is 0.05 or higher.

Admittedly it is difficult at this time to fully comprehend these concepts. Hopefully things will make more sense when we go more deeply into hypothesis testing.

$$P(\textit{Error I}) = P(P_{VALUE} < \alpha \mid H_0) = \alpha$$



$$P(\textit{Error II}) = P(P_{VALUE} > \alpha \mid H_A) = \beta$$

8.6. Two Sample Tests, Paired Sample Tests and Tests for Proportions

Example and Solution 5

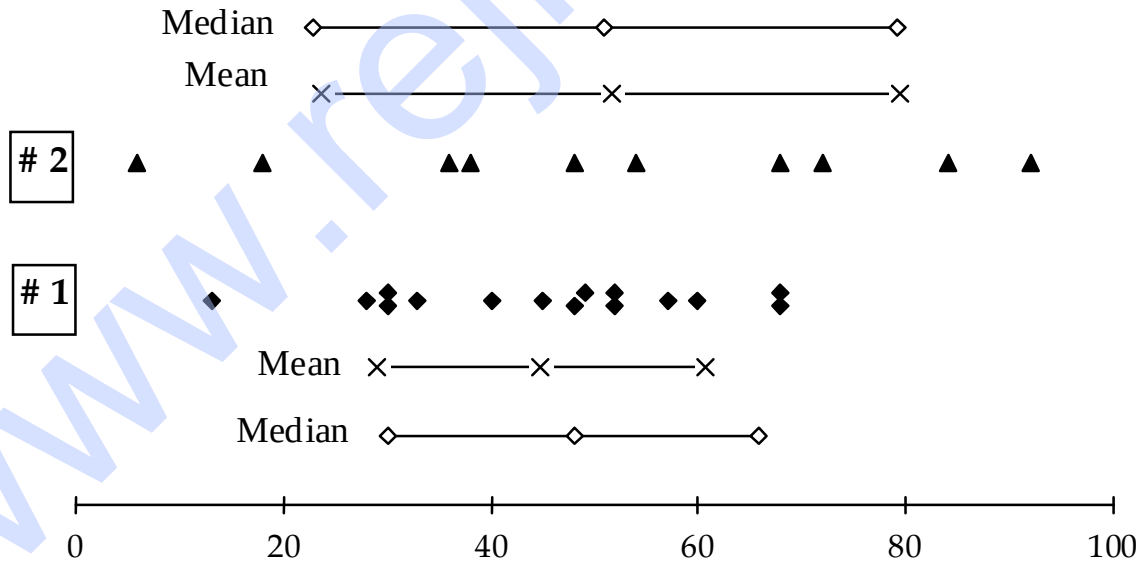
A situation which arises frequently in practice is the two sample test. Two samples have been obtained from different sources and it is necessary to determine whether the two sources have the same mean or median. One source may be a control group and the other an experimental group. For example, to determine the effectiveness of a new teaching method, a controlled experiment may be conducted in which one group of students, the control group, is taught by traditional methods and a second group by the experimental method. The research question in this case is whether the students taught by the experimental method attained higher results.

Sample from population #1

60	49	52	68	68
45	57	52	13	40
33	30	28	30	48

Sample from population #2

38	18	68	84	72
48	36	92	6	54



The sample means and standard deviations are:

$$\#1: \overline{X} = 44.867; \quad s_1 = 15.77$$

$$\#2: \overline{Y} = 51.6; \quad s_2 = 27.93$$

1. We assume that both samples issue from normal distributions:

$$X_i \to N(\mu_1, \sigma_1^2); \quad i = 1,..., n_1 \quad Y_j \to N(\mu_2, \sigma_2^2); \quad j = 1,..., n_2,$$

Let:

$\sigma_1 = 15 \quad \sigma_2 = 25$

Test **H_0:** $\mu_1 = \mu_2$

The test statistics is

$$Z = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \to N(0,1)$$

$$z = \frac{44.867 - 51.6}{\sqrt{\frac{15^2}{15} + \frac{25^2}{10}}} = -0.765$$

$$p_{VALUE} = \Phi(-0.765) = 0.222$$

2. ***Student's t test for difference of means***

The assumption of equality of variance in both populations requites the computation of a single estimate of standard deviation called the pooled sample standard deviation. The pooled standard deviation is the average of squared deviations of all observations from the sample mean of their respective populations. If x_i is the i^{th} sample observation from population #1, and y_j is the j^{th} sample observation from population #2, then the pooled standard deviation is

$$s_p = \sqrt{\frac{\sum_{i=1}^{n_1} (x_i - \bar{x})^2 + \sum_{j=1}^{n_2} (y_j - \bar{y})^2}{n_1 + n_2 - 2}}$$

$$= \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Under the assumption of equal variance in both populations the estimated standard deviation of the difference of sample means will be

$$s_{\bar{x}-\bar{y}} = s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Then the two sample t statistic is computed as:

$$t_{n_1+n_2-2} = \frac{\bar{x} - \bar{y}}{s_{\bar{x}-\bar{y}}} = \frac{\bar{x} - \bar{y}}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

and will have a t distribution with ($n_1 + n_2 - 2$) degrees of freedom.

Although the assumption of equal variances is questionable in light of the difference in sample standard deviations, the two sample t test applied to the IQ data above yields the following result.

$$s_p = \sqrt{\frac{14\left(15.77^2\right) + 9\left(27.93^2\right)}{23}} = 21.37$$

$$t_{23} = \frac{44.867 - 51.6}{21.37\sqrt{\frac{1}{15} + \frac{1}{10}}} = -0.772$$

$$p_{VALUE} = t_{n_1+n_2-2}(-0.772) = 0.224$$

3. ***Mann –Whitney or Wilcoxon rank test for difference of medians***

The two sample rank test is equivalent to ranking the total sample from the two populations and calculating the two sample t test using the ranks rather than the original observations. For the IQ data, this gives the following results.

Ranks for population #1

19	14	15.5	21	21
11	18	15.5	2	10
7	5.5	4	5.5	12.5

Ranks for population #2

9	3	21	24	23
12.5	8	25	1	17

The means and standard deviations of the ranks in each population are

$$\bar{r}_1 = 12.1; \qquad s_{r_1} = 6.29$$
$$\bar{r}_2 = 14.35; \qquad s_{r_2} = 8.89$$

The pooled sample standard deviation of ranks is

$$s_r = \sqrt{\frac{(n_1 - 1)s_{r_1}^2 + (n_2 - 1)s_{r_2}^2}{(n_1 + n_2 - 2)}}$$
$$= \sqrt{\frac{14(6.29^2) + 9(8.89^2)}{23}} = 7.42$$

The test statistic is

$$w = \frac{\bar{r}_1 - \bar{r}_2}{s_r\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = -0.743$$

$$p_{VALUE} = W(-0.743) = \Phi(-0.743) = 0.229$$

Paired sample tests

When testing the effect of some experimental condition or comparing the effects of two different conditions, the experimental design often applies either conditions or both experimental and control conditions to the same sampling units or in this case experimental units. The rationale for this design is that variation in experimental results due to differences in sampling units can be eliminated leaving only measurement variation to obscure the effects of the experimental conditions. However, to secure the benefits of reduction in variation offered by this design, the appropriate methods of data analysis and construction of test statistic must be applied.

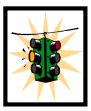
Suppose two observations under different conditions are taken of n sampling units. For example heart rate before and after exercise. Let X_{i0} be the initial observed value for the i^{th} sampling unit and X_{i1} the subsequent observed value for the same sampling unit. Such a design is called a paired sample design. It is possible to analyze this data and test the hypothesis of no difference between the two experimental conditions using the two sample methods discussed above. However, this approach would failure to take advantage of the opportunity to eliminate variation due to differences in individual sampling units.

A statistically more efficient method to analyze this data is to take advantage of the paired nature of the data and create a single value for each sampling unit. In the simplest data model, this value would be the difference of the two observations for each sampling unit.

$$d_i = X_{i1} - X_{i0}$$

The value d_i is the result only of differences in experimental conditions and experimental error. The methods discussed in the section on one sample tests can then be used to test the

hypotheses that the mean or median of d_i is zero which is equivalent to no difference between the two experimental conditions.



Example and Solution 6

Consider the following example of the heart rates of 12 patients at rest and after ten minutes of exercise.

Resting rate	Rate after Exercise	Difference of Rates	Signed Rank of Difference
42	52	10	3.5
173	175	0	1
113	147	34	11
115	83	-32	-10
69	123	54	12
101	119	20	6
94	69	-25	-7
93	123	30	8.5
112	82	-30	-8.5
67	57	-10	-3.5
104	100	-4	-2
76	89	13	5

The sign test is also applicable to this data. For this data, there are 5 negative signs out of 12 observations. If the true median were 0, the number of negative signs has a binomial distribution with parameters n = 12, and p = 0.5 and the probability if this event is:

$$p_{value} = P(Y \le 5) = \sum_{k=0}^{5} \binom{12}{k} \frac{1}{2^{12}} = 0.387$$

For such a small sample with unspecified population variance, we assume that the observations are normally distributed and apply the Student's t-test.

The mean and standard deviation of the paired differences are

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} = 5$$

$$s_d = \sqrt{\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}} = 26.86$$

and the *t*-statistic is

$$t_{11} = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{5}{26.86/\sqrt{12}} = 0.645$$

$$p_{value} = \text{from Definition 2} = t_{11}(0.645) = 0.266$$

Applying the Wilcoxon signed rank test to these data yields the following results. The mean and standard deviation of the paired differences are

$$\bar{r} = \frac{\sum_{i=1}^{n} r_i}{n} = 1.33$$

$$s_r = \sqrt{\frac{\sum_{i=1}^{n} (r_i - \bar{r})^2}{n-1}} = 7.55$$

and the W statistic is

$$W = \frac{\bar{r}}{s_r/\sqrt{n}} = \frac{1.33}{7.55/\sqrt{12}} = 0.611$$

$$p_{value} = 1 - \Phi(0.611) = 0.271$$

Tests for proportions

When testing hypotheses about the proportion of a population having some attribute, the sample size, n, will be large enough in most cases to use the normal approximation to the distribution of the sample proportion. Under the null hypothesis that the population proportion is equal to some specified value,

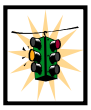
$$H_0: \quad p = p_0$$

The distribution of the sample proportion will be approximately normally distributed for large n.

$$\hat{p} \rightarrow N\left(p_0, \frac{p_0(1-p_0)}{n}\right)$$

and the p-value can be calculated from the z-score of the sample proportion.

$$p_{value} = \Phi\left(\frac{(\hat{p} - p_0)}{\sqrt{\frac{p_0(1-p_0)}{n}}}\right)$$



Example and Solution 7

If the manufacturer's specifications for the defective rate of an item is not to exceed 3%, and 7 defective items are found in a sample of 95, then the p-value for testing the hypothesis that the sampled population meets the manufacturers specification is

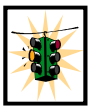
$$z = \frac{(\hat{p} - p_0)}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{7/95 - 0.03}{\sqrt{\frac{(0.03)(0.97)}{95}}} = 2.5$$

$$p_{value} = 1 - \Phi(2.5) = 0.006$$

Reject H_0.

Two sample test for proportions

A two sample test for proportions arises when samples are taken from two populations and the null hypothesis to be tested is that the proportions in both populations are the same. If the samples from each population are large enough, the normal approximation can again to apply to the distribution of the difference of sample proportions. However, since the null hypothesis does not specify a single value for p in each population, the variance is estimated using the total proportion from the samples of both populations which is the maximum likelihood estimate of p under the null hypothesis of equal proportions in both populations.



Example and Solution 8

Let X_1 be the number of items in a sample of n_1 from population # 1 having the attribute and X_2 be the number of items in a sample of n_2 from population # 2 having the attribute. Then

$$\hat{p}_1 = \frac{X_1}{n_1}; \quad \hat{p}_2 = \frac{X_2}{n_2}; \quad \hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

Then under the null hypothesis that the proportions in the two populations are equal,

H_0: $p1 = p2$

the distribution of the difference in sample proportions is:

$$\hat{p}_1 - \hat{p}_2 \rightarrow N\left(0, \hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\right)$$

For example, suppose 12 defective items were found in a sample of 88 from one production run and 8 defective items in a sample 92 from a second run. Then,

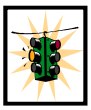
$$\hat{p}_1 = \frac{12}{88};\quad \hat{p}_2 = \frac{8}{92};\quad \hat{p} = \frac{12+8}{88+92}$$

Then the z statistic for testing the hypothesis that the defective rate in both runs is the same is

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{0.136 - 0.087}{\sqrt{(0.11)(0.99)\left(\frac{1}{88}+\frac{1}{92}\right)}} = 1.054$$

$$p_{value} = 1 - \Phi(1.054) = 0.146$$

Accept H_0.



Example and Solution 9

We have two types of floppy disks - Sony and 3M. In any packet are 20 disks. There were found 24 defective disks into 40 Sony packets and there were found 14 defective disks in 30 3M packets. Does difference in the quality of Sony and 3M disks exist?

