MA 8451 PROBABILITY AND RANDOM PROCESSES

UNIT II - TWO DIMENSIONAL RANDOM VARIABLES

Basic formulae:

1. $\sum\sum p_{ij} = 1$ (Discrete Random variable)

   $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ (Continuous Random variable)

2. Conditional probability function X Given Y, $P(X = x / Y = y) = \dfrac{P(x,y)}{P_{*y}}$

   Conditional probability function Y given X $P(Y = y / X = x) = \dfrac{P(x,y)}{P_{x*}}$

3. Conditional density function of X given Y, $f\left(x/y\right) = \dfrac{f(x,y)}{f_Y(y)}$

   Conditional density function of Y given X, $f\left(y/x\right) = \dfrac{f(x,y)}{f_X(x)}$

4. If X and Y are independent random variables then

   $f(x,y) = f_X(x) f_Y(y)$ (for continuous random variable)

   $P(X = x, Y = y) = P(X = x).P(Y = y) = P_{x*}P_{*y}$ (for discrete random variable)

5. $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

6. Marginal density function of X, $f_X(x) = \int f(x,y)dy$

   Marginal density function of Y , $f_Y(y) = \int f(x,y)dx$

7. Correlation coefficient $\rho_{xy} = \dfrac{\text{cov}(x,y)}{\sigma_x \sigma_y}$

   $\rho_{xy} = \dfrac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2 \sum(y-\bar{y})^2}}$ (for discrete case)

8. $COV(X,Y) = E(XY) - E(X)E(Y)$

9. If X and Y are independent random variables, COV(X,Y) = 0

10. $E(X) = \int x\, f_X(x) dx$,

$E(Y) = \int y\, f_Y(y) dy$,

$E(XY) = \int \int xy\, f(x,y) dx\, dy$

11. Regression for Discrete Random variable:

Regression line X on Y is $(x - \bar{x}) = b_{xy}(y - \bar{y})$

Regression line Y on X is $(y - \bar{y}) = b_{yx}(x - \bar{x})$

where the regression coefficients

$$b_{xy} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (y-\bar{y})^2}}, \quad b_{yx} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}},$$

12. Correlation coefficient $\rho_{xy} = b_{xy} \,.\, b_{yx}$

13. Transformation of random variable:

One dimensional random variable $f(y) = \dfrac{f(x)}{\left| dy/dx \right|}$

Two dimensional random variable $g(u,v) = |J|\, f(x,y)$ *where* $J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

Central Limit theorem (CLT)

If $X_1, X_2, X_3, ..., X_n, ...$ be a sequence of independent identically distributed random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2, i = 1, 2, ...,$ and if $S_n = X_1 + X_2 + X_3 + ... + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity

Applications of CLT:

(i) Central limit theorem provides a simple method for computing approximate probabilities of sums of independent random variables.

(ii) It also gives us the wonderful fact that the empirical frequencies of so many natural "populations" exhibit a bell shaped curve.

PART – A

1. Define joint probability density function of two random variables X and Y.

If (X,Y) is a two dimensional continuous random variable such that $P\left[x-\frac{dx}{2}\le X\le x+\frac{dx}{2}\ ,\ y-\frac{dy}{2}\le Y\le y+\frac{dy}{2}\right]=f(x,y)dx\,dy$, then $f(x,y)$ is called the joint pdf of (X,Y), provided $f(x,y)$ satisfies the following conditions

$(i)\ f(x,y)\ge 0\ for\ all\ (x,y)\in R$

$(ii)\iint_R f(x,y)dx\,dy=1$

2. State the basic properties of joint distribution of (X,Y) where X and Y are random variables.

Properties of joint distribution of (X,Y) are

$(i)\ F[-\infty,y]=0=F[x,-\infty]\,and\ F[-\infty,\infty]=1$

$(ii)\ P[a<X<b,Y\le y]=F(b,y)-F(a,y)$

$(iii)\ P[X\le x,c<Y<d]=F[x,d]-F[x,c]$

$(iv)\ P[a<X<b,c<Y<d]=F[b,d]-F[a,d]-F[b,c]+F[a,c]$

$(v)\ At\ points\ of\ continuity\ of\ f(x,y),\ \frac{\partial^2 F}{\partial x\,\partial y}=f(x,y)$

3. Can the joint distributions of two random variables X and Y be got if their marginal distributions are random?

If the random variables X and Y are independent, then the joint distributions of two random variables can be got if their marginal distributions are known.

4. Let X and Y be two discrete random variable with joint pmf $P[X=x,Y=y]=\begin{cases}\frac{x+2y}{18}, & x=1,2;\ y=1,2\\ 0 & ,otherwise\end{cases}$. Find the marginal pmf of X and $E[X]$.

The joint pmf of (X,Y) is given by

	1	2
1	$\frac{3}{18}$	$\frac{4}{18}$
2	$\frac{5}{18}$	$\frac{6}{18}$

Marginal pmf of X is

$$P[X=1]=\frac{3}{18}+\frac{5}{18}=\frac{8}{18}=\frac{4}{9}$$

$$P[X=2]=\frac{4}{18}+\frac{6}{18}=\frac{10}{18}=\frac{5}{9}$$

$$E[X]=\sum x\,p(x)=(1)\left(\frac{4}{9}\right)+(2)\left(\frac{5}{9}\right)=\frac{4}{9}+\frac{10}{9}=\frac{14}{9}.$$

5. Let X and Y be integer valued random variables with $P[X=m, Y=n]=q^2 p^{m+n-2}$, $n, m=1,2,............$ and $p+q=1$. Are X and Y independent?

The marginal pmf of X is

$$p(x)=\sum_{n=1}^{\infty} q^2 p^{m+n-2}=\sum_{n=1}^{\infty} q^2 p^{m-1} p^{n-1}=q^2 p^{m-1}\sum_{n=1}^{\infty} p^{n-1}$$
$$=q^2 p^{m-1}\left[1+p+p^2+p^3+.........\right]=q^2 p^{m-1}(1-p)^{-1}$$
$$=q^2 p^{m-1} q^{-1}=q\,p^{m-1}$$

The marginal pmf of Y is

$$p(y)=\sum_{m=1}^{\infty} q^2 p^{m+n-2}=\sum_{m=1}^{\infty} q^2 p^{m-1} p^{n-1}=q^2 p^{n-1}\sum_{m=1}^{\infty} p^{m-1}$$
$$=q^2 p^{n-1}\left[1+p+p^2+p^3+.........\right]=q^2 p^{n-1}(1-p)^{-1}$$
$$=q^2 p^{n-1} q^{-1}=q\,p^{n-1}$$

$$p(x)\,p(y)=q\,p^{m-1}.q\,p^{n-1}=q^2\,p^{m+n-2}=P[X=m\ \ Y=n]$$

Therefore X *and* Y are independent random variables.

6. The joint probability density function of the random variable (X, Y) is given by $f(x,y)=k\,x\,y\,e^{-(x^2+y^2)}, x>0, y>0$. Find the value of k.

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)\,dx\,dy=1 \qquad \text{put } x^2=t$$
$$\int_0^{\infty}\int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}\,dx\,dy=1 \qquad 2x\,dx=dt$$
$$k\int_0^{\infty}\int_0^{\infty} x\,y\,e^{-x^2}e^{-y^2}\,dx\,dy=1 \qquad x\,dx=\frac{dt}{2}$$
$$k\int_0^{\infty} y\,e^{-y^2}\left[\int_0^{\infty} x\,e^{-x^2}\,dx\right]dy=1$$

$when\,x=0, t=0\,and\,when\,x=\infty, t=\infty$

$$k\int_0^{\infty} y\,e^{-y^2}\left[\int_0^{\infty} e^{-t}\,\frac{dt}{2}\right]dy=1$$
$$\frac{k}{2}\int_0^{\infty} y\,e^{-y^2}\left(-e^{-t}\right)_0^{\infty} dy=1$$
$$\frac{k}{2}\int_0^{\infty} y\,e^{-y^2}(0+1)\,dy=1 \qquad \text{put } y^2=t$$
$$\frac{k}{2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}=1 \qquad 2y\,dy=dt$$

$$\frac{k}{4}\left(e^{-t}\right)_0^{\infty}=1 \qquad y\,dy=\frac{dt}{2}$$

$$\frac{k}{4}(0+1)=1 \qquad when\ y=0,t=0\ and\ when\ y=\infty,t=\infty$$

$$\frac{k}{4}=1$$

Therefore, the value of k is $k=4$.

