MA 8451 PROBABILITY & RANDOM PROCESSES

UNIT I - PROBABILITY AND RANDOM VARIABLES

INTRODUCTION
In daily usage, the word probability means uncertainty about happenings. In mathematics or statics, a numerical measure of uncertainty is practiced by a branch of statistics called "theory of probability".
Applications of probability

- Decision theory
- Solutions of Gambling
- Economic decision making
- Business applications – Investment, new product launch etc.

Let us assume that an experiment is repeated under some conditions, any number of times, it does not give unique result but may result in any one of the several possible outcomes. Then we call the experiment a **random experiment**.
The outcomes of random experiments may be numerical or non-numerical. For example, the outcomes obtained when we throw a die are numerical. But the outcomes we obtain when a coin tossed are non-numerical. For ease of manipulation, we may assign a real number for each of the outcomes using a fixed rule. For example, in the random experiment of tossing two coins, we assign the value 0 to the outcome of getting 2 tails, the value 1 to the outcome of getting1 head and one tail and the value 2 to the outcome of getting 2 heads.

Basics concepts:
Random experiment: Random experiment is one whose results depend on chance that is the result cannot be predicted. Tossing of coins, throwing of dice are some examples of random experiments.

Trial: Performing a random experiment is called a trial

Outcomes:
The results of a random experiment are called its outcomes. When two coins are tossed the possible outcomes are HH, HT, TH, TT.

Event: An outcome or a combination of outcomes of a random experiment is called an event. For example tossing or a coin is a random experiment and getting a head or tail is an event.

Sample space:
Each possible outcome of an experiment is called a sample point. The collection of all sample points is called a sample space and is denoted by S. For example, when a coin is tossed, the sample space is $S = \{H, T\}$. H and T are the sample points of the sample space S.

Equally likely events:
Two or more events are said to be equally likely if each one of them has an equal chance of occurring. For example, in tossing of a coin, the event of getting a head and the event of getting a tail are equally likely events.

Mutually exclusive events:
Two or more events are said to be mutually exclusive, when the occurrence of any one event excludes the occurrence of the other event. Mutually exclusive events cannot occur simultaneously.
For example, when a coin is tossed, either the head or the tail will come up. Therefore the occurrence of head completely excludes the occurrence of the tail. Thus getting head or tail in tossing of a coin is a mutually exclusive event.

Exhaustive events:
Events are said to be exhaustive when their totality includes all the possible outcomes or a random experiment. For example, while throwing a die, the possible outcomes are $\{1,2,3,4,5,6\}$ and the number of cases is 6.

Complementary events:
The event 'A occurs' and the event "A does not occur" are called complementary events to each other. The event "A does not occur" is denoted by A' or A^c. The event and its complements are mutually exclusive events.

Independent events:
Events are said to be independent if the occurrence of one does not affect the others. In the experiment of tossing a fair coin, the occurrence of the event ' head' in the first toss is independent of the occurrence of the event 'head' in the second toss, third toss and subsequent tosses.

Definitions:
Mathematical Probability (Priori Probability)

If the probability of an event can be calculated even before the actual happening of the event is called Mathematical probability.

If the random experiments results in 'n' exhaustive mutually exclusive and equally likely cases, out of which 'm' are favourable to the occurrence of an event A, then the ratio m/n is called the probability of occurrence of event A (written as P(A)) is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of cases favourable to the event A}}{\text{Total number of exhaustive cases}}$$

Statistical probability (Posteriori probability)

If the probability of an event can be determined only after the actual happening of the event, it is called statistical probability.

If an event occurs m times out of n, its relative frequency is m/n. In the limiting case, when n becomes sufficiently large it corresponds to a number which is called the probability of that event.

Axiomatic approach to probability:

Axioms of probability:

Let S be a sample space and A be an event in S and P(A) is the probability satisfying the following axioms:

(i) $0 \le P(A) \le 1$ (ii) $P(S) = 1$

(iii) If $A_1, A_2, ...A_n$ is a sequence of mutually exclusive events in S, then $P(A_1 \cup A_2....) = P(A_1) + P(A_2)....$

Interpretations of statistical statements in terms of set theory:

S- Sample space

$\overline{A}$ - A does not occur

$A \cup \overline{A} = S$

$A \cup B$ - Event A occurs or B occurs or both A and B occur.
(Atleast one of the events A or B occurs)

$A \cap B$ - Both the A and B occur

Addition theorem on probabilities for mutually exclusive events:

If two events A and B are mutually exclusive, the probability of occurrence of either A or B is the sum of the individual probabilities of A and B

$P(A \cup B) = P(A) + P(B)$

Addition theorem on probabilities for non-mutually exclusive events:

If two events A and B are non-mutually exclusive, the probability of occurrence of either A or B is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Compound events

The joint occurrence of two or more events is called the compound events. Thus compound events imply the simultaneous occurrence of two or more simple events.

The compound events are classified into (a) Independent events (b) Dependent events

Independent events:

If two or more events occur in such a way that the occurrence of one does not affect the occurrence of another, they are said to be independent events.

For example if a coin is tossed twice, the results of second throw would no way be affected by the results of the first throw.

Dependent events:

If the occurrence of one event influences the occurrence of the other, then the second event is said to be dependent on the first.

For example if a person draw a card from a full pack and does not replace it, the result of the draw made afterwards will be dependent on the first draw.

Conditional probability:

Let A be any event with P(A) >0. The probability that an event B occurs subject to the condition that A has already occurred is known as the conditional probability of occurrence of the event B on the assumption that the event A has already occurred and is denoted by $P(B / A)$ which is read as probability of B given A.

If two events A and B are dependent, then the $P(B/A) = \frac{P(A \cap B)}{P(A)}$

If the events A and B are independent, then $P(B/A) = P(B)$

Multiplication theorem on probabilities for independent events:
If two events A and B are independent, the probability that both of them occur is equal to the product of their individual probabilities $P(A \cap B) = P(A).P(B)$

Multiplication theorem on probabilities for dependent events:

$$P(B/A).P(A) = P(A \cap B)$$

BAYES' theorem:

If $A_1, A_2, ...A_n$ is a set of n mutually exclusive and collectively exhaustive events and $P(A_1), P(A_2), ...P(A_n)$ are their corresponding probabilities. If B is another event such that P(B) is not zero and the priori probabilities $P(B/A_i)$ for I = 1,2,3,…n also known, then

$$P(A_i/B) = \frac{P(B/A_i)P(A_i)}{\sum_{i=1}^{n} P(B/A_i)P(A_i)}$$

Definition: A **random variable** (RV) is a function that assigns real number $X(s)$ to every element $s \in S$, where S is the space corresponding to the random experiment E.

NOTE:

1. We denote a RV by X and the values it takes by $x_1, x_2, x_3, x_4, \ldots$
2. $P\{X = x\} = P\{S: X(s) = x\}$
3. The range (set of values which the RV X takes) of X is called the spectrum of the RV.

   Example:

   Let E denote the random experiment of tossing an unbiased coin twice.

   Let X denote the number of heads turning up.

   The outcomes are $\{HH, HT, TH, TT\}$.

   $\therefore R_X = \{0,1,2\}$

   $P\{X = 1\} = P\{\text{Number of heads} = 1\} = \frac{3}{4}$

   $P\{X \leq 1\} = P\{X = 0\} + P\{X = 1\} = \frac{3}{4}$

Types of random variable

1. Discrete Random Variable: If X is a RV which can take a finite number of or countably infinite number of values, X is called a discrete random variable.

   Example:

   i. The marks obtained by a student in an exam.
   ii. The number shown when a die is thrown
2. Continuous Random Variable: If X is a RV which can take all values in an interval, then X is called continuous random variable.

   Example:

   i. The density of milk taken for testing at a farm.
   ii. The length of time during which a vacuum tube installed in a circuit functions.

PROBABLITY FUNCTIONS

If X is a discrete RV which can take values $x_1, x_2, x_3, \dots, x_n, \dots$ such that $P\{X = x_i\} = p_i$ is the probability mass function (pmf) or point probability function, provided $p_i \ (i = 1,2,3, \dots)$ satisfies the following conditions:

i. $p_i \geq 0 \ \forall i$

ii. $\sum p_i = 1$

The collection of pairs $\{x_i, p_i\}, i = 1,2,3, \dots$ is called the probability distribution of the random variable X.

If X is a continuous RV such that $P\left\{x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx\right\} = f(x)dx$ then $f(x)$ is called the probability density function (pdf) of X, provided $f(x)$ satisfies the following conditions:

i. $\int_{R_X} dx = 1$

ii. $f(x) \geq 0, \forall x \in R_X$

NOTE:

i. $P(a \leq X \leq b)$ Or $P(a < X < b)$ is defined as $P(a \leq X \leq b) = \int_a^b f(x)dx$.

The curve $f(x)$ is called the probability curve of the RV X.

ii. When X is continuous RV, $P\{X = a\} = P(a \leq X \leq a) = \int_a^a f(x)dx = 0$

$\therefore P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = 0$

CUMULATIVE DISTRIBUTION FUNCTION (cdf)

If X is a RV, discrete or continuous, then $P\{X \leq x\}$ is called the cumulative distribution function of X or distribution function of X and is denoted as $F(x)$.

$i.e. F(x) = P\{X \leq x\}$

If X is discrete,

$$F(x) = \sum_{\substack{j \\ X_j \leq x}} P_j$$

If X is continuous,

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Properties of CDF:

1. $F(-\infty) = 0$ and $F(\infty) = 1$.
2. $F(x)$ is a non decreasing function of x. $i.e.$ if $x_1 \leq x_2$, then $F(x_1) \leq F(x_2)$.
3. If X is a discrete RV taking values $x_1, x_2, x_3, \dots$ where $x_1 < x_2 < x_3 < \cdots < x_{i-1} < x_i < \cdots$ then $P\{X = x_i\} = F(x_i) - F(x_{i-1})$.
4. If X is a continuous RV, then $\frac{d}{dx}F(x) = f(x)$ at all points where $F(x)$ is differentiable.

Examples of some discrete distributions:

Binomial, Poisson, Pascal (negative binomial), Geometric distributions.

Examples of some continuous distributions:

Uniform (rectangular), Normal (Gaussian), Gamma, Erlang, Exponential, Rayleigh, Maxwell distributions.

MOMENT AND MOMENT GENERATING FUNCTION

Let X be a RV. The mean/ expectation/ stochastic average/ ensemble average of X is defined as

$$\mu_X = E(X) = \overline{X} = \sum_{i=1}^{n} x_i p_i$$

provided the series is absolutely convergent (for discrete RV)

For a continuous RV,

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

provided the integral is absolutely convergent.

$$Var(X) = \sigma_X^2 = E\{(X-\mu_X)^2\} = \sum_i (x-\mu_X)^2 p_i \text{ if } X \text{ is discrete}$$

$$= \int_{R_X} (X-\mu_X)^2 f(x) dx \text{ if } X \text{ is continuous}$$

$\sqrt{Var}$ =Standard Deviation (SD)

$$Var(X) = \sigma_X^2 = E\{(X-\mu_X)^2\} = E\{X^2 - 2X\mu_X + \mu_X^2\} = E\{X^2\} - 2\mu_X E(X) + \mu_X^2 E\{1\}$$
$$= E\{X^2\} - 2[E(X)]^2 + [E(X)]^2 Var(X) = E\{X^2\} - [E(X)]^2$$

Properties of Expectation:

i. $E(a) = a$, a is a constant.
ii. $E(aX) = aE(X)$.
iii. $E(aX + b) = aE(X) + b$.
iv. $E(XY) = E(X)E(Y)$ if X and Y are independent RV.
v. $E(X - \overline{X}) = 0$.
vi. $E(X) \geq 0$ if $X \geq 0$.

MOMENTS

To completely define a RV, we need something more than just their average. Moments serve this purpose. We can completely characterize the behavior of a RV through its higher order moments. It is also possible to reconstruct the pdf of a RV using its moments.

Definition: (Moment about origin or Raw Moments)

The n^{th} order moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

$$i.e. E(X^n) = \mu_n' = \sum_i x_i^n p_i, n \geq 1, \text{ if } X \text{ is discrete} E(X^n) = \mu_n'$$

$$= \int_{-\infty}^{\infty} x^n f(x) dx, n \geq 1, \text{ if } X \text{ is continuous}$$

Definition: (Moment about mean or Central Moments)

The n^{th} central moment of a RV X is its moment about its mean value $\overline{X}$ and is defines as

$$i.e. E\left(\left(X-\overline{X}\right)^n\right) = \mu_n = \sum_i \left(x_i - \overline{X}\right)^n p_i, \text{ if } X \text{ is discrete} E\left(\left(X-\overline{X}\right)^n\right) = \mu_n$$

$$= \int_{-\infty}^{\infty} \left(x-\overline{X}\right)^n f(x)dx, \text{ if } X \text{ is continuous}$$

NOTE:

$$Var(X) = \sigma_X^2 = E\left[\left(x-\overline{X}\right)^2\right] = E\left[X^2 - 2X\overline{X} = \overline{X}^2\right] = E[X^2] - 2E\left[X\overline{X}\right] + E\left[\overline{X}^2\right]$$

$$= E[X^2] - 2E[X]E\left[\overline{X}\right] + E\left[\overline{X}^2\right] = \overline{X^2} - 2\overline{X}.\overline{X} + \overline{X}^2 = \overline{X^2} - 2\overline{X}^2 + \overline{X}^2$$

$$= \overline{X^2} - \overline{X}^2 Var(X) = \mu_2' - (\mu_1')^2$$

Properties of Variance

i. $Var(X) \geq 0$
ii. $E(X^2) \geq [E(X)]^2$
iii. $Var(b) = 0$, b is a constant.
iv. $Var(aX+b) = a^2Var(X)$, a & b are constants.
v. If X and Y are independent RV, a and b are constants then $Var(aX+bY) = a^2Var(X) + b^2Var(Y)$

Relationship between Moment about Origin and Central Moments

$$\boldsymbol{\mu_r = \mu_r' - rC_1\mu\mu_{r-1}' + rC_2\mu^2\mu_{r-2}' - rC_3\mu^3\mu_{r-3}' + \cdots}$$

Proof:

$$\boldsymbol{\mu_r} = [E(x-\mu)^r] = E[X^r - rC_1X^{r-1}\mu + rC_2X^{r-2}\mu^2 - rC_3X^{r-3}\mu^3 + \cdots]$$

$$= E[X^r] - rC_1\mu E[X^{r-1}] + rC_2\mu^2E[X^{r-2}] - rC_3\mu^3E[X^{r-3}] + \cdots \boldsymbol{\mu_r}$$

$$= \boldsymbol{\mu_r' - rC_1\mu\mu_{r-1}' + rC_2\mu^2\mu_{r-2}' - rC_3\mu^3\mu_{r-3}' + \cdots}$$

Hence,

$$\mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0 Var = \mu_2 = \mu_2' - 2C_1\mu\mu_1' + 2C_2\mu^2\mu_0'$$

$$= \mu_2' - 2\mu_1'\mu_1' + (\mu_1')^2 = \mu_2' - (\mu_1')^2$$

Similarly

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

MOMENT GENERATING FUNCTION (MGF)

MGF of a RV X is defined as $E(e^{tX})$, where t is a real variable.

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx}p(x), \text{ if } X \text{ is discrete} = \int_{-\infty}^{\infty} e^{tx}f(x)dx, \text{ if } X \text{ is continuous}$$

NOTE: MGF exists only if all of the moments exist.

Properties of MGF

1.

$$M_X(t) = E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right]$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \cdots + \frac{t^r}{r!}E[X^r] + \cdots$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

where μ_r' is the r^{th} moment about the origin.

$\therefore$ The coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r'.

Thus $M_X(t)$ generates the moments about the origin and hence the term MGF.

2. Consider

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

Differentiate with respect to t

$$\frac{d}{dt}[M_X(t)] = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots + \frac{rt^{r-1}}{r!}\mu_r' + \cdots$$

Probability distributions

Discrete Distributions:

Name of the distribution	pmf	Mean	Variance	MGF
Binomial	$nC_x p^x q^{n-x}, \ x = 0,1,2\ldots n$	np	npq	$(q+pe^t)^n$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0,1,2\ldots\infty$	λ	λ	$e^{\lambda(e^t-1)}$
Geometric	$q^{x-1}p, \ x = 1,2,\ldots\infty$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{pe^t}{(q+pe^t)}$

Continuous Distributions:

Name of the distribution	pmf	Mean	Variance	MGF
Uniform (Rectangular)	$\frac{1}{b-a}, \ a<x<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{t(b-a)}$
Exponential	$\lambda e^{-\lambda x}, \ x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$

PART A

1. When two dice are thrown, find the probability of getting doublets (same number on both dice)

   When two dice are thrown, the number of events in the sample space is n(S)=36,

   Getting doublets: $A=\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

   $$P(A)=\frac{6}{36}=\frac{1}{6}$$

2. A ball is drawn at random from the box containing 5 Green, 6 red, and 4 yellow balls. Determine the probability that the ball drawn is (i) green (ii) red (iii) yellow (iv) Green or Red (v) not yellow.