Solution:

$\hat{p}_1 = \frac{24}{40.20} = 0.030$ (proportion of defective Sony disks)

$\hat{p}_2 = \frac{14}{30.20} = 0.023$ (proportion of defective 3M disks)

$\hat{p} = \frac{24+14}{(40+30).20} = 0.027$

1. H_0: $p_1 = p_2$
   H_A: $p_1 > p_2$

2. We select test statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\hat{p} \cdot (1 - \hat{p}) \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \rightarrow N(0,1)$$

$$\frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\hat{p} \cdot (1 - \hat{p}) \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.030 - 0.023}{\sqrt{0.027 \cdot (1 - 0.027)(\frac{1}{800} + \frac{1}{600})}} = 0.80$$

3. p-value = 1- Φ (P_2) = 1 - Φ (0,80) = 0,21 $\qquad$ Φ (0,80) = 0,79

/

(Z has a standard normal distribution)

4. p-value >>> 0.05 $\Rightarrow$ accept H_0

We can't affirm that there exists statistical important difference in quality of Sony and 3M floppy disks.

Summary

The pure significance test answers the question whether given random sample $\underline{X}$ (its observed values) is or is not extreme in relation to some tested hypothesis about population. It consists of 5 steps. The last step concludes about acceptation or rejection H_0. The hypothesis tests are most often used for mean and median: **Student's test**, **Wilcoxon test** for median.

There can be errors in conclusion of the pure significance test because it is not happening in a real situation. In case of rejecting H_0 while it is true, there is the **type 1 error**. If H_A is accepted but H_0 holds it is the **type 2 error**.

The following tests are the most often used: **Student's test for difference of mean**, Wilcoxon rank test for difference of medians and **paired tests**. There are the most often used the **tests for proportions** in engineering practice.

Quiz

1. How do you get the P_{VALUE}?
2. What is the alternate hypothesis?
3. Characterize two sample tests for proportions.

Practical Exercises

Exercise 1: Suppose you want to show that only children have an average higher cholesterol level than the national average. It is known that the mean cholesterol level for all Americans is 190. You test 100 children only and find that the mean is 198 and the standard deviation is 15. Do you have evidence to suggest that only children have an average higher cholesterol level than the national average?

{Answer: rejecting H_0 you can conclude that children do have a higher average cholesterol level then the national average. *}*

Exercise 2: Nine dogs and ten cats were tested to determine if there is a difference in the average number of days that the animal can survive without food. The dogs averaged 11 days with a standard deviation of 2 days while the cats averaged 12 days with a standard deviation of 3 days. What can be concluded?

{*Answer:* You fail to reject the null hypothesis and conclude that there is not sufficient evidence to suggest that there is a difference between the mean starvations time for cats and dogs.}

9. POINT AND INTERVAL ESTIMATION

Study time: 40 minutes

Learning Objectives - you will be able to

- Explain the properties of the point estimation
- Construct interval estimations for mean, standard deviation and variance

Explanation

9.1. Introduction

The estimation problem is distinguished from hypothesis testing. In hypothesis testing we had a preference towards null hypothesis and only rejected it in face of strong contrary evidence. In the case of estimation, all parameter values or potential hypotheses are equal and we want to choose as our estimates those values which are supported by or consistent with the data. An estimate by itself is just a number. Anyone can make an estimate. To be usable, the accuracy of the estimate must also be known. Therefore in addition to deriving estimates, we must also make some assessment of the error of estimation.

9.2. Interval Estimation

The objective of interval estimation is to find an interval of values which have a high likelihood or probability of containing the true parameter values. The strategy used is to find those values which would have a large p-value if they had been chosen as the null hypothesis, i. e. those parameter values which are not inconsistent with the data. In order to give a probability interpretation to the data, we usually choose a fixed p-value, either one-sided or two-sided depending on whether we want a one or two sided interval, and then include in our interval all parameter values whose p-value for the observed data exceeds the chosen minimum p-value, α. The probability of the sample having a p-value which exceeds the selected p-value is $1-\alpha$, and therefore the probability that the interval so constructed will include the true parameter value is also $1-\alpha$. We call the value $1-\alpha$ the confidence level of the interval.

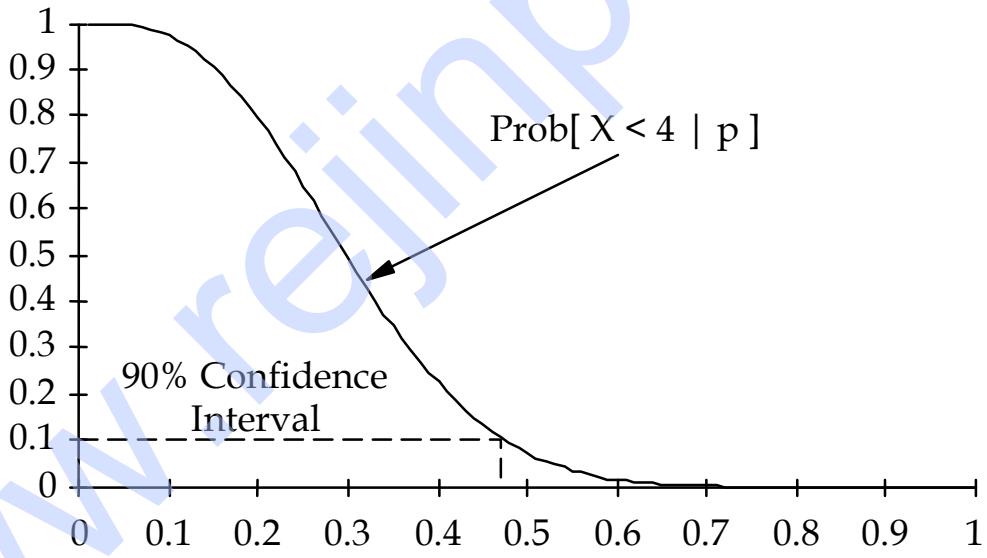
Consider the example of sampling semi-conductor devices to determine the proportion of defective devices produced. In this case, suppose it is a new process and we wish to estimate a maximum value for the proportion of defectives. A sample of 12 devices is selected and tested. Three are found to be defective. Since we are interested in an upper bound, we ask how large the true proportion of defectives can be before our observed sample has a very small probability. For some proportion, p, of defective devices, the probability of obtaining less than 4 defectives in a sample of 12 is

$$\Pr ob\left[X < 4 \middle| p \right] = \sum_{x=0}^{3} \binom{12}{x} p^x (1-p)^{12-x}$$

To obtain a (1-α) upper for p, we find the value of p such that the p-value is exactly α .

$$\alpha = \sum_{x=0}^{3} \binom{12}{x} p^x (1-p)^{12-x}$$

The following diagram illustrates the p-value as a function of the proportion of defectives, finds the value of p whose p-value is α = 0.1, and identifies the 90% upper confidence limit for p.



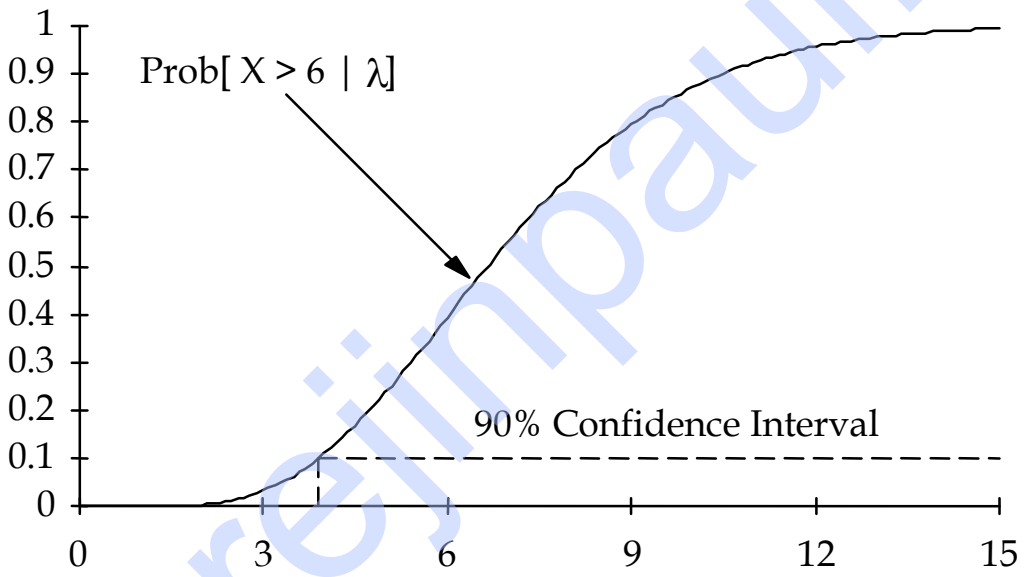
The probability that a population with 47% defectives will have less than 4 defectives in a sample of 12 is 10%. Thus, we are 90% confident that the true proportion of defectives is less than 47%. Another way of expressing this idea is to say that 90% of confidence intervals calculated by this methodology will include the true proportion of defectives.

Consider a second example. A firm which assembles PC's from basic components, loads the software, and tests the system before delivery is interested in estimated how long it takes a worker to complete preparation of a PC for delivery. They observe a worker for 4

hours, one half of his daily work period. In that time, the worker completes 7 PC's. If we assume that the time to complete a single PC is exponentially distributed then the number of PC's completed in 4 hours will have a Poisson distribution. The firm is interested in estimating an upper bound for the mean time to complete a PC or equivalently a lower bound for the rate at which PC's are completed. Therefore we ask how low the rate λ can be before the probability of our sample result, more than 6, has a very small probability. For a given value of λ , the p-value of our sample is

$$\Pr ob\left[X > 6 \middle| \lambda \right] = \sum_{x=7}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!}.$$

The following diagram illustrates the p-value as a function of the completion rate λ, finds the value of p whose p-value is α=0.1, and identifies the 90% upper confidence limit for λ.



If the rate of preparing PC's for delivery is per 4 hour period, then the probability of completing 7 or more PC's in 4 hours is 10%. Therefore, we are 90% confident that the true rate of preparation is at least 3.9 computers per 4 hours, or slightly less than one computer per hour. Alternatively the mean time to complete each PC is no more than 61 minutes 32.3 seconds. This is obviously a conservative estimate since in our sample, computers were completed at the rate of 7 per 4 hours or 1.75 per hour with an equivalent mean preparation time of 34 minutes 17 seconds. To obtain a less conservative estimate at the same confidence level, a larger sample size is required.

This analysis depends on the assumption that the time to complete a PC is exponentially distributed. In practice this is unlikely to be a very good model because in theory according to the exponential distribution, the PC could be completed instantaneously. The Poisson process is a more appropriate model for events which occur randomly such as traffic accidents.

Now consider an example where we wish to estimate both an upper and lower bound for the parameter. In this case, we use the p-value for testing hypotheses against two-sided alternatives. The 1-α confidence interval is the set of all parameter values having a p-value greater than α.

Files transmitted via computer networks are often bundled into groups of files having similar network pathways. Into order to determine the optimal number of files to include in a single bundle, network engineers need some estimate of the distribution of file size. A sample of 15 files is taken with the following result. Sizes are in MB units

4.027	1.887	3.806	7.018	2.753
5.956	8.117	2.857	4.525	7.282
0.140	6.186	5.171	10.558	5.534

The summary statistics for these data are

Mean	5.055	Standard. Deviation.	2.646
Median	5.171	1.483*MAD	2.739
Shorth	3.806	to	7.018

For any hypothetical value of the true mean file size, we can compute the t-statistic for our observed sample.

$$t_{n-1}(\mu) = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

and its associated p-value. Since in this case, we want both upper and lower bounds for our estimate of μ, we use the two-sided definition of p-value.