7. The joint pdf of the random variable (X,Y) is $f(x,y)=\begin{cases} k(x+y), 0<x<2\ ;0<y<2 \\ 0 \qquad ,otherwise \end{cases}$.

Find the value of k .

Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)dx\,dy=1$$

$$\int_0^2\int_0^2 k(x+y)dx\,dy=1$$

$$k\int_0^2\left[\left(\frac{x^2}{2}\right)_0^2+y(x)_0^2\right]dy=1$$

$$k\int_0^2[(2-0)+y(2-0)]dy=1$$

$$k\int_0^2(2+2y)dy=1$$

$$k\left[2(y)_0^2+2\left(\frac{y^2}{2}\right)_0^2\right]=1$$

$$k[2(2-0)+(4-0)]=1$$

$$8k=1$$

$$k=\frac{1}{8}$$

8. The joint pdf of the random variable (X,Y) is $f(x,y)=\begin{cases} c\,x\,y \qquad ,0<x<2\ ;0<y<2 \\ 0 \qquad ,otherwise \end{cases}$.

Find the value of c .

Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)dx\,dy=1$$

$$\int_0^2\int_0^2 c\,x\,y\,dx\,dy=1$$

$$c\int_0^2\int_0^2 x\,y\,dx\,dy=1$$

$$c\int_0^2 y\left(\frac{x^2}{2}\right)_0^2 dy = 1$$

$$c\int_0^2 y(2-0)dy = 1$$

$$2c\left[\frac{y^2}{2}\right]_0^2 = 1$$

$$c[4-0] = 1 \Rightarrow 4c = 1 \Rightarrow c = \frac{1}{4}$$

Therefore the value of c is $c = \frac{1}{4}$.

9. If two random variables $X \, and \, Y$ have probability density function $f(x,y) = k(2x+y)$ for $0 \le x \le 2 \; and \; 0 \le y \le 3$. Evaluate k.

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)dx\,dy = 1$$

$$\int_0^3\int_0^2 k(2x+y)dx\,dy = 1$$

$$k\int_0^3\left[2\left(\frac{x^2}{2}\right)_0^2 + y(x)_0^2\right]dy = 1$$

$$k\int_0^3 (4+2y)dy = 1$$

$$k\left[4(y)_0^3 + 2\left(\frac{y^2}{2}\right)_0^3\right] = 1$$

$$k[12+9] = 1 \Rightarrow 21k = 1 \qquad \Rightarrow k = \frac{1}{21}$$

10. If the function $f(x,y) = c(1-x)(1-y), 0 < x < 1$, $0 < y < 1$ is to be a density function find the value of c.

Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)dx\,dy = 1$$

$$\int_0^1\int_0^1 c(1-x)(1-y)dx\,dy = 1$$

$$c\int_0^1\int_0^1 (1-x-y+xy)dx\,dy = 1$$

$$c\int_0^1\left[(x)_0^1 - \left(\frac{x^2}{2}\right)_0^1 - y(x)_0^1 + y\left(\frac{x^2}{2}\right)_0^1\right]dy = 1$$

$$c\int_0^1\left[1 - \frac{1}{2} - y + \frac{y}{2}\right]dy = 1$$

$$c\int_0^1\left(\frac{1}{2}-\frac{y}{2}\right)dy=1$$

$$c\left[\frac{1}{2}(y)_0^1-\frac{1}{2}\left(\frac{y^2}{2}\right)_0^1\right]=1 \quad \Rightarrow c\left[\frac{1}{2}-\frac{1}{4}\right]=1 \;\Rightarrow \frac{c}{4}=1 \;\Rightarrow c=4$$

Therefore the value of c is $c=4$

11. Find the marginal density functions of $X \text{ and } Y$ if $f(x,y)=\frac{2}{5}(2x+5y), 0\le x\le 1, 0\le y\le 1$.

Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\frac{2}{5}\int_0^1(2x+5y)\,dy=\frac{2}{5}\left[2x(y)_0^1+5\left(\frac{y^2}{2}\right)_0^1\right]$$

$$=\frac{2}{5}\left[2x+\frac{5}{2}\right]=\frac{4}{5}x+1 \;, \quad 0\le x\le 1$$

Marginal density of Y is

$$f_Y(y)=\int f(x,y)\,dx=\frac{2}{5}\int_0^1(2x+5y)\,dx=\frac{2}{5}\left[2\left(\frac{x^2}{2}\right)_0^1+5y(x)_0^1\right]$$

$$=\frac{2}{5}[1+5y]=\frac{2}{5}+2y \quad , \quad 0\le y\le 1$$

12. If $X \text{ and } Y$ have joint pdf $f(x,y)=\begin{cases} x+y \;; 0<x<1, 0<y<1 \\ 0 \quad ;otherwise\end{cases}$. Check whether $X \text{ and } Y$ are independent.

$$f_X(x)=\int f(x,y)\,dy=\int_0^1(x+y)\,dy = x(y)_0^1+\left(\frac{y^2}{2}\right)_0^1=x+\frac{1}{2}, 0<x<1$$

$$f_Y(y)=\int f(x,y)\,dx=\int_0^1(x+y)\,dx = \left(\frac{x^2}{2}\right)_0^1+y(x)_0^1=y+\frac{1}{2}, 0<y<1$$

$$f_X(x).f_Y(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)=xy+\frac{x}{2}+\frac{y}{2}+\frac{1}{4}\ne x+y\ne f(x,y)$$

Therefore $X \text{ and } Y$ are not independent variables.

13. If $X \text{ and } Y$ are random variables having the joint density function $f(x,y)=\frac{1}{8}(6-x-y), 0<x<2, 2<y<4$, find $P[X+Y<3]$.

$$P[X+Y<3]=\iint f(x,y)\,dx\,dy=\frac{1}{8}\int_2^3\int_0^{3-y}(6-x-y)\,dx\,dy=\frac{1}{8}\int_2^3\left[(6-y)(x)_0^{3-y}-\left(\frac{x^2}{2}\right)_0^{3-y}\right]dy$$

$$=\frac{1}{8}\int_2^3\left[(6-y)(3-y)-\frac{1}{2}(3-y)^2\right]dy=\frac{1}{8}\int_2^3\left[18-9y+y^2-\frac{1}{2}(3-y)^2\right]dy$$

$$=\frac{1}{8}\left[18(y)_2^3-9\left(\frac{y^2}{2}\right)_2^3+\left(\frac{y^3}{3}\right)_2^3-\frac{1}{2}\left[\frac{(3-y)^3}{-3}\right]_2^3\right]$$

$$=\frac{1}{8}\left[18(3-2)-\frac{9}{2}(9-4)+\frac{1}{3}(27-8)+\frac{1}{6}(0-1)\right]$$

$$=\frac{1}{8}\left[18-\frac{45}{2}+\frac{19}{3}-\frac{1}{6}\right]=\frac{5}{24}.$$

14. Let $X \; and \; Y$ be continuous random variable with joint pdf $f_{XY}(x,y)=\frac{3}{2}\left(x^2+y^2\right), 0<x<1,\ 0<y<1$. Find $f_{X/Y}(x/y)$.

$$f_Y(y)=\int f(x,y)dx=\frac{3}{2}\int_0^1\left(x^2+y^2\right)dx=\frac{3}{2}\left[\left(\frac{x^3}{3}\right)_0^1+y^2(x)_0^1\right]=\frac{3}{2}\left[\frac{1}{3}+y^2\right]=\frac{3}{2}y^2+\frac{1}{2}$$

$$f_{X/Y}(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{\frac{3}{2}\left(x^2+y^2\right)}{\frac{3}{2}\left(y^2+\frac{1}{3}\right)}=\frac{x^2+y^2}{y^2+\frac{1}{3}}.$$

15. If the joint pdf of (X,Y) is given by $f(x,y)=2-x-y;\ 0\le x<y\le 1$, find $E[X]$.

$$E[X]=\iint x\, f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^y x[2-x-y]\,dx\,dy$$

$$=\int_0^1\int_0^y\left(2x-x^2-xy\right)dx\,dy=\int_0^1\left[2\left(\frac{x^2}{2}\right)_0^y-\left(\frac{x^3}{3}\right)_0^y-y\left(\frac{x^2}{2}\right)_0^y\right]dy$$

$$=\int_0^1\left(y^2-\frac{y^3}{3}-\frac{y^3}{2}\right)dy=\int_0^1\left(y^2-\frac{5}{6}y^3\right)dy=\left(\frac{y^3}{3}\right)_0^1-\frac{5}{6}\left(\frac{y^4}{4}\right)_0^1$$

$$=\frac{1}{3}-\frac{5}{24}=\frac{3}{24}=\frac{1}{8}.$$

16. Let $X \; and \; Y$ be random variable with joint density function $f_{XY}(x,y)=\begin{cases}4xy\ ;\ 0\le x\le 1,\ 0\le y\le 1\\ 0 \quad ,otherwise\end{cases}$. Find $E[XY]$.