   Total number of balls in the box = 5+6+4 =15 balls

   (i) Probability of drawing a green ball = $\frac{5}{15}=\frac{1}{3}$

   (ii) Probability of drawing a red ball = $\frac{6}{15}=\frac{2}{5}$

   (iii) Probability of drawing a yellow ball = $\frac{4}{15}$

   (iv) Probability of drawing a Green or a Red ball $=\frac{5}{15}+\frac{6}{15}=\frac{11}{15}$

   (v) Probability of getting not yellow = $1-P(\text{Yellow})=1-\frac{4}{15}=\frac{11}{15}$

3. Two dice are thrown, what is the probability of getting the sum being 8 or the sum being 10?

   Number of sample points in throwing two dice at a time is n(S) = 36

   Let A= {the sum being 8}

   A= {(6, 2), (5, 3), (4, 4), (3, 5), (2, 6)}; P (A) =5/36

   Let B = {the sum being 10}

   B = {(6, 4), (5, 5) (4, 6)}; P (B) = 3/36

   $A\cap B=\{0\}$; $n(A\cap B)=0$

   $\therefore$ The two events are mutually exclusive

   $$P(AUB) = P(A) + P(B) = \frac{5}{36}+\frac{3}{36}=\frac{2}{9}$$

4. Two persons A and B appeared for an interview for a job. The probability of selection of A is 1/3 and that of B is 1/2. Find the probability that (i) both of them will be selected (ii) only one of them will be selected (iii) none of them will be selected

   $$P(A)=\frac{1}{3},\ P(B)=\frac{1}{2},$$

   $$P(\bar{A})=\frac{2}{3},\ P(\bar{B})=\frac{1}{2}$$

   Selection or non-selection of any one of the candidate is not affecting the selection of the other. Therefore A and B are independent events.

   (i) Probability of selecting both A and B

$P(A\cap B) = P(A).P(B)$

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

(ii) Probability of selecting any one of them

= P (selecting A and not selecting B) + P (not selecting A and selecting B)

i.e $P(A\cap \bar{B}) + P(\bar{A}\cap B) = P(A).P(\bar{B}) + P(\bar{A}).P(B)$

$$= \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{2}$$

$$= \frac{1}{6} + \frac{2}{6}$$

$$= \frac{3}{6} = \frac{1}{2}$$

(iii) Probability of not selecting both A and B

i.e $P(\bar{A}\cap \bar{B})$

$$= P(\bar{A}).P(\bar{B})$$

$$= \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{1}{3}$$

5. There are three T.V programmes A, B and C which can be received in a city of 2000 families. The following information is available on the basis of survey. 1200 families listen to Programme A, 1100 families listen to Programme B, 800 families listen to Programme C, 765 families listen to Programme A and B, 450 families listen to Programme A and C, 400 families listen to Programme B and C, 100 families listen to Programme A, B and C. Find the probability that a family selected at random listens at least one or more T. V Programme.

Total no. of families n(S) = 2000.

Let n(A) = 1200, n(B) = 1100, n(C) = 800, $n(A\cap B) = 765$, $n(A\cap C) = 450$, $n(B\cap C) = 400$, $n(A\cap B\cap C) = 100$.

Let us find first $n(A\cup B\cup C)$.

$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$

$= 1200 + 1100 + 800 - 765 - 450 - 400 - 100$

$= 1585$

Now $P(A\cup B\cup C) = n(A\cup B\cup C) / n(S)$

$= 1585 / 2000$

$= 0.792$

Therefore about 79 % chance that a family selected at random listens to one or more T. V programmes.

6. If a random variable X takes the values 1,2,3,4 such that

$P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution of X.

Solution:

Assume $P(X=3) = \alpha$.

By the given equation $P(X=1) = \frac{\alpha}{2}$ $P(X=2) = \frac{\alpha}{3}$ $P(X=4) = \frac{\alpha}{5}$.

For a probability distribution (and mass function) $\sum P(x) = 1$

$P(1)+P(2)+P(3)+P(4) = 1$

$$\frac{\alpha}{2}+\frac{\alpha}{3}+\alpha+\frac{\alpha}{5}=1 \quad \Rightarrow \quad \frac{61}{30}\alpha=1 \quad \Rightarrow \alpha=\frac{30}{61}$$

$$P(X=1)=\frac{15}{61}\ ;P(X=2)=\frac{10}{61};\ P(X=3)=\frac{30}{61};\ P(X=4)=\frac{6}{61}$$

The probability distribution is given by

X	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

7. Let X be a continuous random variable having the probability density function $f(x)=\begin{cases} 2/x^3, & x\ge 1 \\ 0, & otherwise \end{cases}$ Find the distribution function of x.

Solution:

$$F(x)=\int_1^x f(x)\,dx=\int_1^x \frac{2}{x^3}\,dx=\left[-\frac{1}{x^2}\right]_1^x=1-\frac{1}{x^2}$$

8. If a random variable has the probability density $f(x)=\begin{cases} 2e^{-2x}, & x\ge 0 \\ 0, & otherwise \end{cases}$.

Find the probability that it will take on a value between 1 and 3. Also, find the probability that it will take on value greater than 0.5.

Solution:

$$P(1<X<3)=\int_1^3 f(x)\,dx=\int_1^3 2e^{-2x}\,dx=\left[-e^{-2x}\right]_1^3=e^{-2}-e^{-6}$$

$$P(X>0.5)=\int_{0.5}^{\infty} f(x)\,dx=\int_{0.5}^{\infty} 2e^{-2x}\,dx=\left[-e^{-2x}\right]_{0.5}^{\infty}=e^{-1}$$

9. A random variable X has the probability density function *f(x)* given by $f(x)=\begin{cases} cxe^{-x}, & x\ge 0 \\ 0, & otherwise \end{cases}$. Find the value of ‘c ‘and CDF of X.

Solution:

$$\int_0^{\infty} f(x)\,dx=1$$

$$\int_0^{\infty} cxe^{-x}\,dx=1$$

$$c\left[-xe^{-x}-e^{-x}\right]_0^{\infty}=1$$

$$c(1)=1$$

$$c=1$$

$$F(x)=\int_0^x f(x)\,dx$$

$$=\int_0^x cxe^{-x}\,dx$$

$$=\int_0^x xe^{-x}\,dx$$

$$=\left[-xe^{-x}-e^{-x}\right]_0^x$$

$$=1-xe^{-x}-e^{-x}$$

10. A continuous random variable X has the probability density function *f(x)* given by $f(x) = ce^{-|x|}, -\infty < x < \infty$. Find the value of c and CDF of X.

Solution:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} c\, e^{-|x|}\, dx = 1 \Rightarrow \quad 2\int_{0}^{\infty} c\, e^{-|x|}\, dx = 1$$

$$2\int_{0}^{\infty} c\, e^{-x}\, dx = 1 \Rightarrow 2c\left[-e^{-x}\right]_0^{\infty} = 1$$

$$2c(1) = 1 \Rightarrow c = \frac{1}{2}$$

$Case(i)\ x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\, dx \Rightarrow \quad = c\int_{-\infty}^{x} e^{x}\, dx$$

$$= c\left[e^{x}\right]_{-\infty}^{x} \Rightarrow \quad F(X) = \frac{1}{2} e^{x}$$

$Case(ii)\ x > 0$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\, dx \quad = c\int_{-\infty}^{0} e^{x}\, dx + c\int_{0}^{x} e^{-x} dx$$

$$= c\left[e^{x}\right]_{-\infty}^{0} + c\left[-e^{-x}\right]_0^{x} \Rightarrow c - ce^{-x} + c$$

$$= c\left(2 - e^{-x}\right)$$

$$= \frac{1}{2}\left(2 - e^{-x}\right)$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x}, & x > 0 \\ \frac{1}{2}\left(2 - e^{-x}\right), & x < 0 \end{cases}$$

11. Is the function defined as follows a density function?

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3 + 2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

Solution:

$$\int_{2}^{4} f(x)\ dx = \int_{2}^{4} \frac{1}{18}(3 + 2x)\, dx = \left[\frac{(3+2x)^2}{72}\right]_2^4 = 1$$

Hence it is density function.

12. The cumulative distribution function (CDF) of a random variable X is $F(X) = 1 - (1 + x)e^{-x}$, $x > 0$. Find the probability density function of X.

Solution:

$$f(x) = F'(x)$$

$$= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right]$$

$$= xe^{-x}, \quad x > 0$$

13. The number of hardware failures of a computer system in a week of operations has the following probability mass function:

No of failures:	0	1	2	3	4	5	6
Probability :	0.18	0.28	0.25	0.18	0.06	0.04	0.01

Find the mean of the number of failures in a week.

Solution:

$$E(X) = \sum x\, P(x) = (0)(0.18) + (1)(0.28) + (2)(0.25) + (3)(0.18) + (4)(0.06) + (5)(0.04) + (6)(0.01)$$

$$= 1.92$$

14. Given the p.d.f of a continuous r.v X as follows: $f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

Find the CDF of X.

Solution:

$$F(x) = \int_0^x f(x)\, dx = \int_0^x 6x(1 - x)\, dx = \int_0^x 6x - 6x^2\, dx = \left[3x^2 - 2x^3\right]_0^x = 3x^2 - 2x^3$$

15. A continuous random variable X has the probability function $f(x) = k(1+x), \quad 2 \le x \le 5$. Find P(X<4).

Solution:

$$\int_2^4 f(x)dx = 1 \quad \Rightarrow k\int_2^5 (1+x)dx = 1 \quad \Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1 \quad \Rightarrow k\frac{27}{2} = 1 \quad \Rightarrow k = \frac{2}{27}$$

$$P(X < 4) = \int_2^4 f(x)\, dx = \frac{2}{27}\int_2^4 (1+x)\, dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4 = \frac{1}{25}(25-9) = \frac{16}{27}$$

16. Given the p.d.f of a continuous R.V X as follows

$f(x) = \begin{cases} 12.5x - 1.25 & 0.1 \le x \le 0.5 \\ 0, & elsewhere \end{cases}$. Find P(0.2 < X < 0.3)

Solution:

$$P(0.2 < X < 0.3) = \int_{0.2}^{0.3} (12.5\,x - 1.25)\,dx = \left[12.5\frac{x^2}{2} - 1.25\,x\right]_{0.2}^{0.3}$$

$$= 1.25\left[5(0.3)^2 - 0.3 - 5(0.2)^2 + 0.2\right]$$

$$= 0.1875$$

17. If the MGF of a continuous R.V X is given by $M_X(t) = \frac{3}{3-t}$. Find the mean and variance of X.

Solution:

$$M_X(t) = \frac{3}{3-t} = \frac{1}{1-\frac{t}{3}} = \left(1-\frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \ldots$$

$$E(X) = (coefficient\ \ of\ \ t)\ 1! = \frac{1}{3}\ \ is\ \ the\ \ mean$$

$$E(X^2) = \left(coefficient\ of\ t^2\right)2! = \frac{1}{9}2! = \frac{2}{9}$$

$$Variance\ = E(X^2) - E(X)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

18. If the MGF of a discrete R.V X is given by $M_X(t) = \frac{1}{81}\left(1+2e^t\right)^4$, find the distribution of X.

Solution:

$$M_X(t) = \frac{1}{81}\left(1+2e^t\right)^4 = \frac{1}{81}\left(1 + 4C_1\left(2e^t\right) + 4C_2\left(2e^t\right)^2 + 4C_3\left(2e^t\right)^3 + 4C_4\left(2e^t\right)^4\right)$$

$$= \frac{1}{81} + \frac{8}{81}e^t + \frac{24}{81}e^{2t} + \frac{32}{81}e^{3t} + \frac{16}{81}e^{4t}$$

By definition of MGF ,

$$M_X(t) = \sum e^{tx} p(x) = p(0) + p(1)e^t + p(2)e^{2t} + p(3)e^{3t} + p(4)e^{4t}$$

On comparison with above expansion the probability distribution is

X	0	1	2	3	4
$p(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

19. Find the MGF of the R.V X whose p.d.f is $f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & elsewhere \end{cases}$. Hence its mean.

Solution:

$$M_X(t) = \int_0^{10} \frac{1}{10} e^{tx}\ dx = \frac{1}{10}\left(\frac{e^{tx}}{t}\right)_0^{10} = \frac{1}{10}\left(\frac{e^{10t}-1}{t}\right)$$

$$= \frac{1}{10t}\left(1+10t+\frac{100t^2}{2!}+\frac{1000t^3}{3!}+....-1\right)$$

$$= 1+5t+\frac{1000}{31}t^2+.....$$

Mean = coefficient of t = 5

20. Given the probability density function $f(x) = \frac{k}{1+x^2}, \ -\infty < x < \infty$, find k and C.D.F.

Solution:

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\ dx = 1$$

$$\Rightarrow k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\left[\left[\tan^{-1}\infty\right]-\left[\tan^{-1}-\infty\right]\right] = 1$$

$$\Rightarrow k\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1$$

$$\Rightarrow k = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)\ dx$$

$$= \int_{-\infty}^{\infty} \frac{k}{1+x^2}\ dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\left[\tan^{-1}\infty\right]-\left[\tan^{-1}-x\right]\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}-\tan^{-1}x\right) = \frac{1}{\pi}\cot^{-1}x$$

21. It has been claimed that in 60 % of all solar heat installation the utility bill is reduced by atleast one-third. Accordingly, what are the probabilities that the utility bill will be reduced by atleast one-third in atleast four of five installations?

Soln: Given n=5, p=60 % =0.6 and q=1-p=0.4

$$p(x \ge 4) = p[x=4]+p[x=5]$$

$$= 5c_4(0.6)^4(0.4)^{5-4}+5c_5(0.6)^5(0.4)^{5-5}$$

$$= 0.337$$

22. The no. of monthly breakdowns of a computer is a r.v. having poisson distribution with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Soln: $p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $given\ \lambda = 1.8$

$$p(x=1) = \frac{e^{-1.8}(1.8)^1}{1!} = 0.2975$$

23. In a company 5 % defective components are produced. What is the probability that atleast 5 components are to be examined in order to get 3 defectives?

Soln: To get 3 defectives, 3 or more components must be examined.

p = 5 % = 0.05, q = 1- p=0.95 and k = success = 3

$$p(X=x)=(x-1)c_{k-1}\,p^k\,q^{x-k}\ ,\ x=k,k+1,k+2,...$$

$$p(x\geq 5)=1-p(x<5)$$

$$=1-\left[p(x=3)+p(x=4)\right]$$

$$=1-\left[2c_2(0.05)^3(0.95)^0+3c_2(0.05)^3(0.95)^1\right]$$

$$=1-0.00048=0.9995$$

24. A discrete r.v X has mgf $M_x(t)=e^{2(e^t-1)}$ .Find E(x), Var(x), and p(x=0).

Soln: Given $M_x(t)=e^{2(e^t-1)}$

We know that mgf of poisson is $M_x(t)=e^{\lambda(e^t-1)}$

Therefore λ=2

In poisson E(x) = var(x) = λ

$\therefore$ *Mean* $E(x)=\text{var}\,(x)=2$

$$p(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$$

$$\therefore\ p(X=0)=\frac{e^{-\lambda}\lambda^0}{0!}=e^{-\lambda}=e^{-2}=0.1353$$

25. Find the mean and variance of geometric distribution.

Soln: The pmf of Geometric distribution is given by

$$p(X=x)=p\,q^{x-1}\ ,x=1,2,3,.....$$

$$Mean\quad E(x)=\sum x\ \ p(x)$$

$$=\sum_{x=1}^{\infty} x\ \ p\ \ q^{x-1}$$

$$=p\sum_{x=1}^{\infty} x\ \ q^{x-1}$$

$$=p\left[1\ \ q^{1-1}+2q^{2-1}+3q^{3-1}+.....\right]$$

$$=p\left[1\ \ +2q+3q^2+.....\right]$$

$$=p[1-q]^2$$

$$=p\ \ p^{-2}$$

$$=p^{-1}$$

$$=\frac{1}{p}$$

Mean $= \frac{1}{p}$

$$= \sum_{x=1}^{\infty} \left[x(x+1) - x \right] \; p \; q^{x-1}$$

$$= \sum_{x=1}^{\infty} x(x+1)p \; q^{x-1} - \sum_{x=1}^{\infty} x \; p \; q^{x-1}$$

$$= 1(1+1)p \; q^{1-1} + 2(2+1)pq^{2-1} + 3(3+1)pq^{3-1} + ... - \frac{1}{p}$$

$$= 2p \; + 2(3)pq^{1} + 3(4)pq^{2} + ... - \frac{1}{p}$$

$$= 2p \; \left[1 + 3q \; + 6q^{2} + ... \right] - \frac{1}{p}$$

$$= 2p \; [1-q]^{-3} - \frac{1}{p} = 2 \; p \; p^{-3} - \frac{1}{p}$$

$$= \frac{2}{p^{2}} - \frac{1}{p}$$

Variance $= E(x^2) - [E(x)]^2$

$$= \frac{2}{p^2} - \frac{1}{p} - \left(\frac{1}{p} \right)^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Variance =

26. It has been claimed that in 60 % of all solar heat installation the utility bill is reduced by atleast one-third. Accordingly, what are the probabilities that the utility bill will be reduced by atleast one-third in atleast four of five installations?