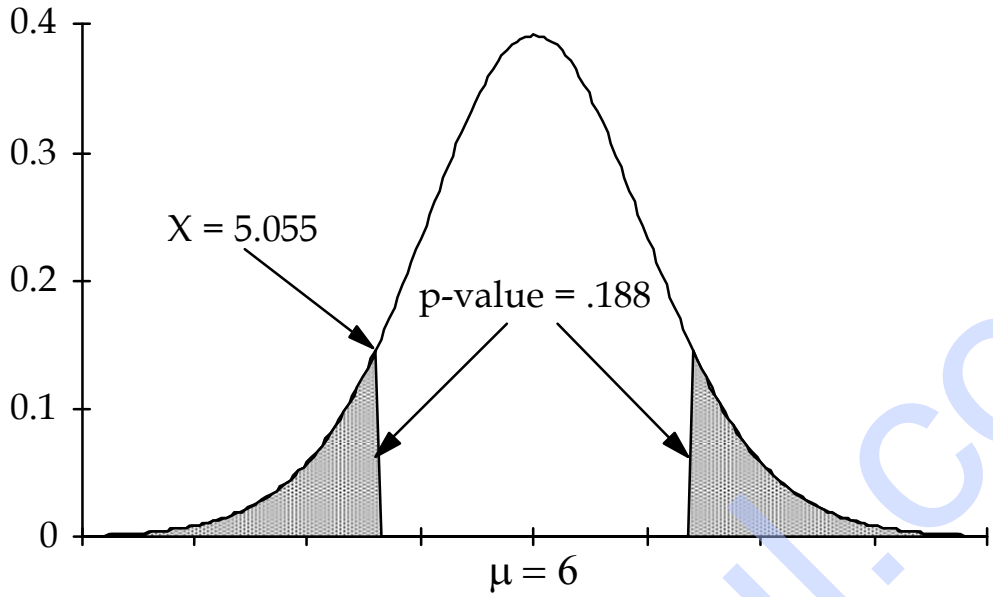
$$p(\bar{x}|\mu) = 2\min\left\{F_{n-1}\left[t(x|\mu)\right], 1 - F_{n-1}\left[t(x|\mu)\right]\right\}$$

For example for a hypothetical value of μ = 6, the observed t-value is

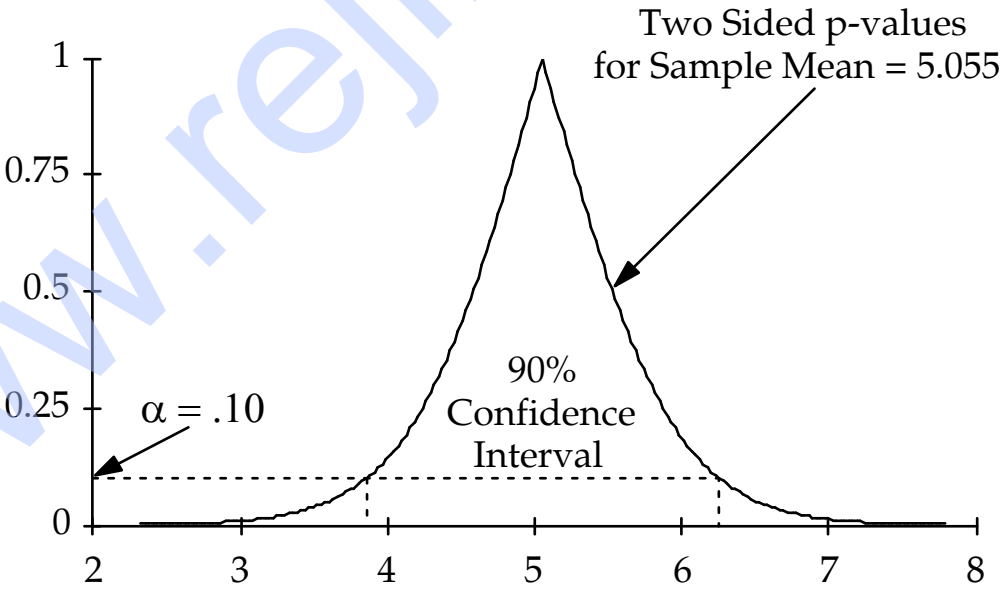
$$t_{14} = \frac{5.055 - 6}{2.646/\sqrt{15}} = -1.384$$

and the associated p-value is

$$p(\bar{x}|\mu) = \Pr ob\left[|t_{14}| \geq 1.384\right] = 0.188$$



Thus, we can determine the p-value associated with any value of μ. The (1-α) confidence interval will consist of all values of whose p-value is greater than α. The following chart shows the two-sided p-value at different values of μ. The p-value reaches its maximum value of 1 when μ = 5.055, the sample mean. If we include in our interval estimate all values of μ having a p-value of at least 10%, then we cane be 90% confident that the interval estimate contains the true mean value in the sense that 90% of interval estimates derived by this procedure contain the true value of the mean. We call such an interval a 90% confidence interval. In this example, the 90% confidence interval for mean file size



is the range (3.852, 6.258). Notice that μ = 6 with a p-value of 0.188 is included in the 90% confidence interval.

9.3. Construction of Confidence Intervals

There is a simple procedure for constructing confidence intervals for parameters whose test statistic has a symmetric distribution, such as the Student's t or the normal. This procedure eliminates the need to compute the p-value for every value of μ. To construct a (1-α) confidence interval, we need only determine the upper and lower bounds of the interval. The lower bound will be that value of μ less than the sample mean whose p-value is exactly α. Therefore the t-value of the sample mean with respect to the lower bound must be equal to the (1-α /2) percentage point of the Student's t distribution.

$$t_{n-1,\alpha/2} = \frac{\bar{x} - \mu_{Lower}}{s/\sqrt{n}}$$

Solving for μ_{Lower}, we have

$$\mu_{Lower} = \bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

Similarly μ_{Upper} must satisfy the equation

$$t_{n-1,\alpha/2} = \frac{\mu_{Upper} - \bar{x}}{s/\sqrt{n}}$$

Hence the (1 - α) confidence interval is given by

$$\left(\mu_{Lower}, \mu_{Upper}\right) \Leftrightarrow \bar{x} \pm t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}$$

In our example of files sizes, the 90% confidence interval is

$$\bar{x} \pm t_{14,.05}\frac{s}{\sqrt{n}} \Leftrightarrow 5.055 \pm 1.761\frac{2.646}{\sqrt{15}} \Leftrightarrow 5.055 \pm 1.203$$

As previously determined graphically, this interval is (3.852, 6.258).

9.4. Sample Size Determination

As well as giving a range of reasonably good parameter values, an interval estimate also provides information about the quality of the estimated values. The quality of an estimate has two aspects,

1) Accuracy

2) Reliability

The accuracy of a interval estimate is equivalent to the length of the interval. The smaller the interval is, the greater the accuracy. Reliability is given by the confidence level of the interval. However, as for Type I and Type II errors of hypothesis tests, accuracy and reliability of a confidence interval are in conflict. For a fixed sample size, the confidence level can only be increased by increasing the length of the interval thereby reducing its accuracy. Increasing both the accuracy and reliability can only be achieved by increasing the sample size.

Determining the sample size required constructing an interval estimate having some fixed reliability and accuracy is a problem which arises commonly in practice. Suppose it is necessary to estimate a mean to an accuracy of Δ with a reliability of (1-α). That is, we require a (1-α) confidence interval of length not exceeding 2Δ. Then the sample size must be chosen large enough to satisfy the following inequality.

$$\Delta \geq t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}$$

and therefore the sample size must exceed

$$n \geq \left[\frac{t_{n-1,\alpha/2}\, s}{\Delta} \right]^2$$

In practice it is common to substitute a conservative estimate for s and substitute the z quantile for the t quantile on the assumption that the required sample size will be large enough that the applicable t distribution will be approximately normal. If we wished to estimate file size to an accuracy of 256 KB, or .25 MB with 90% confidence, using a conservative estimate of 3 for s, we would require a sample size of

$$n \geq \left[\frac{z_{\alpha/2}\, s}{\Delta} \right]^2 = \left[\frac{1.645 * 3}{.25} \right]^2 = 389.67$$

9.5. Point Estimation

Randomness is difficult and unpopular. We are used to have specific answers to questions. An interval of estimates can be unsatisfying. We may ask "What is the single best point or value in the interval?" Such a single value would be a point interval. The single best value is clearly the value which has the highest p-value for the observed data. This point estimate is called the maximum likelihood estimate. But notice that as the p-value increases and the size of the confidence interval decreases, the confidence level decreases accordingly. In most cases, the maximum p-value will be 1, so the confidence level will be zero. That is, the point estimate will never be exactly correct. Therefore in this case it is extremely important to estimate the error of estimation.

While choosing the single point estimate to be the parameter value assigning the maximum p-value to the observed data is a logical consequence of the method of constructing

confidence intervals, it may have indeterminate or non-unique solutions in certain cases, particularly for discrete random variables. Therefore, point estimates are determined by a method similar in spirit to maximizing p-values, the method of maximum likelihood. Rather than maximizing the p-value, maximum likelihood point estimates seek the parameter value which maximizes the probability mass or probability density of the observed sample data.

9.6. Maximum Likelihood Estimation

Because statistics measure general distributional properties such as location and scale, means, medians, and standard deviations can be applied to any distribution. But estimators are associated with parameters for a specific distribution. Developing satisfactory estimators for every individual distributional form would become a daunting task without some general procedure or approach. Fortunately, the idea of likelihood offers such an approach. Intuitively, if the conditional probability or likelihood of the observed data is greater for one parameter value than another, then the first parameter value is a preferred estimate of the population parameter. By extension, the best choice of estimate for the population parameter should be the parameter value whose likelihood is maximum.

$$\hat{\theta} = \max_{\theta} \Pr ob\left[x \mid \theta\right] = \max_{\theta} f(x,\theta)$$

An estimator derived by this criterion is called a maximum likelihood estimator or MLE and is always denoted with a small cap over the parameter symbol as indicated.

Consider the case of sampling for an attribute. If X is the number of items in a sample of size n having the desired attribute, then X will have a binomial distribution with parameters n which is known from the sampling procedure and p which is unknown. The likelihood of p for the observed X is the conditional probability of X given p.

$$f(p \mid x) = \Pr ob[x \mid p] = \binom{n}{x} p^x (1-p)^{n-x}$$

We can find the maximum likelihood estimate of p by finding the point of the likelihood function having zero slope. That is, by solving the following equation.

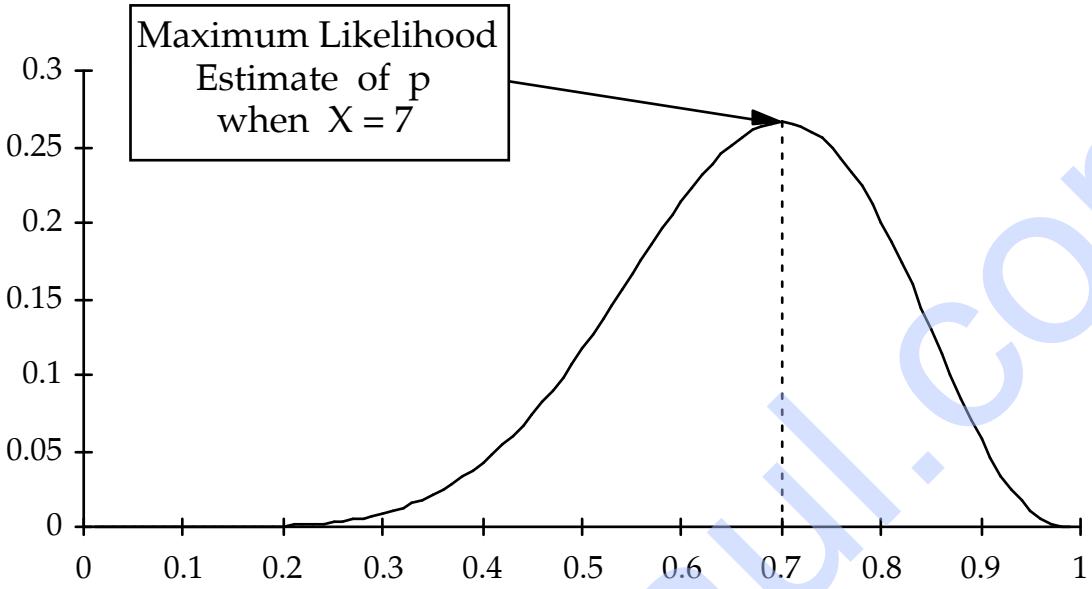
$$\frac{\partial f(p \mid x)}{\partial p} = \frac{\partial \binom{n}{x} p^x (1-p)^{n-x}}{\partial p}$$

$$= \binom{n}{x}\left[xp^{x-1}(1-p)^{n-x} - (n-x)p^x(1-p)^{n-x-1}\right] = 0$$

The solution is the sample proportion. We know from the sampling distribution of this estimator that it is unbiased. It is also consistent, sufficient, and efficient among unbiased estimators.

$$\hat{p} = \frac{x}{n}$$

The maximum likelihood estimate for n = 10 and X = 7 is illustrated below.



The maximum likelihood estimator of λ for the Poisson distribution is derived in the same fashion.

$$f(\lambda \mid x) = \Pr ob[x \mid \lambda] = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

Solving for the value of λ where the slope of the likelihood function is zero

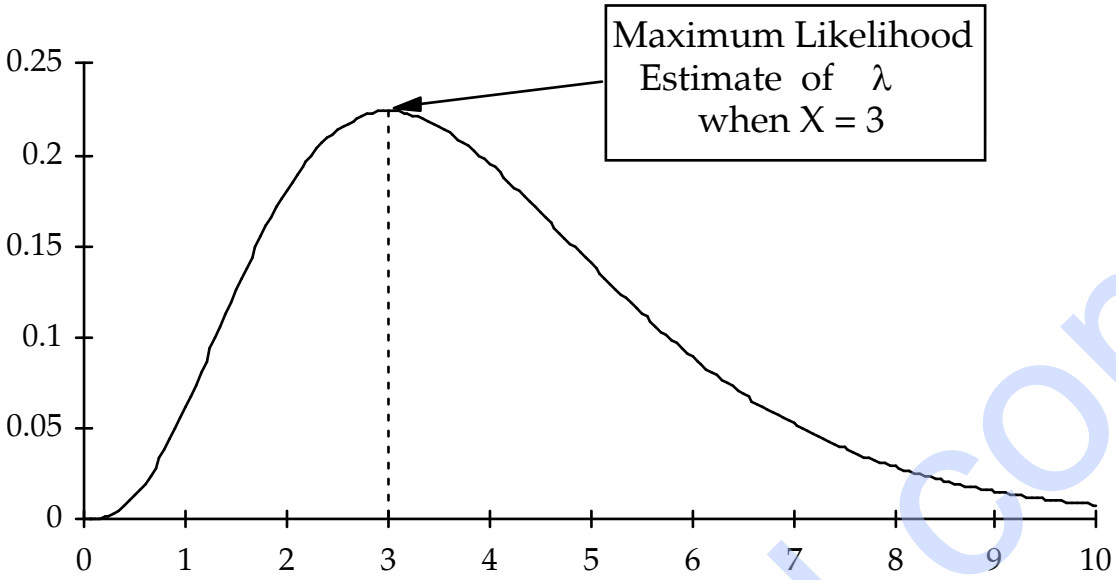
$$\frac{\partial f(\lambda \mid x)}{\partial \lambda} = \frac{\partial \dfrac{(\lambda t)^x e^{-\lambda t}}{x!}}{\partial \lambda}$$

$$= \frac{x(\lambda t)^{x-1} te^{-\lambda t} - (\lambda t)^x te^{-\lambda t}}{x!} = 0$$

we find

$$\hat{\lambda} = \frac{x}{t}.$$

The maximum likelihood estimate of λ when X = 3 and t = 1 is illustrated below.