$$E[XY]=\iint xy\, f(x,y)\,dx\,dy=\int_0^1\int_0^1 xy(4xy)\,dx\,dy=4\int_0^1\int_0^1 x^2y^2\,dx\,dy=4\int_0^1 y^2\left(\frac{x^3}{3}\right)_0^1dy$$

$$=\frac{4}{3}\int_0^1 y^2\,dy=\frac{4}{3}\left(\frac{y^3}{3}\right)_0^1=\frac{4}{3}\left(\frac{1}{3}\right)=\frac{4}{9}.$$

17. Let $X \; and \; Y$ be any two random variables and a,b be constants. Prove that $Cov(aX,bY)=ab\,\mathrm{cov}(X,Y)$.

$$Cov(X,Y)=E[XY]-E[X]E[Y]$$

$$Cov(aX,bY)=E[aX\ bY]-E[aX]E[bY]=ab\,E[XY]-a\,E[X]\,b\,E[Y]$$

$$=ab\left[E(XY)-E(X)E(Y)\right]=ab\,Cov(X,Y)$$

18. If $Y=-2X+3$, find $Cov(X,Y)$.

$$\begin{aligned}Cov(X,Y) &= E[XY]-E[X]E[Y]\\ &= E[X(-2X+3)-E[X]E[-2X+3]]\\ &= E[-2X^2+3X]-E[X](-2E[X]+3)\\ &= -2E[X^2]+3E[X]+2(E[X])^2-3E[X]\\ &= -2\left[E[X^2]-(E[X])^2\right]=-2Var\,X.\end{aligned}$$

19. If X_1 has mean 4 and variance 9 while X_2 has mean -2 and variance 5 and the two are independent, find $Var(2X_1+X_2-5)$.

Given $E[X_1]=4$, $Var[X_1]=9$

$E[X_2]=-2$, $Var[X_2]=5$

$$Var(2X_1+X_2-5)=4Var\,X_1+Var\,X_2 = 4(9)+5=36+5=41.$$

20. Find the acute angle between the two lines of regression.

The equations of the regression lines are

$$y-\bar{y}=r\frac{\sigma y}{\sigma x}(x-\bar{x}) \;-----------(1)$$

$$x-\bar{x}=r\frac{\sigma x}{\sigma y}(y-\bar{y}) \;-----------(2)$$

Slope of line (1) is $m_1=r\frac{\sigma y}{\sigma x}$

Slope of line (2) is $m_2=\frac{\sigma y}{r\sigma x}$

If θ is the acute angle between the two lines, then

$$\tan\theta=\frac{|m_1-m_2|}{1+m_1m_2}=\frac{\left|r\frac{\sigma y}{\sigma x}-\frac{\sigma y}{r\sigma x}\right|}{1+r\frac{\sigma y}{\sigma x}.\frac{\sigma y}{r\sigma x}}=\frac{\left|\frac{(r^2-1)}{r}\frac{\sigma y}{\sigma x}\right|}{1+\frac{\sigma y^2}{\sigma x^2}}=\frac{\left|-\frac{(1-r^2)}{r}\frac{\sigma y}{\sigma x}\right|}{\frac{\sigma x^2+\sigma y^2}{\sigma x^2}}$$

$$=\frac{(1-r^2)\sigma x\sigma y}{|r|(\sigma x^2+\sigma y^2)}$$

21. If X and Y are random variables such that $Y=aX+b$ where a and b are real constants, show that the correlation co-efficient $r(X,Y)$ between them has magnitude one.

Correlation co-efficient $r(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$

$$\begin{aligned}Cov(X,Y) &= E[XY]-E[X]E[Y]\\ &= E[X(aX+b)-E[X]E[aX+b]]\\ &= E[aX^2+bX]-E[X](aE[X]+b)\\ &= aE[X^2]+bE[X]-a(E[X])^2-bE[X]\\ &= a\left[E[X^2]-(E[X])^2\right]=aVar\,X=a\sigma_X^{\ 2}.\end{aligned}$$

$$\sigma_Y{}^2 = E[Y^2] - (E[Y])^2$$
$$= E[(aX+b)^2] - (E[aX+b])^2 = E[a^2 X^2 + 2ab X + b^2] - (a E[X] + b)^2$$
$$= a^2 E[X^2] + 2ab E[X] + b^2 - a^2 (E[X])^2 - 2ab E[X] - b^2$$
$$= a^2 \left[E[X^2] - (E[X])^2 \right] = a^2 Var X = a^2 \sigma_X{}^2$$

Therefore $\sigma_Y = a\sigma_X$ and $r(X,Y) = \dfrac{a\sigma_X{}^2}{\sigma_X . a\sigma_X} = 1$.

Therefore, the correlation co-efficient $r(X,Y)$ between them has magnitude one.

22. State central limit theorem.

If $X_1, X_2, X_3, ..., X_n, ...$ be a sequence of independent identically distributed random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2, i = 1, 2, ...$ and if $S_n = X_1 + X_2 + X_3 + ... + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

23. Write the applications of central limit theorem.

(i) Central limit theorem provides a simple method for computing approximate probabilities of sums of independent random variables.

(ii) It also gives us the wonderful fact that the empirical frequencies of so many natural “populations” exhibit a bell shaped curve.

PART - B

1. Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y).

Solution:

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3 since there are only 3 red balls.

$$P[X=0, Y=0] = P[\text{drawing 3 balls none of which is white or red}]$$
$$= P[\text{all three balls drawn are black}] = \frac{4c_3}{9c_3} = \frac{1}{21}$$

$$P[X=0, Y=1] = P[\text{drawing 3 balls 1 red and 2 black}] = \frac{3c_1 \times 4c_2}{9c_3} = \frac{3}{14}$$

$$P[X=0, Y=2] = P[\text{drawing 3 balls 2 red and 1 black}] = \frac{3c_2 \times 4c_1}{9c_3} = \frac{1}{7}$$

$$P[X=0, Y=3] = P[\text{drawing 3 red balls}] = \frac{3c_3}{9c_3} = \frac{1}{84}$$

$$P[X=1, Y=0] = P[\text{drawing 1 white 2 black}] = \frac{2c_1 \times 4c_2}{9c_3} = \frac{1}{7}$$

$$P[X=1, Y=1] = P[\text{drawing 1 white 1 red and 1 black}] = \frac{2c_1 \times 3c_1 \times 4c_1}{9c_3} = \frac{2}{7}$$

$$P[X=1\,,\,Y=2]=P[drawing\,1\,white\,2\,red\,]=\frac{2c_1\times 3c_2}{9c_3}=\frac{1}{14}$$

$P[X=1\,,\,Y=3]=0$ [Since only 3 balls are drawn]

$$P[X=2\,,\,Y=0]=P[drawing\,2\,white\,1\,black\,]=\frac{2c_2\times 4c_1}{9c_3}=\frac{1}{21}$$

$$P[X=2\,,\,Y=1]=P[drawing\,2\,white\,and\,1\,red\,balls\,]=\frac{2c_2\times 3c_1}{9c_3}=\frac{1}{28}$$

$P[X=2\,,\,Y=2]=0$ [Since only 3 balls are drawn]

$P[X=2\,,\,Y=3]=0$ [Since only 3 balls are drawn]

The joint probability distribution of (X,Y) may be represented in the form of a table as given below

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

2. **The joint probability mass function of (X,Y) is given by $p(x,y)=k(2x+3y)$, $x=0,1,2$, $y=1,2,3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $X+Y$.**

<u>Solution</u>:

The joint probability distribution of (X,Y) is given below

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

Since $p(x,y)$ is a probability mass function, we have

$$\sum\sum p(x,y)=1$$

$$3k+6k+9k+5k+8k+11k+7k+10k+13k=1$$

$$72k=1\Rightarrow k=\frac{1}{72}$$

Marginal probability distribution of X

$$P[X=0]=3k+6k+9k=18k=\frac{18}{72}=\frac{1}{4}$$

$$P[X=1] = 5k+8k+11k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[X=0] = 7k+10k+13k = 30k = \frac{30}{72} = \frac{5}{12}$$