Soln: Given n=5 , p=60 % =0.6 and q=1-p=0.4

$$p(x \geq 4) = p[x=4] + p[x=5]$$

$$= 5c_4 (0.6)^4 (0.4)^{5-4} + 5c_5 (0.6)^5 (0.4)^{5-5}$$

$$= 0.337$$

27. The no. of monthly breakdowns of a computer is a r.v. having poisson distribution with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Soln: $p(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$, $given \; \lambda = 1.8$

$$p(x=1) = \frac{e^{-1.8} (1.8)^{1}}{1!} = 0.2975$$

28. In a company 5 % defective components are produced. What is the probability that atleast 5 components are to be examined in order to get 3 defectives?

Soln: To get 3 defectives, 3 or more components must be examined.

p=5 % =0.05 , q = 1- p=0.95 and k=success=3

$p(X=x)=(x-1)c_{k-1}\, p^k q^{x-k}$, $x=k,k+1,k+2,...$

$$\begin{aligned} p(x\geq 5) &= 1-p(x<5) \\ &= 1-\left[p(x=3)+p(x=4)\right] \\ &= 1-\left[2c_2(0.05)^3(0.95)^0+3c_2(0.05)^3(0.95)^1\right] \\ &= 1-0.00048 \\ &= 0.9995 \end{aligned}$$

29. A discrete r.v X has mgf $M_x(t)=e^{2(e^t-1)}$ .Find E(x), var(x), and p(x=0).

Soln: Given $M_x(t)=e^{2(e^t-1)}$

We know that mgf of poisson is $M_x(t)=e^{\lambda(e^t-1)}$

Therefore λ=2

In Poisson distribution, E(x) = var(x) = λ

$\therefore\ Mean\ E(x)=\text{var}\,(x)=2$

$$p(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$$

$$\therefore\ p(X=0)=\frac{e^{-\lambda}\lambda^0}{0!}=e^{-\lambda}=e^{-2}=0.1353$$

30. Find the mean and variance of geometric distribution.

Soln: The pmf of Geometric distbn is given by

$p(X=x)=p\,q^{x-1}$, $x=1,2,3,.....$

$$\begin{aligned} Mean\quad E(x) &= \sum x\ p(x) = \sum_{x=1}^{\infty} x\ p\ q^{x-1} = p\sum_{x=1}^{\infty} x\ q^{x-1} \\ &= p\left[1\ q^{1-1}+2q^{2-1}+3q^{3-1}+.....\right] \\ &= p\left[1\ +2q+3q^2+.....\right] \\ &= p[1-q]^2 \\ &= p\ p^{-2} \\ &= \frac{1}{p} \end{aligned}$$

$$E(X^2)=\sum x^2 p(x)=\sum_{x=1}^{\infty}\left[x(x+1)-x\right]\ p\ \ q^{x-1}$$

$$=\sum_{x=1}^{\infty} x(x+1)p\ \ q^{x-1}-\sum_{x=1}^{\infty} x\ \ p\ \ q^{x-1}$$

$$=\ 1(1+1)p\ \ q^{1-1}+2(2+1)pq^{2-1}+3(3+1)pq^{3-1}+.....-\frac{1}{p}$$

$$=\ 2p\ \ +2(3)pq^{1}+3(4)pq^{2}+.....-\frac{1}{p}$$

$$=\ 2p\ \left[1+3q\ \ +6q^{2}+.....\right]-\frac{1}{p}$$

$$=\ 2p\ \ [1-q]^{-3}-\frac{1}{p}$$

$$=2\ \ p\ \ p^{-3}-\frac{1}{p}$$

$$=\frac{2}{p^2}-\frac{1}{p}$$

Variance $=E(x^2)-[E(x)]^2$

$$=\frac{2}{p^2}-\frac{1}{p}-\left(\frac{1}{p}\right)^2=\frac{2}{p^2}-\frac{1}{p}-\frac{1}{p^2}$$

$$=\frac{1}{p^2}-\frac{1}{p}=\frac{1-p}{p^2}\ =\frac{q}{p^2}$$

Variance $=\dfrac{q}{p^2}$

31. Find mgf of geometric distribution.

Soln: The pmf of geometric distribution is given by

$$Mgf\ \ M_x(t)=E(e^{tx})=\sum e^{tx}\ p(x)$$

$$=\sum_{x=1}^{\infty}e^{tx}p\,q^{x-1}=\sum_{x=1}^{\infty}e^{tx}p\ q^{x}\ q^{-1}$$

$$=\frac{p}{q}\sum_{x=1}^{\infty}\left(e^{t}\right)^{x}q^{x}=\frac{p}{q}\sum_{x=1}^{\infty}\left(qe^{t}\right)^{x}$$

$$=\ \frac{p}{q}\left[qe^{t}+\left(qe^{t}\right)^{2}+\left(qe^{t}\right)^{3}+...\right]$$

$$=\ \frac{p}{q}\ qe^{t}\left[1+qe^{t}+\left(qe^{t}\right)^{2}+...\right]$$

$$=\ pe^{t}\left(1-qe^{t}\right)^{-1}1-qe^{t}$$

$$\therefore\ \ M_x(t)=1-qe^{t}$$

32. Show that for the uniform distribution $f(x)=\dfrac{1}{2a}, -a<x<a$, the mgf about origin is $\dfrac{\sinh at}{at}$

Soln: Given $f(x)=\dfrac{1}{2a}, -a<x<a$

$$M_x(t)=E\left[e^{tx}\right]=\int_{-\infty}^{\infty} e^{tx} f(x)dx=\int_{-a}^{a} e^{tx}\frac{1}{2a}dx=\frac{1}{2a}\int_{-a}^{a} e^{tx}dx=\frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$=\frac{1}{2at}\left[e^{at}-e^{-at}\right]$$

$$=\frac{1}{2at}2\sinh at$$

$$=\frac{\sinh at}{at}$$

$$M_x(t)=\frac{\sinh at}{at}$$

33. Define exponential density function and find mean and variance of the same.

Soln: The density function of exponential distribution is given by

$$f(x)=\lambda e^{-\lambda x}, x\geq 0$$

$$Mean=E[x]=\int_{-\infty}^{\infty} xf(x)dx=\int_{0}^{\infty} x\lambda e^{-\lambda x}dx=\lambda\int_{0}^{\infty} xe^{-\lambda x}dx=\lambda\left[\frac{-xe^{-\lambda x}}{\lambda}-\frac{e^{-\lambda x}}{\lambda^2}\right]_0^{\infty}$$

$$=\lambda\left[(0-0)-\left(0-\frac{1}{\lambda^2}\right)\right]$$

$$=\lambda\left(\frac{1}{\lambda^2}\right)$$

$$=\frac{1}{\lambda}$$

$$Mean=\frac{1}{\lambda}$$

$$E\left[x^2\right]=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_{0}^{\infty} x^2\lambda e^{-\lambda x}dx=\lambda\int_{0}^{\infty} x^2 e^{-\lambda x}dx$$

$$=\lambda\left[\frac{-x^2e^{-\lambda x}}{\lambda}-\frac{2xe^{-\lambda x}}{\lambda^2}-\frac{2e^{-\lambda x}}{\lambda^3}\right]_0^{\infty}$$

$$=\lambda\left[(0-0-0)-\left(0-0-\frac{2}{\lambda^3}\right)\right]$$

$$=\lambda\left(\frac{2}{\lambda^3}\right)$$

$$=\frac{2}{\lambda^2}$$

$$Variance=E\left(x^2\right)-[E(x)]^2=\frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

34. Given the R.v X with density function $f(x)=\begin{cases}2x & , 0<x<1\\ 0 & , elsewhere\end{cases}$. Find the pdf of $y=8x^3$

Soln: The pdf of y is given by $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$, where $y=8x^3$.

$$x^3=\frac{y}{8}\Rightarrow x=\left(\frac{y}{8}\right)^{\frac{1}{3}}$$

$$\frac{dx}{dy}=\frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\frac{1}{8}=\frac{1}{24}\left(\frac{y}{8}\right)^{\frac{-2}{3}}$$

$$f_Y(y)=2x\,\frac{1}{24}\left(\frac{y}{8}\right)^{\frac{-2}{3}}=\frac{2}{24}\left(\frac{y}{8}\right)^{\frac{1}{3}}\left(\frac{y}{8}\right)^{\frac{-2}{3}}=\frac{2}{24}\left(\frac{y}{8}\right)^{\frac{-1}{3}}=\frac{1}{12}\left(\frac{8}{y}\right)^{\frac{1}{3}}=\frac{1}{6}\left(y\right)^{\frac{-1}{3}}$$

$$f_Y(y)=\frac{1}{6}\left(y\right)^{\frac{-1}{3}}\ ,\ 0<y<8$$

35. If the pdf of X is $f_X(x)=2x\ ,0<x<1$, then find the pdf of Y = 3 x + 1.

Soln: The pdf of Y is given by $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$, where Y = 3 x + 1

$$Y=3x+1\ \Rightarrow x=\frac{y-1}{3}$$

$$\frac{dx}{dy}=\frac{1}{3}\Rightarrow\left|\frac{dx}{dy}\right|=\frac{1}{3}$$

$$\therefore\ f_Y(y)=2x\,\frac{1}{3}=\frac{2}{3}\left(\frac{y-1}{3}\right)=\frac{2}{9}(y-1)\ ,\ 1<y<4$$

36. A r.v. X has pdf $f(x)=\begin{cases}e^{-x} & ,x>0\\ 0 & ,x<0\end{cases}$ find the density function of $\frac{1}{x}$.

Soln: The pdf of Y is given by $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$ where $y=\frac{1}{x}$

$$y=\frac{1}{x}\Rightarrow x=\frac{1}{y}\qquad\qquad \frac{dx}{dy}=-\frac{1}{y^2}\Rightarrow\left|\frac{dx}{dy}\right|=\frac{1}{y^2}$$

$$f_Y(y)=e^{-x}\,\frac{1}{y^2}=\frac{e^{-1/y}}{y^2}\quad ,y>0$$

37. If X has an exponential distribution with parameter α, find the pdf of y = log x.

Soln: The pdf of exponential distribution is $f(x)=\alpha e^{-\alpha x}$

The pdf of Y is given by $f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$ where y = log x

$y = \log x \Rightarrow x = e^{y}$, $\quad \dfrac{dx}{dy} = e^{y} \Rightarrow \left|\dfrac{dx}{dy}\right| = e^{y}$

$\therefore \quad f_Y(y) = \alpha \; e^{-\alpha x} e^{y}$,

$\Rightarrow f_Y(y) = \alpha \; e^{-\alpha e^{y}} e^{y}, \; -\infty < y < \infty$

38. If $Y = x^2$, where x is a Gaussian r.v. with zero mean and variance σ^2, find the pdf of the variable Y.

Soln: $F_Y(y) = p(Y \le y) = p\left[x^2 \le y\right]$

$$= p\left[-\sqrt{y} \le x \le \sqrt{y}\right], \text{ if } y \ge 0$$

$$= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \quad ----------(1)$$

and $F_Y(y) = 0 \;\; \text{if} \;\; y < 0$

Differentiating (1) with respect to y , we have

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right] & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases} \quad --------(2)$$

It is given that X follows $N(0, \sigma)$

$$\therefore \quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \; -\infty < x < \infty$$

Using this value in (2) , we have

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}}\right] = \frac{1}{2\sqrt{y}} \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{y}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}}$$

$$\therefore \quad f_Y(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{y}{2\sigma^2}}, \; y > 0$$

39. If X is a Gaussian r.v. with zero mean and variance σ^2, find the pdf of $Y = |x|$.

Soln: $F_Y(y) = p[Y \le y] = p\,[|x| \le y]$

$$= p[-y \le x \le y]$$

$F_Y(y) = F_X(y) - F_X(-y) \quad -------(1)$

Differentiating (1) both sides w.r.t. y , we have

$$f_Y(y) = f_X(y) + f_X(-y) \quad , y > 0 \quad ----(2)$$

Since $X \to N(0, \sigma)$, the density function is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \; -\infty < x < \infty$$

$$(2) \Rightarrow f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}$$

$$\Rightarrow f_Y(y) = \frac{2}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}}, \; y > 0.$$

40. If X is a r.v. with cdf as F(x), show that the r.v. Y = F(x) is uniformly distributed in (0,1)

Soln: The pdf of Y is given by $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$ where Y = F(x)

Y = F(x) $\frac{dy}{dx} = \frac{d}{dx}F(x) = f(x)$

Then $f_Y(y) = f(x).\frac{1}{f(x)} = 1$

So, Y is uniformly distributed in (0,1).

PART B (8 Marks)

1.

A box containing 5 green balls and 3 red colour balls. Find the probability of selecting 3 green colour balls one by one (i) without replacement (ii) with replacement

Solution:

(i) Selection without replacement

Selecting 3 balls out of 8 balls = $8C_3$ ways

i.e n(S) = $8C_3$

Selecting 3 green balls in $5C_3$ ways

$\therefore$ P(3 green balls) = $\frac{5C_3}{8C_3} = \frac{5\times4\times3}{8\times7\times6} = \frac{5}{28}$

(ii) Selection with replacement

When a ball is drawn and replaced before the next draw, the number of balls in the box remains the same. Also the 3 events of drawing a green ball in each case is independent. $\therefore$ Probability of drawing a green ball in each case is $\frac{5}{8}$

The event of selecting a green ball in the first, second and third event are same,

$\therefore$ Probability of drawing

3 green balls = $\frac{5}{8} \times \frac{5}{8} \times \frac{5}{8} = \frac{125}{512}$

2.

In a certain town, males and females form 50 percent of the population. It is known that 20 percent of the males and 5 percent of the females are unemployed. A research student studying the employment situation selects unemployed persons at random. What is the probability that the person selected is (i) a male (ii) a female?

Out of 50% of the population 20% of the males are unemployed. i.e $\frac{50}{100} \times \frac{20}{100} = \frac{10}{100} = 0.10$

Out of 50% the population 5% of the females are unemployed.

$$\text{i.e } \frac{50}{100} \times \frac{5}{100} = \frac{25}{1000} = 0.025$$

Based on the above data we can form the table as follows:

	Employed	Unemployed	Total
Males	0.400	0.100	0.50
Females	0.475	0.025	0.50
Total	0.875	0.125	1.00

Let a male chosen be denoted by M and a female chosen be denoted by F

Let U denotes the number of unemployed persons then

(i) $P(M/U) = \frac{P(M \cap U)}{P(U)} = \frac{0.10}{0.125} = 0.80$

(ii) $P(F/U) = \frac{P(F \cap U)}{P(U)} = \frac{0.025}{0.125} = 0.20$

3.

In a bolt factory machines A_1, A_2, A_3 manufacture respectively 25%, 35% and 40% of the total output. Of these 5, 4, and 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A_2 ?

$P(A_1) = P($ that the machine A_1 manufacture the bolts$) = 25\%$
$= 0.25$

Similarly $P(A_2) = 35\% = 0.35$ and

$P(A_3) = 40\% = 0.40$

Let B be the event that the drawn bolt is defective.

$P(B/A_1) = P$ (that the defective bolt from the machine A_1)
$= 5\ \% = 0.05$

Similarly, $P(B/A_2) = 4\% = 0.04$

And $P(B/A_3) = 2\% = 0.02$

We have to find $P(A_2/B)$.
Hence by Bayes' theorem, we get

$$P(A_2/B). = \frac{P(A_2)P(B/A_2)}{P(A_1)P(B/A_1)+P(A_2)P(B/A_2)+P(A_3)P(B/A_3)}$$

$$= \frac{(0.35)(0.04)}{(0.25)(0.05)+(0.35)(0.04)+(0.4)(0.02)}$$

$$= \frac{28}{69}$$

$$= 0.4058$$

4.

A company has two plants to manufacture motorbikes. Plant I manufactures 80 percent of motor bikes, and plant II manufactures 20 percent. At Plant I 85 out of 100 motorbikes are rated standard quality or better.

At plant II only 65 out of 100 motorbikes are rated standard quality or better.

(i) What is the probability that the motorbike, selected at random came from plant I. if it is known that the motorbike is of standard quality?

(ii) What is the probability that the motorbike came from plant II if it is known that the motor bike is of standard quality?

Let A_1 be the event of drawing a motorbike produced by plant I.

A_2 be the event of drawing a motorbike produced by plant II.

B be the event of drawing a standard quality motorbike produced by plant I or plant II.

Then from the first information, $P(A_1) = 0.80$, $P(A_2) = 0.20$

From the additional information

$P(B/A_1) = 0.85$

$P(B/A_2) = 0.65$

The required values are computed in the following table.

The final answer is shown in last column of the table.

