PROBABILITY AND QUEUEING THEORY

UNIT- I.

RANDOM VARIABLES.

A real variable "X" whose value is determined by the outcome of a random experiment is called a random variable.

For example,

Toss a coin twice. the Sample Space is $S = \{HH, HT, TH, TT\}$. Let X denote the "number of heads", then X is a random variable with values

$X(HH) = 2$, $X(HT \text{ or } TH) = 1$, $X(TT) = 0$.

(ie) values of X are 0, 1, 2.

$\therefore$ Range Space $R_X = \{0, 1, 2\}$ is a finite set.

Random variables

- DISCRETE RANDOM VARIABLES → Probability mass function $[p(x)]$
- CONTINUOUS RANDOM VARIABLES. → probability density function $[f(x)]$.

Discrete random variable:

A random variable which can assume only a countable number of real values is called a discrete random variable.

For example,

(i) Number of telephone calls per unit time

(ii) Marks obtained in a test.

(iii) Number of printing mistakes in each page of a book.

PROBABILITY MASS FUNCTION:-

If X is a discrete random variable taking atmost a countably infinite number of values $x_1, x_2 \ldots$ then

$$p(x_i) = P(X = x_i), \quad i = 1, 2, \ldots$$

is called the probability mass function of the random variable X. and the number $p(x_i)$; $i = 1, 2, 3 \ldots$ must satisfy the following conditions.

(i) $p(x_i) \geq 0 \quad \forall\, i$ and (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

The set of ordered pairs of numbers $(x_i, p(x_i))$ is called the probability distribution of the random variable X.

Continuous random variable:-

②

A random variable X is said to be Continuous if it can take all possible values between certain limits.

For example,

(1) Heights of people in a population.

(2) The speed of a car.

PROBABILITY DENSITY FUNCTION:-

A function f, defined for all $x \in (-\infty, \infty)$ is called the probability density function of a Continuous random variable X if

(i) $f(x) \geq 0 \;\forall\; x \in (-\infty, \infty)$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

NOTE:- (1) $p(X < a) = \int_{-\infty}^{a} f(x)\,dx$

(2) $p(X > a) = \int_{a}^{\infty} f(x)\,dx$

(3) $p(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$

DISTRIBUTION FUNCTION (OR) CDF:-

Let X be a random variable. the function F defined for all real value x by $F(x) = p(X \leq x)$, $-\infty < x < \infty$, is called the distribution function of the RV X.

Properties of distribution function $F(x)$:

(1) $0 \leq F(x) \leq 1, \quad -\infty < x < \infty.$

(2) $F(x)$ is an Increasing function of x.

(3) $F(-\infty) = 0$ (i) $\lim_{x \to -\infty} F(x) = 0.$

(4) $F(\infty) = 1.$ (ii) $\lim_{x \to \infty} F(x) = 1.$

(5) If X is a Continuous random variable then $F'(x) = f(x).$

(6) If X is a discrete random variable with values $x_1 < x_2 < x_3 \ldots$ then $p(x_i) = F(x_i) - F(x_{i-1}).$

(7) ~~For~~ continuous random variable

$p(X \leq a) = p(X < a), \quad p(X > a) = p(X \geq a)$

$p(a < X \leq b) = p(a \leq X < b) = p(a \leq x \leq b)$
$= p(a < x < b).$

(8) $F(x) = p(X \leq x) = \sum_{x_i \leq x} p(x_i)$ [Discrete RV].

(9) $F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$ [Continuous RV].

(10) $p(a \leq X \leq b) = F(b) - F(a)$

2M ① If the random variable X takes values 1, 2, 3 and 4 such that $2p(X=1) = 3p(X=2) = p(X=3) = 5p(X=4)$. Find the probability distribution of X.

Sol: The values of X are 1, 2, 3 and 4.

Let $p(X=3) = k$.

$2p(X=1) = p(X=3)$

$2p(X=1) = k$

$\boxed{p(X=1) = k/2}$

$3p(X=2) = p(X=3)$

$3p(X=2) = k$

$\boxed{p(X=2) = \frac{k}{3}}$

$p(X=3) = 5p(X=4)$

$k = 5p(X=4)$

$\boxed{\therefore p(X=4) = k/5}$

We have $\sum p(x) = 1$.

$\therefore p(X=1) + p(X=2) + p(X=3) + p(X=4) = 1$.

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$.

$\frac{15k + 10k + 30k + 6k}{30} = 1. \Rightarrow \frac{61k}{30} = 1$.

$\boxed{\therefore k = \frac{30}{61}}$

The probability distribution of X is.

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$.

eg: (2) A random variable X has the following probability function.

$X=x$:	0	1	2	3	4	5	6	7
$p(X=x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$.

find (i) k (ii) $p(X<4)$ (iii) $p(X\geq 4)$

(iv) find the minimum value of λ such that $p(X\leq\lambda)>\frac{1}{2}$.

(v) $p(1.5<X<4.5/X>2)$. (vi) cdf of X. $(F(x))$.

Sol: (i) We have $\sum p(X=x)=1$.

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1.$$

$$10k^2+9k=1.$$

$$10k^2+9k-1=0.$$

$$k=\frac{1}{10} \quad \text{or} \quad k=-1. \quad (\text{By Calc})$$

Since $p(x)\geq 0$ $\therefore$ $k\geq 0$.

$$\boxed{\therefore k=\frac{1}{10}}$$

Hence the probability distribution of X is,

X :	0	1	2	3	4	5	6	7
$p(X=x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$.
(ie) $p(X=x)$:	0	$\frac{10}{100}$	$\frac{20}{100}$	$\frac{20}{100}$	$\frac{30}{100}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$.

(ii) $p(x<4) = p(x = 0, 1, 2, 3)$

$= 0 + k + 2k + 3k = 0 + \frac{10}{100} + \frac{20}{100} + \frac{20}{100}$

$= 5k$ (OR) $= \frac{50}{100}$

$\boxed{p(x<4) = \frac{5}{10}}$ $\boxed{p(x<4) = \frac{1}{2}}$

(iii) $p(x \geq 4) = p(x = 4, 5, 6, 7)$

$= \frac{30}{100} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}$

$= \frac{50}{100}$

$\boxed{p(x \geq 4) = \frac{1}{2}}$

(iv) put $\lambda = 0$.

$p(x \leq 0) = p(x=0) = 0 < \frac{1}{2}$

put $\lambda = 1$.

$p(x \leq 1) = p(x=0) + p(x=1) = 0 + \frac{10}{100} = \frac{1}{10} < \frac{1}{2}$.

put $\lambda = 2$

$p(x \leq 2) = p(x = 0, 1, 2)$

$= 0 + \frac{10}{100} + \frac{20}{100} = \frac{30}{100} = \frac{3}{10} < \frac{1}{2}$.

put $\lambda = 3$.

$p(x \leq 3) = p(x = 0, 1, 2, 3)$

$= 0 + \frac{10}{100} + \frac{20}{100} + \frac{20}{100} = \frac{50}{100} = \frac{1}{2} = \frac{1}{2}$

put $\lambda = 4$.

$$p(X \le 4) = p(X = 0, 1, 2, 3, 4)$$

$$= 0 + \frac{10}{100} + \frac{20}{100} + \frac{20}{100} + \frac{30}{100}$$

$$= \frac{8\cancel{0}}{10\cancel{0}}$$

$$= 0.8 > \frac{1}{2}.$$

$\therefore$ The minimum value of λ is 4.

(v) $p[1.5 < X < 4.5 / X > 2]$

We know that, $p(A/B) = \dfrac{p(A \cap B)}{p(B)}$.

$$\therefore p[1.5 < X < 4.5 / X > 2] = \frac{p[1.5 < X < 4.5 \cap X > 2]}{p(X > 2)}.$$

$$= \frac{p([X = 2, 3, 4] \cap p[X = 3, 4, 5, 6, 7])}{p[X > 2]}$$

$$= \frac{p(X = 3, 4)}{p(X > 2)}$$

$$= \frac{p(X = 3, 4)}{p(X = 3, 4, 5, 6, 7)}$$

$$= \frac{\frac{20}{100} + \frac{30}{100}}{\frac{20}{100} + \frac{30}{100} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}} = \frac{\frac{50}{\cancel{100}}}{\frac{70}{\cancel{100}}} = \frac{5\cancel{0}}{7\cancel{0}}$$

$$= 5/7$$

(vi) cdf of $F(x)$.

$X=x$:	0	1	2	3	4	5	6	7
$p(X=x)$:	0	$\frac{10}{100}$	$\frac{20}{100}$	$\frac{20}{100}$	$\frac{30}{100}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F(x)$:	0	$\frac{10}{100}$	$\frac{30}{100}$	$\frac{50}{100}$	$\frac{80}{100}$	$\frac{81}{100}$	$\frac{83}{100}$	$\frac{100}{100}$

MATHEMATICAL EXPECTATION:

MEAN:-

$$E(x) = \begin{cases} \sum x\, p(x), & x \text{ is discrete RV's} \\ \int_{-\infty}^{\infty} x\, f(x)\, dx, & x \text{ is Continuous RV's} \end{cases}$$

$E(x)$ is called the Mean of the distribution or mean of X and it is denoted by $\bar{X}$ or μ_1'

$$E(x^2) = \begin{cases} \sum x^2 p(x), & x \text{ is discrete RV's.} \\ \int_{-\infty}^{\infty} x^2 f(x)\, dx, & x \text{ is Continuous RV's.} \end{cases}$$

$E(x^2)$ is called second moment about the origin and it is denoted by μ_2'.

NOTE:- (1) $E(c) = c$, where c is a Constant.

(2) $E(aX+b) = aE(X)+b$, where a and b are Constants.

VARIANCE :-

$$Var(X) = E[X - E(X)]^2.$$

(ie) $Var(X) = E(X^2) - [E(X)]^2.$

$Var(x)$ is denoted by σ_X^2 or μ_2.

(ie) $\mu_2 = \mu_2' - (\mu_1')^2$

NOTE :-

(1) $Var(aX) = a^2 Var(X)$

(2) $Var(aX+b) = a^2 Var(X)$

(3) $Var(b) = 0$. where a and b are

Constants.

HW (1) A random variable X has the following probability distribution.

x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	$3k$

(i) find k (ii) $p(X<2)$ and $p(-2<X<2)$

$K = \frac{1}{15}$ 0.5 0.4

(iii) CDF of X (iv) Mean of X.

1.07

0
0.1
0.17
0.37
0.5
0.8
1

eg: ③ Given: $p(X=j) = \frac{1}{2^j}, j = 1, 2, \ldots \infty$ ⑥

Verify that the total probability is 1. and find the mean and variance of the distribution. find also p(X is even), $p(X \geq 5)$ and p(X is divisible by 3).

Sol: (i) To verify: $\sum p(x) = 1$.

Consider LHS = $\sum p(x) = p(1) + p(2) + p(3) + \ldots$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{2}\left[1 - \tfrac{1}{2}\right]^{-1} \qquad \left[\because 1 + x + x^2 + \ldots = (1-x)^{-1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^{-1}$$

$$= \frac{1}{2}(2)$$

$$= 1.$$

$$= \text{RHS}.$$

Hence $p(X=j) = \frac{1}{2^j}$ is a probability mass function.

Mean and Variance:

$$\text{Mean} = E(X) = \sum x\, p(x).$$

$$= \sum x \left(\frac{1}{2^x}\right)$$

$$= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + 4 \cdot \frac{1}{2^4} + \cdots$$

$$= \frac{1}{2}\left[1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2} \quad \left[\because 1 + 2x + 3x^2 + \cdots = (1-x)^{-2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\right]^{-2}$$

$$= \frac{1}{2} \times \left(2^{-1}\right)^{-2}$$

$$= \frac{1}{\cancel{2}} \times \cancel{4}$$

$$\boxed{E(x) = 2}$$

$$E(x^2) = \sum x^2\, p(x)$$

$$= 1^2 \cdot p(1) + 2^2 \cdot p(2) + 3^2\, p(3) + \cdots$$

$$= 1^2 \left(\frac{1}{2}\right) + 2^2 \left(\frac{1}{2^2}\right) + 3^2 \left(\frac{1}{2^3}\right) + \cdots$$

$$= \frac{1}{2}\left[1 + 4\left(\frac{1}{2}\right) + 9\left(\frac{1}{2}\right)^2 + \cdots\right]$$

$$\left[1 + 4x + 9x^2 + 16x^3 + \cdots = (1+x)(1-x)^{-3}\right]$$

$= \frac{1}{2} \left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^{-3}$

$= \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^{-3}$

$= \frac{1}{2}\left(\frac{3}{2}\right)\left(2^{-1}\right)^{-3}$

$= \frac{1}{\cancel{2}}\left(\frac{3}{\cancel{2}}\right)\left(2^{\cancel{3}}\right)$

$= 3 \times 2$

$\boxed{E(x^2) = 6}$

$\therefore Var(x) = E(x^2) - [E(x)]^2$

$= 6 - 2^2$

$= 6 - 4$

$\boxed{Var(x) = 2}$

$p(x \text{ is even}) = p(x=2) + p(x=4) + p(x=6) + \ldots$

$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots$

$= \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right]$

$= \frac{1}{4}\left[1 + \left(\frac{1}{2^2}\right) + \left(\frac{1}{2^2}\right)^2 + \ldots\right]$

$= \frac{1}{4}\left[1 - \frac{1}{2^2}\right]^{-1}$

$$= \frac{1}{4}\left[1-\frac{1}{4}\right]^{-1}$$

$$= \frac{1}{4}\left(\frac{3}{4}\right)^{-1}$$

$$= \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{3}$$

$$\boxed{p(x \text{ is even}) = \frac{1}{3}}$$

$$p(x \geq 5) = p(5) + p(6) + p(7) + \dots$$

$$= \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \dots$$

$$= \frac{1}{2^5}\left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \dots\right]$$

$$= \frac{1}{32}\left[1-\frac{1}{2}\right]^{-1}$$

$$= \frac{1}{32}\left[\frac{1}{2}\right]^{-1}$$

$$= \frac{1}{\cancel{32}} \times \cancel{2}$$

$$\boxed{p(x \geq 5) = \frac{1}{16}}$$

$$p(x \text{ is divisible by } 3) = p(3) + p(6) + p(9) + \dots$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \dots$$

$$= \frac{1}{2^3}\left[1 + \left(\frac{1}{2^3}\right) + \left(\frac{1}{2^3}\right)^2 + \dots\right]$$

$$= \frac{1}{8}\left[1-\frac{1}{2^3}\right]^{-1}$$

$$= \frac{1}{8}\left[1-\frac{1}{8}\right]^{-1}$$

$$= \frac{1}{8}\left(\frac{7}{8}\right)^{-1}$$

$$= \frac{1}{\cancel{8}} \times \frac{\cancel{8}}{7}$$

$$\boxed{p(x \text{ is divisible by } 3) = \frac{1}{7}}$$

2M ~~eg~~: (4) A random variable X has the following probability distribution.

x :	-2	-1	0	1	2	3
$p(x)$:	0.1	k	0.2	$2k$	0.3	$3k$.

find (i) k (ii) Mean of X.

Sol: We have, $\sum p(x) = 1$.

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1.$$

$$6k + 0.6 = 1.$$

$$6k = 1 - 0.6$$

$$6k = 0.4$$

$$k = \frac{0.4}{6} = \frac{\cancel{4}}{\cancel{60}} = \frac{1}{15}$$

$$\boxed{k = \frac{1}{15}} = 0.066$$

Mean of $X = E(x) = \sum x\, p(x)$.

$$= (-2)\,p(-2) + (-1)\cdot p(-1) + 0\cdot p(0) + 1\,p(1) + 2\,p(2) + 3\,p(3).$$

$$= (-2)(0.1) + (-1)k + 0 + (1)(2k) + 2(0.3) + 3(3k)$$

$$= -0.2 - k + 2k + 0.6 + 9k$$

$$= 10k + 0.4$$

$$= 10(0.066) + 0.4$$

$$= 0.6 + 0.4$$

$$= 1.$$

$$\boxed{\therefore\ E(x) = 1}$$

HW ②

x :	-2	-1	0	1
$p(x)$:	0.4	k	0.2	0.3

find the value of k and mean of X.

0.1 $\qquad$ -0.6

HW ③ Given: $p[X = x] = kx$, $x = 1, 2, 3, 4$

find k and $p(2 < x < 4)$

$\frac{1}{10}$ $\qquad$ $\frac{3}{10}$

eg: (5) If X has the distribution function. (9)

(X)

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } 1 \le x < 4 \\ \frac{1}{2} & \text{for } 4 \le x < 6 \\ \frac{5}{6} & \text{for } 6 \le x \le 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

Find (i) the probability distribution of X.

(ii) $P(2 < X < 6)$

(iii) Mean of X

(iv) Var (x).

Sol: (i) To find the probability distribution of X, we have to find the probabilities at the changing points 1, 4, 6, 10

We know that, $P(X=x_i) = F(x_i) - F(x_{i-1})$

$i = 1, 2, 3 \ldots$

$$P(X=1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}.$$

$$P(X=4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}.$$

$$P(X=6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{10-6}{12} = \frac{4}{12} = \frac{1}{3}.$$

$$P(X=10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{6-5}{6} = \frac{1}{6}.$$

$\therefore$ The probability distribution of X is

x :	1	4	6	10
$p(x)$:	$1/3$	$1/6$	$1/3$	$1/6$.

(ii) $p(2 < x < 6) = p(x=4) = 1/3.$

(iii) Mean of $X = E(x) = \sum x\, p(x).$

$$= \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{6}\right)$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{6}{3} + \frac{5}{3}$$

$$= \frac{1+2+6+5}{3}$$

$$\boxed{\therefore E(x) = \frac{14}{3}}$$

$$E(x^2) = \sum x^2 p(x)$$

$$= \left(1^2 \times \frac{1}{3}\right) + \left(4^2 \times \frac{1}{6}\right) + \left(6^2 \times \frac{1}{3}\right) + \left(10^2 \times \frac{1}{6}\right)$$

$$= \frac{1}{3} + \frac{16}{6} + \frac{36}{3} + \frac{100}{6}$$

$$= \frac{1}{3} + \frac{8}{3} + \frac{36}{3} + \frac{50}{3}$$

$$= \frac{1+8+36+50}{3}$$

$$\boxed{E(x^2) = \frac{95}{3}}$$

(iv) $Var(x) = E(x^2) - (E(x))^2 = \frac{95}{3} - \left(\frac{14}{3}\right)^2$

$$= \frac{95}{3} - \frac{196}{9} = \frac{285 - 196}{9}$$

$$\boxed{Var(x) = \frac{89}{9}}$$

Problems based on probability density function.

(2M) eg: 6 The density function of a random variable X is given by $f(x) = kx(2-x), 0 \le x \le 2$. Find the value of k

Sol: Since $f(x)$ is a pdf, then we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x - x^2)\,dx = 1.$$

$$k\left[\cancel{2}\cdot\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = 1.$$

$$k\left[\left(4 - \frac{8}{3}\right) - 0\right] = 1.$$

$$k\left[\frac{12-8}{3}\right] = 1.$$

$$k\left[\frac{4}{3}\right] = 1$$

$$\boxed{k = \frac{3}{4}}$$

(HW) Given: $f(x) = k(1+x), 2 \le x \le 5$. Find $P(X<4)$ and $P(3<X<4)$

$\left(\frac{1}{3}\right)$ $\quad k = \frac{2}{27}$, $\quad \frac{16}{27}$

(2M) Q: (7) If random variable X has pdf

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$ find k such that $p(X > k) = 0.05$.

Sol: Given that $p(X > k) = 0.05$

$$\int_k^\infty f(x)\,dx = 0.05$$

$$\int_k^1 3x^2\,dx = 0.05$$

$$\cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_k^1 = 0.05$$

$$1 - k^3 = 0.05$$

$$1 - 0.05 = k^3$$

$$0.95 = k^3$$

$$\frac{95}{100} = k^3$$

$$\boxed{k = \sqrt[3]{\frac{95}{100}}}$$

(HW2) Given: $f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$ find c. $\left(\frac{3}{8}\right)$

HW(3) $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$ find $p(0 < X \le 1)$

$\left(\frac{1}{9}\right)$

(2M) Ex: 8 A Continuous random variable X is distributed over the Interval [0, 1] with p.d.f $ax^2 + bx$ where a, b are constants. If the arithmetic mean of X is 0.5 find the values of a and b. (11)

Sol: Since f(x) is a p.d.f then

we have $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\Rightarrow \int_0^1 (ax^2 + bx)\,dx = 1.$$

$$\Rightarrow \left[a\frac{x^3}{3} + b\frac{x^2}{2} \right]_0^1 = 1.$$

$$\Rightarrow \left(\frac{a}{3} + \frac{b}{2}\right) - 0 = 1.$$

$$\Rightarrow \frac{2a + 3b}{6} = 1.$$

$$\Rightarrow \boxed{2a + 3b = 6} \longrightarrow (1)$$

Given that, $E(x) = \text{Mean} = 0.5$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \tfrac{1}{2}$$

$$\Rightarrow \int_0^1 x\,(ax^2 + bx)\,dx = \tfrac{1}{2}$$

$$\Rightarrow \int_0^1 (ax^3+bx^2)\,dx = 1$$

$$\Rightarrow \left[a\frac{x^4}{4} + b\frac{x^3}{3}\right]_0^1 = 1.$$

$$\Rightarrow \left(\frac{a}{4} + \frac{b}{3}\right) - 0 = 1.$$

$$\Rightarrow \frac{3a+4b}{12} = 1.$$

$$\Rightarrow \boxed{3a+4b=12} \longrightarrow (2)$$

Solving eq (1) and (2), we get

$\boxed{a=-6}$ and $\boxed{b=6}$ (Use Calculator).

Eg: (9) A Continuous random variable X has the pdf $f(x) = kx^4$, $-1 < x < 0$. Find the value of k and $P[X > -\frac{1}{2} / X < -\frac{1}{4}]$.

Sol: Since $f(x)$ is a pdf, then we have, $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\Rightarrow \int_{-1}^{0} kx^4\,dx = 1$$

$$\Rightarrow k\left[\frac{x^5}{5}\right]_{-1}^{0} = 1.$$

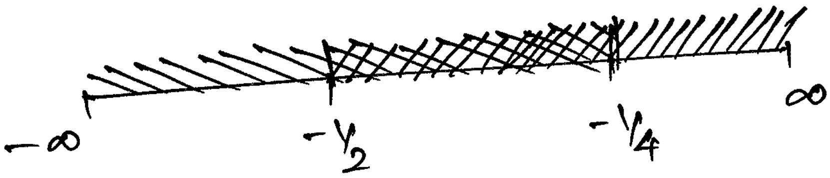
$$k\left[0-\left(-\tfrac{1}{5}\right)\right]=1.$$

$$k\left(\frac{1}{5}\right)=1$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}.$$

$$\boxed{k=5}$$

$$P\left(x>-\tfrac{1}{2}\,/\,x<-\tfrac{1}{4}\right)=\frac{P\left(x>-\tfrac{1}{2}\cap x<-\tfrac{1}{4}\right)}{P\left(x<-\tfrac{1}{4}\right)}$$



$$=\frac{P\left(-\tfrac{1}{2}<x<-\tfrac{1}{4}\right)}{P\left(x<-\tfrac{1}{4}\right)}\longrightarrow (1)$$

Consider $P\left(-\tfrac{1}{2}<x<-\tfrac{1}{4}\right)=\displaystyle\int_{-1/2}^{-1/4} f(x)\,dx$

$$=\int_{-1/2}^{-1/4} kx^4\,dx$$

$$=k\left[\frac{x^5}{5}\right]_{-1/2}^{-1/4}$$

$$=\frac{k}{5}\left[\left(-\frac{1}{4}\right)^5-\left(-\frac{1}{2}\right)^5\right]$$

$$=\frac{\cancel{5}}{\cancel{5}}\left[\frac{-1}{1024}+\frac{1}{32}\right]=\left[\frac{-1+32}{1024}\right]=\frac{31}{1024}.$$

$$P(x < -1/4) = \int_{-\infty}^{-1/4} f(x)\,dx$$

$$= \int_{-1}^{-1/4} 5x^4\,dx$$

$$= \cancel{5}\left[\frac{x^5}{\cancel{5}}\right]_{-1}^{-1/4}$$

$$= \left(\frac{-1}{4}\right)^5 - (-1)^5$$

$$= \frac{-1}{1024} + 1$$

$$= \frac{-1+1024}{1024}$$

$$\boxed{P(x < -1/4) = \frac{1023}{1024}}$$

from (1),

$$P(x > -1/2 \,/\, x < -1/4) = \frac{\frac{31}{\cancel{1024}}}{\frac{1023}{\cancel{1024}}} = \frac{31}{1023} = \frac{1}{33}.$$

(HW) Given: $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$ find,

(i) $P(x < 1/2)$ $(1/4)$ (ii) $P(1/4 < x < 1/2)$ $(3/16)$

(iii) $P(x > 3/4 \,/\, x > 1/2)$ $\left(\frac{7}{12}\right)$ ~~(iv) $P(1/4 < x < 1/2)$~~

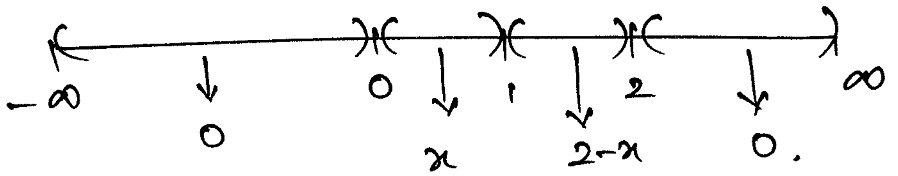
(iv) $P(x < 3/4 \,/\, x > 1/2)$.

$$\frac{5/16}{3/4} = \frac{5}{12}$$

Q:(10) Find the cumulative distribution function (13) of the random variable with p.d.f

$$f(x) = \begin{cases} x & , \text{ if } 0 < x \leq 1 \\ 2-x & , \text{ if } 1 < x \leq 2 \\ 0 & , \text{ otherwise} \end{cases}$$

Sol: We have $F(x) = \int_{-\infty}^{x} f(x)\,dx$.