Marginal probability distribution of Y

$$P[Y=1] = 3k+5k+7k = 15k = \frac{15}{72} = \frac{5}{24}$$

$$P[Y=2] = 6k+8k+10k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[Y=3] = 9k+11k+13k = 33k = \frac{33}{72} = \frac{11}{24}$$

Conditional distribution of X given $Y=1$

$$P[X=0/Y=1] = \frac{P[X=0\ ,\ Y=1]}{P[Y=1]} = \frac{3k}{15k} = \frac{3}{15} = \frac{1}{5}$$

$$P[X=1/Y=1] = \frac{P[X=1\ ,\ Y=1]}{P[Y=1]} = \frac{5k}{15k} = \frac{5}{15} = \frac{1}{3}$$

$$P[X=2/Y=1] = \frac{P[X=2\ ,\ Y=1]}{P[Y=1]} = \frac{7k}{15k} = \frac{7}{15}$$

Conditional distribution of X given $Y=2$

$$P[X=0/Y=2] = \frac{P[X=0\ ,\ Y=2]}{P[Y=2]} = \frac{6k}{24k} = \frac{6}{24} = \frac{1}{4}$$

$$P[X=1/Y=2] = \frac{P[X=1\ ,\ Y=2]}{P[Y=2]} = \frac{8k}{24k} = \frac{8}{24} = \frac{1}{3}$$

$$P[X=2/Y=2] = \frac{P[X=2\ ,\ Y=2]}{P[Y=2]} = \frac{10k}{24k} = \frac{5}{12}$$

Conditional distribution of X given $Y=3$

$$P[X=0/Y=3] = \frac{P[X=0\ ,\ Y=3]}{P[Y=3]} = \frac{9k}{33k} = \frac{9}{33} = \frac{3}{11}$$

$$P[X=1/Y=3] = \frac{P[X=1\ ,\ Y=3]}{P[Y=3]} = \frac{11k}{33k} = \frac{11}{33} = \frac{1}{3}$$

$$P[X=2/Y=3] = \frac{P[X=2\ ,\ Y=3]}{P[Y=3]} = \frac{13k}{33k} = \frac{13}{33}$$

Conditional distribution of Y given $X=0$

$$P[Y=1/X=0] = \frac{P[X=0\ ,\ Y=1]}{P[X=0]} = \frac{3k}{18k} = \frac{3}{18} = \frac{1}{6}$$

$$P[Y=2/X=0] = \frac{P[X=0\ ,\ Y=2]}{P[X=0]} = \frac{6k}{18k} = \frac{6}{18} = \frac{1}{3}$$

$$P[Y=3/X=0] = \frac{P[X=0\ ,\ Y=3]}{P[X=0]} = \frac{9k}{18k} = \frac{9}{18} = \frac{1}{2}$$

Conditional distribution of Y given $X = 1$

$$P[Y=1/X=1]=\frac{P[X=1\ ,\ Y=1]}{P[X=1]}=\frac{5k}{24k}=\frac{5}{24}$$
$$P[Y=2/X=1]=\frac{P[X=1\ ,\ Y=2]}{P[X=1]}=\frac{8k}{24k}=\frac{8}{24}=\frac{1}{3}$$
$$P[Y=3/X=1]=\frac{P[X=1\ ,\ Y=3]}{P[X=1]}=\frac{11k}{24k}=\frac{11}{24}$$

Conditional distribution of Y given $X = 2$

$$P[Y=1/X=2]=\frac{P[X=2\ ,\ Y=1]}{P[X=2]}=\frac{7k}{30k}=\frac{7}{30}$$
$$P[Y=2/X=2]=\frac{P[X=2\ ,\ Y=2]}{P[X=2]}=\frac{10k}{30k}=\frac{10}{30}=\frac{1}{3}$$
$$P[Y=3/X=2]=\frac{P[X=2\ ,\ Y=3]}{P[X=2]}=\frac{13k}{30k}=\frac{13}{30}$$

Probability distribution of $(X+Y)$

$X+Y$	$p(X+Y)$
1	$p_{01}=3k=\frac{3}{72}$
2	$p_{02}+p_{11}=6k+5k=11k=\frac{11}{72}$
3	$p_{03}+p_{12}+p_{21}=9k+8k+7k=24k=\frac{24}{72}$
4	$p_{13}+p_{22}=11k+10k=21k=\frac{21}{72}$
5	$p_{23}=13k=\frac{13}{72}$

3. **The joint distribution of (X,Y) where X and Y are discrete is given in the following table**

Solution:

X \ Y	**0**	**1**	**2**
0	**0.1**	**0.04**	**0.06**
1	**0.2**	**0.08**	**0.12**
2	**0.2**	**0.08**	**0.12**

Verify whether X and Y are independent.

Marginal distribution of X is

$$P[X=0] = 0.1+0.04+0.06 = 0.2$$
$$P[X=1] = 0.2+0.08+0.12 = 0.4$$
$$P[X=2] = 0.2+0.08+0.12 = 0.4$$

Marginal distribution of Y is

$$P[Y=0] = 0.1+0.2+0.2 = 0.5$$
$$P[Y=1] = 0.04+0.08+0.08 = 0.2$$
$$P[Y=2] = 0.06+0.12+0.12 = 0.3$$

X and Y are independent if $P[X=i]\times P[Y=j]=P[X=i\ ,\ Y=j]$ for all i and j

(ie) We have to show that

$$P[X=0]\times P[Y=0]= 0.2\times 0.5=0.1 \text{------------------}(1)$$
$$P[X=0,Y=0]\ 0.1 \quad \text{-------------------------------}(2)$$

From (1) and (2), we have

$$P[X=0]\times P[Y=0]=P[X=0\ ,\ Y=0]$$
$$P[X=0]\times P[Y=1]=0.2\times 0.2=0.04=P[X=0\ ,\ Y=1]$$
$$P[X=0]\times P[Y=2]=0.2\times 0.3=0.06=P[X=0\ ,\ Y=2]$$
$$P[X=1]\times P[Y=0]=0.4\times 0.5=0.2=P[X=1\ ,\ Y=0]$$
$$P[X=1]\times P[Y=1]=0.4\times 0.2=0.08=P[X=1\ ,\ Y=1]$$
$$P[X=1]\times P[Y=2]=0.4\times 0.3=0.12=P[X=1\ ,\ Y=2]$$
$$P[X=2]\times P[Y=0]=0.4\times 0.5=0.2=P[X=2\ ,\ Y=0]$$
$$P[X=2]\times P[Y=1]=0.4\times 0.2=0.08=P[X=2\ ,\ Y=1]$$
$$P[X=2]\times P[Y=2]=0.4\times 0.3=0.12=P[X=2\ ,\ Y=2]$$

Therefore for all i and j, $P[X=i]\times P[Y=j]=P[X=i\ ,\ Y=j]$

Hence, the random variables X and Y are independent.

4. **The joint pdf of a two dimensional random variable (X,Y) is given by $f(x,y)=xy^2+\frac{x^2}{8}, 0\le x\le 2; 0\le y\le 1$. Compute (1) $P[X>1]$ (2) $P\left[Y<\frac{1}{2}\right]$ (3) $P\left[X>1\big/Y<\frac{1}{2}\right]$ (4) $P\left[Y<\frac{1}{2}\big/X>1\right]$ (5) $P[X<Y]$ and (6) $P[X+Y\le 1]$.**

<u>**Solution**</u>:

(1) $$P[X>1]=\int_0^1\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\int_0^1\left[y^2\left(\frac{x^2}{2}\right)_1^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_1^2\right]dy$$
$$=\int_0^1\left[\frac{y^2}{2}(4-1)+\frac{1}{24}(8-1)\right]dy=\int_0^1\left[\frac{3}{2}y^2+\frac{7}{24}\right]dy$$
$$=\frac{3}{2}\left(\frac{y^3}{3}\right)_0^1+\frac{7}{24}(y)_0^1=\frac{1}{2}(1-0)+\frac{7}{24}(1-0)=\frac{1}{2}+\frac{7}{24}=\frac{19}{24}$$