Event	Prior probability $P(A_i)$	Conditional probability of event B given A_i $P(B/A_i)$	Joint probability $P(A_i \cap B) = P(A_i)P(B/A_i)$	Posterior (revised) probability $P(A_i/B) = \frac{P(A\cap B)}{P(B)}$
A_1	0.80	0.85	0.68	$\frac{0.68}{0.81} = \frac{68}{81}$
A_2	0.20	0.65	0.13	$\frac{0.13}{0.81} = \frac{13}{81}$
Total	1.00		P(B) = 0.81	1

Without the additional information, we may be inclined to say that the standard motor bike is drawn from plant I output, since $P(A_1) = 80\%$ is larger than $P(A_2) = 20\%$

5. The density function of a random variable X is given by $f(x) = kx(2-x),\ \ 0 \le x \le 2$. Find k, mean, variance and r^{th} moment.

Solution:

Since $f(x)$ is pdf $\int_0^2 f(x)\,dx = 1$

$$\int_0^2 kx(2-x)\,dx = 1 \Rightarrow k\int_0^2 (2x - x^2)\,dx = 1 \Rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow k\left(4 - \frac{8}{3}\right) = 1 \Rightarrow k = \frac{3}{4}$$

$$\mu_r' = \int_0^2 x^r \frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx = \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2 = \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right] = 6(2^r)\frac{1}{(r+2)(r+3)}$$

put $r = 1,2$ $\quad \mu_1' = \frac{12}{(3)(4)} = 1 \quad \mu_2' = \frac{24}{(4)(5)} = \frac{6}{5}$

Mean = 1 and variance = $\mu_2' - \mu_1'^2 = \frac{6}{5} - 1 = \frac{1}{5}$

6. The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand:	1	2	3	4	5	6	7	8
Probability:	0.08	0.3k	0.19	0.24	k^2	0.1	0.07	0.04

Determine the expected demand for watches. Also, compute the variance.

Solution:

$$\sum P(x) = 1$$

$$(0.08) + (0.3k) + (0.19) + (0.24) + (k^2) + (0.1) + (0.07) +)(0.04) = 1$$

$$k^2 + 0.3k - 0.28 = 0 \quad \Rightarrow k = 0.4$$

$$E(X) = \sum x\, P(x) = (1)(0.18) + (2)(0.12) + (3)(0.19) + (4)(0.24) + (5)(0.16) + (6)(0.1) + (7)(0.07) + (8)(0.04)$$
$$= 4.02 \text{ is the mean}$$
$$E(X^2) = \sum x^2\, P(x) = (1)(0.18) + (4)(0.12) + (9)(0.19) + (16)(0.24) + (25)(0.16) + (36)(0.1) + (49)(0.07) + (64)(0.04)$$
$$= 19.7$$
$$Variance = E(X^2) - E(X)^2 = 19.07 - 4.02^2 = 3.54$$

7. The distribution of a random variable X is given by $F(X) = 1-(1+x)e^{-x},\ x>0$. Find the r^{th} moment, mean and variance.

Solution:

$$(i)\ f(x) = F'(x) = 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right] = xe^{-x},\quad x>0$$

$$(ii)\ E\left(X^r\right) = \mu_r' = \int_0^\infty x^r\ f(x)\,dx = \int_0^\infty x^r\ xe^{-x}\ dx = \int_0^\infty x^{r+1} e^{-x}\ dx = (r+1)!$$

$$(iii)\ E(X) = \mu_1' = (1+1)! = 2$$

$$(iv)\ E\left(X^2\right) = \mu_2' = (2+1)! = 6,\quad \text{Variance} = E(X^2) - E(X)^2 = 2$$

8. Suppose that the duration 'X' in minutes of long distance calls from your home, follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-x/5},\ x>0$. Find p (X > 5), p ($3 \le X \le 6$), mean and variance.

Solution:

$$\text{(i)}\ p(X>5) = \int_5^\infty f(x)\,dx = \int_5^\infty \frac{1}{5} e^{-\frac{x}{5}}dx = \left[-e^{-\frac{x}{5}}\right]_5^\infty = e^{-1}$$

$$\text{(ii)}\ p(3<X<6) = \int_3^6 f(x)\,dx = \int_3^6 \frac{1}{5} e^{-\frac{x}{5}}dx = \left[-e^{-\frac{x}{5}}\right]_3^6 = -e^{-1.2} + e^{-0.5}$$

$$\text{(iii)}\ E(X) = \int_0^\infty xf(x)\,dx = \int_0^\infty \frac{1}{5} xe^{-\frac{x}{5}}dx = \frac{1}{5}\left[-xe^{-\frac{x}{5}}5 - e^{-\frac{x}{5}}25\right]_0^\infty = \frac{1}{5}(0+25) = 5$$

$$\text{(iv)}\ E(X^2) = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty \frac{1}{5} x^2 e^{-\frac{x}{5}}dx = \frac{1}{5}\left[-x^2e^{-\frac{x}{5}}5 - 2xe^{-\frac{x}{5}}25 + 2e^{-\frac{x}{5}}125\right]_0^\infty = 50$$

Variance = $E(X^2) - E(X)^2 = 50 - 25 = 25$

9. A random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
f(x):	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find (i) the value of k (ii) p(1.5 < X < 4.5 | X > 2) and (iii) the smallest value of λ such that p($X \le \lambda$) > $\frac{1}{2}$.

Solution:

(i) $\sum P(x) = 1$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow k = -1, \frac{1}{10}$$

$$k = \frac{1}{10} = 0.1$$

(ii) $A = 1.5 < X < 4.5 = \{2,3,4\}$

$$B = X > 2 = \{3,4,5,6,7\}$$

$$A \cap B = \{3,4\}$$

$$p(1.5 < X < 4.5 | X > 2) = p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(3,4)}{p(3,4,5,6,7)}$$

$$= \frac{2k + 3k}{2k + 3k + k^2 + 2k^2 + 7k^2 + k} = \frac{5k}{10k^2 + 6k} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(iii)

X p(X) F(X)

0 0 0

2 2k = 0.2 0.3

3 2k = 0.2 0.5

4 3k = 0.3 0.8

5 k^2=0.01 0.81

6 $2k^2$ = 0.02 0.83

7 $7k^2$+k = 0.17 1.00

From the table for X = 4,5,6,7 P(X) > $\frac{1}{2}$ and the smallest value is 4

Therefore λ = 4.

10. Find the MGF of triangular distribution whose density function is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & elsehwere \end{cases}$$

.Hence its mean and variance.

Solution:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\, dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2 = \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

Expanding the above in powers of t, we get

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{1}{t^2}\left[\begin{array}{l}\left(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\right) \\ -2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + .\right) - 1\end{array}\right]$$

$$= \frac{1}{t^2}\left(\frac{2t^2}{2!} + \frac{6t^3}{3!} + \frac{14t^4}{4!} + \ldots\right)$$

$$= 1 + t + \frac{7t^2}{12} + \frac{t^3}{4} + \ldots.$$

Mean = E(X) = (coefficient of t) 1! = 1

$E(X^2)$ = (coefficient of t^2) 2! = $\frac{7}{6}$

Variance = $E(X^2) - E(X)^2 = \frac{1}{6}$

11. Find the MGF of the RV X, whose pdf is given by $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$.

Hence find its mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{0} e^{tx} e^{x}\, dx + \int_{0}^{\infty} e^{tx} e^{-x}\, dx$$

$$= \int_{-\infty}^{0} e^{(t+1)x}\, dx + \int_{0}^{\infty} e^{-(1-t)x}\, dx = \left[\frac{e^{(t+1)x}}{(t+1)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \ldots.$$

Mean = E(X) = (coefficient of t) 1! = 0

$E(X^2)$ = (coefficient of t^2)2! =2

Variance = $E(X^2) - E(X)^2$ = 2

12. The p.m.f of a RV X, is given by $p(X = j) = \frac{1}{2^j}$, $j = 1, 2, 3\ldots$ Find MGF, mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=0}^{\infty}\left(\frac{e^t}{2}\right)^x = \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \ldots.$$

$$= \frac{e^t}{2}\left(1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + ..\right) = \frac{e^t}{2}\frac{1}{1 - \frac{e^t}{2}} = \frac{e^t}{2 - e^t}$$

Differentiating twice with respect to t

$$M'_X(t) = \frac{\left(2-e^t\right)\left(e^t\right) - e^t\left(-e^t\right)}{\left(2-e^t\right)^2} = \frac{2e^t}{\left(2-e^t\right)^2}$$

$$M''_X(t) = \frac{\left(2-e^t\right)^2\left(2e^t\right) - 2e^t\,2\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4} = \frac{4e^t + 2e^{2t}}{\left(2-e^t\right)^3}$$

put t = 0 above $E(X) = M'_X(0) = 2$

$E\left(X^2\right) = M''_X(0) = 6$

$Variance = E\left(X^2\right) - E(X)^2 = 6 - 4 = 2$

13. Find MGF of the RV X, whose pdf is given by $f(x) = \lambda e^{-\lambda x}$, $x > 0$ and hence find the first four central moments.
Solution

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} e^{-(\lambda - t)x}\,dx = \lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty} = \frac{\lambda}{(\lambda - t)}$$

Expanding in powers of t

$$M_X(t) = \frac{\lambda}{(\lambda-t)} = \frac{1}{1-\left(\frac{t}{\lambda}\right)} = 1 + \left(\frac{t}{\lambda}\right) + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \ldots$$

Taking the coefficient, we get the raw moments about origin

$E(X) = (coefficient\ of\ t)1! = \frac{1}{\lambda}$, $\quad E\left(X^2\right) = \left(coefficient\ of\ t^2\right)2! = \frac{2}{\lambda^2}$

$E\left(X^3\right) = \left(coefficient\ of\ t^3\right)3! = \frac{6}{\lambda^3}$, $\quad E\left(X^4\right) = \left(coefficient\ of\ t^4\right)4! = \frac{24}{\lambda^4}$

and the central moments are

$\mu_1 = 0$

$\mu_2 = \mu'_2 - 2C_1\mu'_1\mu'_1 + \mu'^2_1$

$= \frac{2}{\lambda^2} - 2\frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$\mu_3 = \mu'_3 - 3C_1\mu'_2\mu'_1 + 3C_2\mu'_1\mu'^2_1 - \mu'^3_1$

$= \frac{6}{\lambda^3} - 3\frac{2}{\lambda^2}\frac{1}{\lambda} + 3\frac{1}{\lambda}\frac{1}{\lambda^2} - \frac{1}{\lambda^3} = \frac{2}{\lambda^3}$

$\mu_4 = \mu'_4 - 4C_1\mu'_3\mu'_1 + 4C_2\mu'_2\mu'^2_1 - 4C_3\mu'^4_1 + \mu'^4_1$

$= \frac{24}{\lambda^4} - 4\frac{6}{\lambda^3}\frac{1}{\lambda} + 6\frac{2}{\lambda^2}\frac{1}{\lambda^2} - 4\frac{1}{\lambda^4} + \frac{1}{\lambda^4} = \frac{9}{\lambda^4}$

14. If the MGF of a (discrete) RV X is $\frac{1}{5-4e^t}$ find the distribution of X and p (X = 5 or 6).

Solution:

$$M_X(t)=\frac{1}{5-4e^t}=\frac{1}{5\left(1-\frac{4e^t}{5}\right)}=\frac{1}{5}\left[1+\left(\frac{4e^t}{5}\right)+\left(\frac{4e^t}{5}\right)^2+\left(\frac{4e^t}{5}\right)^3+\ldots\right]$$

By definition

$$M_X(t)=E\left(e^{tX}\right)=\sum e^{tx}p(x)=1+e^{t0}p(0)+e^t p(1)+e^{t2}p(2)+\ldots$$

On comparison

$$p(0)=\frac{1}{5} \qquad p(1)=\frac{4}{25} \qquad p(2)=\frac{16}{125} \qquad p(3)=\frac{64}{625}$$

In general $p(X=r)=\frac{1}{5}\left(\frac{4}{5}\right)^r, \quad r=0,1,2,3$

$$p(X=5\ or\ 6)=p(X=5)+p(X=6)$$

$$=\frac{1}{5}\left(\frac{4}{5}\right)^5+\frac{1}{5}\left(\frac{4}{5}\right)^6=\frac{1}{5}\left(\frac{4}{5}\right)^5\left(1+\frac{4}{5}\right)=\frac{9}{25}\left(\frac{4}{5}\right)^5$$

15. If X has the probability density function $f(x)=k\,e^{-3x}$, $x>0$. Find (i) k (ii) p (0.5 ≤X≤1) (iii) Mean of X.

Solution:

(i) $p(0.5\le X\le 1)=\int_{0.5}^{1} f(x)\,dx=\int_{0.5}^{1} 3e^{-3x}\,dx=3\left[\frac{-e^{-3x}}{3}\right]_{0.5}^{1}=-e^{-3}+e^{-1.5}$

(iii) $Mean=E(X)=\int_0^\infty x\,f(x)\,dx=\int_0^\infty 3x\,e^{-3x}\,dx=3\left[-x\frac{e^{-3x}}{3}-\frac{e^{-3x}}{9}\right]_0^\infty=3\left(\frac{1}{9}\right)=\frac{1}{3}$

16. If a RV X has the pdf $f(x)=\begin{cases}\frac{1}{4}, & |x|<2\\ 0, & otherwise\end{cases}$. Obtain (i) p(X <1) (ii) p($|X|$>1) (iii) p(2X+3 > 5) (iv) p ($|X|$ < 0.5 | X < 1.

Solution:

(i) p(X < 1) = $\int_{-2}^{1}\frac{1}{4}dx=\frac{1}{4}[x]_{-2}^{1}=\frac{3}{4}$

(ii) p($|X|\le 1$) = $\int_{-1}^{1}\frac{1}{4}dx=\frac{1}{4}[x]_{-1}^{1}=\frac{1}{2}$

Hence p($|X|$>1)= 1 - p($|X|\le 1$) = $\frac{1}{2}$

(iii) p (2X+3 > 5) = p(X > 1) = 1- p(X ≤1) = 1- $\frac{3}{4}=\frac{1}{4}$

$$(iv)\ p(|X|<0.5/X<1)=\frac{p(|X|<0.5\cap X<1)}{p(X<1)}=\frac{p([-0.5<X<0.5]\cap X<1)}{p(X<1)}$$

$$=\frac{p([-0.5<X<0.5])}{p(X<1)}=\frac{\int_{-1}^{1}\frac{1}{4}dx}{\frac{3}{4}}=\frac{[x]_{-0.5}^{0.5}}{3}=\frac{1}{3}$$

17. If X has the distribution function

$$F(x)=\begin{cases}0, & x<1\\ \frac{1}{3}, & 1\le x<4\\ \frac{1}{2}, & 4\le x<6\\ \frac{5}{6}, & 6\le x<10\\ 1, & x>10\end{cases}$$

(1) Probability distribution of X, (2) p (2<X<6), (3) Mean, (4) variance

Solution:

(1) As there is no x term in the distribution function given is a discrete random variable. Hence the probability distribution is given by

X	1	4	6	10
$p(X)$	$\frac{1}{3}$	$\frac{1}{2}-\frac{1}{3}$	$\frac{5}{6}-\frac{1}{2}$	$1-\frac{5}{6}$
	$\frac{1}{3}$	$=\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(2) p(2<X<6) = p(4) = $\frac{1}{6}$

(3) Mean = E(X) = $\sum x\ p(x) = (1)\left(\frac{1}{3}\right)+(4)\left(\frac{1}{6}\right)+(6)\left(\frac{1}{3}\right)+(10)\left(\frac{1}{6}\right) = \frac{14}{3}$

$E(X^2) = \sum x^2\ p(x) = (1)\left(\frac{1}{3}\right)+(16)\left(\frac{1}{6}\right)+(36)\left(\frac{1}{3}\right)+(100)\left(\frac{1}{6}\right) = \frac{95}{3}$

(4) Variance = $E(X^2) - E(X)^2 = \frac{95}{3} - \frac{196}{9} = \frac{89}{9}$

18. A continuous random variable X has the distribution function

$$F(x)=\begin{cases}0 & : & x\le 1\\ k(1-x)^4 & : & 1<x\le 3\\ 0 & : & x>3\end{cases}.$$

Find *k*, the probability density function *f(x)* and *P(X <2)*.

Solution:

Since it is a distribution function

$F(\infty)=F(3)=1 \quad \Rightarrow k(3-1)^4=1 \quad \Rightarrow k=\frac{1}{16}.$

The density function is, p(X < 2) = F(2) = $\frac{1}{16}(2-1)^4 = \frac{1}{16}$.

19. If the cumulative distribution function of a R.V X is given by $F(x) = \begin{cases} 1-\dfrac{4}{x^2}, & x>2 \\ 0, & x \le 2 \end{cases}$ find (i) P(X < 3) (ii) P(4 < X < 5) (iii) P (X ≥3).

Solution:

(i) $P(X < 3) = F(3) = 1-\dfrac{4}{3^2} = \dfrac{5}{9}$.