The maximum likelihood estimator of λ is also unbiased, efficient, consistent, and sufficient for the Poisson distribution.

For the normal distribution, the maximum likelihood estimate of μ is obtained by minimizing the value in the exponent of the density.

$$\min_{\mu} \sum_{i=1}^{n} (x_i - \mu)^2$$

The solution is the sample mean.

9.7. Estimation, Estimators, Properties of Estimator

What is an Estimate or an Estimator?

Formally, an estimator is a statistic defined on the domain of the sample data which is used as an estimate a population parameter. An estimate is the value of that statistic for a particular sample result. Since every statistic has a theoretical probability distribution for every hypothetical probability distribution of the population, it is possible to examine the properties of an estimator through its probability distributions. Since many estimators are based on likelihood functions, many of the properties of the likelihood function will also be exhibited in the probability distribution.

Because of the special requirements of an estimator, criteria particular to the problem of estimation have been proposed as means of evaluating the suitability of a statistic as an estimator, of comparing competing estimators, and of developing new and improved estimators. You will not that the following criteria can only be applied to statistics which are required to be close to some parameter, hence to estimators.

a) Consistency

Consistency is generally agreed to be an essential characteristic of an estimator. As the sample size increases, an estimator which is consistent will have smaller and smaller probability of deviating a specified distance from the parameter being estimated. In the limit of an infinitely large sample, the value of the estimator will be equal to the estimated parameter with probability one. An estimator which does not have this property would give more reliable results for smaller samples and therefore could not be using the information in the sample consistently.

b) Sufficiency

Every statistics is a reduction or summarization of the original sample data and as such discards some information contained in the original sample data. An estimator which is sufficient does not discard any information relevant to the parameter being estimated. This may seem at first to be a rather vague requirement but in fact sufficiency has a very specific probabilistic definition. Any statistic creates a partition of the sample space. All elements within any partition lead to the same value of the statistic. If the relative or conditional probabilities of the individual sample space elements are independent of the parameter being estimated, then the partition and the estimator which created it are sufficient. For example, the binomial random variable partitions the sample space of n Bernoulli trials into subsets all having the same number of successes. The probability of two elements having the same number of successes is always the same no matter what the value of p, the probability of success. Therefore no further information about p can be obtained by knowing which element in the partition actually occurred. Therefore, the sample proportion is a sufficient statistic or estimator of p. We always try to work with sufficient statistics.

c) Bias

Bias or unbiasedness concerns the expected or mean value of the statistic. The statistic should be close to the parameter being estimated and therefore its mean value should be near the parameter value. Bias is the difference between the mean of the estimator and the value of the parameter. Bias should be small. If bias is zero, we say the estimator is unbiased. Unbiased estimators are not always preferable to biased ones if they are not sufficient or have larger variance. An estimator need not be unbiased in order to be consistent.

d) Efficiency

As has be said several times and reinforced by the orientation of the preceding criteria, an estimator should be close to the parameter being estimated. Simply being unbiased will not insure that the estimator is close to the parameter. The variance of the sampling distribution of the estimator must be small as well. For two estimators, the estimator whose sum of variance and squared bias is smaller is said to be more efficient. If the two estimators are unbiased, then the estimator with smaller variance is more efficient. If an unbiased estimator has minimum possible variance for all unbiased estimators, then it is said to be efficient.

Summary

From a methodological point of view we use two kinds of parameters estimations. It is a **point estimation** where distribution parameter is approximated by a number and so called **interval estimation** where this parameter is approximated by an interval where the parameter belongs with a high probability. **Unbiassed, consistent and efficient** estimations of parameters are the most important for a quality of point estimation.

In the case of interval estimation of a parameter we can search for **two-sided** or **one-sided** estimations.

Quiz

1. What is the consistent estimation of parameter?
2. How can you describe 100.(1-α) % two-sided confidence interval for a θ-parameter?

Practical Exercises

Exercise 1: In a random sample of chipsets 10% is not suitable for new quality demands. Find 95% confidence interval for a p-chipset proportion not suitable for a new norm if a sample size is:

a) n = 10
b) n= 25
c) n = 50
d) n = 200

{ *Answer:* a) –0.06<p<0.26, b) 0.00<p<0.20, c) 0.03<p<0.17, d) 0.07<p<0.13}

Exercise 2: Four students were randomly selected from the first parallel group. Their exam results were 64, 66, 89 and 77 points. Three students were randomly selected from the second parallel group. Their exam results were 56, 71 and 53 points. Find 95% confidence interval for difference between mean values of exam results.

{ *Answer:* (-4;32)}

10. ANOVA – One Factor Analysis of Variance

Study time: 60 minutes

Learning Objectives - you will be able to

- Explain the structure of *F-ratio*
- Conclude by the test named Analysis of Variance
- Construct the ANOVA table
- Carry out the post hoc analysis

Explanation

10.1. Introduction

We discussed one-sample and two-sample mean tests in the previous chapters. The analysis of variance (ANOVA) is an extension of those tests. It enables to compare any mean of independent random samples. The analysis of variance (in its parametric form) assumes normality of the distributions and homoscedasticity (identical variances). If these conditions are not executed then we must use nonparametric *Kruskall-Wallis test*. It is an analogy of the one-factor sorting in the analysis of the variance. It doesn't assume distributions normality but its disadvantage is smaller sensitivity.

10.2. Construction of the F-Statistic

Let's have *k*-random samples that are independent on one another. These samples have the standard distribution with the same variation:

$$(X_{11}, X_{12}, ..., X_{1n_1}) \rightarrow N(\mu_1, \sigma^2)$$
$$(X_{21}, X_{22}, ..., X_{2n_2}) \rightarrow N(\mu_2, \sigma^2)$$
$$...$$
$$(X_{k1}, X_{k2}, ..., X_{kn_k}) \rightarrow N(\mu_k, \sigma^2) \ ,$$

$$\sum_{i=1}^{k} n_i = N$$

Let n_i ... number of observations in i-th sample.

***Formulation of the problem*:**

The hypothesis of interest is H_0: $\mu_1 = \mu_2 = ... = \mu_k = \mu$

The alternate hypothesis is: H_A: At least two μ_i's are different.

We want to determine H_0 in one test because we try to find a test statistic that enables not only H_0 implementation but it is also sensitive on the H_0 validity.

Define the **total sum of squares** (or **total variability**) as

$$SS_{TOTAL} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X})^2 \text{ , where } \overline{X} \text{ is the mean of all observations.}$$

- the total sum of squares is the raw measure of variability in the data

This total sum of squares has 2 components:

$$SS_{TOTAL} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X})^2 \Rightarrow SS_{TOTAL} = SS_W + SS_B \quad ,$$

where

SS_W ... the within group variation (the sum of squares within groups) - is the raw variability within samples

$$SS_W = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2 = \sum_{i=1}^{k}(n_i - 1)S_i^2$$

- the degrees of freedom is equal to the sum of the individual degrees of freedom for each sample. Since each sample has degrees of freedom equal to one less their sample sizes, and there are k samples, the total degrees of freedom is k less than the total sample size: N - k

S_i is a sample standard deviation of *i*-th random sample:

$$S_i = \sqrt{\frac{\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2}{n_i - 1}}$$

and

$$\overline{X}_i = \frac{\sum_{j=1}^{n_i} X_{ij}}{n_i}$$ is a the sample mean in the *i*-th sample.

SS_B ... the between group variation (the sum of squares between groups) - is the raw variability between samples:

$$SS_B = \sum_{i=1}^{k} n_i \cdot (\overline{X}_i - \overline{X})^2$$

The within group variance (mean square within groups): $S_W^2 = \frac{SS_W}{N-k}$

The between group variance (mean squares between groups): $S_B^2 = \frac{SS_B}{k-1}$

Properties of these variances:

1. $ES_W^2 = \frac{1}{N-k} E(\sum_{i=1}^{k} (n_i - 1) S_i^2) = \frac{1}{N-k} \sum_{i=1}^{k} (n_i - 1) E(S_i^2) = \sigma^2$

because $E(S_i^2) = \sigma^2$.

The within mean square is an unbiased estimate of the variance, independent of H_0.

2. $ES_B^2 = \sigma^2 + \frac{1}{k-1} \sum_{i=1}^{k} n_i (E\overline{X} - E\overline{X}_i)^2$

$ES_B^2 = \sigma^2 \Leftrightarrow$ when H_0 is true

Therefore the ratio of the two sums of squared variances divided by their degrees of freedom will have an F distribution under the hypothesis of equal population means.

$$F = \frac{SS_B / k-1}{SS_W / N-k} = \frac{S_B^2}{S_W^2}$$

Definition:
We call this F statistic as ***F-ratio***.

Why is useful use *F*-ratio as the test statistic?

We see that if H_0 is true then *F-ratio* is any random number close to 1 ... $F \approx 1$. If H_0 is false then this number is markedly bigger than 1 (see property 2). The statistic *F-ratio* is sensitive to validity of the hypothesis H_0. So we can use it during the following testing as test statistic and we have to determine its statistical behavior which means to determine its propability distribution.

We know that $\frac{S_w^2}{\sigma^2} \cdot (N-k) = \sum_{i=1}^{k} \frac{(n_i - 1) S_i^2}{\sigma^2} \to \chi^2(N-k)$,

because $\frac{(n_i - 1) S_i^2}{\sigma^2} \to \chi^2(n_i - 1)$, and further it is known that the sum of random variables $\chi^2(n_i - 1)$ is again a random variable of the same type with degrees of freedom number same as summarized variables.

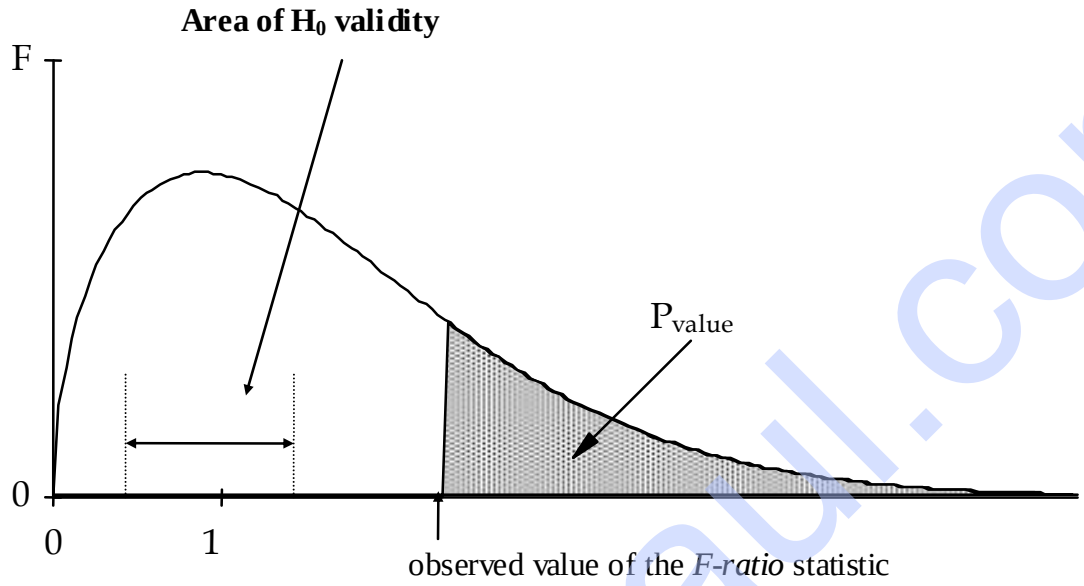
If H_0 is true then: $\frac{S_B^2}{\sigma^2} \cdot (k-1) = \frac{1}{\sigma^2} \sum_{i=1}^{k} n_i \cdot (\overline{X}_i - \overline{X})^2 = \sum_{i=1}^{k} (\frac{\overline{X}_i - \overline{X}}{\frac{\sigma}{\sqrt{n_i}}})^2 \to \chi^2(k-1)$

Then we know (Fisher-Snedecor distribution) that the following ratio:

$$\frac{\frac{S_B^2}{\sigma^2} \cdot (k-1)}{k-1} \Big/ \frac{\frac{S_w^2}{\sigma^2} \cdot (N-k)}{N-k} = \frac{S_B^2}{S_W^2} = F_{k-1, N-k}$$

must have *F-distribution* with *(k-1)* and *(N-k)* degrees of freedom.

If we know a F-ratio statistical behavior we can use it for analysis and determination of previously stated problem in H_0. Following figure illustrates how *F-ratio* is used to determine a hypothesis H_0 validity.