Case (i) If $x \in (-\infty, 0)$ then $F(x) = \int_{-\infty}^{0} 0\,dx = 0$.

Case (ii) If $x \in (0, 1)$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$\boxed{F(x) = \frac{x^2}{2} \quad , \quad 0 \leq x < 1}$$

Case (iii) If $x \in (1, 2)$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^x$$

$$= \left(\frac{1}{2} - 0\right) + \left(2x - \frac{x^2}{2}\right) - \left(2 - \tfrac{1}{2}\right)$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \tfrac{1}{2}$$

$$= 1 + 2x - \frac{x^2}{2} - 2$$

$$\boxed{F(x) = 2x - \frac{x^2}{2} - 1 \,, \quad 1 \le x < 2}$$

<u>Case</u> (iv) If $x \in (2, \infty)$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^{\infty} f(x)\,dx$$

$$= 0 + \int_0^1 x\,dx + \int_1^2 (2-x)\,dx + 0$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left(4 - \frac{4}{2}\right) - \left(2 - \tfrac{1}{2}\right)$$

$$= \frac{1}{2} + \cancel{4} - \cancel{2} - \cancel{2} + \tfrac{1}{2}$$

$$\boxed{F(x) = 1 \,, \quad x \ge 2}$$

$$\therefore \; F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \le x < 1 \\ 2x - \frac{x^2}{2} - 1 & \text{if } \cancel{0} \le x < 2 \\ 1 & \text{if } x \ge 2 . \end{cases}$$

Eg: (11) Given: $f(x) = \begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ is a pdf (14)

find (i) λ (ii) $P(50 < X < 150)$ (iii) $P(X < 500)$.

Sol: (i) Since $f(x)$ is a pdf, then

we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_0^{\infty} \lambda e^{-x/100}\,dx = 1.$$

$$\lambda \left[\frac{e^{-x/100}}{-1/100} \right]_0^{\infty} = 1.$$

$$\lambda \left[0 - \frac{1}{-1/100} \right] = 1.$$

$$\lambda (100) = 1.$$

$$\boxed{\lambda = \frac{1}{100}}$$

(ii) $P(50 < X < 150) = \int_{50}^{150} f(x)\,dx$

$$= \int_{50}^{150} \frac{1}{100} e^{-x/100}\,dx$$

$$= \cancel{\frac{1}{100}} \left[\frac{e^{-x/100}}{\cancel{-1/100}} \right]_{50}^{150}$$

$$= -\left[e^{-x/100} \right]_{50}^{150}$$

$$= -\left[e^{-\frac{150}{100}} - e^{-\frac{50}{100}} \right]$$

$$= -\left[e^{-3/2} - e^{-1/2} \right]$$

$$\boxed{P(50 < x < 150) = e^{-1/2} - e^{-3/2}} = 0.6065 - 0.2231 = 0.3834.$$

(iii) $P(x < 500) = \int_{-\infty}^{500} f(x)\,dx$

$$= \int_{0}^{500} \frac{1}{100} e^{-x/100}\,dx$$

$$= \frac{1}{\cancel{100}} \left[\frac{e^{-x/100}}{-1/\cancel{100}} \right]_0^{500}$$

$$= -\left[e^{-\frac{500}{100}} - e^{0} \right]$$

$$= -\left[e^{-5} - 1 \right]$$

$$= 1 - e^{-5}$$

$$= 1 - 0.0067$$

$$\boxed{P(x < 500) = 0.9933}$$

Q. (12) The cummulative distribution function (15)
of a random variable X is $F(x) = 1-(1+x)e^{-x}$, $x>0$. find the pdf of x, mean and variance.

Sol: Given: $F(x) = 1-(1+x)e^{-x}$.

$u \quad v$

We have

$$f(x) = F'(x) = \frac{d}{dx}[F(x)]$$

$$f(x) = 0 - [(1+x)e^{-x}(-1) + e^{-x}(0+1)]$$

$$= (1+x)e^{-x} - e^{-x}$$

$$= \cancel{e^{-x}} + xe^{-x} - \cancel{e^{-x}}$$

$$\boxed{\therefore f(x) = xe^{-x}, \quad x>0}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \cdot xe^{-x}\,dx$$

$$\boxed{\Gamma n = \int_0^{\infty} x^{n-1}e^{-x}\,dx \qquad \Gamma n = (n-1)!}$$

$$= \int_0^{\infty} x^2 e^{-x}\,dx$$

$$= \Gamma 3$$

$$= 2!$$

$$\boxed{E(x) = 2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2\, x e^{-x}\,dx$$

$$= \int_0^{\infty} x^3 e^{-x}\,dx$$

$$= \Gamma 4$$

$$= 3!$$

$$\boxed{E(x^2) = 6}$$

$$\therefore \text{var}(x) = E(x^2) - [E(x)]^2$$

$$= 6 - 2^2$$

$$= 6 - 4$$

$$\boxed{\text{Var}(x) = 2}$$

(HW)

$$F(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2 & , \frac{1}{2} \le x < 3 \\ 1 & , x \geq 3 \end{cases} \qquad \left(f(x) = \begin{cases} 2x \\ \frac{6}{25}(3-x) \\ 0 \end{cases} \right)$$

find (i) pdf of x (ii) $P(|x| \le 1)$ $\left(\frac{13}{25}\right)$

(iii) $P\left(\frac{1}{3} < x < 4\right)$ using both pdf and distribution function.

$\left(\frac{8}{9}\right)$

HW Verify whether $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ is a probability function of a continuous random variable X. If so find the mean and $Var(X)$. $\left(\frac{1}{3} \text{ and } \frac{2}{9}\right)$

HW

X Given: $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$ $F(x) = \begin{cases} 0 \\ \frac{x^2}{4} \\ \frac{1}{4}(2x-1) \\ \frac{1}{4}(-x^2+6x-5) \\ 1 \end{cases}$

find (i) value of a $\left(\frac{1}{2}\right)$ (ii) cdf of X.

HW Given: $f(x) = C e^{-|x|}, \; -\infty < x < \infty$. $F(x) = \begin{cases} \frac{e^{x}}{2} \\ 1 - \frac{e^{-x}}{2} \end{cases}$

X find (i) c $\left(\frac{1}{2}\right)$ (ii) cdf of X.

MOMENTS AND MOMENT GENERATING FUNCTION

MOMENT:- The expected value of an Integral power of a random variable is called its moments.

Moments about the origin or raw moments

$$\mu'_r = E(x^r)$$

Moments about the mean or Central moments:

$$\mu_r = E(x-\mu)^r$$

RESULTS:-

① For every distribution discrete or continuous, the first moment about its mean is zero.

② $\mu_2 = E(x-\mu)^2 =$ variance of x.

③ The moments about any point 'a' is

$$\mu'_r = E(x-a)^r.$$

MOMENT GENERATING FUNCTION (MGF):

The moment generating function of a random variable X is defined as $E[e^{tX}]$ for all $t \in (-\infty, \infty)$. It is denoted by $M_X(t)$ or $M(t)$.

$$M_X(t) = \begin{cases} \sum_x e^{tx} p(x), & x \text{ is discrete RV} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & x \text{ is Continuous RV.} \end{cases}$$

NOTE:-

① μ'_r = Coefficient of $\frac{t^r}{r!}$ in the series of $M_X(t)$.

② $\mu'_1 = M'_X(0)$ and $\mu'_2 = M''_X(0)$.

(2M) **Properties of Moment generating function of X.**

(i) $M_{cX}(t) = M_X(ct)$

(ii) $M_{X+c}(t) = e^{ct} M_X(t)$

(iii) $M_{aX+b}(t) = e^{bt} M_X(at)$

(iv) $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ if X and Y are Independent random variables.

(2M) ⑬ Find the MGF of $f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

Sol:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-1}^{2} e^{tx} \left(\frac{1}{3}\right) dx$$

$$= \frac{1}{3} \left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

$$= \frac{1}{3} \left[\frac{e^{2t}}{t} - \frac{e^{-t}}{t}\right]$$

$$\boxed{M_X(t) = \frac{1}{3t}\left[e^{2t} - e^{-t}\right]}$$

(2m) Eg: 14 If a random variable x has the MGF

$M_x(t) = \frac{2}{2-t}$ find $var(x)$.

Sol: Given: $M_x(t) = \frac{2}{2-t} = \frac{\cancel{2}}{\cancel{2}(1-t/2)}$

$M_x(t) = (1 - t/2)^{-1}$ $\quad [(1-x)^{-1} = 1 + x + x^2 + \dots]$

$M_x(t) = 1 + (t/2) + (t/2)^2 + (t/2)^3 + \dots$

$M_x(t) = 1 + \frac{1}{2}\left(\frac{t}{1!}\right) + \frac{t^2}{2!}\cdot\left(\frac{2!}{4}\right) + \frac{t^3}{3!}\cdot\left(\frac{3!}{8}\right) + \dots$

$\mu_1' =$ Coeff of $\frac{t^1}{1!}$ in the expansion of $M_x(t) = 1/2$.

$\mu_2' =$ Coeff of $\frac{t^2}{2!}$ in the expansion of $M_x(t) = \frac{2}{4} = 1/2$

$\therefore var(x) = E(x^2) - [E(x)]^2$

$= \mu_2' - (\mu_1')^2$

$= \frac{1}{2} - (1/2)^2$

$= \frac{1}{2} - 1/4$

$= \frac{2-1}{4}$

$\boxed{var(x) = 1/4}$

(HW) $M_x(t) = \frac{3}{3-t}$ find the S.D of x.

$(1/3)$

(2m) eg: (15) Let X be a RV with $E(X) = 10$ and (18)
$Var(X) = 25$. find the positive values of a and b such that $Y = aX - b$ has expection 0 and Var 1.

Sol: Given: $E(X) = 10$ and $Var(X) = 25$.

$Y = aX - b$

$E(Y) = E(aX - b)$

$0 = aE(X) - b$

$0 = a(10) - b$

$\boxed{0 = 10a - b}$ — (1)

$Var(Y) = 1.$

$Var(aX - b) = 1.$

$a^2 Var(X) = 1.$

$a^2 (25) = 1$

$a^2 = 1/25$

$\boxed{a = 1/5}$ sub in eq (1),

$0 = \cancel{10}^{2}\left(\frac{1}{\cancel{5}}\right) - b$

$b = 2$

$\therefore (a, b) = \left(\frac{1}{5}, 2\right)$

(2m) eg: (16) find the MGF of the distribution.

$$p(x) = \begin{cases} \frac{2}{3}, & x = 1 \\ \frac{1}{3}, & x = 2 \\ 0, & \text{elsewhere} \end{cases}$$

Sol:
Given:

x	:	1	2
$p(x)$	:	$2/3$	$1/3$
e^{tx}	:	e^{t}	e^{2t}

$M_X(t) = \sum_x e^{tx} p(x)$

$= e^t\left(\frac{2}{3}\right) + e^{2t}\left(\frac{1}{3}\right)$

$$\boxed{M_X(t) = \frac{e^t}{3}\left[2 + e^t\right]}$$

Q: (17) The density function of a random variable is given by $f(x) = \begin{cases} kx(2-x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

find the value of k, mean, variance and r^{th} moment.

Sol: Given: Since $f(x)$ is a pdf, then we have $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\Rightarrow \int_0^2 k(2x - x^2)\,dx = 1.$$

$$\Rightarrow k\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = 1.$$

$$\Rightarrow k\left[\left(4 - \frac{8}{3}\right) - 0\right] = 1.$$

$$\Rightarrow k\left[\frac{12-8}{3}\right] = 1.$$

$$\Rightarrow k\left(\frac{4}{3}\right) = 1$$

$$\Rightarrow \boxed{k = \frac{3}{4}}$$

r^{th} Moment: $\mu'_r = E[x^r] = \int_{-\infty}^{\infty} x^r f(x)\,dx$

$$= \int_0^2 x^r \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^r (2x - x^2)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right)dx$$

$$= \frac{3}{4}\left[2\cdot\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{2..2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - 0\right]$$

$$= \frac{3}{4}\left[\frac{2^r.2^3}{r+2} - \frac{2^3.2^r}{r+3}\right]$$

$$= \frac{3}{4}\times 2^r \times 2^3 \left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{3}{\cancel{4}}\times 2^r \times \cancel{8}^{2}\left[\frac{(r+3)-(r+2)}{(r+3)(r+2)}\right]$$

$$= 6.2^r\left[\frac{\cancel{r}+3\cancel{-r}-2}{(r+3)(r+2)}\right]$$

$$\boxed{E(x^r) = \frac{6.2^r}{(r+3)(r+2)}}$$

put $r=1$.

$$E(x) = \text{Mean} = \frac{\cancel{6}.\cancel{2}}{(\cancel{4})(\cancel{3})} = 1.$$

$$\boxed{\therefore \text{Mean} = 1}$$

Put $r=2$

$$E(x^2) = \frac{6 \cdot 2^2}{(5)\cdot(4)} = \frac{6 \cdot \cancel{4}}{(5)\cdot\cancel{(4)}} = \frac{6}{5}.$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{6}{5} - 1$$

$$= \frac{6-5}{5}$$

$$\boxed{\text{Var}(x) = \frac{1}{5}}$$

Eg. 18) For the triangular distribution

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the Mean, variance and MGF of X

Sol:
$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x\,(2-x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3}$$

$$= 3 - \frac{6}{3}$$

$$= 3 - 2$$

$$\boxed{E(x) = 1}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2 (2-x)\, dx$$

$$= \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{4} - 0\right) + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4}$$

$$= \cancel{\frac{2}{4}} + \frac{14}{3} - 4$$

$$= \frac{3 + 28 - 24}{6} = \frac{7}{6}. \quad \boxed{E(x^2) = \frac{7}{6}}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{7}{6} - 1$$

$$= \frac{7-6}{6}$$

$$\boxed{Var(x) = \frac{1}{6}}$$

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^1 e^{tx} \cdot x\, dx + \int_1^2 e^{tx} (2-x)\, dx$$

$$= \int_0^1 \underset{u}{\underline{x}}\ \underset{dv}{\underline{e^{tx}\, dx}} + \int_1^2 \underset{u}{\underline{(2-x)}}\ \underset{dv}{\underline{e^{tx}\, dx}}$$

$$= \left[\frac{x e^{tx}}{t} - (1)\frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left(\frac{e^t}{t} - \frac{e^t}{t^2}\right) - \left(0 - \frac{1}{t^2}\right) + \left(0 + \frac{e^{2t}}{t^2}\right) - \left(\frac{e^t}{t} + \frac{e^t}{t^2}\right)$$

$$= \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2} = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$\boxed{\therefore M_x(t) = \frac{(e^t - 1)^2}{t^2}},\ t \neq 0.$$

Eg: 19 A random variable X has the pdf given by 51

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

find (i) MGF (ii) First four moments about the Origin.

Sol: (i) MGF: $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$= \int_0^{\infty} e^{tx}\, 2e^{-2x}\, dx$$

$$= 2\int_0^{\infty} e^{-x(2-t)}\, dx$$

$$= 2\left[\frac{e^{-x(2-t)}}{-(2-t)}\right]_0^{\infty}$$

$$= 2\left[0 - \frac{1}{-(2-t)}\right]$$

$$= 2 \times \frac{1}{2-t}$$

$$= \cancel{2} \times \frac{1}{\cancel{2}(1-t/2)}$$

$$\boxed{M_X(t) = \left(1 - t/2\right)^{-1}}$$

$$M_X(t) = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots$$

$$M_X(t) = 1 + \left(\frac{t}{1}\right)\left(\frac{1}{2}\right) + \left(\frac{t^2}{2!}\right)\left(\frac{2!}{4}\right) + \frac{t^3}{3!}\left(\frac{3!}{8}\right) + \frac{t^4}{4!}\left(\frac{4!}{16}\right) + \cdots$$

We have $\mu_r' =$ Coeff of $\frac{t^r}{r!}$ in the expansion of $M_X(t)$.

$\therefore \mu_1' = \frac{1}{2}$, $\mu_2' = \frac{2}{4} = \frac{1}{2}$, $\mu_3' = \frac{6}{8} = \frac{3}{4}$

and $\mu_4' = \frac{24}{16} = \frac{3}{2}$

(HW) Given: $f(x) = \frac{1}{2} e^{-x/2}, x > 0$. find MGF $\left(\frac{1}{1-2t}\right)$

and Mean, var(X). $(2, 4)$

(HW) $f(x) = \begin{cases} 1/k, & 0 < x < k \\ 0, & \text{otherwise} \end{cases}$ find (i) MGF $\left(\frac{1}{kt}(e^{kt}-1)\right)$ (ii) r^{th} moment $\left(\frac{k^r}{r+1}\right)$

(iii) Mean $\left(\frac{k}{2}\right)$ (iv) var(X). $\left(\frac{k^2}{12}\right)$

(HW) If X represents the out come, when a fair die is tossed then find the MGF and hence $\frac{1}{6}[e^t + e^{2t} + \dots + e^{6t}]$
find $E(X)$ and var(X)
(3.5) (2.9167)

(HW) Given: $f(x) = \begin{cases} \frac{4x(9-x^2)}{81}, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

find the first four moments about the Origin.
$\left(\frac{8}{5}, 3, 6.17, 13.5\right)$

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(X) eg: Let $p(X=x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$, $x = 1, 2, 3 \ldots$ be the probability mass function of X. Compute

(i) $p(X>4)$ (ii) $p(X>4/X>2)$ (iii) $E(X)$ (iv) $Var(X)$.

Sol: Given: $p(X=x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$, $x = 1, 2, 3 \ldots$

(i) $p(X>4) = p(X=5) + p(X=6) + p(X=7) + \ldots$

$$= \frac{3}{4}\left(\frac{1}{4}\right)^4 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^5 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^6 + \ldots$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 \left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots\right]$$

$$\boxed{1 + x + x^2 + \ldots = (1-x)^{-1}}$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 \left[1 - \frac{1}{4}\right]^{-1}$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{-1}$$

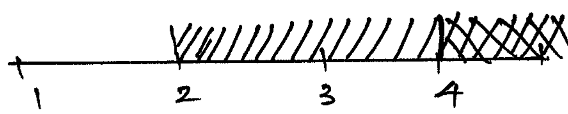
$$= \left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{1}{4}\right)^4 \left(\frac{\cancel{4}}{\cancel{3}}\right)$$

$$\boxed{\therefore\ p(X>4) = \frac{1}{256}}$$

(ii) $p(X>4/X>2)$

$$\boxed{p(A/B) = \frac{p(A\cap B)}{p(B)}}$$

$$= \frac{p(X>4 \cap X>2)}{p(X>2)} \longrightarrow (1)$$



$$= \frac{p(X>4)}{p(X>2)} \longrightarrow (1)$$

Now, $p(X>2) = p(X=3) + p(X=4) + p(X=5) + \ldots$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \ldots$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots\right]$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \left(1 - \frac{1}{4}\right)^{-1}$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{-1}$$

$$= \left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{1}{4}\right)^2 \left(\frac{\cancel{4}}{\cancel{3}}\right)$$

$$= \frac{1}{16}$$

from eq (1) $\quad p(X>4/X>2) = \dfrac{\frac{1}{256}}{\frac{1}{16}} = \dfrac{1}{\cancel{256}} \times \dfrac{\cancel{16}}{1}$

$$\boxed{\therefore\ p(X>4/X>2) = \frac{1}{16}}$$

(iii) $E(X) = \sum_{x=1}^{\infty} x\, p(X=x)$

$$= \sum_{x=1}^{\infty} x \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4} \sum_{x=1}^{\infty} x \left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4}\left[1 \cdot \left(\frac{1}{4}\right)^0 + 2 \cdot \left(\frac{1}{4}\right)^1 + 3\left(\frac{1}{4}\right)^2 + \ldots\right]$$

$$= \frac{3}{4}\left[1 + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right)^2 + \ldots\right]$$

$$\boxed{1 + 2x + 3x^2 + \ldots = (1-x)^{-2}}$$

$$= \frac{3}{4}\left(1-\frac{1}{4}\right)^{-2}$$

$$= \frac{3}{4}\left(\frac{3}{4}\right)^{-2}$$

$$= \frac{3}{4}\left(\frac{4}{3}\right)^{2}$$

$$= \frac{\cancel{3}}{\cancel{4}}\left(\frac{\cancel{16}}{\cancel{9}}\right)$$

$$\boxed{E(X) = \frac{4}{3}}$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \, p(X=x)$$

$$= \sum_{x=1}^{\infty} x^2 \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4}\sum_{x=1}^{\infty} x^2 \left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4}\left[1^2\left(\frac{1}{4}\right)^0 + 4\left(\frac{1}{4}\right) + 9\left(\frac{1}{4}\right)^2 + \dots\right]$$

$$= \frac{3}{4}\left[1 + 4\left(\frac{1}{4}\right) + 9\left(\frac{1}{4}\right)^2 + \dots\right]$$

$$\boxed{1 + 4x + 9x^2 + \dots = (1+x)(1-x)^{-3}}$$

$$= \frac{3}{4}\left(1+\frac{1}{4}\right)\left(1-\frac{1}{4}\right)^{-3}$$

$$= \frac{3}{4}\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)^{-3}$$

$$= \frac{3}{4}\left(\frac{5}{4}\right)\left(\frac{4}{3}\right)^3$$

$$= \left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{5}{\cancel{4}}\right)\left(\frac{\cancel{64}^{4}}{\cancel{27}_{9}}\right)$$

$$\boxed{E(x^2) = \frac{20}{9}}$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$\text{Var}(x) = \frac{20}{9} - \left(\frac{4}{3}\right)^2$$

$$= \frac{20}{9} - \frac{16}{9}$$

$$= \frac{20-16}{9}$$

$$\boxed{\text{Var}(x) = \frac{4}{9}}$$

Eg: If the density function of a Continuous RV X is given by

$$f(x) = \begin{cases} ax & \text{for } 0 \le x \le 1 \\ a & \text{for } 1 \le x \le 2 \\ 3a - ax & \text{for } 2 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

(i) find the value of a

(ii) find the c.d.f of X.

(iii) If X_1, X_2, X_3 are 3 Independent Observations of X, what is the probability that exactly one of these is greater than 1.5?