(ii) $P(4 < X < 5) = F(5) - F(4) = \left(1-\dfrac{4}{5^2}\right) - \left(1-\dfrac{4}{4^2}\right) = \dfrac{21}{25}-\dfrac{3}{4} = \dfrac{9}{100}$

(iii) $P (X \ge 3) = 1- F(3) = 1 - \left(1-\dfrac{4}{3^2}\right) = 1- \dfrac{5}{9} = \dfrac{4}{9}$

20. Prove that Poisson distribution is the limiting case of Binomial distribution.

(or)

Poisson distribution is a limiting case of Binomial distribution under the following conditions

(i) n , the no. of trials is indefinitely large , i.e, $n \to \infty$

(ii) p, the constant probability of success in each trial is very small, i.e $p \to 0$

(iii) $np = \lambda$ *is* inf *inite or* $p = \dfrac{\lambda}{n}$ *and* $q = 1-\dfrac{\lambda}{n}$, λ*is positive real*

Soln: If X is binomial r.v with parameter n & p, then

$$p(X = x) = nc_x p^x q^{n-x}, \quad x = 0,1,2,....n$$

$$= \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))(n-x)!}{(n-x)!x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{n^x\, x!} n.n\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right).....n\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit as $n \to \infty$ on both sides

$$\lim_{n\to\infty} p(X = x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}(1.1....1).\left(e^{-\lambda}\right).1 = \frac{e^{-\lambda}\lambda^x}{x!} \quad , x = 0,1,2,.....$$

$\therefore\ p(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$, $x=0,1,2,.....$ and it is Poisson distribution.

Hence the proof.

21. Prove that the sum of two independent Poisson variates is a Poisson variate, while the difference is not a poisson variate.

Soln: Let X_1 and X_2 be independent r.v.s that follows poisson distribution with parameters λ_1 and λ_2 respectively.

Let $X=X_1+X_2$

$$p(X=n)=p(X_1+X_2=n)$$

$$=\sum_{r=0}^{n}p[X_1=r].p[X_2=n-r] \quad \sin ce\, X_1 \,\&\, X_2 are\, independent$$

$$=\sum_{r=0}^{n}\frac{e^{-\lambda_1}.\lambda_1^{\ r}}{r!}.\frac{e^{-\lambda_2}.\lambda_2^{\ n-r}}{(n-r)!}$$

$$=e^{-\lambda_1}e^{-\lambda_2}\sum_{r=0}^{n}\frac{.\lambda_1^{\ r}}{r!}.\frac{1}{n!}\frac{n!.}{(n-r)!}\lambda_2^{\ n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.\frac{n!.}{r!(n-r)!}\lambda_1^{\ r}\lambda_2^{\ n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.nc_r\lambda_1^{\ r}\lambda_2^{\ n-r}=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^{n}$$

This is poisson with parameter $(\lambda_1+\lambda_2)$

(ii) Difference is not poisson

Let $X=X_1-X_2$

$$E(X)=E[X_1-X_2]=E(X_1)-E(X_2)=\lambda_1-\lambda_2$$

$$E(X^2)=E\left[(X_1-X_2)^2\right]$$

$$=E\left[X_1^{\ 2}+X_1^{\ 2}-2X_1X_2\right]=\left[X_1^{\ 2}\right]+E\left[X_2^{\ 2}\right]-2E[X_1]E[X_2]$$

$$=(\lambda_1^{\ 2}+\lambda_1)+(\lambda_2^{\ 2}+\lambda_2)-2(\lambda_1\lambda_2)=(\lambda_1-\lambda_2)^2+(\lambda_1+\lambda_2)$$

$$\neq(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_2)$$

It is not poisson.

22. If X and Y are two independent poisson variates, show that the conditional distribution of X, given the value of X+Y is Binomial.

Soln: Let X and Y follow poisson with parameters λ_1 and λ_2 respectively.

$$p[X=r/X+Y=n]=\frac{p[X=r\, and\, X+Y=n]}{p[X+Y=n]}=\frac{p[X=r].p[X+Y=n]}{p[X+Y=n]}$$

$$=\frac{\dfrac{e^{-\lambda_1}.\lambda_1^{\ r}}{r!}.\dfrac{e^{-\lambda_2}.\lambda_2^{\ n-r}}{(n-r)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}.(\lambda_1+\lambda_2)^{n}}{n!}}$$

$$=\frac{n!}{r!(n-r)!}\frac{e^{-\lambda_1}.\lambda_1^{\ r}.e^{-\lambda_2}.\lambda_2^{\ n-r}}{e^{-\lambda 1}e^{-\lambda_2}..(\lambda_1+\lambda_2)^{n}}$$

$$= n c_r \frac{\lambda_1^r . \lambda_2^{n-r}}{(\lambda_1+\lambda_2)^r . (\lambda_1+\lambda_2)^{n-r}}$$

$$= n c_r \left[\frac{\lambda_1}{\lambda_1+\lambda_2}\right]^r \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^{n-r}$$

$$= n c_r \, p^r \, q^{n-r}$$

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ *and* $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

This is binomial distribution.

23. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast, exactly, atmost 2 defectives in a consignment of 1000 packets using poisson.

Soln: Give n = 20 , p = 0.05 , N = 1000

Mean $\lambda = n p = 1$

Let X denotes the no. of defectives.

$$p[X = x] = \frac{e^{-\lambda} . \lambda^x}{x!} = \frac{e^{-1} . 1^x}{x!} = \frac{e^{-1} .}{x!} \quad x = 0,1,2,...$$

$$p[x \ge 2] = 1 - p[x < 2] = 1 - [p(x=0) + p(x=1)] = 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!}\right] = 1 - 2e^{-1} = 0.2642$$

Therefore, out of 1000 packets, the no. of packets containing atleast 2 defectives

$$= N . p[x \ge 2] = 1000 * 0.2642 \cong 264 \ \textit{packets}$$

$$p[x = 2] = \frac{e^{-1}}{2!} = 0.18395$$

Out of 1000 packets,=N*p[x=2]=184 packets

$$p[x \le 2] = p[x=0] + p[x=1] + p[x=2] = \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} = 0.91975$$

For 1000 packets = 1000*0.91975=920 packets approximately.

24. The atoms of radioactive element are randomly disintegrating. If every gram of this element, on average, emits 3.9 alpha particles per second, what is the probability during the next second the no. of alpha particles emitted from 1 gram is (i) atmost 6 (ii) atleast 2 (iii) atleast 3 and atmost 6?

Soln: Given $\lambda = 3.9$

Let X denote the no. of alpha particles emitted

$$(i) \ p(x \le 6) = p(x=0) + p(x=1) + p(x=2) + ... + p(x=6)$$

$$= \frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!} + \frac{e^{-3.9}(3.9)^2}{2!} + ... + \frac{e^{-3.9}(3.9)^6}{6!}$$

$$= 0.898$$

$$(ii) \quad p(x \geq 2) = 1 - p(x < 2)$$
$$= 1 - \left[p(x=0) + p(x=1)\right]$$
$$= 1 - \left[\frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!}\right]$$
$$= 0.901$$

$$(iii) \quad p(3 \leq x \leq 6) = p(x=3) + p(x=4) + p(x=5) + p(x=6)$$
$$= \frac{e^{-3.9}(3.9)^3}{3!} + \frac{e^{-3.9}(3.9)^4}{4!} + \frac{e^{-3.9}(3.9)^5}{5!} + \frac{e^{-3.9}(3.9)^6}{6!}$$
$$= 0.645$$

25. Establish the memoryless property of geometric distribution.
Soln: If X is a discrete r.v. following a geometric distribution.

$$\therefore p(X = x) = pq^{x-1} \quad , x = 1, 2, ...$$

$$p(x > k) = \sum_{x=k+1}^{\infty} pq^{x-1} = p\left[q^k + q^{k+1} + q^{k+2} + ...\right] = pq^k\left[1 + q + q^2 + ...\right] = p\,q^k(1-q)^{-1}$$
$$= p\,q^k p^{-1} = q^k$$

Now

$$p\left[x > m+n / x > m\right] = \frac{p\left[x > m+n \text{ and } x > m\right]}{p\left[x > m\right]}$$
$$= \frac{p\left[x > m+n\right]}{p\left[x > m\right]} = \frac{q^{m+n}}{q^m} = q^n = p\left[x > n\right]$$
$$\therefore p\left[x > m+n / x > m\right] = p\left[x > n\right]$$

26. If X_1, X_2 be independent r.v. each having geometric distribution $pq^k, k = 0,1,2,....$ Show that th conditional distribution of X_1 given $X_1 + X_2$ is Uniform distribution.
Soln:

$$p\left[X_1 = r / X_1 + X_2 = n\right] = \frac{p\left[X_1 = r \text{ and } X_1 + X_2 = n\right]}{p\left[X_1 + X_2 = n\right]} = \frac{p\left[X_1 = r \text{ and } X_2 = n - r\right]}{\sum_{s=0}^{n} p\left[X_1 = s \text{ and } X_2 = n - s\right]}$$

$$= \frac{p\left[X_1 = r\right].p\left[X_2 = n - r\right]}{\sum_{s=0}^{n} p\left[X_1 = s\right].p\left[X_2 = n - s\right]} \quad \textit{by independent}$$

$$= \frac{q^r\,p\,q^{n-r}\,p}{\sum_{s=0}^{n} q^s\,p\,q^{n-s}\,p} = \frac{q^n}{\sum_{s=0}^{n} q^n} = \frac{1}{n+1}, \; r = 0,1,2,.....n$$

$(i.e)\ p\left[X_1 = r / X_1 + X_2 = n\right] = \dfrac{1}{n+1}$. This is Uniform distribution.

27. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.7.

(i) What is the probability that the target would be hit in 10^{th} attempt?
(ii) What is the probability that it takes him less than 4 shots?
(iii) What is the probability that it takes him an even no. of shots?
(iv) What is the average no. of shots needed to hit the target?

Soln: Let X denote the no. of shots needed to hit the target and X follows geometric distribution with pmf $p[X = x] = p\,q^{x-1}$, $x = 1,2,....$
Given p=0.7, and q = 1-p = 0.3

$(i)\ p[x=10] = (0.7)(0.3)^{10-1} = 0.0000138$

$(ii)\ p[x<4] = p(x=1) + p(x=2) + p(x=3)$

$= (0.7)(0.3)^{1-1} + (0.7)(0.3)^{2-1} + (0.7)(0.3)^{3-1} = 0.973$

$(iii)\ p[x\, is\, an\, even\, number] = p(x=2) + p(x=4) + ...$

$= (0.7)(0.3)^{2-1} + (0.7)(0.3)^{4-1} + ...$

$= (0.7)(0.3)\left[1 + (0.3)^2 + (0.3)^4.......\right]$

$= 0.21\left[1 + \left((0.3)^2\right) + \left((0.3)^2\right)^2 + ...\right]$

$= 0.21\left[1-(0.3)^2\right]^{-1} = (0.21)(0.91)^{-1} = \dfrac{0.21}{0.91} = 0.231$

(iv) Average no. of shots $= E(X) = \dfrac{1}{p} = \dfrac{1}{0.7} = 1.4286$

28. The number of personel computer (pc) sold daily at a computer world is uniformly distributed with a minimum of 2000 pc and a maximum of 5000 pc. Find (1) The probability that daily sales will fall between 2500 and 3000 pc
(2) What is the probability that the computer world will sell atleast 4000 pc's?
(3) What is the probability that the computer world will sell exactly 2500 pc's?

Soln: Let X~U(a , b) , then the pdf is given by

$$f(x) = \frac{1}{b-a} \ ,\ a < x < b$$

$$= \frac{1}{5000-2000}\ ,\quad 2000 < x < 5000$$

$$= \frac{1}{3000}\ ,\quad 2000 < x < 5000$$

(1) $p[2500 < x < 3000] = \int_{2500}^{3000} f(x)\,dx = \int_{2500}^{3000} \frac{1}{3000}\,dx = \frac{1}{3000}\left[x\right]_{2500}^{3000}$

$= \dfrac{1}{3000}[3000-2500] = 0.166$

(2) $p\left[x \geq 4000\right] = \int_{4000}^{5000} f(x)\,dx = \int_{4000}^{5000} \frac{1}{3000}\,dx = \frac{1}{3000}\left[x\right]_{4000}^{5000}$

$$= \frac{1}{3000}\left[5000 - 4000\right] = 0.333$$

(3) $p\left[x = 2500\right] = 0$, (i.e) it is particular point ,the value is zero.

29. Starting at 5.00 am every half an hour there is a flight from SanFransisco airport to Losangles .Suppose that none of three planes is completely sold out and that they always have room for passengers. A person who wants to fly to Losangles arrive at a random time between 8.45 am and 9.45 am. Find the probability that she waits (a) Atmost 10 min (b) atleast 15 min.

Soln: Let X be the uniform r.v. over the interval (0, 60)

Then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , a < x < b$$

$$= \frac{1}{60} \quad , \quad 0 < x < 60$$

(a) The passengers will have to wait less than 10 min. if she arrives at the airport

$$(5 < x < 15) + p(35 < x < 45) = \int_{5}^{15} \frac{1}{60}\,dx + \int_{35}^{45} \frac{1}{60}\,dx = \frac{1}{60}[x]_{5}^{15} + \frac{1}{60}[x]_{35}^{45} = \frac{1}{3}$$

(b) The probability that she has to wait atleast 15 min.

$$p(15 < x < 30) + p(45 < x < 60) = \int_{15}^{30} \frac{1}{60}\,dx + \int_{45}^{60} \frac{1}{60}\,dx = \frac{1}{60}\left[x\right]_{15}^{30} + \frac{1}{60}\left[x\right]_{45}^{60} = \frac{1}{2}$$

30. Establish the memoryless property of exponential distribution.

Solution:

If X is exponentially distributed, then

$$p\left[x > s+t/x > s\right] = p\left[x > t\right] \; for\, any\, s,t > 0$$

The pdf of exponential distribution is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \; otherwise \end{cases}$$

$$p(x > k) = \int_{k}^{\infty} f(x)\,dx = \int_{k}^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_{k}^{\infty} = -\left[0 - e^{-\lambda k}\right] = e^{-\lambda x} \; ------(1)$$

$$p\left[x > s+t/x > s\right] = \frac{p\left[x > s+t \text{ and } x > s\right]}{p\left[x > s\right]}$$

$$= \frac{p\left[x > s+t\ \right]}{p\left[x > s\right]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = p\left[x > t\right]$$

$$\therefore \quad p\left[x > s+t/x > s\right] = p\left[x > t\right] \; for\, any\, s,t > 0$$

31. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a) What is the probability that the repair time exceeds 2 hrs?

(b) What is the conditional probability that a repair takes atleast 11 hrs given that its direction exceeds 8 hrs?

Soln: If X represents the time to repair the machine, the density function of X is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$
$$= \frac{1}{2} e^{-x/2}, \quad x \geq 0$$

(a) $p(x > 2) = \int_2^\infty f(x)\,dx = \int_2^\infty \lambda e^{-\lambda x}\,dx = \int_2^\infty \frac{1}{2} e^{\frac{-x}{2}}\,dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^\infty = -\left[0 - e^{-1}\right] = 0.3679$

(b) $p\left[x \geq 11/x > 8\right] = p\left[x > 3\right] = \int_3^\infty f(x)\,dx = \int_3^\infty \lambda e^{-\lambda x}\,dx = \int_3^\infty \frac{1}{2} e^{\frac{-x}{2}}\,dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{--\frac{1}{2}}\right]_3^\infty$

$$= -\left[0 - e^{-\frac{3}{2}}\right] = e^{-\frac{3}{2}} = 0.2231$$

32. Derive the mean, variance and MGF of Binomial distribution.

The Probability mass function of Binomial distribution is

$P(X = x) = nC_x p^x q^{n-x}; \; x = 0,1,2,3...n$

MGF $M_X(t) = E(e^{tX})$

$$E(e^{tX}) = \sum_{x=0}^{n} e^{tX} P(X)$$
$$= \sum_{x=0}^{n} e^{tX} nC_x p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \left(pe^t\right)^X nC_x q^{n-x}$$
$$= nC_0 q^n + nC_1 \left(pe^t\right)^1 q^{n-1} + ... + nC_n \left(pe^t\right)^n$$
$$= \left(q + pe^t\right)^n$$

To find Mean form the MGF

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$
$$= n\left(q + pe^t\right)^{n-1} (pe^t)\Big|_{t=0}$$
$$= n(q+p)^n p = np \quad (\because p + q = 1)$$