10.3. ANOVA Table

We summarize the data in an ANOVA table:

Source	Sum of squares	Degrees of freedom	Mean squares	*F-ratios*	P-value
total	$SS_{TOTAL} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X})^2$	$N-1$			
between	$SS_B = \sum_{i=1}^{k} n_i \cdot (\overline{X}_i - \overline{X})^2$	$k-1$	$S_B^2 = \frac{SS_B}{k-1}$	$F = \frac{S_B^2}{S_W^2}$	see definition
within	$SS_W = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i)^2$	$N-k$	$S_W^2 = \frac{SS_W}{N-k}$		

Analysis of variance table - ANOVA

The big values of F-ratio indicate small values of p_{value} which means rejection of H_0. The F-ratio value will be a big number if the within group variation is a negligible part of the total variability and equivalently if the between variation is a significant part of the total variability.

10.4. Examples and Solutions

We assume three data sets for illustration of *F-ratio* statistical behavior. In each data set, the sample means are the same but the variations within groups differ. When the within group variation is small then the the *F-ratio* is large. When the within group variation is large then the *F-ratio* is small. The examples illustrate three cases: a small within group variation; a normal within group variation; and a large within group variation.

Example 1:

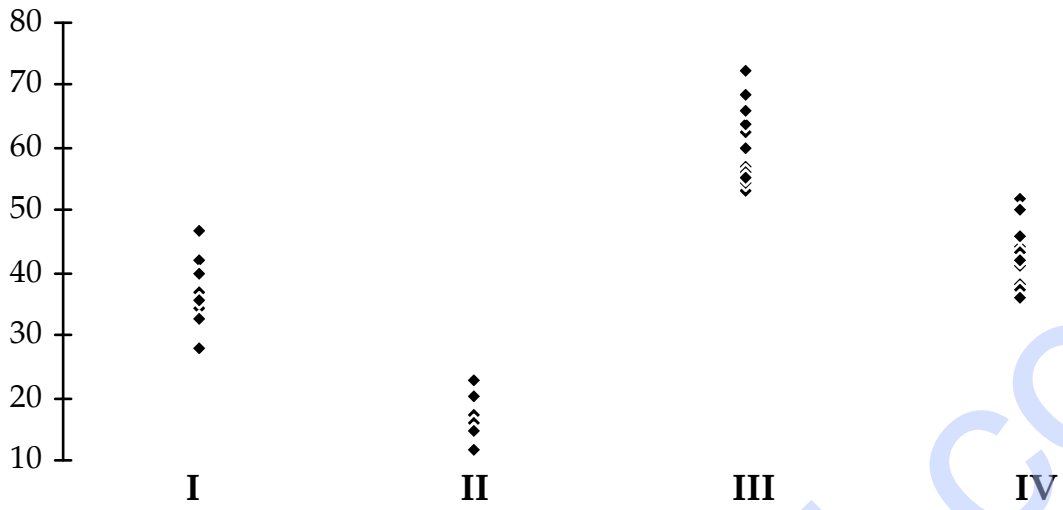
Small within group variation

Groups	I	II	III	IV
	42	17.5	68.5	38
	34.5	12	72	44
	32.5	16	53	52
Data	40	15	64	50
	46.5	20.5	57	43.5
	28	23	56	41
	37	15	54.5	42
	35.5		62.5	46
			63.5	37.5
			60	36
			66	
			55	
Sample size	8	7	12	10
Group means	37	17	61	43
Group standard deviations	5.78	3.71	6.06	5.27

ANOVA Table

	Degrees of freedom	Sum of squares	Mean squares	*F-ratio*
total	36	9872.7027		
between	3	8902.7027	2967.57	100.96
within	33	970	29.39	

P-value = 0.0000



Example 2:

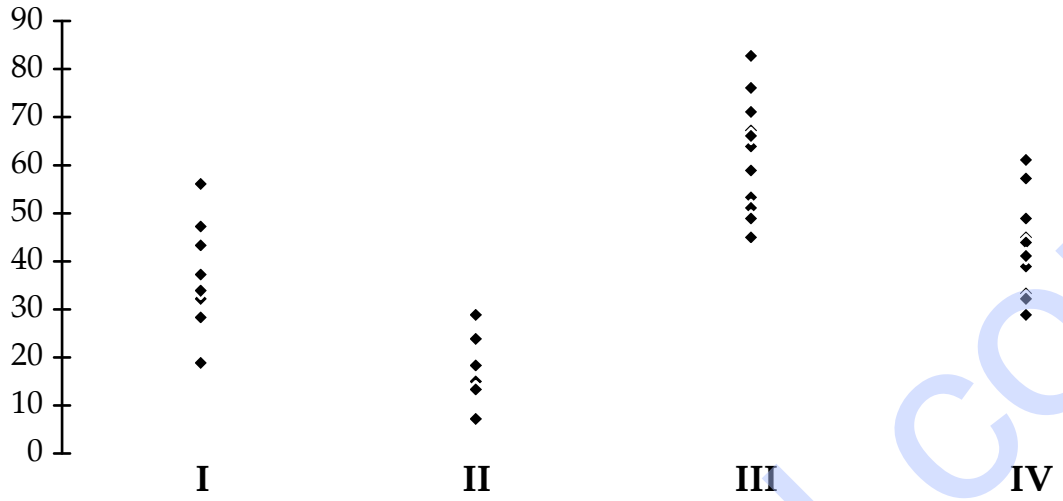
Normal within group variation

Groups	I	II	III	IV
	47	18	76	33
	32	7	83	45
	28	15	45	61
Data	43	13	67	57
	56	24	53	44
	19	29	51	39
	37	13	48	41
	34		64	49
			66	32
			59	29
			71	
			49	
Sample size	8	7	12	10
Group means	37	17	61	43
Group standard deviations	11.56	7.42	12.12	10.53

ANOVA table

	Degrees of freedom	Sum of squares	Mean squares	*F-ratio*
total	36	12782.7027		
between	3	8902.7027	2967.57	25.24
within	33	3880	117.58	

P-value = 0.0000



Example 3:

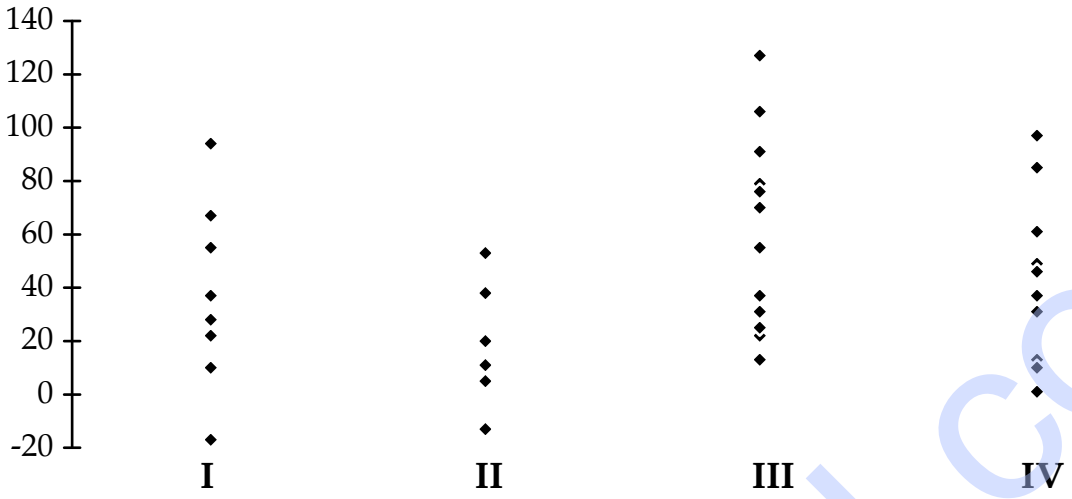
Large within group variation

Groups	I	II	III	IV
	67	20	106	13
	22	-13	127	49
	10	11	13	97
Data	55	5	79	85
	94	38	37	46
	-17	53	31	31
	37	5	22	37
	28		70	61
			76	10
			55	1
			91	
			25	
Sample size	8	7	12	10
Group means	37	17	61	43
Group standard deviations	34.69	22.25	36.36	31.59

ANOVA table

	Degrees of freedom	Sum of squares	Mean squares	*F-ratio*
total	36	43822.7027		
between	3	8902.7027	2967.57	2.804
within	33	34920	1058.18	

P-value = 0.0549



10.5. Post Hoc Analysis

A large *F-ratio* indicates only that some differences exist among the group means, but not where those differences occur. If the *F-ratio* is large, our analysis would be incomplete without identifying which group means differ. This process is called **post hoc** analysis, and consists of comparing the means of all pairs of samples to determine if there is a difference of means.

Several methods are available for post hoc multiple comparisons. We will discuss the simplest method here called Least Significant Differences. The Least Significant Difference or **LSD-method** consists of applying the two-sample *t* test to every pair of sample means. However, there is one adjustment - the square root of mean square within is used rather than the pooled standard deviation from the two samples as the estimate of population standard deviation. Thus for any pair of sample means, the LSD is computed as:

$$(LSD)_{i,j} = \frac{\overline{X}_i - \overline{X}_j}{S_W \cdot \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}} \rightarrow t_{N-k}$$

where $S_W = \sqrt{S_W^2} = \sqrt{\frac{SS_W}{N-k}}$.

This statistic has the Student distribution with *N-k* degrees of freedom.

The LSD method is illustrated for the three examples given previously.

Example 1: Small within group variation

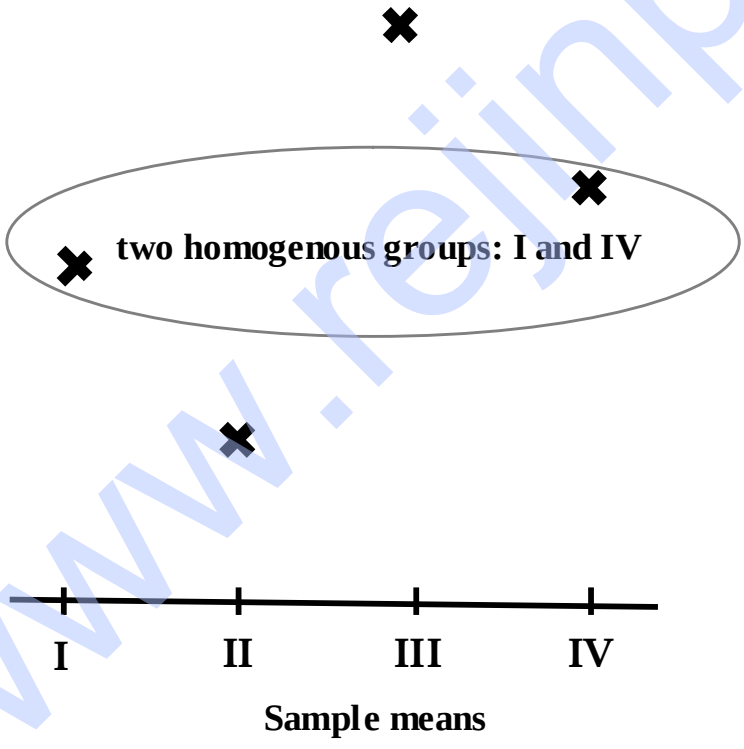
We determine $(LSD)_{i,j}$ for all pairs of given four groups and the obtained values recorded in the following table:

Sample		8	7	12	10
sizes		I	II	III	IV
8	I	0	-7.128	9.698	2.333
7	II	7.128	0	17.064	9.731
12	III	-9.698	-17.064	0	-7.754
10	IV	-2.333	-9.731	7.7541	0

In this case, there is a very strong evidence of differences between all groups except I and IV where the evidence is weaker.

Example 2: Normal within group variation

Sample		8	7	12	10
sizes		I	II	III	IV
8	I	0	-3.564	4.849	1.167
7	II	3.564	0	8.532	4.8656
12	III	-4.849	-8.532	0	-3.877
10	IV	-1.167	-4.866	3.877	0



In this case, even though the sample means are the same, there is no evidence of differences between the means of Groups I and IV. Therefore there are essentially three Groups: II; III; and I and IV together.

Example 3: Large within group variation

Since the *F-ratio* for this example is very small, we would normally conclude that there is no evidence against the null hypothesis of equal group means and not proceed further. Any two-sample *t*-test which produces small p-values should be regarded as spurious. However, for illustration, let's have a look at the table of least significant differences.

Sample		8	7	12	10
sizes		I	II	III	IV
8	I	0	-1.188	1.616	0.389
7	II	1.188	0	2.844	1.622
12	III	-1.616	-2.844	0	-1.292
10	IV	-0.389	-1.622	1.292	0

In this example, the only least significant difference which has a small **p-value** is between groups II and III. However, because the overall *F-ratio* was too small, this difference would be disregarded with conclusion that no differences exist between the means of any of the groups.

Note:

Apart from the LSD method there are also other tests that offer similar multiple comparisons like in the post hoc analysis. More flexible methods were also developed and are accessible through the more advanced software (e.g. Duncan test, Tukey test for significant differences, Scheffe test and Bonferoni test). These tests are based on similar decision strategy and that's on setting of a critical difference requested for determination if two sample means from several groups are different. In many cases these tests are much more effective than the LSD method.

10.6. Kruskal-Wallis Test

The *F-ratio* test statistic used in the standard analysis of variance is known to be very sensitive to the assumption that the original observations are normally distributed. Because the test statistic is based on squared deviations from the mean, it can be badly distorted by outliers. For two-sample analysis, the Wilcoxon/Mann Whitney rank test was introduced as a nonparametric alternative which is less sensitive to outliers than the *t* test. For multiple samples, the Kruskal-Wallis rank test can be used for the same purpose. Like the Wilcoxon/Mann Whitney test, the Kruskal-Wallis test substitutes the ranks of the original data values and performs an analysis of variance on the ranks. For the large deviation data of the previous example the ranks for each group are listed in the following table.