Sol: (i) Since $f(x)$ is a pdf, $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$. (24)

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$a\left(\frac{x^2}{2}\right)_0^1 + a\,(x)_1^2 + \left(3ax - a\frac{x^2}{2}\right)_2^3 = 1$$

$$a\left(\frac{1}{2} - 0\right) + a(2-1) + \left(9a - \frac{9a}{2}\right) - (6a - 2a) = 1$$

$$\frac{a}{2} + a + 9a - \frac{9a}{2} - 6a + 2a = 1$$

$$-\frac{\cancel{8}a}{\cancel{2}} + 6a = 1$$

$$-4a + 6a = 1$$

$$2a = 1$$

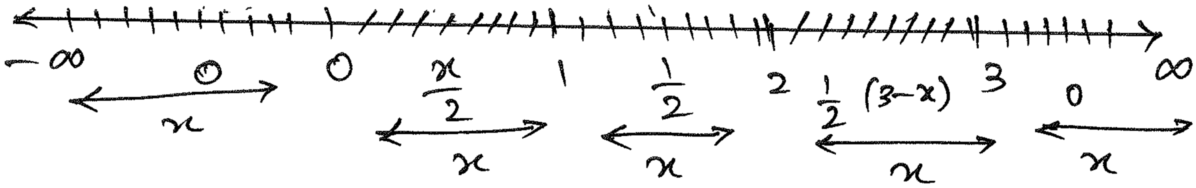
$$\boxed{a = \frac{1}{2}}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(3-x), & 2 \le x \le 3 \\ 0, & \text{elsewhere.} \end{cases}$$

(ii) To find: CDF of X

We know that, $F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(x)\,dx$$



Case (i) If $x \in (\infty, 0)$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0 \,.\, dx = 0.$$

Case (ii) If $x \in (0, 1)$ then

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,.\,dx + \int_{0}^{x} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{2}\left(\frac{x^2}{2}\right) = \frac{x^2}{4}.$$

Case (iii) If $x \in (1, 2)$ then

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,.\,dx + \int_{0}^{1} \frac{x}{2}\,.\,dx + \int_{1}^{x} \frac{1}{2}\,.\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0\right) + \frac{1}{2}(x - 1)$$

$$= \frac{1}{4} + \frac{1}{2}x - \frac{1}{2}$$

$$= \frac{1}{2}x - \frac{1}{4}$$

$$\boxed{F(x) = \frac{1}{4}(2x - 1)}$$

Case (iv) If $x \in (2,3)$ then

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0.\,dx + \int_{0}^{1} \frac{x}{2}.\,dx + \int_{1}^{2} \frac{1}{2}.\,dx + \int_{2}^{x} \frac{1}{2}(3-x)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^x$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(2-1) + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - (6-2)\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3}{2}x - \frac{x^2}{4} - 2$$

$$= \frac{1+2+6x-x^2-8}{4}$$

$$\boxed{F(x) = \frac{6x - x^2 - 5}{4}}$$

Case (v) If $x \in (3,\infty)$ then

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} \frac{x}{2}\,dx + \int_{1}^{2} \frac{1}{2}\,dx + \int_{2}^{3} \frac{1}{2}(3-x)\,dx + 0$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^3$$

$$= \frac{1}{4} + \frac{1}{2}(2-1) + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - (6-2)\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - 2$$

$$= \frac{1+2+18-9-8}{4}$$

$$= \frac{4}{4}$$

$$= 1.$$

$\therefore$ The cdf of x is given by

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{x^2}{4} & , \quad 0 \le x < 1 \\ \frac{1}{4}(2x-1) & , \quad 1 \le x < 2 \\ \frac{1}{4}(6x - x^2 - 5) & , \quad 2 \le x < 3 \\ 1 & , \quad x \geq 3 \end{cases}$$

(iii) Consider, $p(X > 1.5) = \int_{1.5}^{\infty} f(x)\, dx$

$$= \int_{1.5}^{2} f(x)\, dx + \int_{2}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx$$

$$= \int_{1.5}^{2} \frac{1}{2}\, dx + \int_{2}^{3} \frac{1}{2}(3-x)\, dx + 0$$

$$= \frac{1}{2}\left(\cancel{x}\right)_{1.5}^{2} + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_{2}^{3}$$

$$= \frac{1}{2}\left(2 - 1.5\right) + \frac{1}{2}\left[\left(9 - \frac{9}{2}\right) - \left(6 - 2\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{9}{2} - \frac{9}{4} - 2$$

$$= \frac{1+18-9-8}{4}$$

$$= \frac{2}{4}$$

$$\boxed{p(x>1.5) = \tfrac{1}{2}}$$

$p(\text{success}) = p(x>1.5) = \frac{1}{2}$

$\therefore p(\text{Failure}) = 1 - p(s) = 1 - \frac{1}{2} = \frac{1}{2}$

Since x_1, x_2, x_3 are Independent.

Req. probability $= p(SFF) + p(FSF) + p(FFP)$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{3}{8}$$

$\therefore$ probability that exactly one of these is greater than 1.5 is $\frac{3}{8}$.

eg: A Continuous random variable

$f(x) = Kx^3 e^{-x}$, $x > 0$. find the r^{th} moment and MGF of X. Hence find the mean & Var.

Sol: Since $f(x)$ is a pdf, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} k x^3 e^{-x}\,dx = 1$$

$$\boxed{\int_0^{\infty} x^n e^{-ax}\,dx = \frac{\lfloor n}{a^{n+1}}}$$

$$k \int_0^{\infty} x^3 e^{-x}\,dx = 1$$

$$k \frac{\lfloor 3}{1^{3+1}} = 1$$

$$k\left(\frac{6}{1}\right) = 1$$

$$\boxed{k = \frac{1}{6}}$$

(i) r^{th} moment:

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \cdot K x^3 e^{-x}\,dx$$

$$= K \int_0^{\infty} x^{r+3} \cdot e^{-x}\,dx$$

$$E(x^r) = \frac{1}{6}\left(\frac{\lfloor r+3}{1}\right)$$

$$\boxed{E(x^r) = \frac{(r+3)!}{6}}$$

put $r = 1$

$$E(x) = \frac{4!}{6}$$

$$= \frac{24}{6}$$

$$\boxed{E(x) = 4}$$

put $r = 2$

$$E(x^2) = \frac{5!}{6}$$

$$= \frac{120}{6}$$

$$\boxed{E(x^2) = 20}$$

$$\therefore Var(x) = E(x^2) - (E(x))^2$$

$$= 20 - 4^2$$

$$= 20 - 16$$

$$\boxed{Var(x) = 4}$$

To find: MGF

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} K x^3 e^{-x}\, dx$$

$$= K \int_0^{\infty} x^3 e^{-(1-t)x}\, dx$$

$$= \frac{1}{6} \int_0^\infty x^3 e^{-(1-t)x} dx$$

$$= \frac{1}{6} \frac{\lfloor 3}{(1-t)^{3+1}}$$

$$= \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}}{(1-t)^4}$$

$$\boxed{\therefore \quad M_X(t) = \frac{1}{(1-t)^4}}$$

HW ① A continuous RV $f(x) = K x^2 e^{-x}, x > 0$ find the r^{th} moment and MGF of X. Hence find the mean and var.

Ans:

$E(x^r) = \frac{(r+2)!}{2}$

$K = \frac{1}{2}$

$E(x) = 3$, $Var(X) = 3$

$MGF = \frac{1}{(1-t)^3}$

② Let X be a continuous RV with pdf $f(x) = \frac{1}{4}, \; 2 \le x \le 6$ find the expected value and variance of X.

Ans:

$E(X) = 4$, $Var(X) = \frac{4}{3}$

eg: Find the MGF, Mean and variance of the RV X whose pdf is given by (28)

$$f(x) = \frac{x}{4} e^{-x/2}, \quad x > 0.$$

Sol: Given: $f(x) = \frac{x}{4} e^{-x/2}, \quad x > 0.$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \frac{x}{4} e^{-x/2}\, dx$$

$$= \frac{1}{4} \int_0^{\infty} x \, e^{tx - x/2}\, dx$$

$$= \frac{1}{4} \int_0^{\infty} x^1 \, e^{-x\left(\frac{1}{2}-t\right)}\, dx$$

$$= \frac{1}{4} \frac{\underline{|1}}{\left(\frac{1}{2}-t\right)^{1+1}}$$

$$= \frac{1}{4} \frac{1}{\left(\frac{1-2t}{2}\right)^2}$$

$$= \frac{1}{\cancel{4}} \frac{\cancel{4}}{(1-2t)^2}$$

$$\boxed{M_X(t) = \frac{1}{(1-2t)^{+2}}}$$

$$M_X(t) = (1-2t)^{-2}$$

$$M_X(t) = 1 + 2(2t) + 3(2t)^2 + \dots$$

$$= 1 + 4t + 3(4t^2) + \dots$$

$$M_X(t) = 1 + 4\frac{t^1}{1!} + \frac{12}{1}\frac{t^2}{2!}2! + \dots$$

$$E(X) = \text{Coeff of } \frac{t^1}{1!} = 4$$

$$E(X^2) = \text{Coeff of } \frac{t^2}{2!} = 12 \times 2 = 24$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 24 - (4)^2$$

$$= 24 - 16$$

$$\boxed{Var(X) = 8}$$

(HW) Let X be continuous RV with pdf

$$f(x) = \begin{cases} x e^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

find (i) CDF of X (ii) MGF of X (iii) $P(X<2)$ and (iv) $E(X)$.

ADDITIONAL PROBLEMS (PMF, PDF & MGF) (1)

(1) A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs 10 for each white ball and Rs 5 for each black ball. Find his expectation.

Sol: Let X denote the amount gained. 3 balls drawn may be 3 black, 2 black and 1 white, 1 black and 2 white or 3 white. ∴ The amount gained are the values of X.

∴ The values of X are 15, 20, 25, 30 Rs.

$$p(X=15) = P(3 \text{ black balls}) = \frac{7C_3}{12C_3} = \frac{7}{44}. \quad (\text{Use calculator}).$$

$$p(X=20) = P(2 \text{ black } \& \text{ 1 white}) = \frac{7C_2 \cdot 5C_1}{12C_3} = \frac{21}{44}$$

$$p(X=25) = P(1 \text{ black } \& \text{ 2 white}) = \frac{7C_1 \cdot 5C_2}{12C_3} = \frac{14}{44}$$

$$p(X=30) = P(3 \text{ white balls}) = \frac{5C_3}{12C_3} = \frac{2}{44}.$$

∴ The prob distribution of X is

x :	15	20	25	30
$p(x)$:	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{14}{44}$	$\frac{2}{44}$

The Expectation,

$$E(X) = \sum_x x\, p(x) = 15 \times \frac{7}{44} + 20 \times \frac{21}{44} + 25 \times \frac{14}{44} + 30 \times \frac{2}{44}$$

$$= \frac{935}{44} \quad (\text{Using cal})$$

$$= 21.25.$$

So, His expectation is Rs 21.25.

(2) If X has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } 1 \le x < 4 \\ \frac{1}{2} & \text{for } 4 \le x < 6 \\ \frac{5}{6} & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10. \end{cases}$$

Find (1) prob dist of x.

(2) $P(2 < x < 6)$

(3) Mean of x

(4) Variance of x.

Sol: (1) We know that, $P(X = x_i) = F(x_i) - F(x_{i-1})$, $i = 1, 2, 3 \ldots$

Where F is Constant in $x_{i-1} \le x \le x_i$.

The cdf changes values at $x = 1, x = 4, x = 6, x = 10$.

$P(x = 1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}$

$P(x = 4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$.

$P(x = 6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{10-6}{12} = \frac{4}{12} = \frac{1}{3}$.

$P(x = 10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{6-5}{6} = \frac{1}{6}$.

$\therefore$ The prob dist of x is

x :	1	4	6	10
$p(x)$:	$1/3$	$1/6$	$1/3$	$1/6$.

(2) $P(2 < x < 6) = P(x = 4) = \frac{1}{3}$.

(3) Mean of $x = E(x) = \sum_i x\, p(x) = 1 \times \frac{1}{3} + 4 \times \frac{1}{6} + 6 \times \frac{1}{3} + 10 \times \frac{1}{6}$

$$E(x) = \frac{1 + 2 + 6 + 5}{3} = \frac{14}{3}.$$

$$E(x^2) = \sum_i x^2 p(x) = 1^2 \times \frac{1}{3} + 4^2 \times \frac{1}{6} + 6^2 \times \frac{1}{3} + 10^2 \times \frac{1}{6}$$

$$= \frac{1 + 8 + 36 + 50}{3} = \frac{95}{3}.$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{95}{3} - \left(\frac{14}{3}\right)^2$$

$\boxed{Var(x) = 9.8889}$ (Using cal).

(3) The amount of time, in hours that a Computer (2) functions before breaking down is a Continuous random variable with probability density function is given by,

$$f(x) = \begin{cases} \lambda e^{-x/100}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

What is the probability that a) a Computer will function between 50 and 150 hours before breaking down. (b) it will function less than 500 hrs.

Sol:: To find: λ Since $f(x)$ is a pdf of x. $\therefore \int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^{\infty} \lambda e^{-x/100} dx = 1.$$

$$\lambda \left[\frac{e^{-x/100}}{-1/100}\right]_0^{\infty} = 1.$$

$$\lambda \left[0 - \frac{e^0}{-1/100}\right] = 1.$$

$$\lambda \left[+\frac{1}{+1/100}\right] = 1. \quad [\because e^{-\infty} = 0]$$

$$100\lambda = 1$$

$$\boxed{\lambda = \frac{1}{100}}$$

a) $p(50 < x < 150) = \int_{50}^{150} f(x)dx$

$$= \int_{50}^{150} \frac{1}{100} e^{-x/100} dx$$

$$= \frac{1}{\cancel{100}} \left[\frac{e^{-x/100}}{-1/\cancel{100}}\right]_{50}^{150}$$

$$= -\left[e^{-150/100} - e^{-50/100}\right]$$

$$= -\left[e^{-1.5} - e^{-0.5}\right] = e^{-0.5} - e^{-1.5}$$

$$= 0.3834 \text{ (Using cal)}$$

b) $p(x < 500) = \int_0^{500} f(x)dx$

$$= \frac{1}{100} \int_0^{500} e^{-x/100} dx$$

$$= \frac{1}{\cancel{100}} \left[\frac{e^{-x/100}}{-1/\cancel{100}}\right]_0^{500}$$

$$= -\left[e^{-500/100} - e^{-0}\right]$$

$$= -\left[e^{-5} - 1\right]$$

$$= 1 - e^{-5}$$

$$= 1 - 0.0067 \text{ (Using cal).}$$

$$= 0.9933.$$

$$\boxed{\therefore p(x < 500) = 0.9933}$$

(4) The probability density function of a Continuous R.V is $f(x) = C\, e^{-|x|}, -\infty < x < \infty$. find value of C, the cdf of X.

Sol.: Since $f(x)$ is a pdf, $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\int_{-\infty}^{\infty} C\, e^{-|x|}\, dx = 1. \qquad \left[\because \text{We know that } |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\right]$$

$$\int_{-\infty}^{0} C\, e^{x}\, dx + \int_{0}^{\infty} C\, e^{-x}\, dx = 1$$

$$C\left\{ \left(e^{x}\right)_{-\infty}^{0} + \left(\frac{e^{-x}}{-1}\right)_{0}^{\infty} \right\} = 1$$

$$C\left\{ \left(e^{0} - e^{-\infty}\right) + \left[-\left(e^{-\infty} - e^{0}\right)\right] \right\} = 1$$

$$C\left\{ (1-0) - (0-1) \right\} = 1$$

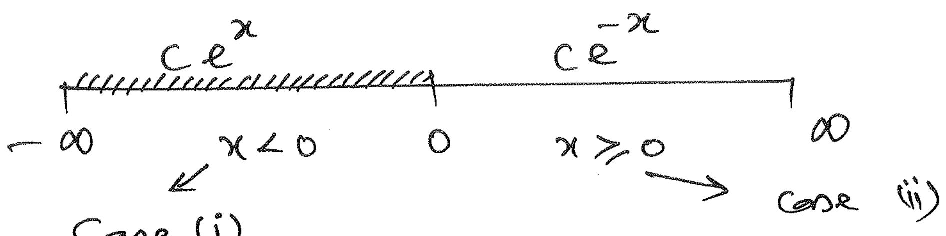
$$C\left\{ 1 + 1 \right\} = 1$$

$$2C = 1$$

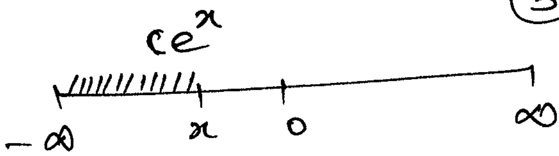
$$\boxed{C = \tfrac{1}{2}}$$

To find: cdf. $(F(x))$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{where } f(x) = C\, e^{x} \text{ if } x < 0$$
$$\text{and } f(x) = C\, e^{-x} \text{ if } x \geq 0$$



Case (i) If $x \in (-\infty, 0)$ then



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} ce^{x}\,dx$$

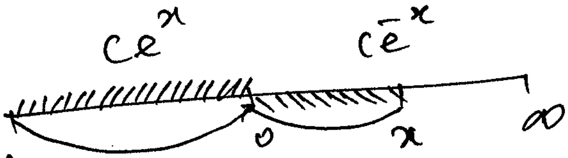
$$= \tfrac{1}{2}\left[e^{x}\right]_{-\infty}^{x} \quad [\because c = \tfrac{1}{2}]$$

$$= \tfrac{1}{2}\left[e^{x} - e^{-\infty}\right]$$

$$= \tfrac{1}{2}\left(e^{x} - 0\right)$$

$$\boxed{F(x) = \frac{e^{x}}{2}}$$

Case (ii) If $x \in (0, \infty)$ then.



$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} ce^{x}\,dx + \int_{0}^{x} ce^{-x}\,dx \Rightarrow \frac{1}{2}\int_{-\infty}^{0} e^{x}\,dx + \tfrac{1}{2}\int_{0}^{x} e^{-x}\,dx$$

$$= \frac{1}{2}\left[e^{x}\right]_{-\infty}^{0} + \frac{1}{2}\left[-e^{-x}\right]_{0}^{x}$$

$$= \frac{1}{2}\left[e^{0} - e^{-\infty}\right] - \frac{1}{2}\left[e^{-x} - e^{0}\right]$$

$$= \frac{1}{2}\left[1 - 0\right] - \tfrac{1}{2}\left[e^{-x} - 1\right] \Rightarrow \frac{1}{2} - \frac{1}{2}e^{-x} + \tfrac{1}{2}$$

$$\boxed{F(x) = 1 - \frac{e^{-x}}{2}}$$

$$\therefore F(x) = \begin{cases} \dfrac{e^{x}}{2} & \text{for } x < 0 \\[2ex] 1 - \dfrac{e^{-x}}{2} & \text{for } x \geq 0. \end{cases}$$

(5) Experience has shown that while walking in a certain park, the time X (in mins), between seeing two people smoking has a density function of the form,

$$f(x) = \begin{cases} \lambda x e^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

① find the value of λ

② find the distribution function of X.

③ what is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes in at least 7 minutes?.

Sol: ① Since $f(x)$ is a pdf, $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_0^{\infty} \lambda x e^{-x}\,dx = 1.$$

$$\lambda \int_0^{\infty} \underbrace{x}_{u}\, \underbrace{e^{-x}dx}_{dv} = 1$$

$$\left[\because \int u\,dv = uv - u'v_1 + u''v_2 - \cdots\right]$$

$$\lambda\left[x\frac{e^{-x}}{-1} - (1)\frac{e^{-x}}{(-1)^2}\right]_0^{\infty} = 1.$$

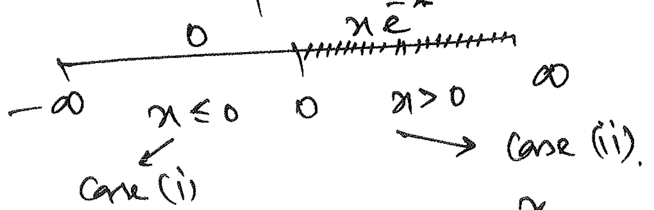
$$\lambda\left\{[0-0] - \left[0 - \frac{1}{1}\right]\right\} = 1$$

$$\lambda(1) = 1$$

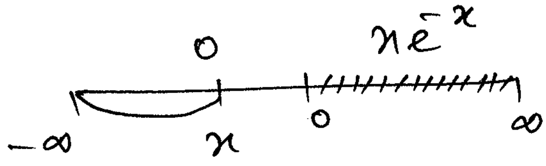
$$\boxed{\lambda = 1}$$

$$\therefore f(x) = \begin{cases} x e^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

(2) To find: Distribution function $(F(x))$.

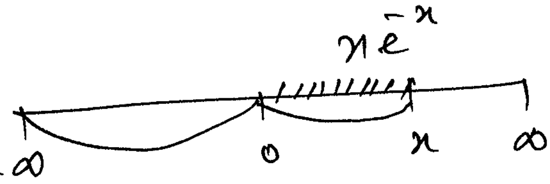


We have, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$



Case (i) If $x \in (-\infty, 0)$ then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,dx = 0.$$



Case (ii) If $x \in (0, \infty)$ then

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} \underset{u}{x}\,\underset{dv}{e^{-x}\,dx}$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right) - (1)\frac{e^{-x}}{(-1)^2}\right]_0^x$$

$$= \left(-xe^{-x} - e^{-x}\right)_0^x$$

$$= \left(-xe^{-x} - e^{-x}\right) - (0 - 1)$$

$$= -xe^{-x} - e^{-x} + 1$$

$$\boxed{F(x) = 1 - e^{-x}(x+1)}$$

The distribution function is

$$\therefore F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-x}(x+1) & \text{if } x > 0. \end{cases}$$

(3) $p(2 < x < 5) = p(2 \le x \le 5)$ $\quad$ ($\because$ Since x is Continuous)

$= F(5) - F(2)$ $\quad$ [$\because$ $p(a \le x \le b) = F(b) - F(a)$]

$= [1 - e^{-5}(5+1)] - [1 - e^{-2}(2+1)]$

$= 0.366$ (Using cal)

$p(x > 7) = 1 - p(x \le 7) = 1 - F(7)$ $\quad$ [$\because$ $F(x) = p(X \le x)$]

$= 1 - [1 - e^{-7}(7+1)] = \cancel{1} - \cancel{1} + 8e^{-7}$

$= 8e^{-7}$

$$\boxed{p(x > 7) = 0.0073}$$ ($\because$ Using cal)

(6) The sales of Convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function of the following form

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \le x < 1 \\ k(4x - x^2) & , 1 \le x < 2 \\ 1 & , x \geq 2 \end{cases}$$

Suppose that this Convenience store's total sales on any given day are less than 2000 dollars.

(1) find the value of K

(2) Let A and B be the events that tomorrow the store's total sales are between 500 and 1500 dollars, and over 1000 dollars respectively. find p(A) and p(B).

(3) Are A and B Independent events?

Sol: We know that, $f(x) = \frac{d}{dx}[F(x)] = F'(x)$.

$\therefore$ The pdf of x is

$$f(x) = F'(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \le x < 1 \\ k(4 - 2x) & , 1 \le x < 2 \\ 0 & , x \geq 2. \end{cases}$$

Since $f(x)$ is a pdf then $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\Rightarrow \int_0^1 x\,dx + \int_1^2 k(4 - 2x)\,dx = 1.$$

$$\Rightarrow \left(\frac{x^2}{2}\right)_0^1 + k\left(4x - \cancel{2}\frac{x^2}{\cancel{2}}\right)_1^2 = 1.$$

$$\Rightarrow \left(\frac{1}{2} - 0\right) + k\left[(8 - 4) - (4 - 1)\right] = 1.$$

$$\Rightarrow \frac{1}{2} + k[4 - 3] = 1.$$

$$\Rightarrow \frac{1}{2} + k(1) = 1.$$

$$k = 1 - \frac{1}{2}$$

$$\boxed{k = \frac{1}{2}}$$

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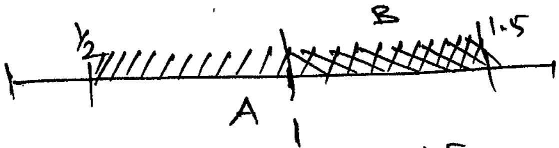
Since the total Sales X is in thousands of dollars the Sales is between 500 and 1500 dollars is the event $A = \frac{1}{2} < X < 1.5$ and Sales over 1000 dollars is the even $B = X > 1$.