To find Variance:

$$E(X^2)=\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}$$

$$=\left(\frac{d}{dt}n\left(q+pe^t\right)^{n-1}(pe^t)\right)\bigg|_{t=0}$$

$$=\left(n(n-1)(q+pe^t)^{n-2}(pe^t)^2+n\left(q+pe^t\right)^{n-1}(pe^t)\right)\bigg|_{t=0}$$

$$=n(n-1)(q+p)^{n-2}(p)^2+n\left(q+p\right)^{n-1}(p)$$

$$=n(n-1)p^2+np \quad (\because p+q=1)$$

$$Var=\sigma^2=E(X^2)-\left(E(X)\right)^2$$

$$=n(n-1)p^2+np-n^2p^2$$

$$=np-np^2$$

$$=np(1-p)=npq$$

33. Find the mean, variance and MGF of Poisson Distribution.

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!};\ \ x=0,1,2,3...$$

MGF $M_X(t)=E(e^{tX})$

$$E(e^{tX})=\sum_{x=0}^{\infty}e^{tX}P(X)$$

$$=\sum_{x=0}^{\infty}e^{tX}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=\sum_{x=0}^{\infty}\frac{e^{-\lambda}\left(\lambda e^t\right)^x}{x!}$$

$$=e^{-\lambda}\left[\frac{\left(\lambda e^t\right)^0}{0!}+\frac{\left(\lambda e^t\right)^1}{1!}+\frac{\left(\lambda e^t\right)^2}{2!}+.....\right]$$

$$=e^{-\lambda}\left(e^{\lambda e^t}\right)=e^{\lambda\left(e^t-1\right)}$$

To find Mean form the MGF:

$$E\left(X\right)=\left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$=\left[\frac{d}{dt}e^{-\lambda}\left(e^{\lambda e^t}\right)\right]_{t=0}$$

$$=e^{-\lambda}e^{\lambda e^t}(\lambda e^t)\Big|_{t=0}$$

$$=e^{-\lambda}e^{\lambda}(\lambda)=\lambda$$

To find Variance:

$$E(X^2)=\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}$$

$$=\left[\frac{d}{dt}e^{-\lambda}e^{\lambda e^t}(\lambda e^t)\right]_{t=0}$$

$$=\left[e^{-\lambda}\frac{d}{dt}\left(e^{\lambda e^t}\lambda e^t\right)\right]_{t=0}$$

$$=e^{-\lambda}\left[e^{\lambda e^t}(\lambda e^t)^2+e^{\lambda e^t}\lambda e^t\right]\Big|_{t=0}$$

$$=e^{-\lambda}\left[e^{\lambda}(\lambda)^2+e^{\lambda}\lambda\right]$$

$$=e^{-\lambda}e^{\lambda}\left[(\lambda)^2+\lambda\right]=\lambda^2+\lambda$$

$$Var=\sigma^2=E(X^2)-\left(E(X)\right)^2$$

$$=\lambda^2+\lambda-\lambda^2=\lambda$$

34. Find the mean, variance and MGF of Geometric distribution.

$$p(X=x)=pq^{x-1} \qquad , x=1,2,.....$$

MGF $M_X(t)=E(e^{tX})$

$$E(e^{tX})=\sum_{x=1}^{\infty}e^{tX}pq^{x-1}$$

$$=\sum_{x=1}^{\infty}e^{tX}pq^{x-1}$$

$$=\sum_{x=1}^{\infty}\frac{p}{q}\left(qe^t\right)^X$$

$$=\frac{p(qe^t)}{q}\left[1+qe^t+(qe^t)^2+.....\right]$$

$$=pe^t\left(1-qe^t\right)^{-1}=\frac{pe^t}{1-qe^t}$$

To find Mean form the MGF :

$$E(X)=\left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$=\left[\frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\right]_{t=0}$$

$$=\left(\frac{\left(1-qe^t\right)pe^t-pe^t\left(-qe^t\right)}{\left(1-qe^t\right)^2}\right)_{t=0}$$

$$=\left(\frac{(1-q)p-p(-q)}{(1-q)^2}\right)=\frac{p}{p^2}=\frac{1}{p}$$

To find Variance:

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{\left(1-qe^t\right)pe^t - pe^t\left(-qe^t\right)}{\left(1-qe^t\right)^2}\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{pe^t}{\left(1-qe^t\right)^2}\right)\right]_{t=0}$$

$$= \left(\frac{\left(1-qe^t\right)^2 pe^t - 2\left(1-qe^t\right)\left(-qe^t\right)\left(pe^t\right)}{\left(1-qe^t\right)^4}\right)\Bigg|_{t=0}$$

$$= \left(\frac{(1-q)\left[(1-q)p - 2(-q)(p)\right]}{(1-q)^4}\right) = \frac{(1-q)\left[p - pq + 2pq\right]}{(1-q)^4}$$

$$= \frac{\left[p+pq\right]}{p^3} = \frac{1}{p^2} + \frac{q}{p^2}$$

$$Var = \sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

35. Find the Mean, variance and MGF of Uniform distribution.

The pdf of Uniform distribution is $f(x) = \frac{1}{b-a}$; $a \le x \le b$

$$E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)}$$

$$= \frac{(b^2 + ab + a^2)}{3}$$

Variance

$$\sigma^2 = E(X^2) - \left(E(X)\right)^2$$
$$= \frac{(b^2+ab+a^2)}{3} - \left(\frac{a+b}{2}\right)^2$$
$$= \frac{(b^2+ab+a^2)}{3} - \left(\frac{a^2+2ab+b^2}{4}\right)$$
$$= \frac{\left(4b^2+4ab+4a^2\right) - 3a^2+6ab+3b^2}{12}$$
$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

MGF

$$M_X(t) = E(e^{tX})$$
$$= \int_a^b e^{tX}\frac{1}{b-a}dx$$
$$= \frac{1}{b-a}\left(\frac{e^{bt}-e^{at}}{t}\right)$$
$$= \frac{e^{bt}-e^{at}}{t(b-a)}$$

36. Find the mean, variance and MGF of the Exponential distribution.

Solution

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^{\infty} e^{-(\lambda-t)x}\,dx = \lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty} = \frac{\lambda}{(\lambda-t)}$$

Expanding in powers of t

$$M_X(t) = \frac{\lambda}{(\lambda-t)} = \frac{1}{1-\left(\frac{t}{\lambda}\right)} = 1+\left(\frac{t}{\lambda}\right)+\left(\frac{t}{\lambda}\right)^2+\left(\frac{t}{\lambda}\right)^3+\ldots$$

Taking the coefficient, we get the raw moments about origin

$$E(X) = (coefficient\ \ of\ \ t)1! = \frac{1}{\lambda}, \qquad E\left(X^2\right) = \left(coefficient\ of\ t^2\right)2! = \frac{2}{\lambda^2}$$

and the central moments are

$$\mu_1 = 0\ (Mean)$$
$$\mu_2 = \mu_2' - \mu_1'^2$$
$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}(Variance)$$

37. Derive the mean, variance and MGF of Normal distribution.

Mean (or Expectation) and Variance of a Normal Distribution :

We have,

$$\text{Mean} = \mu_1' = E(X) \quad \text{... by definition of raw moments}$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \quad \text{... by definition of expectation}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad \text{... using } f(x)$$

put $\frac{x-\mu}{\sigma} = z$ i.e. $x = \mu + \sigma z$

$\therefore$ $dx = \sigma\, dz$

and, when $x \to -\infty$, $z \to -\infty$

and $x \to +\infty$, $z \to +\infty$

$$\therefore \quad \text{Mean} = \int_{-\infty}^{\infty} (\mu + \sigma z) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \sigma dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz + 0 \quad \text{... } \because z e^{-z^2/2} \text{ is an odd function of } z.$$

$$= \mu \int_{-\infty}^{\infty} \phi(z)\, dz \quad \text{where } \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \text{ is the p.d.f. of variate } z$$

$$= \mu\,(1) \quad \text{... property of p.d.f.}$$

i.e Mean $= E(X) = \mu$

Again we have,

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

put $\frac{x-\mu}{\sigma} = z$ i.e. $x = \mu + \sigma z$

$\therefore$ $dx = \sigma\, dz$

and, when $x \to -\infty$, $z \to -\infty$

$x \to \infty$, $z \to \infty$

$$\therefore \quad \mu_2' = E(X^2) = \int_{-\infty}^{\infty} (\mu+\sigma z)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \sigma dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu^2 + 2\mu\sigma z + \sigma^2 z^2) e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\mu^2 \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + 2\mu\sigma \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz + \sigma^2 \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\, dz \right]$$

$$= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz + 0 + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2}\, dz$$

... $\because z e^{-z^2/2}$ is odd and $z^2 e^{-z^2/2}$ is even.

$$= \mu^2 (1) + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2}\, dz$$

... $\because \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is a p.d.f. of z

put $\frac{z^2}{2} = t$ i.e $z = \sqrt{2}\sqrt{t}$

$$\therefore dz = \frac{\sqrt{2}}{2\sqrt{t}} dt = \frac{1}{\sqrt{2t}} dt$$

When $z = 0, t = 0$

$z \to \infty, t \to \infty$

$$\therefore \quad \mu_2' = E(X^2) = \mu^2 + \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2t\, e^{-t} \frac{1}{\sqrt{2t}} dt$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{1/2}\, dt$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right) \qquad \ldots \because \int_0^{\infty} e^{-y} y^{n-1} dy = \Gamma(n)$$

$$= \mu^2 + \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma(1/2) \qquad \ldots \because \Gamma(n+1) = n\Gamma(n)$$

$$= \mu^2 + \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} \qquad \ldots \because \Gamma(1/2) = \sqrt{\pi}$$

$$\therefore \quad \mu_2' = E(X^2) = \mu^2 + \sigma^2$$

$$\therefore \quad Var(X) = E(X^2) - [E(X)]^2 \qquad \ldots \text{standard result}$$

$$= \mu^2 + \sigma^2 - (\mu)^2$$

i.e $\quad Var(X) = \sigma^2$

38. *The daily wages of 1000 workers are normally distributed with average wages of Rs. 70 and standard deviation of Rs. 5. Estimate the number of workers whoes daily wages will be i) between Rs. 70 and 72 ii) Between Rs. 69 and 72 iii) More than Rs. 75 iv) Less than Rs. 63. v) Also find the lowest daily wages of the 100 highest paid workers.*

Solution : Let X denote daily wages in Rs. Thus, X is a normal variate with mean $\mu = 70$ and $\sigma = 5$ i.e. $\sigma^2 = 25$. i.e. $X \sim N(70, 25)$

$\therefore$ The standard normal variate is

$$z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$$

i. Probabilty that a worker will have wages between Rs.70 and 72 = $P(70 < x < 72)$

Now, when $x_1 = 70, \ z_1 = \frac{70-70}{5} = 0$ and

when $x_2 = 72, \ z_2 = \frac{72-70}{5} = 0.4$

$\therefore P(70 < x < 72) = P(0 < z < 0.4)$

$= 0.1554$ ... using tabled value

$\therefore$ Estimated number of workers out of N = 1000 having daily wages between Rs.70 and 72 i.e.

$f(70 < x < 72) = N \times P(70 < x < 72)$

$= 1000 \times (0.1554)$

$= 155.4 \approx 155.$

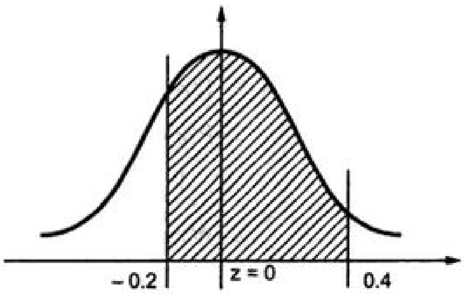


Fig. 3.10

ii. We have to find : $P(69 < x < 72)$

Now, when $x_1 = 69, \ z_1 = \frac{69-70}{5} = -0.2$

and when $x_2 = 72, \ z_2 = 0.4$

$\therefore P(69 < x < 72) = P(-0.2 < z < 0.4)$

$= P(-0.2 < z < 0) + P(0 < z < 0.4)$

$= P(0 < z < 0.2) + P(0 < z < 0.4)$ ... by symmetry

$= 0.0793 + 0.1554$ ... using table

$= 0.2347$

$\therefore$ Number of workers with wages between Rs.69 and Rs.72,

$= 1000 \times (0.2347)$

$= 234.7 \approx 235$

iii) We have to find : $P(x > 75)$

When $x = 75, \ z = \frac{75-70}{5} = 1$

$\therefore \quad P(x > 75) = P(z > 1)$

$= P(1 < z < \infty)$

$= P(0 < z <) - P(0 < z < 1)$

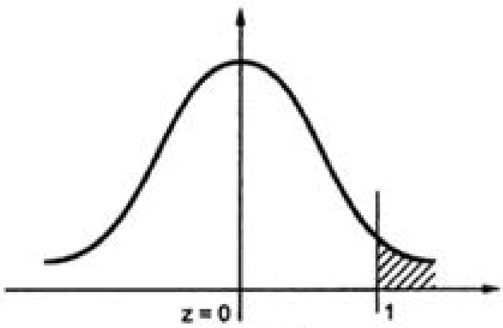


Fig. 3.11

$= 0.5 - 0.3413$ ... using table

$= 0.1587$

$\therefore$ Number of workers with wages more than Rs. 75

$= 1000 \times 0.1587$

$= 158.7 = 159$

iv) We have to find : $P(x < 63)$

When $x = 63$, $z = \dfrac{63-70}{5} = -1.4$

$\therefore \quad P(x < 63) = P(z < -1.4)$

$= P(z > 1.4)$

... by symmetry

$= P(1.4 < z < \infty)$

$= P(0 < z < \infty) - P(0 < z < 1.4)$

$= 0.5 - 0.4192 = 0.0808$

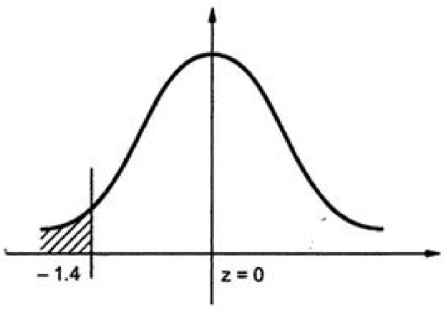


Fig. 3.12

$\therefore$ Number of workers with wages less than Rs. 63

$= 1000 \times 0.0808$

$= 80.8 \approx 81$

v) Let x_1 denote the lowest daily wages of the 100 highest paid workers.

Now, $\because$ Number of workers with wages more than x_1 $= 1000 \times P(x > x_1)$

$\therefore \quad 100 = 1000 \times P(x > x_1)$

$\therefore \quad P(x > x_1) = \dfrac{100}{1000} = 0.1$

Now, when $x = x_1$, $z_1 = \dfrac{x_1 - 70}{5}$

and $\therefore$ $P(z > z_1) = 0.1$

i.e. $P(z_1 < z < \infty) = 0.1$

Now, we know that $P(0 < z < \infty) = 0.5$

i.e. area under the standard normal curve to the right of $z = 0$ is 0.5

and $\because$ $P(z_1 < z < \infty) = 0.1$

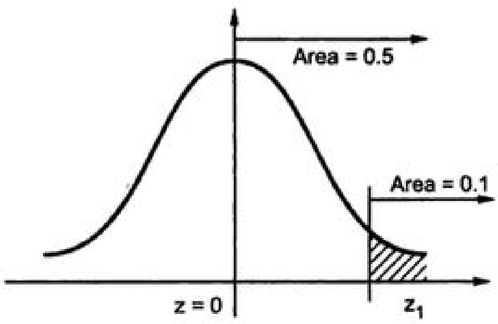


Fig. 3.13

$\therefore$ Area under the standard normal curve to the right of $z = z_1$ is 0.1 which is less than 0.5.

Hence, $P(z_1 < z < \infty) = 0.1 \Rightarrow z_1$ lies in the right half of the curve.

Now $$P(z_1 < z < \infty) = 0.1$$

i.e. $$P(0 < z < \infty) - P(0 < z < z_1) = 0.1$$

i.e. $$0.5 - P(0 < z < z_1) = 0.1$$

$\therefore$ $$P(0 < z < z_1) = 0.5 - 0.1 = 0.4$$

$\therefore$ Reading the value of z corresponding to the probability (i.e. the area under the Standard Normal curve) 0.4 from the table in the reverse order we see that

$$z_1 = 1.28$$

i.e. $$z_1 = \frac{x_1 - 70}{5} = 1.28$$

$\therefore$ $$x_1 - 70 = 1.28\ (5) = 6.4$$

$\therefore$ $$x_1 = 70 + 6.4 = 76.4$$

$\therefore$ The required lowest wages are Rs. 76.4.

39. *In a distribution exactly normal 7 % of the items are under 35 and 89 % are under 63. What is the mean and variance of the distribution ?*

Solution : Let X be the normal variate having mean μ and standard deviation σ.

$\therefore$ Standard Normal Variate is $z = \dfrac{x-\mu}{\sigma}$

Now, if $x_1 = 35$, $z = \dfrac{35-\mu}{\sigma}$ = say z_1 and

if $x_2 = 63$, $z = \dfrac{63-\mu}{\sigma}$ = say z_2

Now $\because$ 7 % of the items have value under 35

$\therefore$ $P(x < 35) = 7\ \% = 0.07$

$\therefore$ $P(z < z_1) = 0.07$

Hence, area to the left of the line $z = z_1$ is 0.07 which is less than the (area = 0.5) to the left of $z = 0$ under the standard normal curve.