(2) $$P\left[Y<\frac{1}{2}\right]=\int_0^{\frac{1}{2}}\int_0^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\int_0^{\frac{1}{2}}\left[y^2\left(\frac{x^2}{2}\right)_0^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_0^2\right]dy$$

$$=\int_0^{\frac{1}{2}}\left[\frac{y^2}{2}(4-0)+\frac{1}{24}(8-0)\right]dy=\int_0^{\frac{1}{2}}\left[2y^2+\frac{1}{3}\right]dy$$

$$=2\left(\frac{y^3}{3}\right)_0^{\frac{1}{2}}+\frac{1}{3}(y)_0^{\frac{1}{2}}=\frac{2}{3}\left(\frac{1}{8}-0\right)+\frac{1}{3}\left(\frac{1}{2}-0\right)=\frac{2}{24}+\frac{1}{6}=\frac{1}{4}$$

(3) $$P\left[X>1\Big/Y<\frac{1}{2}\right]=\frac{P\left[X>1,Y<\frac{1}{2}\right]}{P\left[Y<\frac{1}{2}\right]}$$

$$P\left[X>1,Y<\frac{1}{2}\right]=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\int_0^{\frac{1}{2}}\left[y^2\left(\frac{x^2}{2}\right)_1^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_1^2\right]dy$$

$$=\int_0^{\frac{1}{2}}\left[\frac{y^2}{2}(4-1)+\frac{1}{24}(8-1)\right]dy=\int_0^{\frac{1}{2}}\left[\frac{3}{2}y^2+\frac{7}{24}\right]dy$$

$$=\frac{3}{2}\left(\frac{y^3}{3}\right)_0^{\frac{1}{2}}+\frac{7}{24}(y)_0^{\frac{1}{2}}=\frac{1}{2}\left(\frac{1}{8}-0\right)+\frac{7}{24}\left(\frac{1}{2}-0\right)=\frac{1}{16}+\frac{7}{48}=\frac{5}{24}$$

$$P\left[X>1\Big/Y<\frac{1}{2}\right]=\frac{\frac{5}{24}}{\frac{1}{4}}=\frac{5}{6}$$

(4) $$P\left[Y<\frac{1}{2}\Big/X>1\right]=\frac{P\left[X>1,Y<\frac{1}{2}\right]}{P[X>1]}=\frac{\frac{5}{24}}{\frac{19}{24}}=\frac{5}{19}$$

(5) $$P[X<Y]=\int_0^1\int_0^y\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\int_0^1\left[y^2\left(\frac{x^2}{2}\right)_0^y+\frac{1}{8}\left(\frac{x^3}{3}\right)_0^y\right]dy$$

$$=\int_0^1\left[\frac{y^2}{2}\left(y^2-0\right)+\frac{1}{24}\left(y^3-0\right)\right]dy=\int_0^1\left[\frac{y^4}{2}+\frac{y^3}{24}\right]dy$$

$$=\frac{1}{2}\left(\frac{y^5}{5}\right)_0^1+\frac{1}{24}\left(\frac{y^4}{4}\right)_0^1=\frac{1}{10}(1-0)+\frac{1}{96}(1-0)=\frac{1}{10}+\frac{1}{96}=\frac{53}{480}$$

(6) $$P[X+Y\le 1]=\int_0^1\int_0^{1-y}\left(xy^2+\frac{x^2}{8}\right)dx\,dy=\int_0^1\left[y^2\left(\frac{x^2}{2}\right)_0^{1-y}+\frac{1}{8}\left(\frac{x^3}{3}\right)_0^{1-y}\right]dy$$

$$=\int_0^1\left[\frac{y^2}{2}\left((1-y)^2-0\right)+\frac{1}{24}\left((1-y)^3-0\right)\right]dy$$

$$=\frac{1}{2}\int_0^1 y^2(1-y)^2\,dy+\frac{1}{24}\int_0^1(1-y)^3\,dy$$

$$=\frac{1}{2}\int_0^1 y^2\left(1-2y+y^2\right)dy+\frac{1}{24}\int_0^1(1-y)^3\,dy$$

$$=\frac{1}{2}\int_0^1\left(y^2-2y^3+y^4\right)dy+\frac{1}{24}\int_0^1(1-y)^3\,dy$$

$$=\frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1-2\left(\frac{y^4}{4}\right)_0^1+\left(\frac{y^5}{5}\right)_0^1\right]+\frac{1}{24}\left[\frac{(1-y)^4}{-4}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{1}{3}(1-0)+\frac{1}{2}(1-0)+\frac{1}{5}(1-0)\right]-\frac{1}{96}(0-1)$$

$$=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right]+\frac{1}{96}=\frac{1}{60}+\frac{1}{96}=\frac{13}{480}$$

5. **Given the joint pdf of (X,Y) $f(x,y)=\begin{cases}e^{-(x+y)} & ,x>0,y>0\\ 0 & ,elsewhere\end{cases}$. Find the marginal densities of X and Y. Are X and Y independent?**

<u>**Solution**</u>:

Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\int_0^\infty e^{-x}e^{-y}dy=e^{-x}\int_0^\infty e^{-y}\,dy=e^{-x}\left(-e^{-y}\right)_0^\infty$$

$$=-e^{-x}(0-1)=e^{-x}\ ,x>0$$

Marginal density of Y is

$$f_Y(y)=\int f(x,y)\,dx=\int_0^\infty e^{-x}e^{-y}\,dx=e^{-y}\int_0^\infty e^{-x}\,dx=e^{-y}\left(-e^{-x}\right)_0^\infty$$

$$=-e^{-y}(0-1)=e^{-y}\ ,y>0$$

$$f_X(x).f_Y(y)=e^{-x}.e^{-y}=e^{-(x+y)}=f_{XY}(x,y)$$

Therefore X and Y are independent.

6. **The joint pdf of a two dimensional random variable (X,Y) is given by $f(x,y)=k\left(x^3y+xy^3\right)$,$0\le x\le 2;0\le y\le 2$. Find the value of k and marginal and conditional density functions.**

<u>**Solution**</u>: Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)\,dx\,dy=1$$

$$k\int_0^2\int_0^2\left(x^3y+xy^3\right)dx\,dy=1$$

$$k\int_0^2\left[y\left(\frac{x^4}{4}\right)_0^2+y^3\left(\frac{x^2}{2}\right)_0^2\right]dy=1$$

$$k\int_0^2 \left(4y+2y^3\right)dy=1$$

$$k\left[4\left(\frac{y^2}{2}\right)_0^2+2\left(\frac{y^4}{4}\right)_0^2\right]=1$$

$$k[8+8]=1 \qquad \Rightarrow \quad 16k=1$$

$$k=\frac{1}{16}$$

Therefore, $f(x,y)=\frac{1}{16}\left(x^3\, y+x\, y^3\right); 0\le x\le 2, 0\le y\le 2$

Marginal density of X is

$$f_X(x)=\int f(x,y)\, dy=\int_0^2 \frac{1}{16}\left(x^3y+x\,y^3\right)dy$$

$$=\frac{1}{16}\left[x^3\left(\frac{y^2}{2}\right)_0^2+x\left(\frac{y^4}{4}\right)_0^2\right]=\frac{1}{16}\left[\frac{x^3}{2}(4-0)+\frac{x}{4}(16-0)\right]$$

$$=\frac{1}{16}\left[2x^3+4x\right]=\frac{x^3+2x}{8}, 0\le x\le 2$$

Marginal density of Y is

$$f_Y(y)=\int f(x,y)\, dx=\int_0^2 \frac{1}{16}\left(x^3y+x\,y^3\right)dx$$

$$=\frac{1}{16}\left[y\left(\frac{x^4}{4}\right)_0^2+y^3\left(\frac{x^2}{2}\right)_0^2\right]=\frac{1}{16}\left[\frac{y}{4}(16-0)+\frac{y^3}{2}(4-0)\right]$$

$$=\frac{1}{16}\left[4y+2y^3\right]=\frac{y^3+2y}{8}\ , \ 0\le y\le 2$$

Conditional density of X given Y is

$$f_{X/Y}(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{\frac{1}{16}\left(x^3y+xy^3\right)}{\frac{y^3+2y}{8}}=\frac{8}{16}.\frac{y\left(x^3+xy^2\right)}{y\left(y^2+2\right)}$$

$$f_{X/Y}(x/y)=\frac{\left(x^3+xy^2\right)}{2\left(y^2+2\right)}\ , \ 0\le x\le 2$$

Conditional density of Y given X is

$$f_{Y/X}(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{1}{16}\left(x^3y+xy^3\right)}{\frac{x^3+2x}{8}}=\frac{8}{16}.\frac{x\left(x^2y+y^3\right)}{x\left(x^2+2\right)}$$

$$f_{Y/X}(y/x)=\frac{\left(x^2y+y^3\right)}{2\left(x^2+2\right)}\ ;0\le y\le 2.$$

7. Given the joint pdf of (X,Y) as $f(x,y)=\begin{cases} 8xy \ ; \ 0<x<y<1 \\ 0 \qquad ,otherwise \end{cases}$. Find the marginal and conditional probability density functions of X and Y. Are X and Y are independent?