$\therefore A \cap B = 1 < x < 1.5$

$$p(A) = p(\tfrac{1}{2} < x < 1.5) = \int_{1/2}^{1.5} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$$

$$= \int_{1/2}^{1} x\,dx + \int_{1}^{1.5} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_{1/2}^{1} + \left(2x - \frac{x^2}{2}\right)_{1}^{1.5}$$

$$= \frac{1}{2}\left(1 - \tfrac{1}{4}\right) + \left[2(1.5) - \frac{(1.5)^2}{2}\right] - \left[2(1) - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left(\frac{3}{4}\right) + \left[\frac{4(1.5) - (1.5)^2}{2}\right] - \left(2 - \tfrac{1}{2}\right) = 0.375 + 1.875 - 1.5.$$

$\boxed{p(A) = 0.75}$ (∵ Using cal)

$$p(B) = p(x>1) = \int_{1}^{\infty} f(x)\,dx = \int_{1}^{2} f(x)\,dx = \int_{1}^{2} (2-x)\,dx$$

$$\Rightarrow \left(2x - \frac{x^2}{2}\right)_{1}^{2} \Rightarrow \left(4 - \tfrac{4}{2}\right) - \left(2 - \tfrac{1}{2}\right) = 0.5.$$

$\boxed{P(B) = 0.5}$

$$P(A \cap B) = P(1 < x < 1.5) = \int_{1}^{1.5} f(x)\,dx = \int_{1}^{1.5} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}$$

$$\Rightarrow \left[2(1.5) - \frac{(1.5)^2}{2}\right] - \left[2 - \tfrac{1}{2}\right] = 0.375$$

$\boxed{P(A \cap B) = 0.375}$

Now, $P(A) \cdot P(B) = (0.75)(0.5)$

$= 0.375$

$\boxed{P(A) \cdot P(B) = P(A \cap B)}$

$\therefore$ A and B are Independent events.

7) Let the random variable x have pdf $f(x) = \frac{1}{2} e^{-x/2}, x > 0$ find the MGF and hence find the Mean & variance of x.

Sol: We have, $M_X(t) = E[e^{tx}]$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{tx - x/2}\,dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x(\frac{1}{2}-t)}\,dx$$

$$= \frac{1}{2} \left[\frac{e^{-x(\frac{1}{2}-t)}}{-(\frac{1}{2}-t)} \right]_{\infty}^{\infty}$$

$$= \frac{1}{2} \left[0 + \frac{1}{+(\frac{1}{2}-t)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{\frac{1-2t}{2}} \right]$$

$$= \frac{1}{\cancel{2}} \left(\frac{\cancel{2}}{1-2t} \right)$$

$$\boxed{M_X(t) = \frac{1}{1-2t}}$$

To find: Mean & var

$$M_X(t) = (1-2t)^{-1}$$

$$= 1 + (2t) + (2t)^2 + (2t)^3 + \dots$$

$$[(1-x)^{-1} = 1 + x + x^2 + \dots]$$

$$= 1 + 2t + 2^2t^2 + 2^3t^3 + \dots$$

$$\text{Mean} = E(x) = [\text{coeff of } t^1 \text{ in } M_X(t)]\,1!$$

$$= 2 \times 1! = 2$$

$$\boxed{\text{Mean} = E(x) = 2}$$

$$E(x^2) = [\text{coeff of } t^2 \text{ in } M_X(t)] \times 2!$$

$$= 2^2 \times 2!$$

$$= 4 \times 2$$

$$\boxed{E(x^2) = 8}$$

$$\therefore \text{var}(x) = E(x^2) - [E(x)]^2$$

$$= 8 - (2)^2$$

$$= 8 - 4$$

$$\boxed{\text{Var}(x) = 4}$$

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(8) A random variable X has the pdf given by

$$f(x) = \begin{cases} 2e^{-2x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

find (i) MGF (ii) The first four moments about the Origin.

Ans: (i) $M_X(t) = \frac{2}{2-t}$ (ii) $\mu_1' = \frac{1}{2}$, $\mu_2' = \frac{1}{2}$, $\mu_3' = \frac{3}{4}$ & $\mu_4' = \frac{3}{2}$.

(9) If a random variable X has the pdf,

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & , -1 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

find the mean and Variance.

Ans: Mean $= E(x) = \frac{1}{3}$ & $Var(x) = \frac{2}{9}$.

(10) Let X be a random variable with p.d.f,

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

find a) $p(X>3)$ b) MGF c) $E(X)$ and $Var(X)$.

Ans: a) 0.3679 b) $M_X(t) = \frac{1}{1-3t}$ c) $E(x) = 3$ & $Var(x) = 9$.

(11) If $p(X=x) = \begin{cases} kx & , x = 1, 2, 3, 4, 5 \\ 0 & , \text{otherwise} \end{cases}$ represent the pmf of X

(i) find k (ii) p(X being a prime Number)

(iii) find $p\left(\frac{1}{2} < X < \frac{5}{2} \,/\, X > 1\right)$

(iv) find the distribution function.

Ans: (i) $k = \frac{1}{15}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{7}$ (iv) $F(x) = \begin{cases} 0 & , x < 1 \\ \frac{1}{15} & , 1 \leq x < 2 \\ \frac{3}{15} & , 2 \leq x < 3 \\ \frac{6}{15} & , 3 \leq x < 4 \\ \frac{10}{15} & , 4 \leq x < 5 \\ 1 & , x \geq 5. \end{cases}$

(12) Let X be a discrete RV whose cdf is

$$F(x) = \begin{cases} 0, & x < -3 \\ \frac{1}{6}, & -3 \le x < 6 \\ \frac{1}{2}, & 6 \le x < 10 \\ 1, & x \geq 10. \end{cases}$$

(i) Find $p(X \le 4)$, $p(-5 < X \le 4)$

(ii) find the probability distribution of X.

Ans: (i) $p(X \le 4) = F(4) = \frac{1}{6}$

(ii) $p(-5 < X \le 4) = F(4) - F(-5) = \frac{1}{6} - 0 = 1/6.$

(iii)

x :	-3	6	10
$p(x)$:	$1/6$	$1/3$	$1/2$

(13) find the MGF and hence find its mean & variance of $f(x) = \frac{x}{4} e^{-x/2}, \ x > 0.$

Ans: $M_X(t) = \frac{1}{(1-2t)^2}$, $E(X) = 4$ & $Var(X) = 8.$

(14) The probability mass function of a random variable is given by $p[X=j] = \frac{1}{2^j}, \ j = 1, 2, 3 \ldots$ find the MGF, mean & variance.

Ans: $M_X(t) = \frac{e^t}{2-e^t}$, Mean $= E(x) = 2$, $E(x^2) = 6$, $Var(x) = 2.$

(15) Suppose that the duration 'X' in minutes of long distance calls from your home follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-x/5}, \ x > 0$. find $p(X > 5)$, $p(3 \le X \le 6)$, mean and variance

Ans: $p(X > 5) = 0.3678$ $\quad p(3 \le X \le 6) = 0.247$, Mean $= E(x) = 5$

$E(x^2) = 50$ & $\boxed{Var(X) = 25.}$

BINOMIAL DISTRIBUTION.

Def: A Random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by

$$p(X=x) = p(x) = \begin{cases} nC_x \, p^x q^{n-x} & ; x = 0,1,2 \ldots n, \; q = 1-p \\ 0 & ; \text{otherwise.} \end{cases}$$

The two Independent constants n and p in the distribution are known as the parameters of the distribution. 'n' is also known as the degree of the binomial distribution.

BINOMIAL FREQUENCY DISTRIBUTION.

Let us suppose that N trials constitute an experiment. then if this experiment is repeated N times, the frequency function of the binomial distribution is given by,

$$f(x) = N\,p(x) = N.\, nC_x \, p^x q^{n-x} \; ; x = 0,1,2 \ldots n.$$

PROBLEMS BASED ON BINOMIAL DISTRIBUTION.

eg ①: The Mean of a Binomial distribution is 20 and S.D is 4. Determine the parameters of the distribution.

Sol: Given: Mean = 20. & S.D = 4

$$\boxed{\therefore np = 20} \rightarrow ① \qquad \boxed{\sqrt{npq} = 4}$$

$$\boxed{npq = 16} \rightarrow ②$$

$\frac{(2)}{(1)} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{16}{20}$

$q = 4/5$

$\boxed{\therefore p = 1/5}$ sub in (1)

$np = 20.$

$n\left(\frac{1}{5}\right) = 20$

$\boxed{n = 100}$

$\therefore$ The parameters are $\boxed{p = 1/5}$ & $\boxed{n = 100}$

Eg: (2) Comment the following statement:

The mean of a B.D is 3 and variance is 4

Sol:: Given: Mean $= 3$ | Variance $= 4$

$\boxed{np = 3} \rightarrow (1)$ | $\boxed{\therefore npq = 4} \rightarrow (2)$

$\frac{(2)}{(1)} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{4}{3}$

$q = 4/3 > 1$

which is impossible.

$\therefore$ The given statement is wrong.

Eg: (3) The Mean & Variance of a binomial variate are 8 and 6. find $p[x \geq 2]$.

Sol:: Given: Mean $= 8$ | Var $= 6$

$\boxed{\therefore np = 8} \rightarrow (1)$ | $\boxed{\therefore npq = 6} \rightarrow (2)$

$\frac{(2)}{(1)} \Rightarrow \frac{\cancel{np}q}{\cancel{np}} = \frac{6}{8}$

$\boxed{q = 3/4} \Longrightarrow p = 1/4$ sub in (1) from ① $np = 8$

$$n \cdot \left(\frac{1}{4}\right) = 8$$

$$\boxed{n = 32}$$

$\therefore$ The probability distribution is

$$p[x=x] = nCx\, p^x q^{n-x} \; ; x = 0, 1, 2 \ldots n.$$

$$p[x=x] = 32Cx \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x} \; ; x = 0, 1, 2 \ldots 32.$$

$$p[x \geq 2] = 1 - p[x < 2]$$

$$= 1 - \{p(x=0) + p(x=1)\}$$

$$= 1 - \left\{32C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{32-0} + 32C_1 \left(\frac{1}{4}\right)^1 \cdot \left(\frac{3}{4}\right)^{32-1}\right\}$$

$$= 1 - \left(\frac{3}{4}\right)^{31} \left[\frac{3}{4} + \frac{32}{4}\right]$$

$$\boxed{p[x \geq 2] = 1 - \frac{35}{4}\left(\frac{3}{4}\right)^{31}}$$

eg: ④ Four coins are tossed simultaneously. What is the probability of getting (i) 2 heads (ii) atleast 2 heads (iii) atmost 2 heads.

Sol: We have, $p[x=x] = nCx\, p^x q^{n-x}$

Here $p = 1/2$; $q = 1/2$ & $n = 4$.

$\therefore$ The probability distribution is,

$$p[x=x] = 4Cx \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$\boxed{p[x=x) = 4Cx \left(\frac{1}{2}\right)^4}$$

(i) $p[2 \text{ heads}] = p(x=2) = {}^4C_2 \left(\frac{1}{2}\right)^4$

$$= \frac{4\times 3}{2\times 1} \left(\frac{1}{2}\right)^4$$

$$= 6 \times \frac{1}{16}$$

$$\boxed{p[x=2] = 3/8.}$$

(ii) $p(\text{atleast } 2 \text{ heads}) = p(x \geq 2)$

$$= p(x=2) + p(x=3) + p(x=4)$$

$$= {}^4C_2 \left(\frac{1}{2}\right)^4 + {}^4C_3 \left(\frac{1}{2}\right)^4 + {}^4C_4 \left(\frac{1}{2}\right)^4$$

$$= \left(\frac{1}{2}\right)^4 [6 + 4 + 1]$$

$$\boxed{p(x \geq 2) = \frac{11}{16}.}$$

(iii) $p(\text{atmost } 2 \text{ heads}) = p(x \leq 2)$

$$= p(x=0) + p(x=1) + p(x=2)$$

$$= {}^4C_0 \left(\frac{1}{2}\right)^4 + {}^4C_1 \left(\frac{1}{2}\right)^4 + {}^4C_2 \left(\frac{1}{2}\right)^4$$

$$= \left(\frac{1}{2}\right)^4 [1 + 4 + 6]$$

$$\boxed{p(x \leq 2) = \frac{11}{16}}$$

(x) eg: find the MGF of Binomial distribution and hence finds its mean and variance. (37)

Sol:

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} p(X=x)$$

Here, $p(X=x) = nC_x \, p^x q^{n-x}, \quad x = 0,1,2\ldots n.$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} \cdot nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x \, (pe^t)^x q^{n-x}$$

W.K.T, $\boxed{\sum_{r=0}^{n} nC_r \, x^{n-r} \cdot a^r = (x+a)^n}$

$$M_X(t) = (q+pe^t)^n \longrightarrow (1)$$

Diff eq (1) w.r.to 't'

$$M'_X(t) = n(q+pe^t)^{n-1} \cdot pe^t$$

$$M'_X(t) = np\,(q+pe^t)^{n-1} \cdot e^t \longrightarrow (2)$$

Diff again w.r.to 't',

$$M''_X(t) = np\left[(q+pe^t)^{n-1} \cdot e^t + e^t (n-1)(q+pe^t)^{n-2} \cdot pe^t\right] \longrightarrow (3)$$

put $t=0$ in eq (2)

$$M'_X(0) = np\,(q+p)^{n-1} \cdot (1)$$

$$E(X) = np\,(1)$$

$$\boxed{E(X) = np}$$

put $t=0$ in eq ③

$$M''_X(0) = np\left[(q+p)^{n-1}(1) + (1)(n-1)(q+p)^{n-2}\cdot p(1)\right]$$

$$= np\left[1+(n-1)p\right]$$

$$= np\left[1+np-p\right]$$

$$= np\left[1-p+np\right]$$

$$= np\left[q+np\right] \qquad (\because q = 1-p)$$

$$\boxed{E(X^2) = npq + (np)^2}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= npq + (np)^2 - (np)^2$$

$$\boxed{Var(X) = npq}$$

Additive property:

Let X_1 and X_2 be two Independent binomial RV with parameter (n_1, p) and (n_2, p) then X_1+X_2 is a binomial RV with parameter (n_1+n_2, p).

pf: W.K.T

$$M_{X_1}(t) = (q+pe^t)^{n_1}$$

$$M_{X_2}(t) = (q+pe^t)^{n_2}$$

$$M_{X_1+X_2}(t) = M_{X_1}(t)\cdot M_{X_2}(t)$$

$$M_{X_1+X_2}(t) = (q+pe^t)^{n_1}\cdot(q+pe^t)^{n_2}$$

$$= (q+pe^t)^{n_1+n_2}$$

Which is a MGF of Binomial RV.

Hence X_1+X_2 is a binomial RV.

Eg: ⑤ Let 'x' follows a binomial distribution. Suppose $p(x=0) = 1 - p(x=1)$. If $E(X) = 3 \cdot Var(X)$. find $p(x=0)$.

Sol:- Given: $E(X) = 3\ Var(X)$

$$\cancel{np} = 3\cancel{np}q$$

$$1 = 3q$$

$$\boxed{q = 1/3} \Rightarrow \boxed{p = 2/3}$$

Also, given that, $p(x=0) = 1 - p(x=1)$

$$nC_0 \cdot \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{n-0} = 1 - nC_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{n-1}$$

$$(1)(1) \cdot \left(\frac{1}{3}\right)^n = 1 - n \cdot \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right)^{n-1}$$

$$\left(\frac{1}{3}\right)^n = 1 - \frac{2n}{3} \left(\frac{1}{3}\right)^{n-1}$$

This true only for $n=1$.

$$\therefore p(x=0) = nC_0\ p^0 q^{n-0}$$

$$= (1)(1)\ q^n$$

$$= \left(\frac{1}{3}\right)^1$$

$$\boxed{p(x=0) = 1/3}$$

Eg: ⑥ A pair of dice is thrown 4 times. If getting a doublet is considered a success. find the probability of 2 success.

Sol:: Let X be the R.V of getting a doublet.

$$\therefore p(\text{getting a doublet}) = p = 6/36 = 1/6 \Rightarrow \boxed{q = \frac{5}{6}}$$

Here, $\boxed{n=4}$

We have, $p[X=x] = nC_x\, p^x . q^{n-x}$

$\therefore p[X=x] = 4C_x \left(\frac{1}{6}\right)^x . \left(\frac{5}{6}\right)^{4-x}$

$p(\text{getting } 2 \text{ success}) = p(X=2)$

$= 4C_2 \left(\frac{1}{6}\right)^2 . \left(\frac{5}{6}\right)^2$

$= \frac{25}{216} = 0.1157.$

---

eg: (7) The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge find the probability that the bridge is destroyed.

Sol: Let 'X' be the R.V of hitting the target.

$\therefore p(\text{hitting the target}) = \boxed{p = 1/5} \Rightarrow \boxed{q = \frac{4}{5}}$ and $n=6$

Here, $n=6$.

We have, $p[X=x] = nC_x\, p^x . q^{n-x}$

$\therefore p[\text{the bridge is destroyed}] = p(X \geq 2) =$

$= 1 - p(X < 2)$

$= 1 - [p(X=0) + p(X=1)]$

$= 1 - \left[6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^6 + 6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5\right]$

$= 1 - (0.2621 + 0.3932)$

$= 1 - 0.6553 = 0.3447$ (using Calculator).

Eg: 8 6 dice are thrown 729 times. How many times do you expect atleast three dice to show a five or a six?

Sol:- Let 'X' be the R.V denoting number of successes when 6 dice are thrown.

p = probability of getting 5 or 6 with one die

$p = \frac{2}{6} = \frac{1}{3} \Rightarrow \boxed{q = \frac{2}{3}}$

Here $n = 6$.

we have, $P[X = x] = nC_x\, p^x . q^{n-x}$; $x = 0, 1, 2 \ldots n$.

$P[X = x] = 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$; $x = 0, 1, 2 \ldots 6$.

P(atleast three dice showing five or six) $= P(X \geq 3)$.

$= P(X \geq 3)$.

$= 1 - P(X < 3)$.

$= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\}$

$= 1 - \left\{6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^6 + 6C_1 \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^5 + 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4\right\}$

$= 1 - \left(\frac{2}{3}\right)^4 \left[\frac{4}{9} + 6 \times \frac{2}{9} + \frac{\cancel{6}^{3} \times 5}{\cancel{2} \times 1} \times \frac{1}{9}\right]$

$= 1 - \left(\frac{16}{81}\right) \left[\frac{4}{9} + \frac{12}{9} + \frac{15}{9}\right]$

$= 1 - \left(\frac{16}{81}\right) \times \left(\frac{31}{9}\right) = 1 - \frac{496}{729} = \frac{233}{729}$.

For 729 times,

$= N . P(X = x) = N . \left(\frac{233}{729}\right) = \cancel{729}\left(\frac{233}{\cancel{729}}\right)$

$= 233$ times

eg: 9 In 256 sets of twelve tosses of a coin how many cases may one expect eight heads and four tails?

Sol: Let 'x' be the R.V denoting the number of successes in twelve tosses of a coin.

p = probability of getting a head in a single toss

$\boxed{p = \frac{1}{2}} \Rightarrow \boxed{q = \frac{1}{2}}$.

Here, Number of tosses $n = 12$.

Number of sets $N = 256$.

We have $p[x=x] = nC_x \, p^x q^{n-x}; x = 0, 1, 2 \ldots n$.

$\therefore p[x=x] = 12C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{12-x}$

$$\boxed{p[x=x] = 12C_x \left(\frac{1}{2}\right)^{12}.}$$

p (getting eight heads & four tails) = $p(x=8)$

$= 12C_8 \left(\frac{1}{2}\right)^{12}$

$= 12C_4 \left(\frac{1}{2}\right)^{12}$

$= \frac{\cancel{12} \times 11 \times \cancel{10}^{5} \times 9}{\cancel{4 \times 3 \times 2} \times 1} \times \frac{1}{32} \times \frac{1}{32} \times \frac{1}{4}$

$= \frac{11 \times 45}{32 \times 32}$

For 256 sets $= N \cdot p(x=x) = \cancel{256}^{8} \times \frac{11 \times 45}{\cancel{32} \times \cancel{32}_{4} \times 4}$

$= \frac{495}{16} = 30.9375$

$= 31$ times.

Eg 10: An irregular 6 faced die is thrown such that the probability that it gives 3 even numbers in 5 throws is twice the probability that it gives 2 even numbers in 5 throws. How many sets of exactly 5 trials can be expected to give no even number out of 2500 sets.

Sol:: Let X be the R.V denoting the number of even numbers.

We have $p(X=x) = nC_x \, p^x q^{n-x}$; $x = 0, 1, 2 \ldots n$.

Here, $p \rightarrow$ probability of getting even numbers.

Given that,

p(getting 3 even numbers in 5 throws) = 2P(getting 2 even numbers in 5 throws).

(ie) $p[X=3] = 2p[X=2]$. & Here $n=5$.

$$5C_3 \, p^3 q^2 = 2\left(5C_2 \, p^2 q^3\right)$$

$$\frac{\cancel{5\times4\times3}}{3\times\cancel{2\times1}} \cdot p^3 q^2 = 2\times\frac{\cancel{5\times4}}{\cancel{2\times1}} \cdot p^2 q^3$$

$$10p^3q^2 = 20p^2q^3$$

$$p = 2q$$

$$p = 2(1-p)$$

$$p = 2-2p$$

$$3p = 2$$

$$\boxed{p = 2/3} \Rightarrow \boxed{q = 1/3}$$

$\therefore p(\text{getting no even number}) = p(x=0)$

$$= nC_0\, p^0 q^{n-0}$$

$$= 5C_0 \left(\tfrac{2}{3}\right)^0 \left(\tfrac{1}{3}\right)^5$$

$$= (1)\,(1)\cdot\left(\tfrac{1}{3}\right)^5$$

$$\boxed{p(x=0) = \frac{1}{3^5}}$$

Out of 2500 set, we have,

$$= N \cdot p(X=x)$$

$$= 2500 \times \frac{1}{3^5}$$

$$= 10 \text{ (app)}.$$

---

eg: (11) In a certain town, 20% samples of the population is literate and assume that 200 investigators take samples of ten individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less are literates in the samples?

Sol: Let X be the R.V denoting the number of literate persons.

$$p = 20\% = \frac{20}{100} = 0.2$$

$$q = 1-p = 1-0.2$$

$$\boxed{q = 0.8} \quad \& \quad \boxed{n = 10}$$

we have, $p[X=x] = nC_x p^x q^{n-x}$; $x=0,1,2\ldots n$.

$p\left[\begin{array}{c}\text{an investigator reporting} \\ \text{3 or less as literate}\end{array}\right] = p[X \leq 3]$

$= p(x=0) + p(x=1) + p(x=2) + p(x=3)$.

$= (0.8)^7 \left[(0.8)^3 + (10)(0.2)(0.8)^2 + 45(0.2)^2(0.8) + 120(0.2)^3\right]$

$= (0.2097)(0.512 + 1.28 + 1.44 + 0.96)$

$= 0.8790$.

$\therefore$ P(200 investigator reporting 3 or less as literate) $= N.p(x=x)$.

$= 200 \times 0.8790$

$= 175.72$

$\simeq 176$

Eg: 12 If the independent R.V's X, Y are binomially distributed respectively with $n=3, p=1/3$ and $n=5, p=1/3$ find $p(X+Y \geq 1)$.

Sol: The R.V 'X' follows binomial with $n=3, p=\frac{1}{3}; q=\frac{2}{3}$.

The R.V 'Y' follows binomial with $n=5; p=1/3; q=\frac{2}{3}$.

Since X & Y are Independent, we have by additive property of binomial distribution with $P_1 = 1/3$; $P_2 = 1/3$ X+Y follows binomial with $n = 3+5 = 8$.

$$\therefore P[X+Y=x] = nC_x\, p^x q^{n-x}$$

$$= 8C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{8-x}$$

$$P(X+Y \geq 1) = 1 - P(X+Y < 1)$$

$$= 1 - P(X+Y = 0)$$

$$= 1 - 8C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{8-0}$$

$$= 1 - \left(\frac{2}{3}\right)^8$$

$$= 1 - \frac{2^8}{3^8}$$

$$= \frac{3^8 - 2^8}{3^8}$$

$$\boxed{P(X+Y \geq 1) = 0.96.}$$

eg: 13 In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good bulbs (ii) At most there are 3 defective bulbs (iii) Exactly there are three defective bulbs.

Ans: (i) 0.1216 (ii) 0.8666 (iii) 0.19

eg: Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) at most 2 girls (iv) children of both sexes. 42

Sol: Given: $N = 800$ and $n = 4$

$p(\text{boys}) = \frac{1}{2}$ and $p(\text{Girls}) = \frac{1}{2}$.

Let X denote No of boys.