$\therefore$ $z = z_1$ lies in the left portion

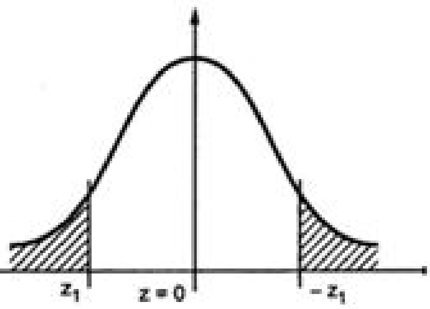


Fig. 3.14

Now, $\because P(z < z_1) = 0.07$

$\therefore$ $$P(z > -z_1) = 0.07$$ ... by symmetry and $(-z_1)$ lies in the right portion

i.e. $$P(-z_1 < z < \infty) = 0.07$$

i.e. $$P(0 < z < \infty) - P(0 < z < -z_1) = 0.07$$

i.e. $$0.5 - P(0 < z < -z_1) = 0.07$$

$\therefore$ $$P(0 < z < -z_1) = 0.5 - 0.07 = 0.43$$

∴ Using the table for area under the normal curve in reverse order we get

$$-z_1 = 1.48 \text{ (approximately)}$$

$$\therefore \quad z_1 = -1.48$$

$$\text{i.e.} \quad \frac{35-\mu}{\sigma} = -1.48 \quad \ldots (1)$$

Also, since 89 % of items are under 63

$$\therefore \quad P(x < 63) = 89\ \% = 0.89$$

$$\therefore \quad P(z < z_2) = 0.89$$

∴ Area to the left of line $(z = z_2)$ is 0.89 which is more than the (area = 0.5) to the left of (z = 0). Hence, $(z = z_2)$ will lie in the right portion.

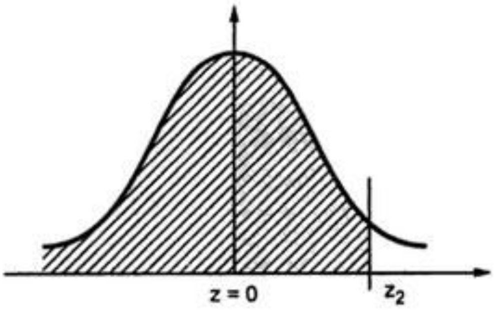


Fig. 3.15

$$\text{Now} \because \quad P(z<z_2) = 0.89$$

$$\text{i.e.} \quad P(-\infty<z<z_2) = 0.89$$

$$\text{i.e.} \quad P(-\infty<z<0)+P(0<z<z_2) = 0.89$$

$$\text{i.e.} \quad 0.5+P(0<z<z_2) = 0.89$$

$$\therefore \quad P(0<z<z_2) = 0.89-0.5 = 0.39$$

∴ From the table,

$$z_2 = 1.23$$

$$\text{i.e.} \quad \frac{63-\mu}{\sigma} = 1.23 \quad \ldots (2)$$

∴ Dividing (2) by (1) we get

$$\frac{\frac{63-\mu}{\sigma}}{\frac{35-\mu}{\sigma}} = \frac{1.23}{-1.48}$$

$$\therefore \quad \frac{63-\mu}{35-\mu} = \frac{-1.23}{1.48} = -\frac{123}{148}$$

$$\therefore \quad 148\,(63-\mu) = -123\,(35-\mu)$$

$$\text{i.e.} \quad 9324 - 148\mu = -4305 + 123\mu$$

$$\therefore \quad 123\mu + 148\mu = 9324 + 4305$$

$$\text{i.e.} \quad 271\mu = 13629$$

$$\therefore \quad \mu = \frac{13629}{271} = 50.3 = \text{mean}$$

$$\therefore \text{From (2),} \quad \frac{63-50.3}{\sigma} = 1.23$$

$$\therefore \quad \sigma = \frac{12.7}{1.23} = 10.33$$

$$\therefore \quad \text{Variance} = \sigma^2$$

$$= (10.33)^2$$

Problem on general Probability rules, Independence, conditional Probability.

① If A, B, C are mutually Independent, with $P(A) = P(B) = P(C) = 0.1$.

Find (a) $P(A \cup B)$ (b) $P(A \cup B \cup C)$

(c) $P(A \setminus (B \cup C))$

Soln

Given $P(A) = P(B) = P(C) = 0.1$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A and B are Independent

$\therefore P(A \cap B) = P(A) \cdot P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$
$$= 0.1 + 0.1 - (0.1)(0.1)$$
$$= 0.2 - 0.01$$
$$= \underline{0.19}$$

(ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$\therefore$ A, B, C are Independent

$$= P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) P(C) - P(C) \cdot P(A) + P(A) \cdot P(B) \cdot P(C)$$
$$= 0.1 + 0.1 + 0.1 - (0.1)(0.1) - (0.1)(0.1) - (0.1)(0.1) + (0.1)(0.1)(0.1)$$
$$= 0.3 - 0.01 - 0.01 - 0.01 + 0.001$$
$$= 0.3 - 0.03 + 0.001$$
$$= 0.271$$

(iii) $P(A \setminus (B \cup C))$

$= P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

$= P(A) - P(A)P(B) - P(A)P(C) + P(A).P(B).P(C)$

$= 0.1 - (0.1)(0.1) - (0.1)(0.1) + (0.1)(0.1)(0.1)$

$= 0.1 - 0.01 - 0.01 + 0.001$

$= 0.081$

---

(2) Given that $P(A) = 0.3$, $P(A/B) = 0.4$ and $P(B) = 0.5$. compute

(i) $P(A \cap B)$ (ii) $P(B/A)$ (iii) $P(A'/B)$

(iv) $P(A/B')$.

solution:

(i) $P(A \cap B) = P(A/B).P(B)$

$= (0.4)(0.5)$

$= 0.2$

(ii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.3} = 0.667$

(iii) $P(A'/B) = \frac{P(A' \cap B)}{P(B)}$

$= \frac{P(B) - P(A \cap B)}{P(B)}$

$= \frac{0.5 - 0.2}{0.5} = \frac{0.3}{0.5}$

$= 0.6$

(iv) $P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

$= \frac{0.3 - 0.2}{1 - 0.5} = \frac{0.1}{0.5} = 0.2$

(3) Given that A and B are Independent with $P(A \cup B) = 0.8$ and $P(B') = 0.3$ find $P(A)$.

Soln Given $P(A \cup B) = 0.8$

$P(B') = 0.3$

$P(B) = 1 - 0.3 = 0.7$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\because$ A and B are Independent

$P(A \cup B) = P(A) + P(B) - P(A).P(B)$

$0.8 = P(A) + 0.7 - P(A).0.7$

$0.8 - 0.7 = P(A) - 0.7P(A)$

$0.1 = P(A)[1 - 0.7]$

$0.1 = P(A)(0.3)$

$\Rightarrow P(A) = \frac{0.1}{0.3} = 0.333$

---

(4) Given that $P(A \cup B) = 0.7$ and $P(A \cup B') = 0.9$ find $P(A)$.

Soln De morgan's Law

$P(A' \cap B') = P((A \cup B)')$

$= 1 - P(A \cup B)$

$= 1 - 0.7$

$= 0.3$

Similarly

$P(A' \cap B) = 1 - P(A \cup B')$

$= 1 - 0.9 = 0.1$

Thus

$\boxed{P(A') = P(A' \cap B') + P(A' \cap B)}$

$= 0.3 + 0.1 = 0.4$

$\therefore P(A) = 1 - P(\bar{A}) = 1 - 0.4 = 0.3$

(5) Given $P(A) = 0.2$, $P(B) = 0.7$ and $P(A/B) = 0.15$ find $P(A' \cap B')$

Soln Given $P(A) = 0.2$

$P(B) = 0.7$

$P(A/B) = 0.15$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(A \cap B) = P(B) \cdot P(A/B)$$

$$= (0.7)(0.15)$$

$$= 0.105$$

De morgan's Law

$$P(A' \cap B') = P((A \cup B)')$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - 0.2 - 0.7 + 0.105$$

$$= 0.205$$

---

(6) Given that A and B are Independent with $P(A) = 2P(B)$ and $P(A \cap B) = 0.15$ Find $P(A' \cap B')$

Soln If A and B are Independent

$$P(A \cap B) = P(A) \cdot P(B)$$

$$0.15 = 2P(B) \cdot P(B)$$

$$\frac{0.15}{2} = [P(B)]^2$$

$$0.075 = [P(B)]^2$$

$$\Rightarrow P(B) = \sqrt{0.075} = 0.274$$

$P(B) = 0.274$

$P(A) = 2\,P(B)$

$P(A) = 2(0.274) = 0.548$

$P(A' \cap B') = P(A').P(B')$

$\because$ A and B are Independent

$= [1 - P(A)]\,[1 - P(B)]$

$= (1 - 0.548)\,(1 - 0.274)$

$= (0.452)\,(0.726)$

$= \underline{\underline{0.3282}}$

(7) If $P(A) = 0.35$, $P(B) = 0.73$, $P(A \cap B) = 0.14$ find $P(\bar{A} \cup \bar{B})$

Soln

$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})$

$= 1 - P(A \cap B)$

$= 1 - 0.14$

$= \underline{\underline{0.86}}$

(8) If $P(A) = 0.5$ $P(B) = 0.3$ and $P(A \cap B) = 0.15$ find $P(A/\bar{B})$

Soln

$$P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{0.5 - 0.15}{1 - 0.3} = \frac{0.35}{0.7} = \underline{\underline{0.5}}$$

(9) If A and B are independent and $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, find $P(A \cap B)$

Soln

since A and B Independent

$P(A \cap B) = P(A).P(B) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

(10) If $P(A) = 0.9$, $P(B/A) = 0.8$ find $P(A \cap B)$.

Soln

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A).P(B/A)$$

$$= 0.9 \times 0.8$$

$$= \underline{\underline{0.72}}$$

(11.) If A and B are independent events, prove that

(i) $\bar{A}$ and B are Independent

(ii) A and $\bar{B}$ are Independent

(iii) $\bar{A}$ and $\bar{B}$ are Independent

Soln

(i) Since A and B are Independent

$$\boxed{P(A \cap B) = P(A).P(B)}$$

W.K.T $B = (A \cap B) \cup (\bar{A} \cap B)$

Now $A \cap B$ and $\bar{A} \cap B$ are disjoint

$$P(B) = P((A \cap B) \cup (\bar{A} \cap \bar{B})]$$

$$\boxed{P(B) = P(A \cap B) + P(\bar{A} \cap B)}$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

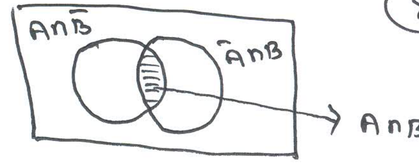
$$= P(B) - P(A).P(B)$$

$$= P(B)[1 - P(A)$$

$$= P(B).P(\bar{A})$$

$$\Rightarrow \boxed{P(\bar{A} \cap B) = P(\bar{A}).P(B)}$$

$\therefore$ $\bar{A}$ and B are Independent



(ii) Now $A = (A \cap B) \cup (A \cap \bar{B})$

$$P(A) = P[A \cap B] + P(A \cap \bar{B})$$

$\because$ $A \cap B$ and $A \cap \bar{B}$ are disjoint

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= P(A) - P(A)\,P(B)$$
$$= P(A)\,[1 - P(B)]$$
$$\underline{P(A \cap \bar{B}) = P(A).\,P(\bar{B})}$$

$\therefore$ A and $\bar{B}$ are Independent

(iii)

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$
$$= 1 - P[A \cup B]$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) + P(A)\,P(B)$$
$$= \underline{1 - P(A)} - P(B) + P(A).\,P(B)$$
$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$
$$= [1 - P(A)]\,[1 - P(B)]$$
$$\underline{\therefore P(\bar{A} \cap \bar{B}) = P(\bar{A}).\,P(\bar{B})}$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent

(12) Prove that Probability of an impossible event is zero. (ie) $P(\phi) = 0$.

<u>Soln</u> The sample space S and impossible even ϕ are mutually exclusive

$$S \cup \phi = S$$
$$P[S \cup \phi] = P(S)$$
$$P(S) + P(\phi) = P(S)$$

(13) If A, B and C are any 3 events such that $P(A) = P(B) = P(C) = 1/4$ and $P(A \cap B) = P(B \cap C) = 0$; $P[C \cap A] = 1/8$ Find the probability that atleast 1 of the event A, B and C occurs.

Soln

P[atleast one of A, B and C occurs)

$P[A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Since $P(A \cap B) = 0 \quad P(B \cap C) = 0$
$P(A \cap B \cap C) = 0$

$\therefore P(A \cup B \cup C) = \frac{3}{4} - 0 - 0 - \frac{1}{8}$

$= \frac{6-1}{8} = 5/8$

---

(14) In a shooting test, the probability of hitting the target is 1/2 for A, 2/3 for B and 3/4 for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.

Soln

A = Event of hitting the target
B = Event of hitting the target
C = Event of hitting the target

$P(A) = \frac{1}{2} \qquad P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(B) = 2/3 \qquad P(\bar{B}) = 1 - P(A) = 1 - 2/3 = \frac{1}{3}$

$P(C) = 3/4 \qquad P(\bar{C}) = 1 - P(C) = 1 - 3/4 = 1/4$

$\therefore$ P[none of them hits the target)

$P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$

$\because$ A, B, C are Independent.

$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$ //

P[atleast one hits the target)

$= 1 - P($none of them hits the target$)$

$= 1 - \frac{1}{24} = \frac{24-1}{24} = \frac{23}{24}$

---

(15) A, B and C shot to hit a target. If A hits the target 3 times in 5 trials, B hits it 2 times in 3 trials, and C hits 5 times in 8 trials, what is the probability that the target is hit by atleast two persons?

Soln

Let A = event of hitting the target

B = event of hitting the target

C = event of hitting the target

$P(A) = \frac{3}{5}$ $\quad P(\bar{A}) = \frac{2}{5}$

$P(B) = \frac{2}{3}$ $\quad P(\bar{B}) = \frac{1}{3}$

$P(C) = \frac{5}{8}$ $\quad P(\bar{C}) = 3/8$

The Target can be hit atleast by two persons in the following ways.

i) A and B hit the target and C does not hit

ii) A and C hit the target and B does not hit

iii) B and C hit the target and A does not hit

iv) A, B and C hit the target.

$= P(\cancel{A \cap B \cap \bar{C}}) + P(\cancel{A \cap \bar{B} \cap C}) + P(\cancel{\bar{A} \cap B \cap C})$
$+ P(\cancel{A \cap B \cap C})$

$= P(A)P(B)P(\bar{C}) + P(A)\, P(\bar{B})\, P(C)$
$+ P(\bar{A}).P(B)P(C) + P(A \cap B \cap C)$

$= \frac{\cancel{3}}{5}\cdot\frac{2}{\cancel{3}}\cdot\frac{3}{8} + \frac{\cancel{3}}{5}\cdot\frac{1}{\cancel{3}}\cdot\frac{5}{8} + \frac{2}{\cancel{8}}\cdot\frac{2}{3}\cdot\frac{\cancel{8}}{8} + \frac{3}{\cancel{8}}\cdot\frac{2}{3}\cdot\frac{\cancel{8}}{8}$

$= \frac{6}{40} + \frac{5}{40} + \frac{4}{24} + \frac{6}{24}$

$= \frac{11}{40} + \frac{10}{24} = \underline{\underline{0.6916}}$

(16) A Problem is given to 3 students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that the (i) Problem will be solved (ii) Exactly Two of them will ~~be~~ solve the problem.

Soln

Let A = Event of solve the problem by first student

B = Event of solve the problem by second student

C = Event of solve the problem by third student

$P(A) = \frac{1}{2}$ $\quad P(\bar{A}) = \frac{1}{2}$

$P(B) = \frac{1}{3}$ $\quad P(\bar{B}) = \frac{2}{3}$

$P(C) = \frac{1}{4}$ $\quad P(\bar{C}) = 3/4$

(i) The problem will be solved if at last on of them solves the problem. That is

$$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$
$$= 1 - P(\bar{A}) . P(\bar{B}) P(\bar{C})$$
$$= 1 - \frac{1}{2} . \frac{2}{3} . \frac{3}{4}$$
$$= 1 - \frac{1}{4}$$
$$= 3/4$$

(ii) P[Exactly two of them solve the problem]

$$= P[A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$$
$$= P(A) P(B) . P(\bar{C}) + P(A) . P(\bar{B}) . P(C) + P(\bar{A}) P(B) . P(C)$$
$$= \frac{1}{2} . \frac{1}{3} . \frac{3}{4} + \frac{1}{2} . \frac{2}{3} . \frac{1}{4} + \frac{1}{2} \frac{1}{3} . \frac{1}{4}$$
$$= \frac{1}{8} + \frac{1}{12} + \frac{1}{24}$$
$$= \frac{3+2+1}{24} = \frac{06}{24} = \frac{6}{24} = \frac{1}{4}$$

(17) Two coins are tossed. Let A denote the event "at most one head on the two tosses" and let B denote the event "one head ~~on the~~ and one tail in both tosses". Are A and B Independent events?

Soln

The sample space of the experiment is $S = \{HH, HT, TH, TT\}$.

A = Event of at most one head on the two tosses (HT, TH, TT)

B = Event of one head and tail in both tosses (HT, TH)

$A = \underline{HT}, \underline{TH}, TT$

$B = \underline{HT} \quad \underline{TH}$

$A \cap B = HT, TH$

$P(A) = \frac{3}{4}, \quad P(B) = \frac{2}{4}, \quad P(A \cap B) = \frac{2}{4}$

If A and B are Independent

$P(A \cap B) = P(A) \cdot P(B)$

$\frac{2}{4} = \frac{3}{\cancel{4}_2} \cdot \frac{\cancel{2}}{4} = 3/8$

$\frac{2}{4} \neq 3/8$

We conclude that ~~P(A)~~ A and B are not independent.