Solution:

Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\int_x^1 8xy\,dy=8x\int_x^1 y\,dy=8x\left(\frac{y^2}{2}\right)_x^1$$
$$=4x(1-x^2),\ 0<x<1$$

Marginal density of Y is

$$f_Y(y)=\int f(x,y)\,dx=\int_0^y 8xy\,dx=8y\int_0^y x\,dx=8y\left(\frac{x^2}{2}\right)_0^y$$
$$=4y(y^2-0)=4y^3,\ 0<y<1$$

$$f_X(x).\,f_Y(y)=4x(1-x^2).4y^3\neq 8xy\neq f_{XY}(x,y)$$

Therefore X and Y are not independent.

Conditional density of X given Y is

$$f_{X/Y}(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{8xy}{4y^3}=\frac{2x}{y^2},\ 0<x<y$$

Conditional density of Y given X is

$$f_{Y/X}(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{8xy}{4x(1-x^2)}=\frac{2y}{1-x^2},\ x<y<1$$

8. Given $f_{XY}(x,y)=\begin{cases} cx(x-y) ;0<x<2,-x<y<x \\ 0 \qquad ;otherwise \end{cases}$. (1) Evaluate c, find (2) $f_X(x)$ (3) $f_{Y/X}(y/x)$ and (4) $f_Y(y)$.

Solution:

(1) Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)dx\,dy=1$$

$$c\int_0^2\int_{-x}^x (x^2-xy)\,dy\,dx=1$$

$$c\int_0^2\left[x^2(y)_{-x}^x-x\left(\frac{y^2}{2}\right)_{-x}^x\right]dx=1$$

$$c\int_0^2\left[x^2(x-(-x))-\frac{x}{2}(x^2-x^2)\right]dx=1$$

$$c\int_0^2(2x^3-0)dx=1 \quad\Rightarrow 2c\int_0^2 x^3\,dx=1 \quad\Rightarrow 2c\left[\frac{x^4}{4}\right]_0^2=1$$

$$\frac{c}{2}[16-0]=1 \quad \Rightarrow 8c=1 \quad \Rightarrow c=\frac{1}{8}$$

Therefore $f(x,y)=\frac{1}{8}\left(x^2-xy\right); 0<x<2, -x<y<x$

(2) $$f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy=\frac{1}{8}\int_{-x}^{x}\left(x^2-xy\right)dy=\frac{1}{8}\left[x^2(y)_{-x}^{x}-x\left(\frac{y^2}{2}\right)_{-x}^{x}\right]$$

$$=\frac{1}{8}\left[x^2(x-(-x))-\frac{x}{2}\left(x^2-x^2\right)\right]=\frac{1}{8}\left[x^2(2x)-0\right]=\frac{x^3}{4}, 0<x<2.$$

(3) $$f_{Y/X}(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{1}{8}\left(x^2-xy\right)}{\frac{x^3}{4}}=\frac{4}{8}\,\frac{x(x-y)}{x^3}=\frac{x-y}{2x^2}, -x<y<x.$$

(4) $f_Y(y)=\int f(x,y)\,dx$

$$=\begin{cases}\frac{1}{8}\int_{-y}^{2}\left(x^2-xy\right)dx & in\ -2\le y\le 0\\ \frac{1}{8}\int_{y}^{2}\left(x^2-xy\right)dx & in\ 0\le y\le 2\end{cases}$$

$$=\begin{cases}\frac{1}{8}\left[\left(\frac{x^3}{3}\right)_{-y}^{2}-y\left(\frac{x^2}{2}\right)_{-y}^{2}\right] & in\ -2\le y\le 0\\ \frac{1}{8}\left[\left(\frac{x^3}{3}\right)_{y}^{2}-y\left(\frac{x^2}{2}\right)_{y}^{2}\right] & in\ 0\le y\le 2\end{cases}$$

$$=\begin{cases}\frac{1}{8}\left[\frac{1}{3}\left(8+y^3\right)-\frac{y}{2}\left(4-y^2\right)\right] & in\ -2\le y\le 0\\ \frac{1}{8}\left[\frac{1}{3}\left(8-y^3\right)-\frac{y}{2}\left(4-y^2\right)\right] & in\ 0\le y\le 2\end{cases}$$

$$=\begin{cases}\frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3 & in\ -2\le y\le 0\\ \frac{1}{3}-\frac{y}{4}+\frac{1}{48}y^3 & in\ 0\le y\le 2\end{cases}$$

9. Suppose the pdf $f(x,y)$ of (X,Y) is given by $f(x,y)=\begin{cases}\frac{6}{5}\left(x+y^2\right); 0\le x\le 1,\ 0\le y\le 1\\ 0 \quad ; otherwise\end{cases}$.

Obtain the marginal pdf of X, the conditional pdf of Y given $X=0.8$ and then $E[Y/X=0.8]$.

Solution: Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\int_0^1\frac{6}{5}\left(x+y^2\right)dy=\frac{6}{5}\left[x(y)_0^1+\left(\frac{y^3}{3}\right)_0^1\right]=\frac{6}{5}\left[x+\frac{1}{3}\right], 0\le x\le 1$$

Conditional density of Y given $X = 0.8$ if

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{6}{5}(x+y^2)}{\frac{6}{5}\left(x+\frac{1}{3}\right)} = \frac{x+y^2}{x+\frac{1}{3}}$$

$$f_{Y/X=0.8} = \frac{0.8+y^2}{0.8+\frac{1}{3}} = \frac{3\left[0.8+y^2\right]}{3.4}$$

$$E[Y/X=x] = \int y\, f_{Y/X}(y/x)dy$$

$$= \int_0^1 y\left(\frac{x+y^2}{x+\frac{1}{3}}\right)dy = \frac{1}{x+\frac{1}{3}}\int_0^1 (xy+y^3)dy = \frac{3}{3x+1}\left[x\left(\frac{y^2}{2}\right)_0^1 + \left(\frac{y^4}{4}\right)_0^1\right]$$

$$= \frac{3}{3x+1}\left[\frac{x}{2}+\frac{1}{4}\right] = \frac{3(2x+1)}{4(3x+1)}$$

$$E[Y/X=0.8] = \frac{3[2(0.8)+1]}{4[3(0.8)+1]} = \frac{7.8}{13.6} = 0.5735$$

10. Find $Corr(X,Y)$ for the following discrete bivariate distribution

Y \ X	5	15
10	0.2	0.4
20	0.3	0.1

Solution:

Correlation co-efficient $= \dfrac{E[XY]-E[X]E[Y]}{\sigma_X\,\sigma_Y}$

Marginal distribution of X is $P[X=5] = 0.2+0.3 = 0.5$

$P[X=15] = 0.4+0.1 = 0.5$

Marginal distribution of Y is $P[Y=10] = 0.2+0.4 = 0.6$

$P[Y=20] = 0.3+0.1 = 0.4$

$E[X] = \sum x\,p(x) = 5\times0.5+15\times0.5 = 2.5+7.5 = 10$

$E[Y] = \sum y\,p(y) = 10\times0.6+20\times0.4 = 6+8 = 14$

$E[X^2] = \sum x^2\,p(x) = (5)^2\times0.5+(15)^2\times0.5 = 25\times0.5+225\times0.5 = 125$

$E[Y^2] = \sum y^2\,p(y) = (10)^2\times0.6+(20)^2\times0.4 = 100\times0.6+400\times0.4 = 220$

$\sigma_X^{\,2} = E[X^2]-(E[X])^2 = 125-(10)^2 = 125-100 = 25$

$\sigma_X = 5$

$\sigma_Y^{\,2} = E[Y^2]-(E[Y])^2 = 220-(14)^2 = 220-196 = 24$

$\sigma_Y = 4.89$

$E[XY] = \sum xy\, p(x,y)$

$= 5\times10\times0.2 + 15\times10\times0.4 + 5\times20\times0.3 + 15\times20\times0.1 = 130$

Correlation co-efficient $= \dfrac{E[XY]-E[X]E[Y]}{\sigma_X\ \sigma_Y}$

$$Corr(X,Y) = \frac{130-(10)(14)}{5\times4.89} = \frac{-10}{24.45} = -0.4089 .$$

11. If $f(x,y)=\begin{cases}2-x-y\ ,\ 0\le x\le1,\ 0\le y\le1\\ 0 \qquad ;elsewhere\end{cases}$ is the joint pdf of the random variables X and Y, find the correlation co-efficient of X and Y.