$\therefore p = \frac{1}{2}$ $\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$$p(X = x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, \ldots 4$$

$$p(X = x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$= 4C_x \left(\frac{1}{2}\right)^4$$

$$\boxed{p(X = x) = \frac{1}{16} \cdot 4C_x}$$

(i) $p(2 \text{ boys and } 2 \text{ girls}) = p(X = 2)$

$$= \frac{1}{16}\, 4C_2$$

$$= \frac{1}{16}(6)$$

$$= \frac{3}{8}.$$

$\therefore$ Out of 800, No of families having 2 boys and 2 girls $= \frac{3}{\cancel{8}} \times \cancel{800} = 300$/-

(ii) p (at least one boys) $= p(x \geq 1)$

$= 1 - p(x < 1)$

$= 1 - [p(x = 0)]$

$= 1 - \frac{1}{16}(4C_0)$

$= 1 - \frac{1}{16}$

$$\boxed{p(x \geq 1) = \frac{15}{16}}$$

$\therefore$ Out of 800 families $= \cancel{800}^{50} \times \frac{15}{\cancel{16}} = 750.$

(iii) p (at most 2 girls) $= p$ (at most 2 boys)

$= p(x \leq 2)$

$= p(x=0) + p(x=1) + p(x=2)$

$= \frac{1}{16} 4C_0 + \frac{1}{16} 4C_1 + \frac{1}{16} 4C_2$

$= \frac{1}{16}[1 + 4 + 6]$

$= \frac{11}{16}.$

Out of 800 families $= \frac{11}{\cancel{16}} \times \cancel{800}^{50}$

$= 550$

(iv) p(children of both sexes)

$= 1 - p$(children of same sexes)

$= 1 - \{ p(\text{all boys}) + p(\text{all girls}) \}$

$= 1 - \{ p(X=4) + p(X=0) \}$

$= 1 - \left[\frac{1}{16} \cdot {}^4C_4 + \frac{1}{16} {}^4C_0 \right]$

$= 1 - \frac{1}{16} [1+1]$

$= 1 - \frac{2}{16}$

$= \frac{14}{16}$

$= \frac{7}{8}$

$\therefore$ Out of 800 families $= \frac{7}{8} \times 800$

$= 700.$

(HW) A coin is biased so that a head is twice as likely to appear as a tail. If the coin is tossed 6 times, find the probabilities of getting
(i) exactly 2 heads (ii) at least 3 heads
(iii) at most 4 heads

Ans:
(0.0823, 0.8998, 0.6488)

Eg: A man hitting the target is $\frac{1}{4}$. If he fires 7 times what is the probability that he hits the target atleast twice? How many times does he need to fire in order that probability of him hitting the target atleast once is greater than $\frac{2}{3}$.

Sol: Given: $p = \frac{1}{4}$, $n = 7$

$$\therefore q = 1 - p = 1 - \frac{1}{4} = \frac{4-1}{4} = \frac{3}{4}.$$

$$p(X = x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n.$$

$$p(X = x) = 7C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{7-x} \longrightarrow (1)$$

(i) $p(\text{at least twice}) = p(X \geq 2)$

$$= 1 - p(X < 2)$$

$$= 1 - [p(X = 0) + p(X = 1)]$$

$$= 1 - \left[7C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^7 + 7C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^6\right]$$

$$= 1 - [0.13348 + 0.31146]$$

$$= 1 - 0.44494$$

$$= 0.55506.$$

$$\boxed{\therefore p(X \geq 2) = 0.55506}$$

(ii) To find: n such that $p(x \geq 1) > \frac{2}{3}$ (44)

$$p(x \geq 1) > \frac{2}{3}$$

$$1 - p(x < 1) > \frac{2}{3}$$

$$1 - p(x = 0) > \frac{2}{3}$$

$$1 - nC_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$1 - \frac{2}{3} > \left(\frac{3}{4}\right)^n$$

$$\frac{3-2}{3} > \left(\frac{3}{4}\right)^n$$

$$\frac{1}{3} > \left(\frac{3}{4}\right)^n$$

$$0.3333 > (0.75)^n \quad \text{--- (2)}$$

put $n=1 \Rightarrow 0.33 \not> 0.75$

$n=2 \Rightarrow 0.33 \not> 0.56250$

$n=3 \Rightarrow 0.33 \not> 0.42188$

$n=4 \Rightarrow 0.33 > 0.31641$

$\therefore n=4$ satisfied eq (2).

Hence the required value of n is 4.

Poisson Distribution.

A Random variable X is said to follow poisson distribution if it assumes only non-negative values and its probability mass function is given by,

$$P[X=x] = p(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \; ; \; x = 0, 1, 2 \ldots, \; \lambda > 0.$$

λ is known as the parameter of the poisson distribution.

Poisson Frequency Distribution.

Let a poisson experimental consist of n independent trials. Let this experiment, under similar conditions be repeated N times, the frequency function of the poisson distribution is given by,

$$f(x) = N\,p(x) = N \cdot \frac{e^{-\lambda} \lambda^{x}}{\lfloor x}.$$

Poisson distribution is a limiting case of binomial distribution under the following assumptions

(i) The number of trails 'n' should be indefinitely large (ie) $n \to \infty$.

(ii) The probability of successes 'p' for each trail is indefinitely small.

(iii) $np = \lambda$, should be finite, where λ is a constant.

(7) Eg ①: Derive the poisson distribution as a limiting case of binomial distribution.

Sol:: We know that the binomial distribution is.

$$p[X=x] = nC_x \, p^x q^{n-x}$$

We have $np=\lambda$ and $q=1-p$.

$$\boxed{p=\lambda/n}$$

$$p(x=x] = \frac{\lfloor n}{\lfloor n-x \, \lfloor x} \cdot p^x (1-p)^{n-x}$$

$$= \frac{\cancel{1.2.3.\ldots(n-x)}(n-x+1)\ldots n}{\cancel{1.2.3.\ldots(n-x)}.\lfloor x} \cdot \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{\lfloor x} \cdot \left(\frac{\lambda}{n}\right)^x \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n.\ n.(1-1/n).\ n.(1-2/n)\ldots n\left(1-\frac{x-1}{n}\right)}{\lfloor x} \cdot \frac{\lambda^x}{n^x} \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\cancel{n^x}\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]\ldots\left[1-\frac{(x-1)}{n}\right]}{\lfloor x} \cdot \frac{\lambda^x}{\cancel{n^x}} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{\lfloor x} \cdot \lambda^x \cdot \left(1-\frac{\lambda}{n}\right)^{n-x}$$

W.K.T $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$, $\lim_{n\to\infty}\frac{1}{n}=0$

As $n \to \infty$

$$p[X=x] = \frac{\lambda^x}{\lfloor x} \cdot (1) \cdot (1)\, (1) \ldots (1) \cdot e^{-\lambda}$$

$$p[X=x] = \frac{e^{-\lambda} \cdot \lambda^x}{\lfloor x} \quad ; \; x = 0, 1, 2 \ldots \infty.$$

where λ is known as the parameter of the distribution

Eg: ② Find the moment generating function of the poisson distribution.

Sol:: We know that the m.g.f of a random variable 'X' is given by.

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot p(x) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} \left(\frac{e^{-\lambda} \cdot \lambda^x}{\lfloor x} \right)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot (\lambda e^t)^x}{\lfloor x}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{\lfloor x}$$

$$= e^{-\lambda} \left[1 + \lambda e^t + \frac{(\lambda e^t)^2}{\lfloor 2} + \frac{(\lambda e^t)^3}{\lfloor 3} + \ldots \right]$$

$$\left[e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots \right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\boxed{\therefore M_X(t) = e^{\lambda(e^t - 1)}}$$ is the required M.g.f.

eg: (3) If X is a poisson variate such that $p(X=1)=\frac{3}{10}$ and $p(X=2)=\frac{1}{5}$. Find $p(X=0)$ & $p(X=3)$.

Sol:: Given: $p(X=1)=\frac{3}{10}$; $p(X=2)=\frac{1}{5}$.

We have, $p(X=x)=\frac{e^{-\lambda}.\lambda^{x}}{\lfloor x}$; $x=0,1,2\ldots$

$$p(X=1)=\frac{3}{10}$$

$$\frac{e^{-\lambda}.\lambda^{1}}{\lfloor 1}=\frac{3}{10}$$

$$\lambda e^{-\lambda}=\frac{3}{10}\longrightarrow (1)$$

$$p(X=2)=\frac{1}{5}$$

$$\frac{e^{-\lambda}.\lambda^{2}}{\lfloor 2}=\frac{1}{5}$$

$$e^{-\lambda}.\lambda^{2}=\frac{2}{5}\longrightarrow (2)$$

$$\frac{(2)}{(1)}\Rightarrow \frac{\cancel{e^{-\lambda}}.\lambda^{2}}{\cancel{e^{-\lambda}}.\lambda}=\frac{2/5}{3/10}\Rightarrow \lambda=\frac{2}{\cancel{5}}\times\frac{\cancel{10}^{2}}{3}=4/3$$

$$\boxed{\therefore \lambda=4/3}$$

$$p(X=0)=\frac{e^{-\lambda}.\lambda^{0}}{\lfloor 0}$$

$$=\frac{e^{-4/3}(1)}{(1)}$$

$$\boxed{p(X=0)=e^{-4/3}}$$

$$p(X=1)=\frac{e^{-\lambda}.\lambda^{1}}{\lfloor 1}$$

$$=\frac{e^{-4/3}.(4/3)^{1}}{\lfloor 1}$$

$$\boxed{p(X=1)=e^{-4/3}.\left(\frac{4}{3}\right)}$$

eg: (4) Write down the probability mass function of the poisson distribution which is approximately equivalent to $B(100, 0.02)$.

Sol:: Given: $n=100$ & $p=0.02$

$\therefore q=1-0.02=0.08.$

we have $\lambda = np$

$= 100 \times 0.02$

$$\boxed{\lambda = 2}$$

$\therefore$ The probability Mass function is

$$p(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{\lfloor x}\ ; x = 0, 1, 2 \ldots$$

$$\boxed{p(X=x) = \frac{e^{-2} \cdot 2^x}{\lfloor x}\ ; x = 0, 1, 2 \ldots}$$

eg: (5) If X is a poisson variate $p(x=2) = 9p(x=4) + 90\,p(x=6)$. find (i) Mean of X (ii) variance of X.

Sol: Given: $p(x=2) = 9\,p(x=4) + 90\,p(x=6)$.

$$\frac{e^{-\lambda} \cdot \lambda^2}{\lfloor 2} = 9\left(\frac{e^{-\lambda} \cdot \lambda^4}{\lfloor 4}\right) + 90\left(\frac{e^{-\lambda} \cdot \lambda^6}{\lfloor 6}\right).$$

$$\frac{\cancel{e^{-\lambda}} \cdot \cancel{\lambda^2}}{2} = \cancel{e^{-\lambda}} \cdot \cancel{\lambda^2}\left[\frac{9\lambda^2}{24} + \frac{9\cancel{0}\lambda^4}{72\cancel{0}}\right]$$

$$\frac{1}{2} = \frac{9\lambda^2}{24} + \frac{9\lambda^4}{72}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

Let $\lambda^2 = t$

$t^2 + 3t - 4 = 0$

$$\lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = 1 \text{ (or) } -4$$

$\lambda^2 = 1$ (or) $\lambda^2 = -4$.

Since $\lambda > 0$ $\boxed{\therefore \lambda = 1}$

$\therefore$ Mean $= \lambda = 1$.

Variance $= \lambda = 1$.

Eg: (6) If X and Y are Independent poisson variate such that $p(x=1) = p(x=2)$ and $p(y=2) = p(y=3)$ find the variance of $X-2Y$.

Sol:

We know that, $p(x=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{\lfloor x}$.

$p(x=1) = p(x=2)$.

$$\frac{e^{-\lambda} \cdot \lambda^{x=1}}{\lfloor 1} = \frac{e^{-\lambda} \cdot (\lambda)^{2}}{\lfloor 2}$$

$$\cancel{e^{-\lambda}} \cdot \lambda = \frac{\cancel{e^{-\lambda}} \cdot \lambda^{\cancel{2}}}{2}$$

$$1 = \frac{\lambda}{2}$$

$$\boxed{\lambda = 2}$$

$p(y=2) = p(y=3)$.

$$\frac{\cancel{e^{-\lambda_1}} \cdot (\lambda_1)^{2}}{\lfloor 2} = \frac{\cancel{e^{-\lambda_1}} \cdot (\lambda_1)^{3}}{\lfloor 3}$$

$$\frac{\cancel{\lambda_1^{2}}}{2} = \frac{\lambda_1^{\cancel{3}}}{6}$$

$$\boxed{3 = \lambda_1}$$

$\therefore Var(X) = \lambda = 2$.

$\therefore Var(Y) = \lambda_1 = 3$.

$Var(X-2Y)$

$= Var(X) + Var(-2Y)$

$= Var(X) + 4\,Var(Y)$

$= 2 + 4(3)$

$= 2 + 12$

$= 14$.

$$\boxed{\therefore Var(X-2Y) = 14}$$

eg: (7) If 3% of the electric bulbs manufactured by a Company are defective, find the probability that in a Sample of 100 bulbs exactly 5 bulbs are defective ($e^{-3} = 0.0498$).

Sol:: Let 'X' be the R.V denoting the number of defective electric bulbs.

$p(\text{a bulb is defective}) = \frac{3}{100} = 0.03$

and $n = 100$.

we have, $\lambda = np$

$\lambda = (100)(0.03)$

$\boxed{\lambda = 3}$

we have, $p(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{\lfloor x}$; $x = 0, 1, 2 \ldots$

$\therefore p(\text{exactly 5 bulbs are defective}) = p(X = 5)$.

$= \frac{e^{-3} \cdot 3^5}{\lfloor 5} = \frac{0.0498 \times 243}{120}$.

$\boxed{\therefore p(X = 5) = 0.1008}$

eg: (8) A manufacturer of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 and gurantees that not more than 4 pins will be defective what is the probability that a box will fail to meet the guranted quality. ($e^{-2} = 0.13534$).

Sol:: Let 'X' be the R.V denoting the defective pins. Here, $p = 2\% = \frac{2}{100} = 0.02$ & $n = 100$.

we have, $\lambda = np = 100 \times 0.02 = 2 \Rightarrow \boxed{\lambda = 2}$

$\therefore$ The poisson distribution is,

$$p(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{\lfloor x}; \quad x = 0, 1, 2 \ldots$$

p(a box will fail to meet the gurantee quality)

$= p$ (more than 4 pins will be defective).

$= p(X > 4)$

$= 1 - p(X \leq 4)$

$= 1 - [p(X=0) + p(X=1) + p(X=2) + p(X=3) + p(X=4)]$

$$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right]$$

$$= 1 - e^{-2}\left[1 + 2 + 2 + \frac{4}{3} + \frac{2}{3}\right]$$

$$= 1 - e^{-2}(7)$$

$$= 1 - (7 \times 0.13524)$$

$$= 1 - 0.9473$$

$$\boxed{p(X>4) = 0.0527}$$

Eg: (9) Wireless sets are manufactured with 25 soldered joints each on the average 1 joint in 500 defective. How many sets can be expected to be free from defective joints in a consignment of 10,000 sets. $[e^{-0.05} = 0.95122]$

Sol: Let 'X' be the R.V denoting the number of defective joints.

Here, $p = \frac{1}{500}$ & $n = 25$.

$$\text{Mean} = \lambda = np = 25 \times \frac{1}{500} = 0.05$$

$$\boxed{\lambda = 0.05}$$

We have, $p[X=x] = \frac{e^{-\lambda} \cdot \lambda^x}{\lfloor x} = \frac{e^{-0.05} \cdot (0.05)^x}{\lfloor x}$

$\therefore$ p (No joint is defective) = $p(X=0)$.

$$= \frac{e^{-0.05} \cdot (0.05)^0}{\lfloor 0}$$

$$= e^{-0.05} = 0.95122$$

Hence expected number of sets free from defects in 10,000 sets is,

$$= 10{,}000 \times 0.95122$$

$$= 9512.2 = 9512 \text{ (app)}$$

Eg: 10 If X and Y are independent poisson R.V show that the conditional distribution of X given X+Y is a binomial distribution.

Sol: Let X and Y be independent poisson R.V's with parameters λ_1 and λ_2 respectively.

$$p[X=r / X+Y=n] = \frac{p[X=r \text{ and } X+Y=n]}{p[X+Y=n]}.$$

$$= \frac{p(X=r \text{ and } Y=n-r)}{p(X+Y=n)}$$

$$= \frac{p(X=r) \cdot p(Y=n-r)}{p(X+Y=n)}$$

$$= \frac{\dfrac{e^{-\lambda_1} \cdot \lambda_1^r}{\lfloor r} \cdot \dfrac{e^{-\lambda_2} \cdot (\lambda_2)^{n-r}}{\lfloor n-r}}{\dfrac{e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)^n}{\lfloor n}}$$

$$= \frac{\cancel{e^{-(\lambda_1+\lambda_2)}} \cdot \lambda_1^r \cdot \lambda_2^{n-r}}{\cancel{e^{-(\lambda_1+\lambda_2)}} \cdot (\lambda_1+\lambda_2)^n} \times \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

$$= \frac{\lambda_1^r \cdot \lambda_2^{n-r}}{(\lambda_1+\lambda_2)^r (\lambda_1+\lambda_2)^{n-r}} \times \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$= nC_r \, p^r q^{n-r}$ which is a p.d.f of Binomial distribution.

where $p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$; $q = \dfrac{\lambda_2}{\lambda_1+\lambda_2}$

and $nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$ also $p+q=1$.

eg: Messages arrive at a switch board in a poisson manner at an average rate of six per hour. Find the probability for each of the following events. (i) Exactly two messages arrive within one hour (ii) No message arrive with one hour (iii) at least three message arrive within one hour.

Sol: Given: average rate = Mean = 6/hr

$$P[X=x) = \frac{e^{-\lambda}.\lambda^x}{\lfloor x}, \quad x = 0,1,2\dots\infty.$$

$$P[X=x] = \frac{e^{-6}.6^x}{\lfloor x} \longrightarrow (1)$$

(i) $$P[X=2] = \frac{e^{-6}.6^2}{\lfloor 2} = 0.0446.$$

(ii) $$P[X=0] = \frac{e^{-6}.6^0}{\lfloor 0} = e^{-6} = 0.0025$$

(iii) $$P(\text{atleast } 3) = P(X \geq 3) = 1 - P(X<3)$$

$$= 1 - (P[X=0] + P[X=1] + P[X=2])$$

$$= 1 - \left[\frac{e^{-6}.6^0}{\lfloor 0} + \frac{e^{-6}.6^1}{\lfloor 1} + \frac{e^{-6}.6^2}{\lfloor 2}\right]$$

$$= 1 - e^{-6}[1+6+18]$$

$$= 1 - e^{-6}(25)$$

$$= 0.9380.$$

Eg: The probability that an Individual suffers from a bad reaction from a certain Injection is 0.001. Determine the probability that out of 2000 Individuals (i) Exactly 3 (ii) more than 2 Individuals will suffer from a bad reaction.

Sol: Given: $p = 0.001$, $n = 2000$

We have $\lambda = np = (0.001)(2000)$

$$\boxed{\lambda = 2}$$

For a poisson distribution,

$$P[X = x] = \frac{e^{-\lambda} \cdot \lambda^{x}}{\lfloor x}, \quad x = 0, 1, 2 \ldots \infty.$$

$$P[X = x] = \frac{e^{-2} \cdot 2^{x}}{\lfloor x} \longrightarrow (1)$$

(i) $P[X = 3] = \dfrac{e^{-2} \cdot 2^{3}}{\lfloor 3} = 0.1804.$

(ii) $P[X > 2] = 1 - P[X \leq 2]$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left[\dfrac{e^{-2} \cdot 2^{0}}{\lfloor 0} + \dfrac{e^{-2} \cdot 2^{1}}{\lfloor 1} + \dfrac{e^{-2} \cdot 2^{2}}{\lfloor 2}\right]$

$= 1 - e^{-2}[1 + 2 + 2]$

$= 1 - e^{-2}(5)$

$$\boxed{P(X>2) = 0.3233}$$

eg:: The number of accidents in a year attributed to taxi drivers in a city follows poisson distribution with mean equal to 3. Out of 1000 taxi drivers, find approximately the number of drivers with (i) No accidents in a year (ii) more than 3 accidents in a year. (51)

Sol:: Given: Mean $= \lambda = 3.$, $N = 1000$

$$P[X=x] = \frac{e^{-\lambda} \cdot \lambda^{x}}{\lfloor x}, \quad x = 0, 1, 2 \ldots \infty$$

$$P[X=x] = \frac{e^{-3} \cdot 3^{x}}{\lfloor x} \longrightarrow (1)$$

(i) $P(\text{No accidents}) = P(X=0) = \frac{e^{-3} \cdot 3^{0}}{\lfloor 0} = 0.0498$

Out of 1000 taxi drivers $= 1000 \times 0.0498$
$= 49.78$
≈ 50 drivers.

(ii) $P(\text{More than 3 accidents}) = P(X>3)$

$= 1 - P(X \leq 3)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3)\}$

$= 1 - \left\{ \frac{e^{-3} \cdot 3^{0}}{\lfloor 0} + \frac{e^{-3} \cdot 3^{1}}{\lfloor 1} + \frac{e^{-3} \cdot 3^{2}}{\lfloor 2} + \frac{e^{-3} \cdot 3^{3}}{\lfloor 3} \right\}$

$= 1 - e^{-3} \left[1 + 3 + \frac{9}{2} + \frac{27}{6}\right] = 1 - 0.6472$

$= 0.3528.$

Out of 1000 drivers $= 1000 \times 0.3528 \approx 353$ drivers

(HW) ① The number of monthly breakdown of a Computer is a random variable having a poisson distribution with mean equal to 1.8. Find the probability that this Computer will function for a month (i) without breakdown (ii) with only one breakdown (iii) with at least one breakdown.

Ans: [0.1653, 0.2975, 0.8347].

② A manufacture of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective, then what is the probability that a box will fail to meet the guaranteed quality.?

Ans: (0.053)

③ The atoms of a radioactive element are randomly disintegrating. If every gram of this element on average emits 3.9 alpha particles per Second, then what is the probability that during the next Second the number of alpha particles emitted from 1 grams is (i) at most 6 (ii) at least 2 (iii) at least 3 and at most 5?

Ans: (0.8994, 0.901, 0.5474)

GEOMETRIC DISTRIBUTION.

Def:- A discrete random variable 'X' is said to follow Geometric distribution, if it assumes only non-negative values and its probability mass function is given by,

$$p[X=x] = q^{x-1} \cdot p \; ; x = 1, 2 \ldots, 0 < p \leq 1.$$

where $q = 1 - p$.

(OR)

$$p[X=x] = q^{x} \cdot p \; ; x = 0, 1, 2 \ldots, 0 < p \leq 1.$$

where $q = 1 - p$.

---

eg ①: Find the moment generating function of Geometric distribution.

Sol:- The Moment generating function is given by,

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} e^{tx} p[X=x] = \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x} \cdot q^{-1} \cdot p = \frac{p}{q} \sum_{x=1}^{\infty} e^{tx} \cdot q^{x}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^{t})^{x}$$

$$= p/q \left[qe^{t} + (qe^{t})^2 + (qe^{t})^3 + \ldots \right]$$

$$= p/\cancel{q} \cdot (\cancel{q}e^{t}) \left[1 + (qe^{t}) + (qe^{t})^2 + \ldots \right]$$

$$= pe^{t} (1 - qe^{t})^{-1}$$

$$\boxed{\therefore M_X(t) = \frac{pe^{t}}{1 - qe^{t}}}$$

Eg: ② Find the Mean and variance of a geometric distribution.

Sol: We have, $M_x(t) = \frac{pe^t}{(1-qe^t)}$.

$$M_x'(t) = \frac{d}{dt}[M_x(t)] = p\left[\frac{(1-qe^t)e^t - e^t(-qe^t)}{(1-qe^t)^2}\right]$$

$$= p\left[\frac{e^t}{(1-qe^t)^2}\right] \longrightarrow ①$$

$$E(x) = M_x'(0) = \frac{p}{(1-q)^2} = p/p^2$$

$$\boxed{E(x) = \frac{1}{p}}$$

$$M_x''(t) = \frac{d}{dt}[M_x'(t)] = p\left[\frac{(1-qe^t)^2 e^t - e^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]$$

$$\therefore E(x^2) = M_x''(0) = p\left[\frac{(1-q)^2 + 2q(1-q)}{(1-q)^4}\right]$$

$$= p\left[\frac{p^2 + 2pq}{p^4}\right] \quad [\because 1-q = p]$$

$$= \frac{p+q+q}{p^2} = \frac{1}{p^2} + \frac{q}{p^2}$$

$$\boxed{\therefore E(x^2) = \frac{1}{p^2} + \frac{q}{p^2}}$$

$$\therefore Var(x) = E(x^2) - [E(x)]^2 = \cancel{\frac{1}{p^2}} + \frac{q}{p^2} - \cancel{\frac{1}{p^2}}$$

$$\boxed{\therefore Var(x) = \frac{q}{p^2}}$$

eg: 3 Let one copy of a magazine out of 10 copies bears a special prize following geometric distribution. Determine its mean and variance.