Groups	I	II	III	IV
	28	11	36	9.5
Ranks	12.5	2	37	23
of original	6.5	8	9.5	35
data	25.5	4.5	31	32
	34	21	19	22
	1	24	16.5	16.5
	19	4.5	12.5	19
	15		29	27
			30	6.5
			25.5	3
			33	
			14	
Sample size	8	7	12	10
Mean rank	17.6875	10.7143	24.4167	19.35
Standard deviation	11.1674	8.5919	9.6668	10.6538

The test statistic is a modification of calculating the *F-ratio* for the ranks. In this example, the test statistic and p-value are:

K-W test statistics = 7.24325 p-value = 0.0645

The p-value for the Kruskal-Wallis test is slightly higher than for the *F-test*, but the conclusions are the same in both cases. The null hypothesis of equal group means is not rejected.

Summary

Analysis of variance (ANOVA) is an extension of the two-sample tests for means and it enables to compare any mean of independent random samples. ***F-ratio*** is the test statistic in analysis of variance. *F-ratio* statistic is sensitive to validity of the hypothesis H_0, which is formulated as an equality of the samples means. Particular interresults (that we execute during analysis of variance) are recorded into **ANOVA table**. The second step (in ANOVA) is **post hoc** analysis, and it consists of comparing the means of all pairs of groups of purpose to choose homogenous groups. **LSD-statistics** is a criterion for assignment to homogenous groups.

Described procedure ANOVA is sensitive to the assumption that the original observations are normally distributed. If this condition is not executed then we must use nonparametric ***Kruskall-Wallis*** *rank test.*

Quiz

1. Describe construction and statistical behavior of the *F-ratio* statistic.
2. What is the usual output from the analysis of variance?
3. What is the post hoc analysis?

Practical Exercises

Exercise 1: A research assessing dependency of earnings on achieved education has been carried out. In the table there are earnings in thousands of CZK of 7 randomly selected men at each level of education. (B - basic, H - high, U - university).

	B	H	U
1	10.9	8.9	11.2
2	9.8	10.3	9.7
3	6.4	7.5	15.8
4	4.3	6.9	8.9
5	7.5	14.1	12.2
6	12.3	9.3	17.5
7	5.1	12.5	10.1

Do a simple sorting and determine if education influences earnings.

{*Answer:* p-value = 0.057}

Exercise 2: From a large set of homes you randomly selected 5 single-occupant homes, 8 homes occupied by couples, 10 homes with three family members, 10 homes with four family members and 7 homes with five family members. Then their monthly spending for food and drinks per one family member (in CZK) was recorded. Confirm by analysis of variance if a monthly spending for food and drinks depends on a number of family members.

	Spending for one family member (in CZK)				
Number of family members	**1**	**2**	**3**	**4**	**5**
	3.440	2.350	2.529	2.137	2.062
	4.044	3.031	2.325	2.201	2.239
	4.014	2.143	2.731	2.786	2.448
	3.776	2.236	2.313	2.132	2.137
	3.672	2.800	2.303	2.223	2.032
		2.901	2.565	2.433	2.101
		2.656	2.777	2.224	2.121
		2.878	2.899	2.763	
			2.755	2.232	
			3.254	2.661	

{*Answer:* Use suitable software package.}

11. SIMPLE LINEAR REGRESSION

Study time: 60 minutes

Learning Objectives - you will be able to

- Explain a general linear model
- Explain a linear regression model principle
- Use regression analysis results
- Verify a regression model by determination index

Explanation

11.1. Introduction

Mathematical formulation of statistical models

Symbolically, the basic additive formulation of statistical models can be expressed as

$$\underline{Y} = f(X) + \zeta(\varepsilon)$$

where *Y* is the observed value, *f(X)* is the systematic component and ζ(ε) is the random component. This schematic model explicitly identifies three types of variables.

Y – Response, Criterion, dependent Variable (observed value of primary interest)
X– Predictor, Stimulus, Independent Variable (those factors to which the value of the systematic component may be attributed)
ε - random error

Only *Y* and *X* are observable. Random error is always unobservable.
ζ(ε) is always estimated as the residua difference between the estimated systematic component and the observed response, *Y*.

$$\overline{\zeta(\varepsilon)} = Y - \overline{f(X)}$$

Therefore the estimated split of the observed response into its systematic and random components is as much a consequence of the choice of models, f and ζ, and the method of estimation as it is of the observed stimulus and response, X and Y.

11.2. General Linear Model

The general linear statistical model is a special simple case of the schematic statistical models discussed above. The so-called linear statistical model stipulates that the systematic component is a linear combination of the systematic factors or variables, and the random component is the identity function of random error.

- **random component:** $\zeta(\varepsilon) = \varepsilon$
- **systematic component**: $f(X) = \beta_0 + \sum_{i=1}^{p} \beta_i X_i$

Why use a Linear Systematic Function?

Linear systematic components have three fundamental properties which are desirable for statistical models – *simplicity, estimability* and *stability*.

Linear functions represent or give algebraic expression to the simplest kind of relationship. Linear functions postulate either:

- stimulus and response tend to increase and decrease together
- response decreases as stimulus increases

For the simple linear model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

if $\beta_1 < 0$; the relation is negative => Y decreases as X increases
if $\beta_1 > 0$; the relation is positive => Y and X increase together

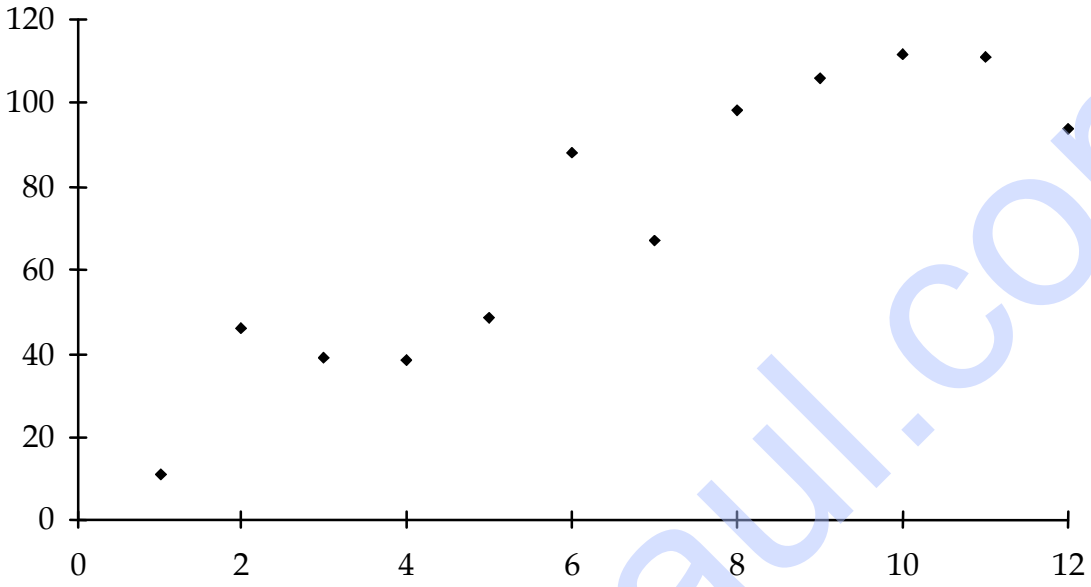
Assumptions about the random component

In decreasing order of impact on results and interpretation, the following three assumptions about the behavior of the random component of a linear statistical model are widely adopted.

1. *Independence* – the random errors ε_i and ε_j are independent for all pairs of observations i and j
2. *Equal Variance* – the random errors ε_i all have the same variance σ^2 for all observations
3. *Normality* – the random errors ε_i are normally distributed

11.3. Estimation of Parameters for the Simple Linear Regression Model

The following scatter plot illustrates the type of data which is typically described by a simple linear model.



From the formulation of the general linear model, the special case of the simple linear model in which the systematic component is a linear function of a single variable, that is a straight line, may be expressed as:

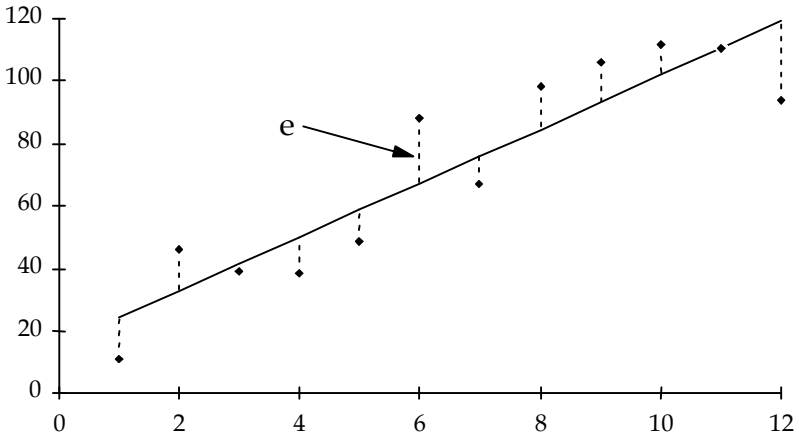
$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$
$$\varepsilon_i \to N\left(0, \sigma^2\right)$$

and all ε_i are mutually independent.

For any estimates of the parameters, β_0 and β_1, say b_0 and b_1, the residual errors of estimation are:

$$e_i = Y_i - b_0 - b_1 X_i$$

as illustrated below.



The least squares parameter estimates are those values of b_0 and b_1 which minimize the sum of squared residual errors.

$$S(b_0, b_1) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i)^2$$

To find the parameter estimates which minimize the sum of squared residuals, we compute the derivatives with respect to b_0 and b_1 and equate them to zero.

$$\frac{\partial S(b_0, b_1)}{\partial b_0} = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i) = \sum_{i=1}^{n} e_i = 0$$
$$\frac{\partial S(b_0, b_1)}{\partial b_0} = \sum_{i=1}^{n} x_i (y_i - b_0 - b_1 x_i) = \sum_{i=1}^{n} x_i e_i = 0$$

The solutions to the above equations are the least squares parameter estimates. Notice that the first equation insures that the residuals for the least squares estimates of β_0 and β_1 always sum to zero.

Because the least squares estimates are also maximum likelihood estimates under the assumption of normally distributed errors, they are usually denoted by the symbols β_0 and β_1. The solutions to the least squares equations are:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = r_{xy} \frac{s_y}{s_x}$$

The intercept parameter, $\hat{\beta}_0$, merely places the vertical position of the line at the point where the residual errors sum to zero. The operative parameter is the slope estimate, $\hat{\beta}_1$, which has a

particularly simple form in terms of the correlation and relative standard deviations of the response *Y* and the explanatory variable *X*.

$$\hat{\beta}_1 = \overbrace{r_{xy}}^{\text{Relation Between X and Y}} \overbrace{\frac{s_y}{s_x}}^{\text{Scale Factor}}$$

The residual sum of squares for the simple regression model is

$$S(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^{n} \left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2 = \left(1 - r_{xy}^2\right)\sum_{i=1}^{n} (y_i - \bar{y})^2 = \left(1 - r_{xy}^2\right)(n-1)s_y^2$$

which like the least squares slope estimate, $\hat{\beta}_1$, has a simple expression in terms of the correlation between *X* and *Y* and the variance of *Y*.

The residual sum of squares for a regression model measures how well the model fits the data. A smaller residual sum of squares indicates a better fit. Because a higher squared correlation between *X* and *Y* is associated with a smaller residual sum of squares as a proportion of the variance of *Y*, the squared correlation between *X* and *Y* is usually used as a measure of the goodness of fit of the regression model. When $r_{xy} = \pm 1$, the sample observations of *X* and *Y* all lie on a straight line and the residual sum of squares is zero. When $r_{xy} = 0$, *X* and *Y* are independent and the residual sum of squares will equal the sum of squared deviations of *Y* about its mean.

If the residual sum of squares measures the size of the random component of the regression model, then the remainder, the difference between the original sum of squared deviations of *Y* about its mean and the residual sum of squares of *Y* about the regression line must represent the systematic component of the model. To better understand what this systematic component measures, let the point on the regression line or predicted value of *Y* for the i^{th} observation of *X* be

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i .$$

Firstly, note that the least squares estimates of $\hat{\beta}_0$ and $\hat{\beta}_1$ insure that the mean of the predicted value of *Y* will always equal the mean of the original observations of *Y*. That is,

$$\frac{\sum_{i=1}^{n} \hat{y}_i}{n} = \frac{\sum_{i=1}^{n} \bar{y} - \hat{\beta}_1(x_i - \bar{x})}{n} = \bar{y} .$$

Then as was the case in the analysis of variance, the total sum of squared deviations of *Y* from its mean

$$SS_{Total} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

can be partitioned into the sum of squared residual errors,

$$SS_{Error} = \sum_{i=1}^{n}\left(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

and the sum of squared deviations of the predicted values of Y from their mean.

$$SS_{\text{Re}\,gression} = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$$

We see that

$$SS_{Total} = \sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$
$$= SS_{\text{Re}\,gression} + SS_{Error}$$

The sum of squares due to regression is often called the explained variation and conversely the sum of squared residual errors, the unexplained variation. The partitioning of the total variation of Y into these two components is due to the fact that the least squares estimates must satisfy the condition,

$$\sum_{i=1}^{n} x_i e_i = 0$$

That is, the residual errors must be orthogonal to the predictor variable.