---

(18) A red die and blue die are rolled together. What is the probability that we obtain 4 on the red die and 2 on the blue die?

Soln

Let R = Event of 4 on the red die

B = Event of 2 on the blue die

To find $P(R \cap B)$

~~P(A) = 1~~

$P(R) = \frac{1}{6}$, $P(B) = \frac{1}{6}$

Since the event R and B are independent out come of one die does not affect the outcome of the other

$$P(R \cap B) = P(R) \cdot P(B)$$

$$= \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

(19) Flip a coin and then independently cost a die, what is the probability of observing heads on the coin and a 2 or 3 on the die?

Sol

A = Event of head on the coin

B = Event of observing a 2 or 3 on the die

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cap B) = P(A) \cdot P(B)$$

$$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

(20) A and B toss a fair coin alternately with the understanding that the one who obtains the head first win. If A starts, what is his chance of winning?

Sol

$P(A) = \frac{1}{2}$ $P(\bar{A}) = \frac{1}{2}$

$P(B) = \frac{1}{2}$ $P(\bar{B}) = \frac{1}{2}$

A win if he obtains head in the first toss or in the third toss, or in the fifth toss etc.

$= P(A) + P(\bar{A}).P(\bar{B}).P(A) + P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A)$
$+ \dots$

$= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \dots$

$= \frac{1}{2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \dots\right]$

$a = 1 \quad r = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}}$

$= \frac{1}{2} \frac{1}{\frac{4-1}{4}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}2}{3} = \frac{2}{3}$ //

---

(21) If A and B are independent events of a random experiment such that $P(A \cap B) = \frac{1}{5}$ and $P(\bar{A} \cap \bar{B}) = \frac{1}{4}$, find $P(A)$.

Soln

Given $P(A \cap B) = \frac{1}{5}$

$P(\bar{A} \cap \bar{B}) = \frac{1}{4}$

Since A and B are independent.

$P(A \cap B) = P(A).P(B) = \frac{1}{5}$ —(1)

$P(\bar{A} \cap \bar{B}) = P(\bar{A}).P(\bar{B}) = \frac{1}{4}$ —(2)

$(2) \Rightarrow [1 - P(A)][1 - P(B)] = \frac{1}{4}$

$1 - P(B) - P(A) + P(A).P(B) = \frac{1}{4}$

$1 - P(A) - P(B) + \frac{1}{5} = \frac{1}{4}$

$- P(A) - P(B) = \frac{1}{4} - \frac{1}{5} - 1$

$= \frac{5-4-20}{20}$

$= \frac{-19}{20}$

$$-P(A) - P(B) = -\frac{19}{20}$$

$$P(A) + P(B) = \frac{19}{20} \quad — (3)$$

WKT $(A-B)^2 = (A+B)^2 - 4AB$

$$[P(A) - P(B)]^2 = [P(A) + P(B)]^2 - 4\,P(A).P(B)$$

$$= \left(\frac{19}{20}\right)^2 - 4\,\frac{1}{5}$$

$$= \frac{361}{400} - \frac{4 \times 80}{5 \times 80}$$

$$= \frac{361 - 320}{400} = \frac{41}{400} = \cancel{0.32} = 0.1025$$

$$[P(A) - P(B)]^2 = \cancel{0.32}\ 0.1025$$

$$P(A) - P(B) = 0.320 \quad — (4)$$

solve (3) and (4)

$$\begin{aligned} P(A) + \cancel{P(B)} &= 0.95 \\ P(A) - \cancel{P(B)} &= 0.320 \\ \hline 2P(A) &= 1.27 \end{aligned}$$

$$2P(A) = 1.27$$

$P(A) = 0.635$ put (4)

$\therefore$ ~~P(B) = 0.365~~

$$P(A) - P(B) = 0.320$$

$$P(B) = P(A) - 0.320$$

$$= 0.635 - 0.320$$

$$P(B) = \underline{\underline{0.315}}$$

$$\boxed{\therefore P(A) = 0.635. \quad P(B) = 0.315}$$

Bayes Problems

1. An urn contains 10 white and 3 black balls. Another urn contain 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and than 1 ball is taken at random from the latter what is the probability that it is a white ball?

Soln

Two balls Transferred may be both white or both black are 1 white and 1 black.

Let B_1 = event of drawing 2 white balls from the first urn.

B_2 = event of drawing 2 black balls from the first urn.

B_3 = event of drawing 1 white ball and 1 black ball from the first urn.

Urn 1

W	B
10	3

total = 13 balls

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{\frac{10\times 9}{1\times 2}}{\frac{13\times 12}{1\times 2}} = \frac{90}{156}$$

$$P(B_2) = \frac{3C_2}{13C_2} = \frac{\frac{3\times 2}{1\times 2}}{\frac{13\times 12}{1\times 2}} = \frac{6}{156}$$

$$P(B_3) = \frac{10C_1\times 3C_1}{13C_2} = \frac{10\times 3}{\frac{13\times 12}{1\times 2}}$$

$$= \frac{30}{78} = \frac{60}{156}$$

E = event of drawing a white ball from the second urn after Transfer

$P(E/B_1)$

$$= \frac{5C_1}{10C_1} = \frac{5}{10}$$

$P(E/B_2)$

$$= \frac{3C_1}{10C_1} = \frac{3}{10}$$

$$P(E/B_3) = \frac{4C_1}{10C_1}$$

$$= \frac{4}{10}$$

Before

3	5

W B

after 2 white

W B

3+2	5

after 2 black

W B

3	5

3	5+2

after 1 white 1 black

W B

3	5

3	1	5	1

By theorem of total probability

$$P(E) = P(B_1)\,P(E/B_1) + P(B_2)\,P(E/B_2) + P(B_3)\,P(E/B_3)$$

$$= \frac{90}{156} \times \frac{5}{10} + \frac{6}{156} \times \frac{3}{10} + \frac{60}{156} \times \frac{4}{10}$$

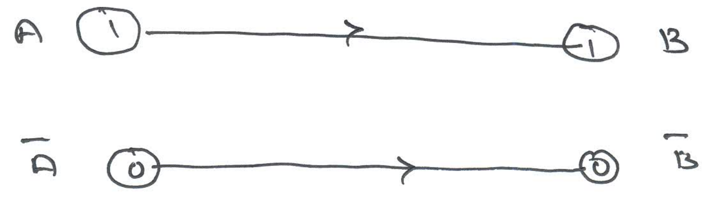
$$= \frac{450}{1560} + \frac{18}{1560} + \frac{240}{1560}$$

$$= \frac{708}{1560} = \frac{59}{130} = 0.4538$$

---

(2) For a certain binary communication channel, the probability that a Transmitted '0' is received as a '0' is 0.95. and the probability that a Transmitted '1' is received as a '1' is 0.90. If the probability that a '0' is transmitted is 0.4 Find the probability that (i) a '1' is received (ii) '1' was Transmitted given that a

Soln



Let A = Event of Transmitting '1'

$\bar{A}$ = Event of Transmitting '0'

B = Event of receiving '1'

$\bar{B}$ = Event of receiving '0'

Given

Transmitted '0' is received as a '0'

$$P(\bar{B}/\bar{A}) = 0.95$$

Transmitted '1' is received as a '1'

$$P(B/A) = 0.90$$

Probability of '0' is Transmitted

$$P(\bar{A}) = 0.4$$

$$P(A) = 1 - P(\bar{A})$$
$$= 1 - 0.4 = 0.6$$

$$P(B/\bar{A}) = 1 - P(\bar{B}/\bar{A})$$
$$= 1 - 0.95$$
$$= \underline{\underline{0.05}}$$

By the theorem of total probability

$$P(B) = P(A).P(B/A) + P(\bar{A}).P(B/\bar{A})$$
$$= 0.6 \times 0.9 + 0.4 \times 0.05$$
$$= 0.54 + 0.02$$
$$= \underline{\underline{0.56}}$$

By Bayes theorem

$$P(A/B) = \frac{P(A).\,P(B/A)}{P(A).\,P(B/A) + P(\bar{A}).\,P(B/\bar{A})}$$

$$= \frac{0.6 \times 0.9}{0.56}$$

$$= \frac{0.54 \times 100}{0.56 \times 100}$$

$$= \frac{54}{56} = \frac{27}{28}$$

---

(3) A student buys 1000 integrated circuits [IC's] from supplier A, 2000 IC's from supplier B, and 3000 IC's from supplier C. He tested ICs and found that the conditional probability of an IC being defective depends on the supplier from whom it was bought. Specifically, Given that that IC came from supplier A, the probability that it is defective is 0.05; Given that an IC came from supplier B, the probability that it is defective is 0.10; and given that an IC came from supplier C, the probability that it is defective is 0.10. If the ICs from the three suppliers are mixed together and one is selected at random, (1) what is the probability that it is defective? (2) Given that a randomly selected IC is defective, what is the probability that it came from supplier A?

Soln

Let $P(A)$, $P(B)$, $P(C)$ denote the probability that a randomly selected IC came from supplier A, B, and C

D = Event drawing ~~a~~ IC is defective

~~P(D/A~~

$$P(A) = \frac{1000}{6000} = \frac{1}{6}$$

$$P(B) = \frac{2000}{6000} = \frac{2}{6}$$

$$P(C) = \frac{3000}{6000} = \frac{3}{6}$$

$P(D/A)$ denote the conditional probability that an IC is defective, given that it came from supplier A

$$P(D/A) = 0.05 = \frac{5}{100}$$

$P(D/B)$ denote the conditional probability that an IC is defective given that it came from supplier B

$$P(D/B) = 0.10 = \frac{10}{100}$$

$P(D/C)$ denote the conditional probability that an IC is defective given that it came from supplier C

$$P(D/C) = 0.10 = \frac{10}{100}$$

$P(D)$ denote the unconditional probability that a randomly selected IC is defective.

By total probability

$$P(D) = P(A).P(D/A) + P(B).P(D/B) + P(C).P(D/C)$$

$$= \frac{1}{6} \times \frac{5}{100} + \frac{2}{6} \times \frac{10}{100} + \frac{3}{6} \times \frac{10}{100}$$

$$= \frac{5 + 20 + 30}{\ldots} = \frac{55}{600}$$

By Bayés theorem:

randomly selected IC came from supplier A, given that it is defective

$$P(A/D) = \frac{P(A \cap D)}{P(D)}$$

$$= \frac{P(D/A).P(A)}{P(A).P(D/A) + P(B).P(D/B) + P(C).P(D/C)}$$

$$= \frac{\frac{1}{6} \times \frac{5}{100}}{\frac{1}{6} \times \frac{5}{100} + \frac{2}{6} \times \frac{10}{100} + \frac{3}{6} \times \frac{10}{100}}$$

$$= \frac{\cancel{\frac{1}{600}}}{\cancel{\frac{1}{600}}} \left[\frac{5}{5+20+30} \right]$$

$$= \frac{5}{55} = \frac{1}{11}$$

---

(4) The quarterback for a certain football team has a good game with probability 0.6 and a bad game with probability 0.4. When he has a good game, he throws at least one interception with a probability of 0.2; and when he had a bad game, he throws at least one interception with a probability of 0.5. Given that he threw at least one interception in a particular game, what is the probability that he had a good game?

Soln

G = Event that the quarterback has a good game

B = Event that the quarterback has a bad game.

$P(G) = 0.6$

$P(B) = 0.4$

Let I = Event that he throws at least one interception

$P(I/G) = 0.2$

$P(I/B) = 0.5$

According to Bay's theorem.

at least one interception given that he had a good game

$$P(G/I) = \cancel{\frac{P(I/G)\,P(G)}{P(G).\,P(}}$$

$$= \frac{P(G).\,P(I/G)}{P(G).\,P(I/G) + P(B).\,P(I/B)}$$

$$= \frac{0.6 \times 0.2}{0.6 \times 0.2 + 0.4 \times 0.5}$$

$$= \frac{0.12}{0.12 + 0.2} = \frac{0.12}{0.32}$$

$$= \frac{12}{32} = 3/8 = \underline{\underline{0.375}}$$

(5) In a bolt factory, machines A, B and C manufacture 25%, 35%, and 40% of the total bolts respectively. Of their output 5, 4 and 2 percent respectively are defective bolts. A bolt is drawn at random and found to be defective. What is the probability that the bolt came from machines A, B and C?

<u>Soln</u>

Let A = Event bolts manufactured by machine A

B = Event bolts manufactured by machine B

C = Event bolts manufactured by machine C.

$$P(A) = \frac{25}{100} \quad P(B) = \frac{35}{100}, \quad P(C) = \frac{40}{100}.$$

Let D = Event drawing a defective bolts

$$P(D/A) = \frac{5}{100} \quad P(D/B) = \frac{4}{100}$$

$$P[D/C) = \frac{2}{100}.$$

(i) The Probability that the bolt came from machine A can be obtained using Baye's

$$P[A/D] = \frac{P(A)\,P[D/A)}{P(A)\,P(D/A) + P(B).\,P(D/B) + P(C).\,P(D/C)}$$

$$= \frac{\frac{25}{100} \times \frac{5}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}}$$

$$= \frac{\frac{1}{\cancel{10000}}\,[125]}{\frac{1}{\cancel{10000}}\,[125 + 140 + 80]} = \frac{125}{345}$$

Similarly came from B

$$\cancel{P[B/A]} = P\left[B/D\right] = \frac{P(B).P(D/B)}{P(A)P[D/A]+P(B).P(D/B)+P[C].P(D/C)}$$

$$P\left[B/D\right] = \frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$= \frac{\frac{1}{\cancel{10000}}}{\frac{1}{\cancel{10000}}}\left[\frac{140}{125+140+80}\right] = \frac{140}{345}$$

$$P\left[C/D\right] = \frac{P[C]\,P[D/C]}{P(A).P[D/A]+P(B)\,P[D/B]+P[C].P[D/C]}$$

$$= \frac{\frac{40}{100}\times\frac{2}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$= \frac{\frac{1}{\cancel{10000}}}{\frac{1}{\cancel{10000}}}\left[\frac{80}{125+140+80}\right]$$

$$= \frac{80}{345}$$

---

(9) Three car brands A, B and C, have all the market share in a certain city. Brand A has 20% of the market share, brand B has 30%, and brand C has 50%. The Probability that a brand A car needs a major repair during the first year of Purchase is 0.05, The Probability that a brand B car needs a major repair during the first year of purchase is 0.10, and the Probability that a brand C

car needs a major repair during the first year of purchase is 0.15.

(i) what is the probability that a randomly selected car in the city needs a major repair during its first year of purchase?

(ii) If a car in the city needs a major repair during its first year of purchase, what is the probability that it is a brand A car?

Soln

Let

A = Event of market share in city A

B = Event of market share in city B

C = Event of market share in city C.

$P(A) = \frac{20}{100} = 0.20$. $P(B) = \frac{30}{100} = 0.30$, $P(C) = \frac{50}{100} = 0.50$.

Let D = Event of major repair during the first year of purchase

$$P[D/A] = 0.05 = \frac{5}{100}$$

$$P[D/B] = 0.10 = \frac{10}{100}$$

$$P[D/C] = 0.15 = \frac{15}{100}.$$

Probability that a randomly selected car in the city need major repair during the first year. (total probability)

$$P[D] = P(A)\,P(D/A) + P(B)\,P(D/B) + P(C).\,P(D/C)$$

$$= 0.20 \times 0.05 + 0.30 \times 0.10 + 0.50 \times 0.15$$

$$= 0.01 + 0.03 + 0.075$$

$$= \underline{\underline{0.115}}$$

(ii) Probability of major repair during its first year of purchase it came from brand A car

$$P[A/D] = \frac{P(A)P(D/A)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{0.20 \times 0.05}{0.115} \quad \text{by ((i))}$$

$$= \frac{0.01}{0.115} = \underline{\underline{0.086}}$$

(10) A group of student consists of 60% men and 40% women. Among the men, 30% are foreign students, and among the women, 20% are foreign students. A student is randomly selected from the group and found to be a foreign student. What is the probability that the student is woman?

Soln

Let A = Event consists of men

B = Event consists of women

$$P(A) = \frac{60}{100} \qquad P(B) = \frac{40}{100}$$

Let D = Event of foreign student

$$P[D/A] = \frac{30}{100}. \qquad P[D/B] = \frac{20}{100}.$$

Probability selected foreign student it came from foreign women.

$$P[B/D] = \frac{P(B).\,P[D/B]}{P(A).\,P[D/A] + P(B).\,P[D/B]}$$

$$P[B/D] = \frac{\frac{40}{100} \times \frac{20}{100}}{\frac{60}{100} \times \frac{30}{100} + \frac{40}{100} \times \frac{20}{100}}$$

$$= \frac{\frac{1}{\cancel{100}}[4 \times 2]}{\frac{1}{\cancel{100}}[6 \times 3 + 4 \times 2]}$$

$$= \frac{8}{18+8} = \frac{8}{26} = \frac{4}{13}$$