Sol:: Given: $p = \frac{1}{10} \Rightarrow q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$.

$\therefore$ Mean of the geometric distribution $= \frac{1}{p} = 10$.

$$\text{Variance} = \frac{q}{p^2} = \frac{9}{10} \times 10^2 = 90.$$

eg: 4 If X is a Geometric variate taking values $1, 2, \ldots \infty$ find $p(X \text{ is odd})$.

Sol:: We have, $p[X=x] = q^{x-1} \cdot p \; ; x = 1, 2 \ldots$

$$\begin{aligned} p[x = \text{odd}] &= p[x = 1, 3, 5 \ldots] \\ &= p[x=1] + p[x=3] + p[x=5] + \ldots \\ &= q^{1-1} \cdot p + q^{3-1} \cdot p + q^{5-1} \cdot p + \ldots \\ &= p + q^2 p + q^4 p + \ldots \\ &= p\,(1 + q^2 + q^4 + \ldots) \\ &= p\,[1-q^2]^{-1} \\ &= \frac{p}{1-q^2} = \frac{p}{(1+q)(1-q)} = \frac{\cancel{1-q}}{(1+q)\cancel{(1-q)}} \end{aligned}$$

$$\boxed{p[x = \text{odd}] = \frac{1}{1+q}}$$

eg: 5 If the probability that a target is destroyed on any one shot is 0.5. What is the probability that it would be destroyed on 6^{th} attempt.

Sol:: Let 'X' be the R.V denoting the number of attempts required for the first success.

Given: $p = 0.5 \quad \therefore q = 1-p = 1-0.5 = 0.5$

$$\boxed{q = 0.5}$$

We know that, $p[X=x] = q^{x-1}p.$

$$p[X=6] = q^{6-1}p$$

$$= (0.5)^5 (0.5)$$

$$\boxed{p[X=6] = 0.0156}$$

---

eg: (6) State and prove the memoryless property of the Geometric distribution.

Sol:

Statement: Let 'X' has a geometric distribution, then for any two positive integers 'm' and 'n',

$p[X > m+n / X > m] = p[X > n].$

proof:

$$p[X > m+n / X > m] = \frac{p[X > m+n \cap X > m]}{p[X > m]}$$

$$= \frac{p[X > m+n]}{p[X > m]} \longrightarrow (1)$$

We have, $p[X=x] = q^{x-1} p \; ; x = 1, 2 \ldots$

$$p[X > k] = \sum_{x=k+1}^{\infty} q^{x-1} p$$

$$= q^k p + q^{k+1} p + q^{k+2} p + \ldots$$

$$= q^k p [1 + q + q^2 + \ldots]$$

$$= q^k p (1-q)^{-1}$$

$$= q^k p . p^{-1} = q^k.$$

$\therefore$ we have, $p[x > m+n] = q^{m+n}$

and $p[x > m] = q^{m}$.

from (1), $p[x > m+n / x > m] = \dfrac{q^{m+n}}{q^{m}}$

$$= \frac{\cancel{q^{m}} \cdot q^{n}}{\cancel{q^{m}}}$$

$$= q^{n}$$

$$= p[x > n].$$

Hence proved.

---

eg: (7) Let X_1, X_2 be independent random variables each having geometric distribution $q^{k} \cdot p$, $k = 0, 1, 2 \ldots$. Show that the conditional distribution of X_1 given $X_1 + X_2$ is uniform.

Sol: Given, $p[X_1 = k] = p[X_2 = k] = pq^{k}$, $k = 0, 1, 2 \ldots$

(ii) $p[X_1 = k] = p[X_2 = k] = pq^{k-1}$, $k = 1, 2, 3 \ldots$

and $X_1 + X_2 = n$ is a discrete uniform distribution.

$$p[X_1 = k / X_1 + X_2 = n] = \frac{p[X_1 = k \text{ and } X_1 + X_2 = n]}{p[X_1 + X_2 = n]}$$

Since $X_1 + X_2 = n$.

$k + X_2 = n$

$\boxed{X_2 = n - k}$

$$= \frac{P[X_1 = k] \cdot P[X_2 = n-k]}{\sum_{k=0}^{n} P[X_1 = k]\, P[X_2 = n-k]}$$

$$= \frac{(p \cdot q^k)(p \cdot q^{n-k})}{\sum_{k=0}^{n} (p \cdot q^k)(p q^{n-k})}$$

$$= \frac{p^2 q^n}{\sum_{k=0}^{n} p^2 q^n .}$$

$$= \frac{\cancel{p^2 q^n}}{\cancel{p^2 q^n} \sum_{k=0}^{n} 1}$$

$$= \frac{1}{1 + 1 + 1 + \dots + (n+1) \text{ times.}}$$

$$= \frac{1}{n+1} \text{ which is a Constant.}$$

$\therefore \frac{X_1}{X_1 + X_2}$ follows a discrete uniform distribution with parameter $\frac{1}{n+1}$.

eg: A Coin is tossed until the first heads occurs. Assuming that the tosses are Independent and the probability of a head occuring each time is 'p'. find the value of p if an odd number of tosses required for getting probability 0.6. Can you find a value of p if that's probability is 0.5? (55)

Sol: Given: probability of getting head = p.

$$p(x \text{ is odd}) = 0.6$$

$$p(x = 1, 3, 5, \ldots) = 0.6$$

For a geometric distribution

$$p[x = x] = q^{x-1} \cdot p, \quad x = 1, 2, 3 \ldots$$

$$p + q^2 p + q^4 p + \ldots = 0.6$$

$$p(1 + q^2 + (q^2)^2 + \ldots) = 0.6$$

$$p(1 - q^2)^{-1} = 0.6$$

$$p\left(\frac{1}{1^2 - q^2}\right) = 0.6$$

$$\frac{p}{(1+q)(1-q)} = 0.6$$

$$\frac{\cancel{1-q}}{(1+q)(\cancel{1-q})} = 0.6$$

eg: A die is cast until 6 appears. what is the probability that it must be cast more than 5 times?

Sol: Getting 6 is Success.

P = probability of 6 = $\frac{1}{6}$.

$$\boxed{P = \frac{1}{6}} \Rightarrow q = 1 - p$$

$$q = 1 - \frac{1}{6}$$

$$\boxed{q = \frac{5}{6}}$$

For a Geometric distribution

$$p(X = x) = q^{x-1} \cdot p \quad , x = 1, 2, 3 \ldots$$

$$p(X = x) = \left(\frac{5}{6}\right)^{x-1} \cdot \left(\frac{1}{6}\right)$$

$$P(X > 5) = q^5 \qquad \boxed{\because p(X > k) = q^k}$$

$$= \left(\frac{5}{6}\right)^5$$

$$\boxed{p(X > 5) = 0.4019}$$

HW If the probability that a target is destroyed by anyone shot is 0.6. what is probability that it should be destroyed on the 5th attempt?

Ans: 0.0154.

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$$\frac{1}{1+q} = 0.6 \quad \text{--- (1)}$$

$$1+q = \frac{1}{0.6}$$

$$1+q = \frac{5}{3}$$

$$q = \frac{5}{3} - 1$$

$$q = \frac{2}{3}$$

$$1 - p = \frac{2}{3}$$

$$1 - \frac{2}{3} = p$$

$$\boxed{\frac{1}{3} = p}$$

Suppose $p(x \text{ is odd}) = 0.5$

$$\frac{1}{1+q} = 0.5 \qquad [\because \text{by (1)}]$$

$$1+q = 2$$

$$q = 2-1$$

$$\boxed{q = 1}$$

$\Rightarrow \quad p = 0$ which is meaningless.

Since $p(x = \text{odd}) = 0$

$\therefore$ So value of p cannot be obtained.

Q: If the probability that an applicant for a driver's license will pass the road test on any given trial is 0.8. what is the probability that he will finally pass the test (i) On the fourth trail (ii) in less than 4 trails?

Sol: Given: $p = 0.8$

$q = 1 - p$

$q = 1 - 0.8$

$\boxed{q = 0.2}$

For a geometric distribution

$P(X = x) = q^{x-1} \cdot p \quad , x = 1, 2, 3 \ldots$

$P(X = x) = (0.2)^{x-1} \cdot (0.8) \longrightarrow (1)$

(i) $P(\text{on the } 4^{th} \text{ trail}) = P(X = 4)$

$= (0.2)^3 (0.8)$

$= 0.0064.$

(ii) $P(\text{less than } 4 \text{ trails}) = P(X < 4)$

$= P(X = 1, 2, 3)$

$= (0.2)^0 (0.8) + (0.2)^1 (0.8) + (0.2)^2 (0.8)$

$= (0.8) [1 + 0.2 + 0.04]$

$= 0.9920$

Q: The probability that a Candidate can pass in an examination is 0.6. what is the probability that he will pass (i) in the third trail (57)

(ii) Before the third trail.

Sol: Given: $p = 0.6$

$$q = 1 - p$$

$$q = 1 - 0.6$$

$$\boxed{q = 0.4}$$

$$p(X = x) = q^{x-1} \cdot p, \quad x = 1, 2, 3 \ldots$$

$$p(X = x) = (0.4)^{x-1}(0.6) \quad \text{—— (1)}$$

(i) $P(\text{in the } 3^{rd} \text{ trail}) = p(X = 3)$

$$= (0.4)^2 (0.6)$$

$$= 0.0960.$$

(ii) $P(\text{Before the } 3^{rd} \text{ trail}) = p(X < 3)$

$$= p(X = 1) + p(X = 2)$$

$$= (0.4)^0 \cdot (0.6) + (0.4)^1 (0.6)$$

$$= (0.6)[1 + 0.4]$$

$$= 0.8400$$

eg: Suppose that a trainee Soldier Shoots a target in an Independent fashion. If the probability that the target is shot on any one shot is 0.7. (i) what is the prob. that it takes him in 10th attempt? (ii) What is the prob that it takes him an even no of shots? (iii) what is the prob that it takes him less than 4 shots? (iv) what is the average no of shots needed to hit the target?

Sol: Given: $\boxed{p=0.7}$, $q=1-p$, $\boxed{q=0.3}$

$$P(X=x) = q^{x-1}\cdot p, \quad x=1,2,3\ldots$$

$$P(X=x) = (0.3)^{x-1}\cdot(0.7) \quad \text{—— (1)}$$

(i) $P(X=10) = (0.3)^9(0.7) = 0.000014.$

(ii) $P(X=\text{even}) = P(X=2,4,6,8\ldots)$

$$= qp + q^3p + q^5\cdot p + \ldots$$

$$= qp(1+q^2+q^4+\ldots)$$

$$= qp(1-q^2)^{-1}$$

$$= qp\cdot\left(\frac{1}{1-q^2}\right)$$

$$= qp\cdot\frac{1}{(1+q)(1-q)} = \frac{q(\cancel{1-q})}{(1+q)(\cancel{1-q})}$$

$$P(X = \text{even}) = \frac{q}{1+q}$$

$$= \frac{0.3}{1.3}$$

$$\boxed{P(X \text{ is even}) = 0.2308}$$

(iii) $P(\text{less than } 4 \text{ shots}) = P(X<4)$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= (0.3)^0 (0.7) + (0.3)^1 (0.7) + (0.3)^2 (0.7)$$

$$= (0.7)\left[1 + 0.3 + 0.09\right]$$

$$\boxed{P(X<4) = 0.9730}$$

(iv) Average number of shots needed to hit the target $= E(X)$

$$= \text{Mean}$$

$$= \frac{1}{p}$$

$$= \frac{1}{0.7}$$

$$= 1.4286$$

CONTINUOUS DISTRIBUTIONS

(1) Uniform (Rectangular) distribution.

(2) Exponential distribution.

(3) Gamma distribution.

(4) Normal distribution.

UNIFORM (RECTANGULAR) DISTRIBUTIONS:

A random variable 'X' is said to have a continuous uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Note: 1 The distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, & -\infty < x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b < x < \infty. \end{cases}$$

Note: 2 The p.d.f of a uniform variate 'X' in $(-a, a)$ is given by $f(x) = \begin{cases} \frac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise.} \end{cases}$

Eg ①: Show that for a uniform distribution in (a, b) $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$

Sol:

Since the total probability is One, we have,

$$\int_a^b f(x)\,dx = 1.$$

$$\int_a^b k\,dx = 1$$

$$k\int_a^b dx = 1$$

$$k[x]_a^b = 1$$

$$k(b-a) = 1$$

$$k = \frac{1}{b-a}$$

$$\therefore f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

Eg: (2) Find the moment generating function of uniform distribution.

Sol:: We have,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{t(b-a)} \cdot (e^{bt} - e^{at})$$

$$\boxed{\therefore M_X(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$

Eg: (3) Find the Mean & variance of uniform distribution.

Sol::

$$E(X) = \mu_1' = \int_a^b x \, f(x)dx$$

$$= \int_a^b x \cdot \frac{1}{b-a} \cdot dx$$

$$= \frac{1}{b-a} \int_a^b x dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2} \right)_a^b$$

$$= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{\cancel{(b-a)}(b+a)}{2}$$

$$\boxed{E(X) = \frac{a+b}{2}}$$

$$E(X^2) = \mu_2' = \int_a^b x^2 f(x)dx$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a} \cdot dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b$$

$$= \frac{1}{b-a} \left(\frac{b^3 - a^3}{3} \right)$$

$$= \frac{1}{\cancel{b-a}} \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3}$$

$$\boxed{E(X^2) = \frac{b^2+ab+a^2}{3}}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4b^2+4ab+4a^2 - 3a^2 - 3b^2 - 6ab}{12}$$

$$= \frac{b^2+a^2-2ab}{12}$$

$$\boxed{Var(X) = \frac{(b-a)^2}{12}}$$

Eg: Let X be uniformly distributed over $(-5,5)$. Determine (i) $p(X\le 2)$ (ii) $p(|X|>2)$ (iii) cummulative distribution of X (iv) var(X).

Sol:- Here, $f(x) = \frac{1}{5+5}$, $-5<x<5$

$$\boxed{f(x) = \frac{1}{10}, \quad -5<x<5}$$

(i) $p(X\le 2) = \int_{-5}^{2} f(x)\,dx = \int_{-5}^{2} \frac{1}{10}\cdot dx$

$$= \frac{1}{10}(x)_{-5}^{2}$$

$$= \frac{1}{10}(2+5)$$

$$\boxed{p(X\le 2) = \frac{7}{10}}$$

(ii) $p(|X|>2) = 1 - p(|X|\le 2)$

$$= 1 - p(-2\le x\le 2)$$

$$= 1 - \int_{-2}^{2} f(x)\,dx$$

$$= 1 - \int_{-2}^{2} \frac{1}{10}dx$$

$$= 1 - \frac{1}{10}(x)_{-2}^{2}$$

$$= 1 - \frac{1}{10}(2+2)$$

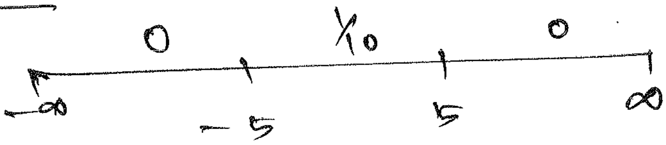
$$= 1 - \frac{4}{10}$$

$$= \frac{10-4}{10}$$

$$= \frac{6}{10}$$

$$\boxed{P(X>2) = \frac{3}{5}}$$

(iii) <u>To find</u>: CDF of X.



$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

<u>Case</u> (i) If $x \in (-\infty, -5)$

$$F(x) = \int_{-\infty}^{x} 0 \cdot dx = 0$$

<u>Case</u> (ii) If $x \in (-5, 5)$

$$F(x) = \int_{-\infty}^{-5} f(x)\,dx + \int_{-5}^{x} f(x)\,dx = 0 + \int_{-5}^{x} \frac{1}{10}\,dx$$

$$= \frac{1}{10}\left(x\right)_{-5}^{x}$$

$$\boxed{F(x) = \frac{1}{10}(x+5)}$$

<u>Case</u> (iii): If $x \in (5, \infty)$

$$F(x) = \int_{-\infty}^{-5} f(x)\,dx + \int_{-5}^{5} f(x)\,dx + \int_{5}^{x} f(x)\,dx$$

(61)

$$F(x) = 0 + \int_{-5}^{5} \frac{1}{10}\, dx + 0$$

$$= \frac{1}{10}\, (x)_{-5}^{5}$$

$$= \frac{1}{10}\, (5+5)$$

$$= \frac{1}{\cancel{10}}\, (\cancel{10})$$

$$\boxed{F(x) = 1}$$

$$\therefore F(x) = \begin{cases} 0 & , \ x < -5 \\ \frac{1}{10}(x+5) & , \ -5 < x < 5 \\ 1 & , \ x > 5. \end{cases}$$

(iii) <u>To find:</u> Var(X).

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-5}^{5} x \cdot \frac{1}{10}\, dx = \frac{1}{10}\left(\frac{x^2}{2}\right)_{-5}^{5}$$

$$= \frac{1}{10}\left(\frac{25-25}{2}\right) = \frac{1}{10}(0) = 0$$

$$\boxed{\therefore E(X) = 0}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-5}^{5} x^2 \cdot \frac{1}{10}\, dx = \frac{1}{10}\left(\frac{x^3}{3}\right)_{-5}^{5}$$

$$= \frac{1}{10}\left(\frac{125+125}{3}\right) = \frac{250}{30}.$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{250}{30} - 0^2$$

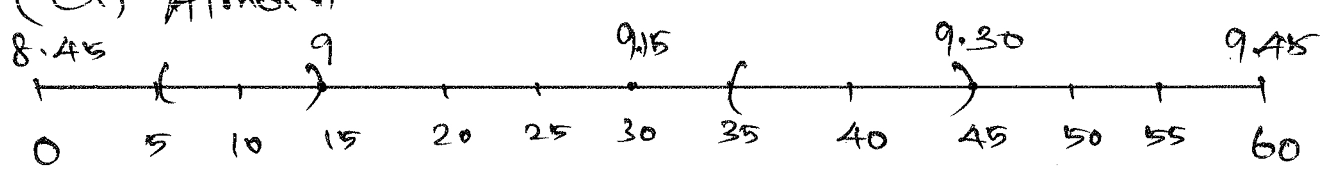
$$\therefore Var(X) = \frac{250}{30} = \frac{25}{3}.$$

eg: Starting at 5.00 am every half an hour there is a flight from San Francisco airport to Losangles. Suppose that none of the planes is completely sold out and that they always have room for passengers. A person who wants to fly to Losangles arrive at a random time between 8.45 Am and 9.45 Am. find the probability that she waits (a) Atmost 10 min (b) atleast 15 min.

Sol: Here X is a uniform RV over the Interval (0, 60).

$$\therefore f(x) = \frac{1}{60}, \quad 0 < x < 60.$$

(a) Almost 10 min.



$$p(\text{almost } 10\text{ min}) = p(5 < X < 15) + p(35 < X < 45)$$

$$= \int_5^{15} f(x)\,dx + \int_{35}^{45} f(x)\,dx$$

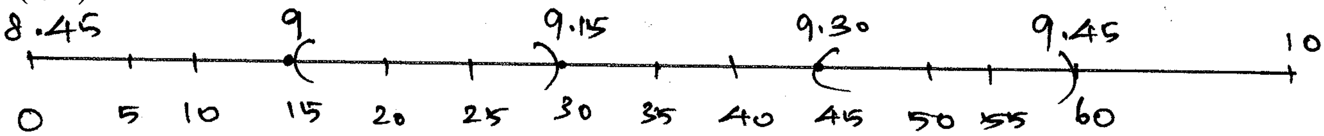
$$= \int_5^{15} \frac{1}{60}\,dx + \int_{35}^{45} \frac{1}{60}\,dx$$

$$= \frac{1}{60}\left[x\right]_5^{15} + \frac{1}{60}\left[x\right]_{35}^{45}$$

$$= \frac{1}{60}[15-5] + \frac{1}{60}(45-35)$$

$$= \frac{10}{60} + \frac{10}{60} = \frac{2\cancel{0}}{6\cancel{0}} = \frac{2}{6} = \frac{1}{3}.$$

(b) Atleast 15 min.



$$P(\text{Atleast } 15 \text{ min}) = P(15 < x < 30) + P(45 < x < 60)$$

$$= \int_{15}^{30} f(x)\,dx + \int_{45}^{60} f(x)\,dx$$

$$= \int_{15}^{30} \frac{1}{60}\,dx + \int_{45}^{60} \frac{1}{60}\,dx$$

$$= \frac{1}{60}\left(x\right)_{15}^{30} + \frac{1}{60}\left(x\right)_{45}^{60}$$

$$= \frac{1}{60}(30-15) + \frac{1}{60}(60-45)$$

$$= \frac{15}{60} + \frac{15}{60}$$

$$= \frac{3\cancel{0}}{6\cancel{0}}$$

$$= \frac{1}{2}$$

HW Trains arrive at a station 15 minutes Interval starting at 4 am. If a passenger arrive at a station at a time that is uniformly distributed between 9.00am to 9.30am. find the probability that he has to wait for the train for (i) less than 6 min (ii) more than 10 min.

$$\left[\frac{2}{5}, \frac{1}{3}\right]$$

eg: If x is uniformly distributed over $(-\alpha, \alpha)$, $\alpha > 0$. Find α so that (i) $p(x>1) = 1/3$

(ii) $p(|x|<1) = p(|x|>1)$

Sol: Here x is uniformly distributed over $(-\alpha, \alpha)$

$$\therefore f(x) = \frac{1}{2\alpha}, \quad -\alpha < x < \alpha.$$

(i) $p(x > 1) = \frac{1}{3}$

$$\int_1^{\infty} f(x)\, dx = \frac{1}{3}$$

$$\int_1^{\alpha} f(x)\, dx = \frac{1}{3}$$

$$\int_1^{\alpha} \frac{1}{2\alpha} \cdot dx = \frac{1}{3}$$

$$\frac{1}{2\alpha} (x)_1^{\alpha} = \frac{1}{3}$$

$$\frac{1}{2\alpha} (\alpha - 1) = \frac{1}{3}$$

$$3(\alpha - 1) = 2\alpha$$

$$3\alpha - 3 = 2\alpha$$

$$3\alpha - 2\alpha = 3$$

$$\boxed{\alpha = 3}$$

(ii) $p(|x|<1) = p(|x|>1)$

$p(|x|<1) = 1 - p(|x| \leq 1)$

$p(|x|<1) + p(|x| \leq 1) = 1$

$2\, p(|x|<1) = 1$

$2\, p(-1 < x < 1) = 1$

$\int_{-1}^{1} f(x)\, dx = \frac{1}{2}$

$\int_{-1}^{1} \frac{1}{\cancel{2}\alpha} \cdot dx = \frac{1}{\cancel{2}}$

$\frac{1}{\alpha} (x)_{-1}^{1} = 1$

$\frac{1}{\alpha} (1+1) = 1$

$\frac{2}{\alpha} = 1$

$\boxed{\alpha = 2}$

(64)

Eg: (4) Show that for the uniform distribution $f(x) = \frac{1}{2a}$, $-a < x < a$ the m.g.f about origin is $\frac{\sinh at}{at}$

Sol: We have,

$$M_X(t) = \int_a^b e^{tx} f(x)dx$$

$$= \int_{-a}^{a} e^{tx} \cdot \frac{1}{2a} \cdot dx$$

$$= \frac{1}{2a} \cdot \left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$= \frac{1}{2a}\left(\frac{e^{at} - e^{-at}}{t}\right)$$

[We have $\sinh x = \frac{e^{x} - e^{-x}}{2}$]

$$= \frac{1}{\cancel{2}at} \cdot \cancel{2}\sinh at$$

$$\boxed{M_X(t) = \frac{\sinh at}{at}}$$

Eg: (5) A random variable 'X' has a uniform distribution over $(-3, 3)$ Compute.

(i) $P(X < 2)$, $P(|X| < 2)$, $P(|X-2| < 2)$.

(ii) Find k for which $P(X > k) = \frac{1}{3}$.