The partitioning of the total sum of squared deviations of the response, Y, about its mean into the systematic component, explained variation, and the random component, sum of squared residuals is frequently presented as an Analysis of Variance table. The F-test computed by this ANOVA Table tests the null hypothesis that the systematic component of the model is zero.

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F-ratio
Total	n-1	$(n-1)s_y^2$		
Regression	1	$r_{xy}^2(n-1)s_y^2$	$r_{xy}^2(n-1)s_y^2$	$\dfrac{(n-2)r_{xy}^2}{(1-r_{xy}^2)}$
Error	n-2	$(1-r_{xy}^2)(n-1)s_y^2$	$\dfrac{(1-r_{xy}^2)(n-1)s_y^2}{(n-2)}$	

Thus, the F-test for testing the significance of the regression model depends only on the correlation between response and explanatory variables and on the sample size. In practice, the null hypothesis of no regression effect is almost always rejected, but even if rejected does not imply that the regression model will provide satisfactory predictions.

As in the case of analysis of variance for factorial models, the estimate of the error variance, σ^2, is the mean squared error.

$$\hat{\sigma}^2 = SS_{Error}/n-2 = \left(1-r_{xy}^2\right)\left(\frac{n-1}{n-2}\right)s_y^2$$

This estimated error variance for the regression line is also called the conditional variance of *Y* given *X*, that is, the variance of *Y* remaining after the effect of *X* has been removed.

$$s_{y|x}^2 = \hat{\sigma}^2 = \left(1-r_{xy}^2\right)\left(\frac{n-1}{n-2}\right)s_y^2$$

A second consequence of least squares estimates of β_0 and β_1 is that the least squares line will always pass through the point of means $\left(\overline{X},\overline{Y}\right)$. In fact the z-value of the prediction for *Y* is simply the correlation between *X* and *Y* times the corresponding z-value for *X*. That is,

$$\hat{y}_i = \bar{y} - \left(r\frac{s_y}{s_x}\right)\bar{x} + \left(r\frac{s_y}{s_x}\right)x_i$$

$$\left(\hat{y}_i - \bar{y}\right) = \left(r\frac{s_y}{s_x}\right)\left(x_i - \bar{x}\right)$$

$$\left(\frac{\hat{y}_i - \bar{y}}{s_y}\right) = r\left(\frac{x_i - \bar{x}}{s_x}\right).$$

Clearly when $x_i = \bar{x}$, then $\hat{y}_i = \bar{y}$.

11.4. Distribution of Least Squares Parameter Estimates

If the predictor or explanatory variable *X* is assumed to be a fixed constant rather than a random variable, then both $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear combinations of the normally distributed criterion or response variable, *Y*, and hence are normally distributed themselves. The mean and variance of the slope parameter estimate are

$$E\left[\hat{\beta}_1\right] = \beta_1$$

$$V\left[\hat{\beta}_1\right] = \frac{\sigma^2}{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} = \frac{\sigma^2}{(n-1)s_x^2}$$

These results can readily be established by noting that the least squares estimate of β_1 may be expressed as the following linear combination of the observations of *Y*.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{(n-1)s_x^2}y_i$$

Then the expected value of the slope parameter estimate is

$$E(\hat{\beta}_1) = \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{(n-1)s_x^2}\mathrm{E}(y_i) = \sum_{i=1}^{n}\frac{(x_i - \bar{x})(\beta_0 + \beta_1 x_i)}{(n-1)s_x^2}$$

and the variance of the slope estimate is

$$V(\hat{\beta}_1) = \sum_{i=1}^{n}\frac{(x_i - \bar{x})}{(n-1)s_x^2}\mathrm{V}(y_i) = \sum_{i=1}^{n}\frac{(x_i - \bar{x})^2}{(n-1)^2 s_x^4}\sigma^2 .$$

The significance of these results is that the least squares estimate of the slope parameter is unbiased and its variance becomes smaller as the sample size increases. In addition, the variance of the estimate becomes smaller when the variance or range of *X* becomes larger.

By substitution of the mean squared error estimate of σ^2 into the expression of the variance of the slope parameter estimate, the following estimated variance of the slope parameter is obtained

$$\hat{\sigma}^2_{\hat{\beta}_1} = \frac{SS_{Error}}{(n-1)(n-2)s_x^2} = \frac{(1-r_{xy}^2)s_y^2}{(n-2)s_x^2}$$

Because $\hat{\beta}_1$ is unbiased, substitution into the least squares determining equation for $\hat{\beta}_0$ readily shows that the least squares intercept estimate, $\hat{\beta}_0$, is also unbiased.

$$E[\hat{\beta}_0] = \beta_0$$

The variance of $\hat{\beta}_0$ is obtained by again noting that from the least squares determining equation, $\hat{\beta}_0$ can be expressed as the following linear combination of the observations of *Y*.

$$\hat{\beta}_0 = \sum_{i=1}^{n}\left[\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{(n-1)s_x^2}\right]y_i$$

By squaring and summing constant terms in this linear combination, the variance is found to be

$$V\left[\hat{\beta}_0\right] = \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{n\sum_{i=1}^{n}(x_i - \bar{x})^2} = \sigma^2\left[\frac{1}{n} + \frac{n\,\bar{x}^2}{(n-1)s_x^2}\right]$$

This expression for the variance of the least squares estimate of the intercept consists of two parts, the reciprocal of the sample size, *n*, which is the usual factor for the variance of a mean, and the ratio of the square of the mean of *X* to its variance. As for $\hat{\beta}_1$, an estimate of the variance of $\hat{\beta}_0$ can be obtained by substituting the mean squared error estimate of σ^2.

$$\hat{\sigma}^2_{\hat{\beta}_0} = \frac{\mathrm{SS}_{\mathrm{Error}}}{\mathrm{n-2}}\left[\frac{1}{n} + \frac{n\,\bar{x}^2}{(n-1)s_x^2}\right] = \frac{\left(1 - \mathrm{r}_{\mathrm{xy}}^2\right)(n-1)s_y^2}{\mathrm{n-2}}\left[\frac{1}{n} + \frac{n\,\bar{x}^2}{(n-1)s_x^2}\right]$$

Since the predicted value, $\hat{y}_i$, is also a linear combination of the least squares parameter estimates, it too will be normally distributed.

$$\hat{y}_i = \sum_{k=1}^{n}\left[\frac{1}{n} + \frac{(x_i - \bar{x})(x_k - \bar{x})}{(n-1)s_x^2}\right] y_k$$

The expected value of $\hat{y}_i$ is obtained by direct substitution into the linear prediction equation.

$$E\left[\hat{y}_i\right] = E\left[\hat{\beta}_0\right] + E\left[\hat{\beta}_1\right] x_i = \beta_0 + \beta_1 x_i$$

As for $\hat{\beta}_0$ and $\hat{\beta}_1$, the variance of $\hat{y}_i$ is derived by squaring and summing the constant terms in the expression of $\hat{y}_i$ as a linear combination of the observations of *Y*.

$$V\left[\hat{y}_i\right] = \sigma^2\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{k=1}^{n}(x_k - \bar{x})^2}\right) = \sigma^2\left(\frac{1}{n} + \frac{(x_i - \bar{x})^2}{(n-1)s_x^2}\right)$$

Again notice that the expression for the variance of the predicted value for the i^{th} observation of *Y* consists of two components. The first component, the reciprocal of the sample size, is the usual factor for the variance of a mean. The second component is a normalized squared distance of x_i, the i^{th} observation of the explanatory variable, from its mean. Thus the variances of predictions of *Y* for values of x_i near its mean will be close to the variance of an ordinary sample mean. But for values of x_i far from its mean, the variances of the predictions will increase linearly with the squared normalized distance from the mean.

$$V[\hat{y}_i] = \sigma^2 \{ \overbrace{\frac{1}{n}}^{\text{Variance of Mean of } Y} + \overbrace{\frac{(x_i - \bar{x})^2}{(n-1)s_x^2}}^{\text{Normed Distance From Mean of } X} \}$$

The foregoing expression for the variance of $\hat{y}_i$ is the variance of the estimate of the regression line, which is the conditional mean of Y given X. But the variance of a prediction for single observation of Y at X will be much greater. This prediction error will be the original variance of Y, σ^2, plus the variance of error due to estimation of the regression line. Therefore, the estimated variance of a single observation or prediction at X_i is

$$\hat{\sigma}^2_{\hat{y}_i} = \hat{\sigma}^2 \{ \overbrace{1}^{\text{Single Observation}} + \overbrace{\frac{1}{n} + \frac{(x_i - \bar{x})^2}{(n-1)s_x^2}}^{\text{Regression Line}} \}$$

Both estimates of the regression line and predictions from the regression line will be more accurate for values of X near the mean.

11.5. Inference for the Regression Line

It is often of interest to test hypotheses about the parameters of the regression model or to construct confidence intervals for various quantities associated with the model. There may be theoretically prescribed values for the parameters. Confidence intervals for predictions from the regression model are frequently required. Inferential procedures follow in a natural way from the fact that least squares parameter estimates and hence the estimated regression line is all linear combination of the response variable, Y, and like Y will be normally distributed. In addition, the estimated variances of these parameters are derived from the sum of squared deviations of Y and hence will have a χ^2 distribution. Therefore, the following statistics all have Student's t distributions with (n-2) degrees of freedom.

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\hat{\beta}_1}} = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma} / \sqrt{(n-1)s_x}} \Rightarrow t_{n-2}$$

$$\frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}_{\hat{\beta}_0}} = \frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}\sqrt{\left[\frac{1}{n} + \frac{n\bar{x}^2}{(n-1)s_x^2}\right]}} \Rightarrow t_{n-2}$$

$$\frac{\hat{y}_i - \beta_0 - \beta_1 x_i}{\hat{\sigma}_{\hat{y}_i}} = \frac{y_i - \beta_0 - \beta_1 x_i}{\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_i - \bar{x})^2}{(n-1)s_x^2}}} \Rightarrow t_{n-2}$$

where β_0 and β_1 are the true or hypothesized values of the regression parameters. The most commonly tested null hypothesis is that the slope and intercept equal zero. This test is the t-

test produced by most regression software. For the slope parameter, this test has a particularly interesting interpretation.

$$\frac{\hat{\beta}_1}{\hat{\sigma}_{\hat{\beta}_1}} = \frac{r_{xy}\dfrac{s_y}{s_x}}{\sqrt{\dfrac{\left(1-r_{xy}^2\right)}{(n-2)}\dfrac{s_y}{s_x}}} = \frac{r_{xy}}{\sqrt{\dfrac{\left(1-r_{xy}^2\right)}{(n-2)}}}$$

which is simply the square root of the F-test for the regression model derived earlier. Thus, testing whether the systematic component is zero is equivalent to testing whether the slope of the regression line is zero. If $\beta_1 = 0$, then the regression line will be horizontal at the mean of *Y*, that is, the mean of *Y* will be predicted at every value of *X* and *X* will have no effect on predictions of *Y*.

Confidence intervals for the intercept, slope, regression line, and predictions from the regression line are calculated in the usual manner.

$$\hat{\beta}_1 \pm t_{\alpha/2,n-2} \Big/ \hat{\sigma}\sqrt{(n-1)s_x}$$

$$\hat{\beta}_0 \pm t_{\alpha/2,n-2}\hat{\sigma}\sqrt{\left[\frac{1}{n}+\frac{n\,\bar{x}^2}{(n-1)s_x^2}\right]}$$

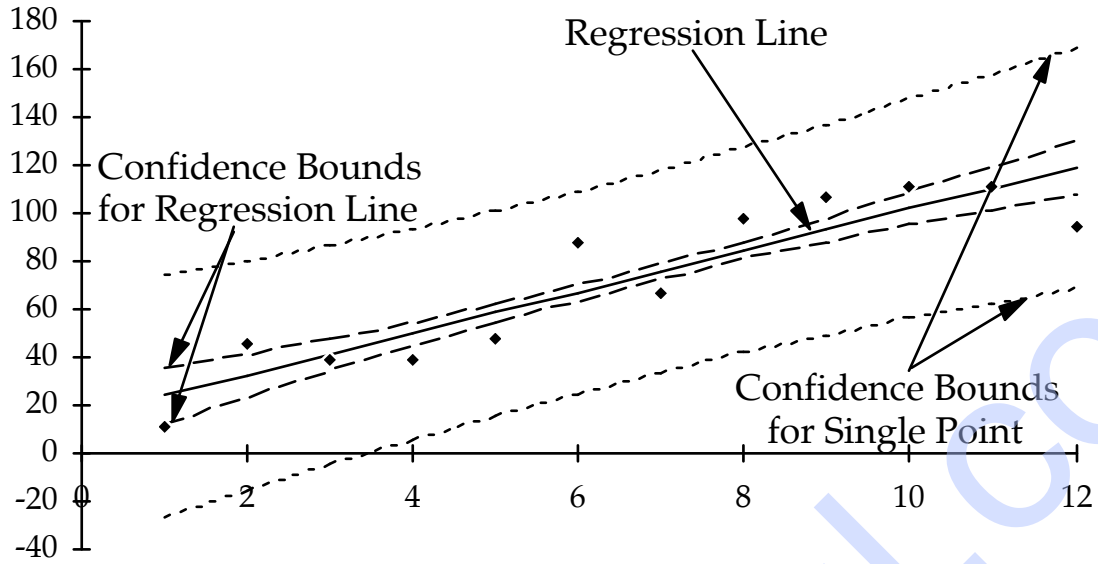
The confidence interval for the regression line is

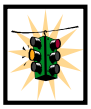
$$\hat{\beta}_0 + \hat{\beta}_1 x_i \pm t_{\alpha/2,n-2}\hat{\sigma}\sqrt{\frac{1}{n}+\frac{(x_i-\bar{x})^2}{(n-1)s_x^2}}$$

And the confidence interval for predictions from the regression line is

$$\hat{\beta}_0 + \hat{\beta}_1 x_i \pm t_{\alpha/2,n-2}\hat{\sigma}\sqrt{1+\frac{1}{n}+\frac{(x_i-\bar{x})^2}{(n-1)s_x^2}}$$

The following chart displays 95% confidence intervals for both the regression line and individual predictions from the regression line. Notice that the confidence bounds for the regression line are very narrow and include very few of the original data points. This is because the correlation between predictor and criterion variables is high and the fit of the regression line is good. On the other hand, all original observations are included within the confidence bounds for single points.