<u>Solution</u>:

Correlation co-efficient $= \dfrac{E[XY]-E[X]E[Y]}{\sigma_X\ \sigma_Y}$

$$E[X] = \iint x\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x(2-x-y)\,dx\,dy = \int_0^1\int_0^1 (2x-x^2-xy)\,dx\,dy$$

$$= \int_0^1\left[2\left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1 - y\left(\frac{x^2}{2}\right)_0^1\right]dy = \int_0^1\left[1-\frac{1}{3}-\frac{y}{2}\right]dy = \int_0^1\left(\frac{2}{3}-\frac{y}{2}\right)dy$$

$$= \frac{2}{3}(y)_0^1 - \frac{1}{2}\left(\frac{y^2}{2}\right)_0^1 = \frac{2}{3}-\frac{1}{4} = \frac{5}{12}$$

$$E[Y] = \iint y\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 y(2-x-y)\,dx\,dy = \int_0^1\int_0^1 (2y-xy-y^2)\,dx\,dy$$

$$= \int_0^1\left[2y(x)_0^1 - y\left(\frac{x^2}{2}\right)_0^1 - y^2(x)_0^1\right]dy = \int_0^1\left[2y-\frac{y}{2}-y^2\right]dy$$

$$= \int_0^1\left(\frac{3}{2}y - y^2\right)dy = \frac{3}{2}\left(\frac{y^2}{2}\right)_0^1 - \left(\frac{y^3}{3}\right)_0^1 = \frac{3}{4}-\frac{1}{3} = \frac{5}{12}$$

$$E[X^2] = \iint x^2\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^2(2-x-y)\,dx\,dy = \int_0^1\int_0^1 (2x^2-x^3-yx^2)\,dx\,dy$$

$$= \int_0^1\left[2\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 - y\left(\frac{x^3}{3}\right)_0^1\right]dy = \int_0^1\left[\frac{2}{3}-\frac{1}{4}-\frac{y}{3}\right]dy = \int_0^1\left(\frac{5}{12}-\frac{y}{3}\right)dy$$

$$= \frac{5}{12}(y)_0^1 - \frac{1}{3}\left(\frac{y^2}{2}\right)_0^1 = \frac{5}{12}-\frac{1}{6} = \frac{1}{4}$$

$$E[Y^2] = \iint y^2\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 y^2\left(2-x-y\right)dx\,dy = \int_0^1\int_0^1\left(2y^2-xy^2-y^3\right)dx\,dy$$

$$= \int_0^1\left[2y^2\left(x\right)_0^1 - y^2\left(\frac{x^2}{2}\right)_0^1 - y^3\left(x\right)_0^1\right]dy = \int_0^1\left[2y^2 - \frac{y^2}{2} - y^3\right]dy$$

$$= \int_0^1\left(\frac{3}{2}y^2 - y^3\right)dy = \frac{3}{2}\left(\frac{y^3}{3}\right)_0^1 - \left(\frac{y^4}{4}\right)_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\sigma_X{}^2 = E\left[X^2\right] - \left(E\left[X\right]\right)^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\sigma_X = \frac{\sqrt{11}}{12}$$

$$\sigma_Y{}^2 = E\left[Y^2\right] - \left(E\left[Y\right]\right)^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\sigma_Y = \frac{\sqrt{11}}{12}$$

$$E\left[XY\right] = \iint xy\,f\left(x,y\right)dx\,dy$$

$$= \int_0^1\int_0^1 xy\left(2-x-y\right)dx\,dy = \int_0^1\int_0^1\left(2xy - x^2y - xy^2\right)dx\,dy$$

$$= \int_0^1\left[2y\left(\frac{x^2}{2}\right)_0^1 - y\left(\frac{x^3}{3}\right)_0^1 - y^2\left(\frac{x^2}{2}\right)_0^1\right]dy = \int_0^1\left[y - \frac{y}{3} - \frac{y^2}{2}\right]dy$$

$$= \int_0^1\left(\frac{2}{3}y - \frac{1}{2}y^2\right)dy = \frac{2}{3}\left(\frac{y^2}{2}\right)_0^1 - \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

$$Corr\left(X,Y\right) = \frac{\frac{1}{6} - \left(\frac{5}{12}\right)\left(\frac{5}{12}\right)}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{\frac{1}{6} - \frac{25}{144}}{\frac{11}{144}} = \frac{\frac{-1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

12. Find the correlation between X and Y if the joint probability density of X and Y is $f\left(x,y\right) = \begin{cases} 2 & for\ x>0\ ,\ y>0\ ,\ x+y<1 \\ 0 & ;elsewhere \end{cases}$.

Solution:

Correlation co-efficient $= \dfrac{E\left[XY\right] - E\left[X\right]E\left[Y\right]}{\sigma_X\,\sigma_Y}$

$$E\left[X\right] = \iint x\,f\left(x,y\right)dx\,dy = \int_0^1\int_0^{1-y} x\left(2\right)dx\,dy = 2\int_0^1\left(\frac{x^2}{2}\right)_0^{1-y}dy = \int_0^1\left(1-y\right)^2dy$$

$$= \left[\frac{\left(1-y\right)^3}{-3}\right]_0^1 = -\frac{1}{3}\left(0-1\right) = \frac{1}{3}$$

$$E[Y] = \iint y\, f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} y(2)\,dx\,dy = 2\int_0^1 y(x)_0^{1-y}\,dy = 2\int_0^1 y(1-y)\,dy$$

$$= 2\int_0^1 (y - y^2)\,dy = 2\left[\left(\frac{y^2}{2}\right)_0^1 - \left(\frac{y^3}{3}\right)_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

$$E\left[X^2\right] = \iint x^2 f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} x^2(2)\,dx\,dy = 2\int_0^1\left(\frac{x^3}{3}\right)_0^{1-y} dy = \frac{2}{3}\int_0^1 (1-y)^3\,dy$$

$$= \frac{2}{3}\left[\frac{(1-y)^4}{-4}\right]_0^1 = -\frac{1}{6}(0-1) = \frac{1}{6}$$

$$E\left[Y^2\right] = \iint y^2 f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} y^2(2)\,dx\,dy = 2\int_0^1 y^2\,(x)_0^{1-y}\,dy = 2\int_0^1 y^2\,(1-y)\,dy$$

$$= 2\int_0^1 (y^2 - y^3)\,dy = 2\left[\left(\frac{y^3}{3}\right)_0^1 - \left(\frac{y^4}{4}\right)_0^1\right]$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{6}$$

$$\sigma_X{}^2 = E\left[X^2\right] - \left(E[X]\right)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \quad \Rightarrow \sigma_X = \frac{1}{\sqrt{18}}$$

$$\sigma_Y{}^2 = E\left[Y^2\right] - \left(E[Y]\right)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18} \quad \Rightarrow \sigma_Y = \frac{1}{\sqrt{18}}$$

$$E[XY] = \iint xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^{1-Y} xy(2)\,dx\,dy = 2\int_0^1 y\left(\frac{x^2}{2}\right)_0^{1-y} dy = \int_0^1 y(1-y)^2\,dy$$

$$= \int_0^1 y\left(1 - 2y + y^2\right)dy = \int_0^1 \left(y - 2y^2 + y^3\right)dy$$

$$= \left(\frac{y^2}{2}\right)_0^1 - 2\left(\frac{y^3}{3}\right)_0^1 + \left(\frac{y^4}{4}\right)_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

$$Corr(X,Y) = \frac{\frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}{\frac{1}{\sqrt{18}}\cdot\frac{1}{\sqrt{18}}} = \frac{\frac{1}{12} - \frac{1}{9}}{\frac{1}{18}} = \frac{\frac{-1}{36}}{\frac{1}{18}} = -\frac{1}{2}$$

13. If the independent random variables X and Y have the variances 36 and 16 respectively, find the correlation co-efficient between $X+Y$ and $X-Y$.