Sol: Here $f(x) = \begin{cases} \frac{1}{6} & ; -3 < x < 3 \\ 0 & ; \text{otherwise}. \end{cases}$

(i) (a) $P[X < 2] = \int_{-3}^{2} f(x)dx$

$$= \int_{-3}^{2} \frac{1}{6} \cdot dx$$

$$= \frac{1}{6}(x)_{-3}^{2}$$

$$= \frac{1}{6}(2+3)$$

$$\boxed{P(X<2) = \frac{5}{6}}$$

(b) $P[|X| < 2] = P(-2 < X < 2)$

$$= \int_{-2}^{2} f(x)dx$$

$$= \int_{-2}^{2} \frac{1}{6} \cdot dx$$

$$= \frac{1}{6}(x)_{-2}^{2}$$

$$= \frac{1}{6}(2+2)$$

$$= 4/6$$

$$= 2/3$$

(c) $p(|x-2|<2) = p(-2<x-2<2)$

$= p(0<x<4)$

$= \int_0^4 f(x)\,dx$

$= \int_0^3 \frac{1}{6}\,dx$

$= \frac{1}{6}\cdot(x)_0^3$

$= \frac{3}{6} = \frac{1}{2}$

(ii) $p[x > k] = \frac{1}{3}$.

$\int_k^3 f(x)\,dx = \frac{1}{3}$.

$\int_k^3 \frac{1}{6}\,dx = \frac{1}{3}$

$\frac{1}{6}\,(x)_k^3 = \frac{1}{3}$

$\frac{1}{6}[3-k] = \frac{1}{3}$

$3-k = 2$

$k = 3-2$

$\boxed{k=1}$

Eg: ⑥ Subway trains on a certain line run every half hour between mid-night and six in the morning. what is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?

Sol: Let 'x' be the random variable which denotes the waiting time for the next train. Assume that a man arrives at the station at random, the random variable 'x' is distributed uniformly in (0, 30) with p.d.f

$$f(x) = \begin{cases} \frac{1}{30-0}, & 0<x<30 \\ 0 & ;\text{otherwise} \end{cases}$$

$\therefore p(\text{atleast } 20 \text{ minutes}) = p(x \geq 20)$

$= \int_{20}^{30} f(x)\,dx$

$$= \int_{20}^{30} \frac{1}{30} \cdot dx$$

$$= \frac{1}{30} (x)_{20}^{30}$$

$$= \frac{1}{30} (30-20)$$

$$\boxed{p(x \geq 20) = \frac{10}{30}}$$

eg: 7: A random variable Y is defined as $\cos \pi x$ where X has uniform p.d.f over $(-\frac{1}{2}, \frac{1}{2})$ find Mean & S.D.

Sol: we have,

$$f(x) = \begin{cases} 1 ; & -\frac{1}{2} < x < \frac{1}{2} \\ 0 ; & \text{otherwise} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(x) dx = \int_{-1/2}^{1/2} \cos \pi x . 1 . dx$$

$$= \left[\frac{\sin \pi x}{\pi}\right]_{-1/2}^{1/2}$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right)$$

$$= \frac{1}{\pi}(1+1) = \frac{2}{\pi}$$

$$\boxed{E(Y) = 0.636}$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(x) dx$$

$$= \int_{-1/2}^{1/2} y^2 . 1 . dx$$

$$= \int_{-1/2}^{1/2} \cos^2 \pi x . dx$$

$$= \int_{-1/2}^{1/2} \left(\frac{1 + \cos 2\pi x}{2}\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2\pi x}{4\pi}\right]_{-1/2}^{1/2}$$

$$= \left(\frac{1}{4} + \frac{\sin \pi}{4\pi}\right) - \left(\frac{-1}{4} + \frac{\sin(-\pi)}{4\pi}\right)$$

$$= \frac{1}{4} + 0 + \frac{1}{4} - 0$$

$$= \frac{2}{4}$$

$$\boxed{E(Y^2) = \frac{1}{2}}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= 0.5 - (0.636)^2$$

$$= 0.5 - 0.4045$$

$$Var(Y) = 0.096$$

$$S.D = \sqrt{Var(Y)}$$

$$= \sqrt{0.096}$$

$$\boxed{S.D = 0.31}$$

EXPONENTIAL DISTRIBUTION.

Def: Let 'X' be a continuous random variable is said to follow an exponential distribution with parameter $\lambda > 0$ if its probability density function is given by,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Eg ①: Find the Moment generating function of exponential distribution.

Sol: We have,

$$M_X(t) = \int_0^\infty e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} \cdot dx$$

$$= \lambda \int_0^\infty e^{-x(\lambda - t)} \cdot dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{t - \lambda} \right]_0^\infty$$

$$= \lambda \left[0 - \frac{1}{t - \lambda} \right]$$

$$= \frac{\lambda}{\lambda - t}$$

$$\boxed{\therefore M_X(t) = \frac{\lambda}{\lambda - t}}, \quad \lambda > t$$

Eg: ② Find the Mean and Variance of exponential distribution.

Sol: We have,

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

$$= \frac{\lambda}{\lambda(1 - t/\lambda)}$$

$$= (1 - t/\lambda)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \dots + \frac{t^r}{\lambda^r} + \dots$$

$$M_X(t) = \sum_{r=0}^\infty \left(\frac{t}{\lambda}\right)^r$$

$$\therefore \mu_r' = \frac{r!}{\lambda^r}; \quad r = 1, 2 \dots$$

$$\text{Mean} = \mu_1' = \frac{1}{\lambda}$$

$$E(X^2) = \mu_2' = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{Var}(X) = \frac{1}{\lambda^2}}$$

$$\boxed{S.D = \frac{1}{\lambda}}$$

eg: The daily Consumption of milk in excess of 20,000 gallons is approximately exponentially distributed with $\theta = 3000$. The city has a daily stock of 35,000 gallons. what is the probability that of two days selected at random, the stock is Insufficient for both days? (66)

Sol: Let X denote the daily Consumption of milk

Let $Y = X - 20{,}000$, the excess, is exponentially distributed with $\theta = 3000$.

$$\therefore \text{Mean} = 3000$$

$$\frac{1}{\lambda} = 3000$$

$$\boxed{\lambda = \frac{1}{3000}} \quad \text{and} \quad X = Y + 20{,}000$$

probability density function of Y is

$$f(y) = \lambda e^{-\lambda y}, \quad \lambda \geq 0, \ y \geq 0$$

$$= \frac{1}{3000} e^{\frac{-y}{3000}}, \quad y \geq 0$$

$\therefore$ prob of stock Insufficient for a day

$$= P(X > 35000)$$

$$= P(Y + 20{,}000 > 35000)$$

$$= P(Y > 35000 - 20000)$$

$$= P(Y > 15{,}000)$$

$$= \int_{15000}^{\infty} f(y)\, dy$$

$$= \frac{1}{3000} \int_{15000}^{\infty} e^{-y/3000}\, dy$$

$$= \frac{1}{3000} \left[\frac{e^{-y/3000}}{-1/3000} \right]_{15,000}^{\infty}$$

$$= \frac{1}{\cancel{3000}} \times \frac{\cancel{3000}}{-1} \left[0 - e^{-5} \right]$$

$$= -(-e^{-5})$$

$$= e^{-5}$$

$\therefore$ The probability for Insufficient stock for 2 days $= e^{-5} \cdot e^{-5}$

$$= e^{-10}$$

$$= 0.000045$$

(67)

Eg: The failure of an electronic system is known to follow exponential distribution with mean time to failure of 500 hr. find the probability that the system failure occurs within 300 hrs.

Sol: Given that X is exponential with mean 500 hrs

$$\therefore \frac{1}{\lambda} = 500 \Rightarrow \boxed{\lambda = \frac{1}{500}} \quad \therefore f(x) = \lambda e^{-\lambda x}$$

$$f(x) = \frac{1}{500} e^{-\frac{1}{500} x}, \quad x \geq 0$$

Required $p(X < 300) = 1 - p(X \geq 300)$

We know that $\boxed{p(X > m) = e^{-\lambda m}}$

$$= 1 - e^{-\lambda \cdot 300}$$

$$= 1 - e^{-\frac{300}{500}}$$

$$= 1 - e^{-3/5}$$

$$= 1 - 0.5488$$

$$\boxed{\therefore p(X < 300) = 0.4512}$$

(HW) The length of time a person speaks over phone follows exponential distribution with mean 6 mins. What is the probability that the person will talk for (i) more than 8 mins (ii) Between 4 and 8 mins.

[0.2636, 0.2498]

eg: A Component has an exponential time to failure distribution with mean of 10000 hours. (i) The Components has already been in Operation for its mean life. what is the probability that it will fail by 15,000 hours. (ii) At 15,000 hours the components is still in Operation. what is the probability that it will operate for another 5000 hours?

Sol:- Let X be exponential with mean 10,000 hours.

$$\therefore \frac{1}{\lambda} = 10,000 \Rightarrow \boxed{\lambda = \frac{1}{10,000}}$$

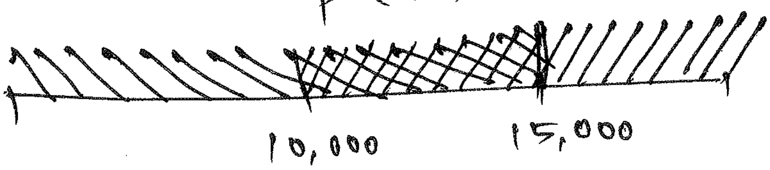
The pdf of x is given by,

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

$$f(x) = \frac{1}{10,000} e^{-\frac{1}{10,000} \cdot x} \longrightarrow (1)$$

(i) $P(X < 15,000 / X > 10,000)$

$$= \frac{P(X < 15000 \cap X > 10,000)}{P(X > 10,000)}$$



$$= \frac{P(10,000 < X < 15,000)}{P(X > 10,000)} \longrightarrow (2)$$

(68)

Now,

$$p(10{,}000 < X < 15{,}000) = \int_{10{,}000}^{15{,}000} f(x)\,dx$$

$$= \int_{10{,}000}^{15{,}000} \frac{1}{10{,}000} e^{\frac{-1}{10{,}000}\cdot x}\,dx$$

$$= \frac{1}{\cancel{10{,}000}} \left[\frac{e^{\frac{-x}{10{,}000}}}{\frac{-1}{\cancel{10{,}000}}} \right]_{10{,}000}^{15{,}000}$$

$$= -1 \left[e^{-3/2} - e^{-1} \right]$$

$$= e^{-1} - e^{-3/2}$$

$$= 0.3679 - 0.2231$$

$$= 0.1448.$$

Now, $p(X > 10{,}000) = e^{-\frac{1}{\cancel{10{,}000}}(\cancel{10000})} = e^{-1} = 0.3679.$

since $\boxed{p(X > m) = e^{-\lambda m}}$

from (2)

$$p(X < 15{,}000 / X > 10{,}000) = \frac{0.1448}{0.3679} = 0.3936.$$

(ii) By Memoryless property,

$$p(X > 15{,}000 + 5000 / X > 15{,}000) = p(X > 5000)$$

$$= e^{-\frac{1}{10{,}000}(5000)} = e^{-1/2} = 0.6065.$$

Eg: (3) Let 'X' be a random variable with p.d.f

$$f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Find (i) $P[X>3]$ (ii) M.g.f of 'X.'

Sol: We know that,

$f(x) = \lambda e^{-\lambda x}; x > 0$.

Here $\lambda = 1/3$.

$E(X) = \text{Mean} = \frac{1}{\lambda} = 3$.

$\text{Var}(X) = \frac{1}{\lambda^2} = (3)^2 = 9$.

$$P(X>3) = \int_3^\infty f(x)\,dx$$

$$= \int_3^\infty \frac{1}{3} \cdot e^{-x/3} \cdot dx$$

$$= \frac{1}{3} \cdot \left[\frac{e^{-x/3}}{-1/3}\right]_3^\infty$$

$$= -(0 - e^{-1})$$

$$\boxed{\therefore P[X>3] = e^{-1} = 1/e}$$

M.g.f is $M_X(t) = \frac{\lambda}{\lambda - t}$

$$= \frac{1/3}{1/3 - t} = \frac{1}{3} \times \frac{3}{1-3t}$$

$$= \frac{1}{1-3t}.$$

Eg: (4) State and prove the Memoryless property of the exponential Distribution.

Statement:

If X is exponentially distributed, then

$P[X > s+t / X > s] = P[X > t]$, for any $s, t > 0$.

proof:

We have, $P[X > k] = \int_k^\infty \lambda e^{-\lambda x} dx$

$$= \lambda \left(-\frac{e^{-\lambda x}}{\lambda}\right)_k^\infty$$

$$= 0 + e^{-\lambda k}$$

$$\boxed{\therefore P(X > k) = e^{-\lambda k}}$$

$$P[X>s+t / X>s] = \frac{P[X>s+t \,\&\, X>s]}{P[X>s]}$$

$$= \frac{P[X > s+t]}{P[X>s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{\cancel{e^{-\lambda s}} \cdot e^{-\lambda t}}{\cancel{e^{-\lambda s}}}$$

$$= e^{-\lambda t}$$

$$= P[X > t]$$

$\therefore P[X>s+t / X>s] = P[X>t]$.

Hence proved

eg: (5) The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ (a) what is probability that the repair time exceeds 2 hrs? (b) what is the conditional probability that a repair takes at 11 hrs given that its duration exceeds 8 hrs?

Sol:: Let 'x' be the R.V which represents the time to repair the machine. then the density function of x is given by.

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-\frac{1}{2} x}, \quad x > 0.$$

(a) Here $t = 2$.

$$p(X > 2) = e^{-\frac{1}{2} \times 2} \qquad \left[\because p[X > k] = e^{-\lambda k} \right]$$

$$= e^{-1}$$

(b) $$p(X > 11 / X > 8) = p(X > 8+3 / X > 8)$$

$$= p(X > 3).$$

$$= e^{-\frac{1}{2} \times 3}$$

$$= e^{-1.5}$$

---

eg: (6) If 'X' is R.V which follows an exponential distribution with parameter λ with $p(X \leq 1) = p(X > 1)$, find Var(X).

Sol:: Since 'X' follows an exponential distribution with parameter λ,

we have, $$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Given: $p(X \le 1) = p(X > 1)$

$1 - p(X > 1) = p(X > 1)$

$1 = p(X > 1) + p(X > 1)$

$2\,p(X > 1) = 1$

$$\boxed{p(X > 1) = \tfrac{1}{2}} \longrightarrow (1)$$

Since 'X' is an exponential distribution, we have

$$p(X > 1) = e^{-\lambda} \longrightarrow (2)$$

$[\because p(X > k) = e^{-\lambda k}]$

from (1) & (2),

$e^{-\lambda} = \tfrac{1}{2}$

$\frac{1}{e^{\lambda}} = \tfrac{1}{2} \Rightarrow e^{\lambda} = 2$

$$\boxed{\lambda = \log_e 2}$$

$\therefore \text{var}(X) = \frac{1}{\lambda^2}$

$$\boxed{\text{var}(X) = \frac{1}{(\log_e 2)^2}}$$

eg: (7) The mileage which car owners get with certain kind of radial tyre is a R.V having an exponential distribution with mean 4000 km. Find the probabilities that one of these tyres will last (i) at least 2000 km (ii) atmost 3000 km.

Sol: Let 'X' be the R.V which denote the mileage obtained with the tyre.

Here $\lambda = \frac{1}{4000}$ then

$$f(x) = \frac{1}{4000} e^{\frac{-x}{4000}}, \; x > 0.$$

(i) $p(X > 2000) = e^{-\frac{1}{4000} \times 2000}$

$= e^{-1/2}$

$= e^{-0.5} = 0.6065$

(ii) $p(X \le 3000) = 1 - p(X > 3000)$

$= 1 - e^{-\frac{1}{4000} \times 3000}$

$= 1 - e^{-0.75}$

$= 0.5276.$

GAMMA DISTRIBUTION.

Def: The continuous random variable 'X' is said to follow a Gamma distribution with parameter 'λ' if its probability function is given by,

$$f(x) = \begin{cases} \dfrac{e^{-x} \cdot x^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0,\ 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

and $\Gamma(\lambda) = \int_0^{\infty} e^{-x} x^{\lambda-1}\, dx$

NOTE: A continuous random variable 'X' whose probability density function is,

$$f(x) = \frac{\lambda^{k} \cdot e^{-\lambda x} \cdot x^{\lambda-1}}{\Gamma(k)}, \lambda > 0, k > 0, 0 < x < \infty.$$

is called a Gamma distribution with two parameter 'λ' and 'k'.

Eg ①: Find the moment generating function of Gamma distribution.

Sol: we have,

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{tx} \cdot \frac{e^{-x} \cdot x^{\lambda-1}}{\Gamma(\lambda)} \cdot dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{tx} \cdot e^{-x}\, x^{\lambda-1}\, dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-(1-t)x} \cdot x^{\lambda-1}\, dx$$

put $(1-t)x = u$ | when $x = 0 \Rightarrow \boxed{u=0}$

$(1-t)\, dx = du$ | $x = \infty \Rightarrow \boxed{u=\infty}$

(71)

$$= \frac{1}{\Gamma\lambda}\int_0^\infty e^{-u}\left(\frac{u}{1-t}\right)^{\lambda-1}\left(\frac{du}{1-t}\right)$$

$$= \frac{1}{\Gamma\lambda}\int_0^\infty \frac{u^{\lambda-1}e^{-u}}{(1-t)^{\lambda-1}(1-t)}\,du$$

$$= \frac{1}{\Gamma\lambda}\int_0^\infty \frac{u^{\lambda-1}e^{-u}}{(1-t)^{\lambda}\cancel{(1-t)^{-1}}\cancel{(1-t)}}\,du$$

$$= \frac{1}{(1-t)^\lambda}\cdot\frac{1}{\Gamma\lambda}\int_0^\infty u^{\lambda-1}e^{-u}\,du$$

$$= \frac{1}{\cancel{\Gamma\lambda}\,(1-t)^\lambda}\cdot\cancel{\Gamma\lambda}$$

$$\boxed{M_X(t) = \frac{1}{(1-t)^\lambda}} \quad ; |t| < 1.$$

---

eg: ②. find the Mean and variance of Gamma distribution.

Sol: we have,

$$M_X(t) = (1-t)^{-\lambda}$$

$$M'_X(t) = -\lambda(1-t)^{-\lambda-1}(-1)$$

$$= \lambda(1-t)^{-\lambda-1}$$

$$\boxed{\mu'_1 = M'_X(0) = \lambda}$$

$$M''_X(t) = \lambda(-\lambda-1)(1-t)^{-\lambda-2}(-1)$$

$$\mu'_2 = M''_X(0) = \lambda(\lambda+1)$$

$$Var = \mu_2 = \mu'_2 - \mu'^2_1$$

$$= \lambda(\lambda+1) - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\boxed{Var = \lambda}$$

For a Gamma random variable X with parameters (k, λ) derive the moment generating function and hence obtain its mean and variance. (72)

Sol: Let X be a gamma RV with parameter (k, λ)

then $f(x) = \dfrac{\lambda^k \cdot x^{k-1} e^{-\lambda x}}{\Gamma k}$, $k = 1, 2, \dots$, $x \geq 0$.

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \frac{\lambda^k x^{k-1} \cdot e^{-\lambda x}}{\Gamma k} \cdot dx$$

$$= \frac{\lambda^k}{\Gamma k} \int_0^{\infty} x^{k-1} e^{-(\lambda - t)x} dx$$

W.K.T $\displaystyle\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

$$= \frac{\lambda^k}{\cancel{(k-1)}!} \frac{\cancel{(k-1)}!}{(\lambda - t)^{k \cancel{-1} \cancel{+1}}} \quad \left(\because \Gamma n = (n-1)!\right)$$

$$= \frac{\lambda^k}{(\lambda - t)^k}$$

$$\boxed{M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^k} \longrightarrow (1)$$

$$M_x(t) = \lambda^k \cdot (\lambda - t)^{-k}$$

Diff w.r.to 't'

$$M'_x(t) = \lambda^k \left(-k(\lambda - t)^{-k-1}\right)(-1)$$

$$M'_x(t) = \lambda^k \cdot k\,(\lambda - t)^{-k-1} \longrightarrow (2)$$

Diff again w.r.to 't'

$$M''_x(t) = \lambda^k \cdot k\,(-k-1)(\lambda - t)^{-k-2} \cdot (-1)$$

$$M''_x(t) = k(k+1) \cdot \lambda^k \cdot (\lambda - t)^{-k-2} \longrightarrow (3)$$

put $t=0$ in eq (2)

$$M'_x(0) = \lambda^k \cdot k\,\lambda^{-k-1}$$

$$E(x) = \cancel{\lambda^k} \cdot k \cdot \cancel{\lambda^{-k}} \cdot \lambda^{-1}$$

$$\boxed{E(X) = \frac{k}{\lambda}}$$

put $t=0$ in eq (3)

$$M''_x(0) = k(k+1) \cdot \cancel{\lambda^k}\, \cancel{\lambda^{-k}}{}^{-2}$$

$$\boxed{E(x^2) = \frac{k(k+1)}{\lambda^2}}$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{k^2+k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{\cancel{k^2}+k-\cancel{k^2}}{\lambda^2}$$

$$\boxed{\text{Var}(X) = \frac{k}{\lambda^2}}$$

eg: In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be considered as a (random variable) central gamma distribution with $\lambda = \frac{1}{2}$, $k = 3$. If the power plant has a daily capacity of 12 million kilowatt hours. What is the probability that the power supply will be inadequate on any given day. 73

Sol: Given X follows central gamma distribution with $\lambda = \frac{1}{2}$ and $k = 3$.

The prob density function of x is given by.

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{k-1}}{\Gamma k}, \quad x \geq 0.$$

$$f(x) = \frac{\frac{1}{2} e^{-x/2} \left(\frac{x}{2}\right)^2}{\Gamma 3} \longrightarrow (1)$$

$$\Gamma n = (n-1)!$$

$$= \frac{\frac{1}{2} e^{-x/2} \left(\frac{x^2}{4}\right)}{2}$$

$$\boxed{f(x) = \frac{1}{16} x^2 e^{-x/2}}, \quad x \geq 0$$

prob for Insufficient supply $= P(X > 12)$

$$= \int_{12}^{\infty} f(x)\, dx$$

$$= \frac{1}{16}\int_{12}^{\infty} \underbrace{x^2}_{u} \underbrace{e^{-x/2}\,dx}_{dv}$$

$$= \frac{1}{16}\left\{ x^2 \left[\frac{e^{-x/2}}{-1/2}\right] - (2x)\left[\frac{e^{-x/2}}{1/4}\right] + (2)\left[\frac{e^{-x/2}}{-1/8}\right]\right\}_{12}^{\infty}$$

$$= \frac{1}{16}\left[-2x^2 e^{-x/2} - 8x e^{-x/2} - 16 e^{-x/2}\right]_{12}^{\infty}$$

$$= \frac{1}{16}\left\{(0-0-0) - (-288 e^{-6} - 96 e^{-6} - 16 e^{-6})\right\}$$

$$= \frac{1}{16}\left[288 e^{-6} + 96 e^{-6} + 16 e^{-6}\right]$$

$$= \frac{1}{\cancel{16}}\left(\cancel{400}\, e^{-6}\right)$$

$$= 25 e^{-6}$$

$$= 25(0.002479)$$

$$= 0.061969$$

eg: The daily consumption of milk in a city in excess of 20,000 litres is approximately distributed as a gamma distribution with parameter $\lambda = \frac{1}{10000}$, $k = 2$. The city has a daily stock of 30,000 litres. what is the probability that the stock is Insufficient on a particular day.

Sol: Let X - daily consumption of milk.