Example and Solution

A company repairs desktop calculators and cash registers. The data from 18 repairs are written in the table. Each repair has 2 important sets of data. The first one is a number of repaired calculators (X) and the second one is a total repair time (Y).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
x	7	6	5	1	5	4	7	3	4	2	8	5	2	5	7	1	4	5
Y	97	86	78	10	75	62	101	39	53	33	118	65	25	71	105	17	49	68

a) Find parameter estimates of the regression line.
b) Draw data and regression function.
c) Use t-tests for the values of all parameters of regression function.

Solution

– you can use STATGRAPHIC software:

Linear regression – Repair time vs. Number

Regression Analysis - Linear model: Y = b_0 + b_1*x

Dependent variable: Repair Time
Independent variable: Number

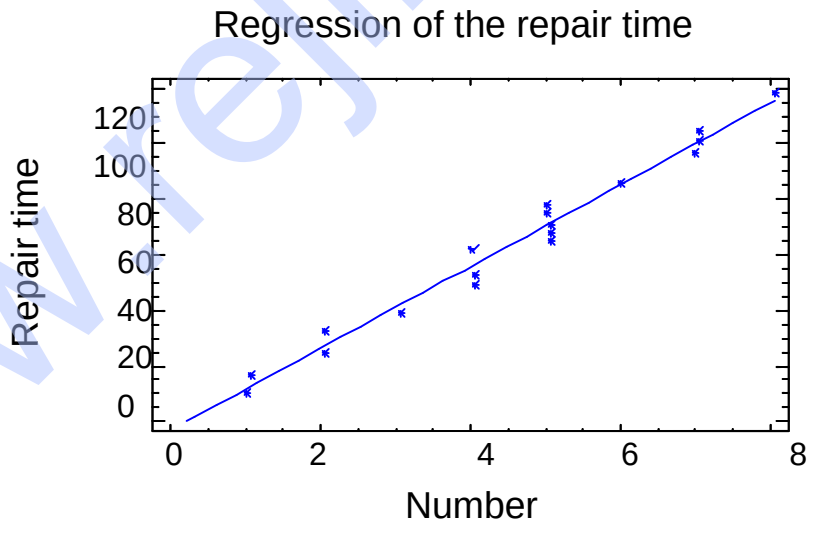
Parameter	Estimate	Standard Error	T Statistic	P-Value
b_0 - Intercept	-2,32215	2,56435	-0,905549	0,3786
b_1 - Slope	14,7383	0,519257	28,3834	0,0000

---

b_0 = *Intercept*, b_1 = *Slope*, the results of these values may be found in the second column.
The following function introduces an equation for the estimate of predicted value:

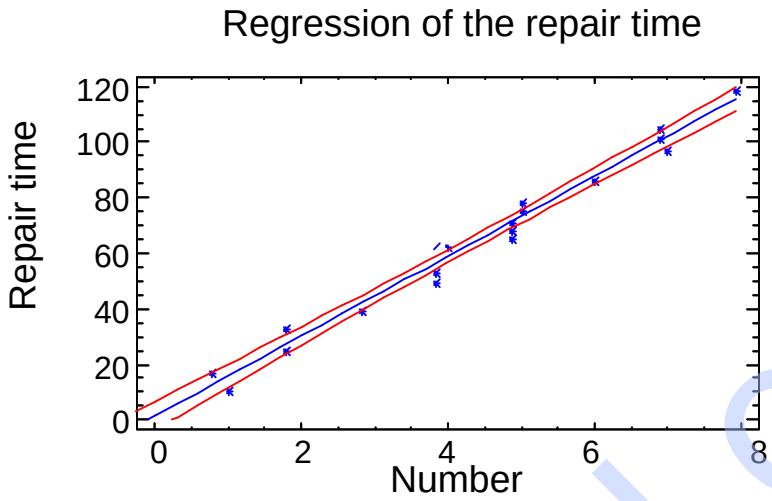
Repair Time = -2,32215 + 14,7383 . Number

The observed values of the t-tests are shown in the fourth column (T Statistic) and corresponding p-values are displayed in the last one. It is obvious that hypothesis H_0: $\beta_0=0$ will not be rejected considering the important value in p-value column. Based on this we can say that regression line passes through the beginning what is a logical conclusion, considering the data nature. The second of particular test says that Slope is a value that significantly differs from zero since we have rejected H_0 hypothesis H_0: $\beta_1=0$.



d) Let's find the 95% confidential interval for the repair time in dependence on the number of calculators.
e) Let's find point and interval estimation for an expected repair time for 5 calculators.

Solution



For value $x=5$:

$$\hat{Y}(x) = b_0 + b_1 x = \sum_{i=1}^{n} \left(\frac{1}{n} + \frac{(x_i - \bar{x}) \cdot (x - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right) \cdot Y_i = 71.3691$$

$$\mathrm{E(Y|\,x)} = \beta_0 + \beta_1 x \in < \hat{Y}(x) - S_{\hat{Y}} \cdot t_{1-\frac{\alpha}{2}}(n-2), \hat{Y}(x) + S_{\hat{Y}} \cdot t_{1-\frac{\alpha}{2}}(n-2) >= < 69.063,\ 73.6752 >$$

f) Consider the quality of examined model of linear regression for the repair time in dependence on a number of calculators using a coefficient of determination

Solution

$$SS_Y = SS_{\hat{Y}} + SS_R$$

Source		Sum of Squares
Regression	$SS_{\hat{Y}}$	16182,6
Error	SS_R	321,396
Total	SS_Y	16504,0

$$I^2 = \frac{SS_{\hat{Y}}}{SS_Y} = 98.0526\ \%$$

Summary

Regression model is a special case of general linear model. The basic assumptions are independence, homoscedasticity and normality.

Dependent variable is the variable of a regression model that is random and we try to describe and explain its behavior by mathematical curve.

Independent (explanatory) variables are the variables in the regression model of which behavior explains the behavior of the dependent variable.

Linear regression model with one explanatory variable is a basic model and it is based on the **Least-Squares Method**. By this method model parameters can be determined. The sum of squared deviations of the real values from modelled values is called the residual sum of squares.

We can obtain interval estimation for the expected value of the dependent variable. These interval bounds form **confidential interval** of the regression line.

Quiz

1. Describe and explain equation of linear regression.
2. What means p-value in the ANOVA table for linear regression?
3. What property describes a coefficient of determination?

Practical Exercises

Exercise 1: During control measurements of industrial components size you randomly chose 8 components showing mostly positive divergences from normal values in the length and height:

Length divergence [mm]	3	4	4	5	8	10	6	3
Height divergence [mm]	4	6	5	6	7	13	9	4

Let's find the linear regression model of dependency between the length divergence and height divergence.

{*Answer:* Use a suitable software package.}

Exercise 2: In the years 1931-1961, water flow in profile of Šance and Morávka water reservoirs were measured. Averages per year (in m^3/s) are given by the following table:

Year	**Šance**	**Morávka**
1931	4,130	2,476
1932	2,386	1,352
1933	2,576	1,238
1934	2,466	1,725
1935	3,576	1,820
1936	2,822	1,913
1937	3,863	2,354
1938	3,706	2,268
1939	3,710	2,534
1940	4,049	2,308
1941	4,466	2,517
1942	2,584	1,726
1943	2,318	1,631
1944	3,721	2,028
1945	3,290	2,423

Year	**Šance**	**Morávka**
1946	2,608	1,374
1947	2,045	1,194
1948	3,543	1,799
1949	4,055	2,402
1950	2,224	1,019
1951	2,740	1,552
1952	3,792	1,929
1953	3,087	1,488
1954	1,677	0,803
1955	2,862	1,878
1956	3,802	1,241
1957	2,509	1,165
1958	3,656	1,872
1959	2,447	1,381
1960	2,717	1,679

Let's assume that in one of following years, the average value of the whole year water flow of Morávka reservoir is missing. In this year, the average water flow for Šance reservoir was 2,910 m^3/s. Based on linear regression, try to determine the average water flow in Morávka reservoir.

{*Answer:* Use a suitable software package.}

12. SOLUTION KEYS

12.1. Lecture 1

1. Exploratory data analysis is often a first step in revealing information hidden in large amount variables and their variants.
2. The basic kinds of variables are quantitative variables (nominal, ordinal) and qualitative variables (discrete, continuous).
3. The frequency table is concerned with absolute and relative frequencies (for a qualitative variable).
4. The outliers are the variable values which significantly differ from other values.
5. b)
6. qualitative variable – bar chart, pie chart
   quantitative variable – box plot, stem and leaf plot
7. b) 14 thousand, d) cca (9;19)

12.2. Lecture 2

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A \cap B) = P(A) \cdot P(B)$
3. Two events are independent if intersection probability of these two events is equal to a product of individual event probabilities.

12.3. Lecture 3+4

1. $F(x) = \sum_{x_i < x} P(X = x_i)$
2. $F(x) = \int_{-\infty}^{x} f(t)dt$ for $-\infty < x < \infty$
3. 50% quantile is called a median

   A mode is a value in which the discrete RV comes with the biggest probability.
4. The conditional distribution is the distribution of one variable at a fixed value of the other jointly distributed random variable.
5. $X_1 \ldots X_n$ are mutually independent $\Leftrightarrow$ $F(x_1,\ldots,x_n) = F_1(x_1) \ldots F_n(x_n)$.
6. The correlation coefficient measures the strength of the relation between two random variables.

12.4. Lecture 5

1. Discrete distribution – binomial, geometric, negative binomial
   Continuous distribution – poisson, exponential, Weibull, Gamma
2. A sequence of Bernoulli trials is defined as a sequence of random events which are mutually independent and which have only two possible outcomes. The probability of

each outcome is assumed to be the same for all trials. On the basis of these expectations we can define the following random variables: binomial, geometric and negative binomial mean of the binomial random variable: EX=np

3. A Gamma distribution describes a time to k-th event occurrence in a Poisson process
4. β=2

12.5. Lecture 6

1. X ... RV with $N(\mu,\sigma^2)$ => $Z = \frac{X-\mu}{\sigma}$ ... N(0,1)
2. Chebyshev's inequality puts an upper boundary on the probability that an observation should be far from its mean.
3. The law of large number is a theorem about convergence of means in the sequence of the random variables.
4. Chí-square distribution is a distribution derived from the sum of squared standard normal random variables.

12.6. Lecture 7

1. Inferential statistics or statistical induction comprises the use of statistics to make inferences concerning some unknown aspect of a population.
2. A random sample is a set of items that have been drawn from a population in such a way that each time an item was selected, every item in the population had an equal opportunity to appear in the sample.

12.7. Lecture 8

1. The p-value calculation depends on defined null hypothesis:
   a) $H0: \mu < \mu_0$ => p-value=$F(x_{obs})$
   b) $H0: \mu > \mu_0$ => p-value=1-$F(x_{obs})$
   c) $H0: \mu = \mu_0$ => p-value=2{$F(x_{obs})$, 1-$F(x_{obs})$}
2. It is a hypothesis that is accepted in case the rejection of null hypothesis.
3. $Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\cdot(1-\hat{p})}\cdot\left(\frac{1}{n_1}-\frac{1}{n_2}\right)}$ ... N(0,1) , where $\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$

12.8. Lecture 9

1. An $\hat{\theta}$ estimation is consistent if
   a) $\hat{\theta}$ is asymptotically unbiassed, $E\hat{\theta} \to \theta$
   b) $\lim_{n \to \infty} D\hat{\theta} = 0$

2. $P(T_D(\overline{X}) \le \theta \le T_H(\overline{X})) \ge 1-\alpha$

12.9. Lecture 10

1. $F = \frac{S_B^2}{S_W^2}$ ... F-distribution with *(k-1)* and *(N-k)* degrees of freedom
2. ANOVA table
3. The post hoc analysis is a second step of ANOVA and consists of comparing the means of all pairs of groups of purpose to choose homogenous groups.

12.10. Lecture 11

1. y=a+b*x, where y is a dependent variable and x is an independent variable. The values a (intercept) and b (slope) are estimates of regression line parameters
2. x and y are independent variables, in the case that p-value > 0,05
3. Coefficient of determination predicate about suitability of used model