Let $Y = X - 20000$, the excess, follows a gamma distribution with parameter $\lambda = \frac{1}{10000}$, $k=2$

$\therefore$ The pdf of Y is given by,

$$f(y) = \frac{\lambda e^{-\lambda y} (\lambda y)^{k-1}}{\Gamma k}, \quad y \geq 0.$$

$$f(y) = \frac{\frac{1}{10000} e^{-y/10000} \left(\frac{1}{10000} y\right)^{1}}{\Gamma 2} \qquad \Gamma 2 = 1$$

$$f(y) = \frac{1}{10000} e^{-y/1000} \left(\frac{y}{10000}\right), \quad y \geq 0.$$

The prob of Insufficient stock

$= P(X > 30,000)$

put $X = Y + 20,000$

$$= P(y + 20{,}000 > 30000)$$

$$= P(y > 30{,}000 - 20{,}000)$$

$$= P(y > 10{,}000)$$

$$= \int_{10}^{\infty} f(y)\,dy$$

$$= \int_{10000}^{\infty} \frac{1}{10{,}000} \cdot \left(\frac{y}{10{,}000} e^{-y/10{,}000} \right) dy$$

$$= \frac{1}{(10{,}000)^2} \int_{10000}^{\infty} \underset{u}{y}\ \underbrace{e^{-y/10{,}000}\, dy}_{dv}$$

$$= \frac{1}{(10{,}000)^2} \left\{ y \left[\frac{e^{-y/10{,}000}}{\frac{-1}{10{,}000}} \right] - (1) \left[\frac{e^{-y/10{,}000}}{\frac{1}{(10{,}000)^2}} \right] \right\}_{10000}^{\infty}$$

$$= \frac{1}{(10{,}000)^2} \left\{ -10{,}000y\, e^{-y/10{,}000} - (10{,}000)^2 e^{-\frac{y}{10{,}000}} \right\}_{10000}^{\infty}$$

$$= \frac{1}{(10{,}000)^2} \left\{ (0 - 0) - \left(-(10000)^2 e^{-1} - (10000)^2 e^{-1} \right) \right\}$$

$$= \frac{1}{(10{,}000)^2} \left[(10000)^2 e^{-1} + (10000)^2 e^{-1} \right]$$

$$= \frac{1}{\cancel{(10{,}000)^2}}\ 2\, \cancel{(10000)^2} e^{-1} = 2e^{-1} = 0.735759$$

NORMAL DISTRIBUTION.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$
$$-\infty < \mu < \infty, \ \sigma > 0.$$

μ - Mean = Mode = Median

σ - standard deviation

---

eg: find the MGF of Normal distribution and hence find its mean and variance

Sol: MGF $M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . dx$$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} . dx$$

put $z = \frac{x-\mu}{\sigma}$

$\sigma z = x - \mu$

$x = \mu + \sigma z$

$\boxed{dx = \sigma dz}$

x	$-\infty$	∞
z	$-\infty$	∞

$$M_X(t) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z) - \frac{z^2}{2}} . \cancel{\sigma} dz$$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu + t\sigma z - \frac{z^2}{2}} . dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} . e^{-\frac{z^2}{2} + \sigma z t} . dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z t)} . dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-1/2 [z^2 - 2\sigma z t + \sigma^2 t^2 - \sigma^2 t^2]} . dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-1/2 [(z - \sigma t)^2 - \sigma^2 t^2]} . dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-1/2 (z - \sigma t)^2 + \frac{\sigma^2 t^2}{2}} . dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-1/2 (z - \sigma t)^2} . e^{\frac{\sigma^2 t^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} . e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 (z - \sigma t)^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 (z - \sigma t)^2} dz$$

put $z - \sigma t = u$

$dz = du$

z	$-\infty$	∞
u	$-\infty$	∞

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \cdot u^2} du$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du$$

W.K.T $\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$

$$M_X(t) = \frac{1}{\cancel{\sqrt{2\pi}}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \cancel{\sqrt{2\pi}}$$

$$\boxed{M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}} \longrightarrow (1)$$

<u>To find:</u> Mean and variance

Diff eq (1) w.r.to 't'.

$$M'_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\mu + \frac{\sigma^2 \cancel{2} t}{\cancel{2}} \right)$$

$$M'_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\mu + \sigma^2 t \right) \longrightarrow (2)$$

put $t = 0$

$$E(X) = M'_X(0) = e^{0+0} (\mu + 0) = 1(\mu) = \mu$$

$$\boxed{\therefore \text{Mean} = E(X) = \mu}$$

Diff eq ② w. r. to "t"

$$M''_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\sigma^2) + (\mu + \sigma^2 t) . e^{\mu t + \frac{\sigma^2 t^2}{2}} . (\mu + \sigma^2 t)$$

$$E(X^2) = M''_X(0) = e^0 \sigma^2 + (\mu + 0) e^0 (\mu + 0)$$

$$E(X^2) = \sigma^2 + \mu^2$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= \sigma^2 + \cancel{\mu^2} - \cancel{\mu^2}$$

$$\boxed{Var(X) = \sigma^2}$$

NOTE:-

For a Standard Normal distribution,

Mean $= \mu = 0$ and $Var = \sigma^2 = 1$.

The pdf is given by,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty$$

where $z = \dfrac{x - \mu}{\sigma}$.

eg: The peak temperature T, as measured in degrees Fahrenheit on a particular day is the Gaussian (85, 10) random variable. What is $p(T > 100)$, $p(T < 60)$ and $p(70 \le T \le 100)$.

<u>Sol:</u> Given: $N(85, 10)$ $\quad\quad X \sim N(\mu, \sigma)$

Here $\mu = 85$, $\sigma = 10$

put $Z = \dfrac{X - \mu}{\sigma}$

$$Z = \frac{X - 85}{10}$$

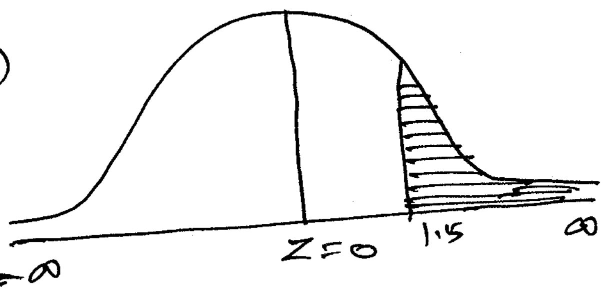
(i) $p(T > 100) = p(X > 100)$

When $X = 100 \Rightarrow Z = \dfrac{100 - 85}{10} = \dfrac{15}{10} = 1.5$

$p(X > 100) = p(z > 1.5)$

$= p(0 < z < \infty) - p(0 < z < 1.5)$

$= 0.5 - 0.4332$

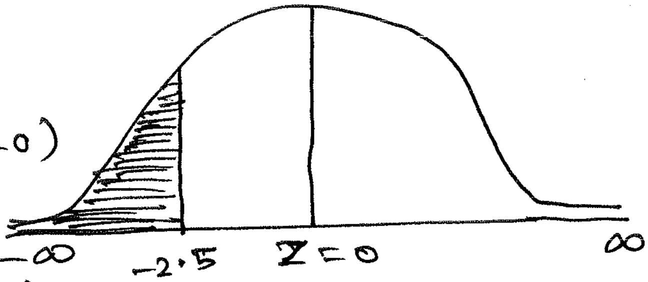


$\boxed{\therefore p(T > 100) = 0.0668}$

(ii) $p(T < 60) = p(X < 60)$

When $X = 60 \Rightarrow z = \dfrac{60 - 85}{10} = \dfrac{-25}{10} = -2.5$

$p(x < 60) = p(z < -2.5)$



$= p(-\infty < z < 0) - p(-2.5 < z < 0)$

$= p(0 < z < \infty) - p(0 < z < 2.5)$

$= 0.5 - 0.4938$

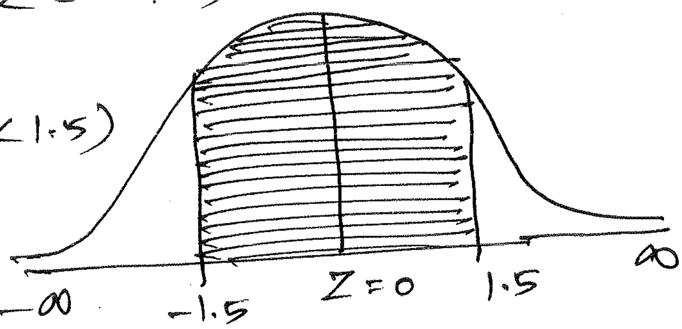
$$\boxed{p(x < 60) = 0.0062.}$$

(iii) $p(70 \le T \le 100) = p(70 \le x \le 100)$

When $x = 70 \Rightarrow z = \frac{70-85}{10} = \frac{-15}{10} = -1.5$

When $x = 100 \Rightarrow z = \frac{100-85}{10} = \frac{15}{10} = 1.5$

$= p(-1.5 < z < 0) + p(0 < z < 1.5)$



$= p(0 < z < 1.5) + p(0 < z < 1.5)$

$= 2\, p(0 < z < 1.5)$

$= 2\,(0.4332)$

$= 0.8664$

$$\boxed{\therefore\ p(70 \le T \le 100) = 0.8664}$$

Eg: The annual rainfall in inches in a certain region has a normal distribution with mean of 40 and variance 16. What is the probability that the rainfall in a given year is between 30 and 48 inches?

Sol: Given: Mean $= \mu = 40$, Var $= \sigma^2 = 16$. $\boxed{\therefore \sigma = 4}$

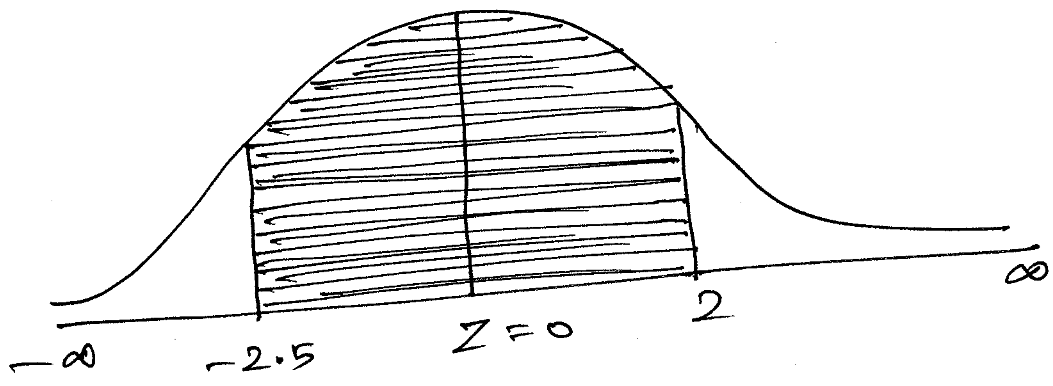
$$Z = \frac{X-\mu}{\sigma} = \frac{X-40}{4} \rightarrow (1)$$

To find: $P(30 < X < 48)$.

When $X = 30 \Rightarrow Z = \frac{30-40}{4} = \frac{-10}{4} = -2.5$

When $X = 48 \Rightarrow Z = \frac{48-40}{4} = \frac{8}{4} = 2$

$$P(30 < X < 48) = P(-2.5 < Z < 2)$$



$= P(-2.5 < Z < 0) + P(0 < Z < 2)$

$= P(0 < Z < 2.5) + P(0 < Z < 2)$

$= 0.4938 + 0.4773$

$$\boxed{P(30 < X < 48) = 0.9711}$$

eg: In a normal population with mean 15 and Standard deviation 3.5, it is found that 647 observations exceeds 16.25. what is the total number of observations in the population.

Sol: Given: Mean $= \mu = 15$, $S.D = \sigma = 3.5$

and $n = 647$

Let 'N' be the total number of observation in the population.

Given that $p(X > 16.25) \times N = 647.$

$$\therefore N = \frac{647}{p(X > 16.25)} \longrightarrow (1)$$

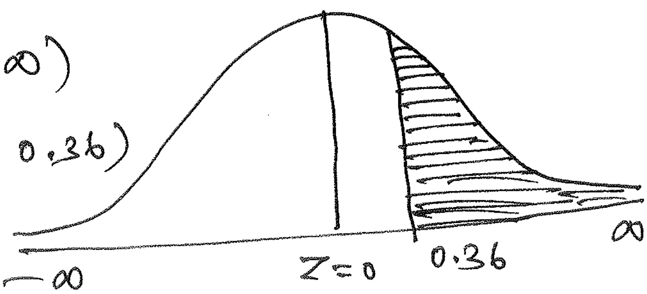
$$Z = \frac{X-\mu}{\sigma} = \frac{X-15}{3.5}$$

When $X = 16.25 \Rightarrow Z = \frac{16.25-15}{3.5} = 0.3571$

$$\boxed{Z = 0.36}$$

$$p(X > 16.25) = p(z > 0.36)$$

$$p(z > 0.36) = p(0 < z < \infty) - p(0 < z < 0.36)$$



$= 0.5 - 0.1406$

$$\boxed{\therefore p(X > 16.25) = 0.3594}$$

from ①

$$N = \frac{647}{0.3594}$$

$$= 1800.22$$

$$\boxed{\therefore N = 1800}$$

eg: The average percentage of marks of candidates in an examination is 42 with a standard deviation of 10. If the minimum mark for pass is 50% and 1000 candidates appear for the examination, how many candidates can be expected to get the pass mark if the marks follows normal distribution. If it is required, that double the number of the candidates should pass, what should be the minimum mark for pass?

Sol: Let X - denote the marks of the candidates

Given: Mean = $\mu = 42$ and S.D = $\sigma = 10$.

$$Z = \frac{X-\mu}{\sigma} = \frac{X-42}{10} \longrightarrow ①$$

Since the minimum pass mark is 50%.

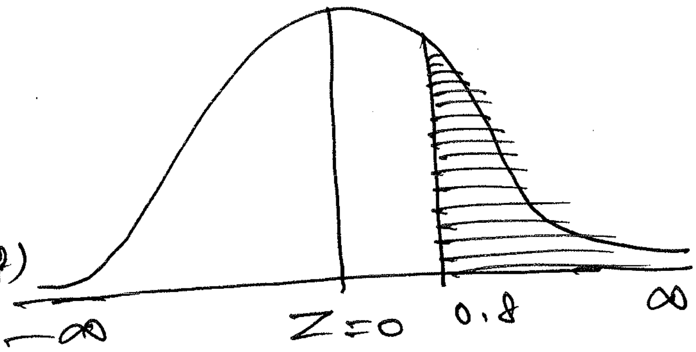
$$P(X \geq 50) = P(Z \geq 0.8)$$

When $X = 50 \Rightarrow Z = \frac{50-42}{10} = \frac{8}{10} = 0.8$

$p(z \geq 0.8)$

$= p(0 \leq z \leq \infty)$

$\quad - p(0 \leq z \leq 0.8)$



$= 0.5 - 0.2881$

$\therefore p(x \geq 50) = p(z \geq 0.8) = 0.2119.$

$\therefore$ Out of 1000 students,

Number of students gets pass mark $= 1000 \times 0.2119$

≈ 211.9

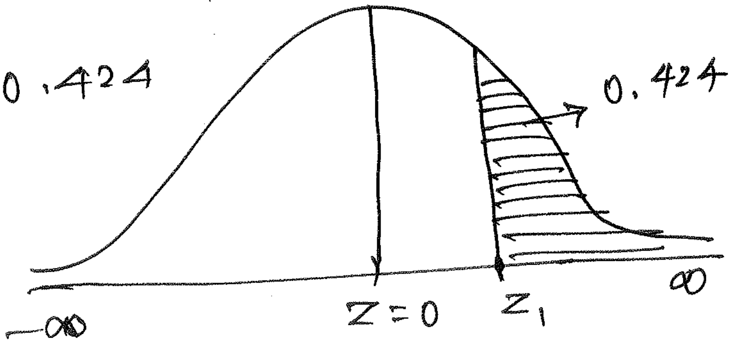
$= 212$ (app)

If double the number of candidates should pass, then the number of pass should be $212 \times 2 = 424.$

To find: Z_1 such that $p[z \geq z_1] = 0.424$

$p(0 < z < z_1) = 0.5 - 0.424$

$= 0.076$



From the table

$\boxed{z_1 = 0.19}$

from eq (1)

$$Z_1 = \frac{X_1 - 42}{10}$$

$$0.19 = \frac{X_1 - 42}{10}$$

$$0.19 \times 10 = X_1 - 42$$

$$1.9 = X_1 - 42$$

$$X_1 = 42 + 1.9$$

$$X_1 = 43.9$$

$$\boxed{X_1 \approx 44 \text{ (app)}}$$

$\therefore$ The Minimum mark for pass is 44.

PROBLEMS BASED ON NORMAL DISTRIBUTIONS. (87)

Eg ①: An electrical firm manufactures light bulbs that have a life before burn-out, that is normally distributed with mean equal to 800 hrs and S.D of 40 hrs. Find,

(i) The probability that a bulb burns more than 834 hrs.

(ii) The probability that a bulb burns between 778 and 834 hrs.

Sol: Let X be the RV denoting the life time of a light bulb.

Given, $\mu = 800$ hrs ; $\sigma = 40$ hrs.

$$Z = \frac{X-\mu}{\sigma} = \frac{X-800}{40} \longrightarrow ①$$

(i) p(a bulb burns more than 834 hrs) = $p(X > 834)$

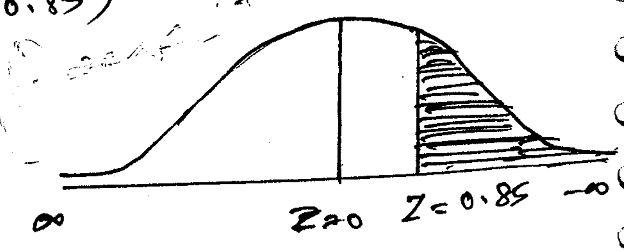
When $X = 834$, $Z = \frac{834-800}{40} = 0.85.$

$$p(X > 834) = p(Z > 0.85).$$

$$= 1 - p(Z \leq 0.85)$$

$$= 1 - 0.3023$$

$$= 0.6977$$



(ii) To find: $p(778 < X < 834).$

When $X = 778$, $Z = \frac{778-800}{40} = -0.55$

$X = 834$, $Z = \frac{834-800}{40} = 0.85.$

$\therefore\ p(778 < X < 834) = p(-0.55 < Z < 0.85)$

$= p(-0.55 < Z < 0) + p(0 < Z < 0.85)$

$= p(0 < Z < 0.55) + p(0 < Z < 0.85)$

$= 0.2088 + 0.3023$

$= 0.5111.$

---

eg: (2) A normal distribution has mean $\mu = 20$ & S.D = 10 find $p(15 \leq X \leq 40)$.

Sol: Given: $\mu = 20$ & $\sigma = 10$.

We have, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-20}{10} \longrightarrow (1)$

from (1)

when $X = 15 \Rightarrow Z = \dfrac{15-20}{10} = -0.5$

$X = 40 \Rightarrow Z = \dfrac{40-20}{10} = 2.$

$p(15 \leq X \leq 40) = p(-0.5 \leq Z \leq 2).$

$= p(-0.5 \leq Z \leq 0) + p(0 \leq Z \leq 2).$

$= p(0 \leq Z \leq 0.5) + p(0 \leq Z \leq 2)$

$= 0.1915 + 0.4772$

$= 0.6687.$

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eg: (3) The weekly wages of 1000 workmen are normally distributed around a mean of Rs 70 with a S.D of Rs 5. Estimate the number of workers whose weekly wages will be

(i) between Rs 69 and Rs 72. (ii) less than Rs 69. (82)

(iii) More than 72.

<u>Sol</u>: Let X be the RV denoting the weekly wages of a worker.

<u>Given</u>: $\mu = 70$ & $\sigma = 5$.

We have, $Z = \frac{X-\mu}{\sigma} = \frac{X-70}{5}$

(i) $p(69 < x < 72)$.

when $x = 69 \Rightarrow Z = \frac{69-70}{5} = -0.2$

$x = 72 \Rightarrow Z = \frac{72-70}{5} = 0.4$

$\therefore p(69 < x < 72) = P(-0.2 < z < 0.4)$

$= P(-0.2 < z < 0) + p(0 < z < 0.4)$

$= p(0 < z < 0.2) + p(0 < z < 0.4)$

$= 0.0793 + 0.1554$

$= 0.2347.$

Out of 1000 workmen, the number of workers whose wages lies between Rs 69 & Rs 72.

$= 1000 \times p(69 < x < 72)$

$= 1000 \times 0.2347$

$= 234.7$

$= 235.$

(ii) p(less than Rs 69) $= p(x < 69)$.

when $X = 69 \Rightarrow z = \frac{69-70}{5} = -0.2$

$\therefore p(X < 69) = p(z < -0.2)$

$= p(-\infty < z < 0) + p(0 < z < 0.2)$

$= p(0 < z < \infty) - p(0 < z < 0.2)$

$= 0.5 - 0.0793$

$= 0.4207$

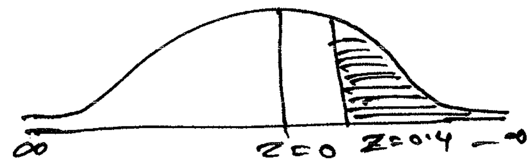
Out of 1000, $= 1000 \times 0.4207$

$= 420.7 = 421.$

(iii) p(More than Rs 72) $= p(X > 72)$.

When $X = 72 \Rightarrow z = \frac{X-\mu}{\sigma} = \frac{72-70}{5} = 0.4$

$\therefore p(X > 72) = p(z > 0.4)$



$= p(0 < z < \infty) - p(0 < z < 0.4)$

$= 0.5 - 0.1554$

$= 0.3446.$

$\therefore$ Out of 1000, $= 1000 \times 0.3446$

$= 344.6$

$\simeq 345.$

Eg: ④ A manufacturer produces air mail envelopes whose weight is normal with mean $\mu = 1.950$gm and SD $\sigma = 0.025$gm. The envelopes are sold in lots of 1000. How many envelopes in a lot may be heavier than 2 grams.

Sol: Let 'X' be the R.V which denotes the weight of an envelope.

Given: $\mu = 1.950$ & $\sigma = 0.025.$

p(the envelopes heavier than 2) = p(X > 2) (83)

$$Z = \frac{X-\mu}{\sigma} = \frac{X-1.950}{0.025}$$

when $X = 2 \Rightarrow \frac{2-1.950}{0.025} = 2.$

$$\therefore p(X>2) = p(Z>2)$$

$$= p(0<Z<\infty) - p(0<Z<2)$$

$$= 0.5 - 0.4772$$

$$= 0.0228$$

Out of 1000, the number of envelopes heavier than 2,

$$= 1000 \times p(X>2)$$

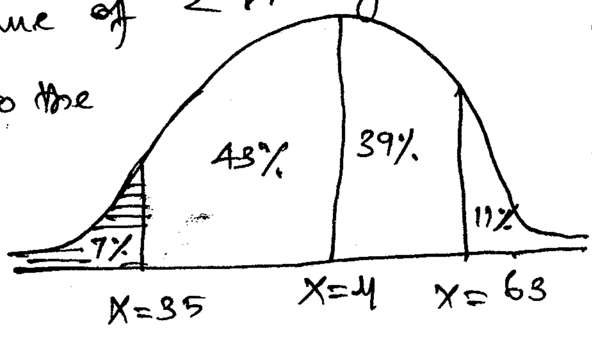
$$= 1000 \times 0.0228 = 22.8.$$

$$= 23 \text{ (app)}.$$

Eg ⑤: In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution.

Sol: Let μ and σ be the Mean and S.D of the normal distribution.

The area lying to the left of the ordinate at $x = 35$ is 0.07. The corresponding value of z is negative



The value of z corresponding to the area 0.43 is 1.4757, 1.48

We have, $z = \frac{x-\mu}{\sigma}$

When $X = 35$,

$$-1.4757 = \frac{35-\mu}{\sigma}$$

$$\frac{\mu-35}{\sigma} = 1.4757$$

$$\mu - 35 = 1.4757\sigma$$

$$\mu - 1.4757\sigma = 35 \rightarrow (1)$$

The corresponding value of z to the area 0.39 is 1.2263

When $x = 63$,

$$\frac{63-\mu}{\sigma} = 1.2263$$

$$63 - \mu = 1.2263\sigma$$

$$\mu + 1.2263\sigma = 63 \rightarrow (2)$$

Solving (1) & (2) we get,

Mean $\mu = 50.288$

S.D $\sigma = 10.36$.

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eg: (6) If X is a normal variate with mean 1 and variance 4, Y is another normal variate independent of X with mean 2 and variance 3. What is the distribution of $X + 2Y$.

Sol: Given X and Y are independent normal variates.

$\therefore$ $X + 2Y$ is also a normal variate by additive property.

$$\therefore \text{Mean of } (X+2Y) = E(X+2Y)$$

$$= E(X) + E(2Y)$$

$$= E(X) + 2E(Y)$$

$$= 1 + 2(2)$$

$$= 1 + 4$$

$$\boxed{\text{Mean} = 5}$$

$$\text{Var}(X+2Y) = \text{Var}(X) + \text{Var}(2Y)$$

$$= \text{Var}(X) + 4\,\text{Var}(Y)$$

$$= 4 + (4 \times 3) = 16$$

$\therefore$ $X + 2Y$ follows normal distribution with mean 5 and variance 16.