<u>Solution</u>:

Let $U = X + Y$ and $V = X - Y$

Given $Var\,X = \sigma_X{}^2 = 36 \;\Rightarrow\; \sigma_X = 6$

$$Var\,Y = \sigma_Y{}^2 = 16 \;\Rightarrow\; \sigma_Y = 4$$

Correlation co-efficient $= \rho_{UV} = \dfrac{E[UV]-E[U]E[V]}{\sigma_U\,\sigma_V}$

$$E[U] = E[X+Y] = E[X]+E[Y]$$
$$E[V] = E[X-Y] = E[X]-E[Y]$$
$$E[UV] = E[(X+Y)(X-Y)] = E[X^2-Y^2] = E[X^2]-E[Y^2]$$
$$E[U^2] = E[(X+Y)^2] = E[X^2+2XY+Y^2]$$
$$= E[X^2]+2E[XY]+E[Y^2]$$
$$= E[X^2]+2E[X]E[Y]+E[Y^2]$$ **[since X and Y are independent]**

$$E[V^2] = E[(X-Y)^2] = E[X^2-2XY+Y^2]$$
$$= E[X^2]-2E[XY]+E[Y^2]$$
$$= E[X^2]-2E[X]E[Y]+E[Y^2]$$ **[since X and Y are independent]**

$$E[U]E[V] = (E[X]+E[Y])(E[X]-E[Y]) = (E[X])^2-(E[Y])^2$$
$$\sigma_U{}^2 = E[U^2]-(E[U])^2$$
$$= (E[X^2]+2E[X]E[Y]+E[Y^2]) - (E[X]+E[Y])^2$$
$$= E[X^2]+2E[X]E[Y]+E[Y^2]-(E[X])^2-2E[X]E[Y]-E[Y^2]$$
$$= \left[E[X^2]-(E[X])^2\right]+\left[E[Y^2]-(E[Y])^2\right]$$
$$= \sigma_X{}^2+\sigma_Y{}^2 = 36+16 = 52$$
$$\sigma_U = \sqrt{52}$$
$$\sigma_V{}^2 = E[V^2]-(E[V])^2$$
$$= (E[X^2]-2E[X]E[Y]+E[Y^2]) - (E[X]-E[Y])^2$$
$$= E[X^2]-2E[X]E[Y]+E[Y^2]-(E[X])^2+2E[X]E[Y]-E[Y^2]$$
$$= \left[E[X^2]-(E[X])^2\right]+\left[E[Y^2]-(E[Y])^2\right]$$
$$= \sigma_X{}^2+\sigma_Y{}^2 = 36+16 = 52$$
$$\sigma_V = \sqrt{52}$$
$$\rho_{UV} = \frac{\left[E[X^2]-E[Y^2]\right]-\left[(E[X])^2-(E[Y])^2\right]}{\sqrt{52}.\sqrt{52}}$$
$$= \frac{\left[E[X^2]-(E[X])^2\right]-\left[E[Y^2]-(E[Y])^2\right]}{52} = \frac{\sigma_X{}^2-\sigma_Y{}^2}{52} = \frac{36-16}{52}$$
$$= \frac{20}{52} = \frac{5}{13}.$$

14. Find the correlation co-efficient for the following data

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution: **Here** $n = 6$

X	Y	XY	X^2	Y^2
10	**18**	**180**	**100**	**324**
14	**12**	**168**	**196**	**144**
18	**24**	**432**	**324**	**576**
22	**6**	**132**	**484**	**36**
26	**30**	**780**	**676**	**900**
30	**36**	**1080**	**900**	**1296**
120	**126**	**2772**	**2680**	**3276**

$$\overline{X} = \frac{\sum X}{n} = \frac{120}{6} = 20$$

$$\overline{Y} = \frac{\sum Y}{n} = \frac{126}{6} = 21$$

$$\sigma_X = \sqrt{\frac{1}{n}\sum X^2 - \left(\overline{X}\right)^2} = \sqrt{\frac{2680}{6} - (20)^2} = 6.83$$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum Y^2 - \left(\overline{Y}\right)^2} = \sqrt{\frac{3276}{6} - (21)^2} = 10.25$$

$$Correlation\ co-efficient = \rho_{XY} = \frac{\frac{1}{n}\sum XY - \left(\overline{X}\right)\left(\overline{Y}\right)}{\sigma_X\ \sigma_Y}$$

$$= \frac{\frac{2772}{6} - (20)(21)}{(6.083)\,(10.25)} = 0.59 = 0.6.$$

15. From the following data, find (1) the two regression equations (2) the co-efficient of correlation between the marks in Mathematics and Statistics (3) the most likely marks in Statistics when marks in Mathematics are 30.

Marks in Mathematics	**25**	**28**	**35**	**32**	**31**	**36**	**29**	**38**	**34**	**32**
Marks in Statistics	**43**	**46**	**49**	**41**	**36**	**32**	**31**	**30**	**33**	**39**

<u>Solution</u>: Here $n = 10$

x	y	xy	x^2	y^2
25	**43**	**1075**	**625**	**1849**
28	**46**	**1288**	**784**	**2116**
35	**49**	**1715**	**1225**	**2401**
32	**41**	**1312**	**1024**	**1681**
31	**36**	**1116**	**961**	**1296**
36	**32**	**1152**	**1296**	**1024**
29	**31**	**899**	**841**	**961**
38	**30**	**1140**	**1444**	**900**
34	**33**	**1122**	**1156**	**1089**
32	**39**	**1248**	**1024**	**1521**
320	**380**	**12067**	**10380**	**14838**

$$\bar{x} = \frac{\sum x}{n} = \frac{320}{10} = 32$$

$$\bar{y} = \frac{\sum y}{n} = \frac{380}{10} = 38$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - (\bar{x})^2} = \sqrt{\frac{10380}{10} - (32)^2} = 3.74$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - (\bar{y})^2} = \sqrt{\frac{14838}{10} - (38)^2} = 6.31$$

$$Correlation\ co-efficient = r_{XY} = \frac{\frac{1}{n}\sum xy - (\bar{x})(\bar{y})}{\sigma_x \sigma_y}$$

$$= \frac{\frac{12067}{10} - (32)(38)}{(3.74)(6.31)} = -0.39 = -0.4$$

The line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 38 = -0.4 \times \frac{6.31}{3.74}(x - 32)$$

$$y - 38 = -0.67(x - 32)$$

$$y - 38 = -0.67x + 21.44$$

$$y = -0.67x + 21.44 + 38$$

$$y = -0.67x + 59.44$$

The line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$$x - 32 = -0.4 \times \frac{3.74}{6.31}(y - 38)$$

$$x - 32 = -0.24(y - 38)$$

$$x - 32 = -0.24y + 9.12$$

$$x = -0.24y + 9.12 + 32$$

$$x = -0.24y + 41.12$$

When marks in Mathematics are 30 (ie) when $x = 30$, we have

$$y = -0.67(30) + 59.44 = -20.1 + 59.44 = 39.34$$

Therefore marks in Statistics = 39.34

16. If $y = 2x - 3$ and $y = 5x + 7$ are the two regression lines, find the mean values of x and y. Find the correlation co-efficient between x and y. Find an estimate value of x when $y = 1$.

<u>Solution</u>: Given $y = 2x - 3$ **-----------------(1)**

$y = 5x + 7$ **-----------------(2)**

Since both the lines of regression passes through the mean values $\bar{x}$ and $\bar{y}$, the point $(\bar{x}\ ,\ \bar{y})$ must satisfy the two given regression lines.

$$\bar{y} = 2\bar{x} - 3 \text{-------------(3)}$$

$$\bar{y} = 5\bar{x} + 7 \text{-------------(4)}$$

Subtracting the equations (3) and (4), we have

$$3\bar{x} = -10 \quad \Rightarrow \quad \bar{x} = \frac{-10}{3}$$

$$\bar{y} = 2\left(\frac{-10}{3}\right) - 3 = \frac{-29}{3}$$

Therefore mean values are $\bar{x} = \frac{-10}{3}$ and $\bar{y} = \frac{-29}{3}$.

Let us suppose that equation (1) is the line of regression of y on x and equation (2) is the equation of the line of regression of x on y, we have

$$(1) \Rightarrow y = 2x - 3$$

$$b_{yx} = 2$$

$$(2) \Rightarrow 5x = y - 7$$

$$x = \frac{1}{5}y - \frac{7}{5}$$

$$b_{xy} = \frac{1}{5}$$

$$r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{1}{5} \times 2} = \pm 0.63$$

Since both the regression co-efficients are positive, r must be positive.

Correlation co-efficient = $r = 0.63$

Substituting $y = 1$ in (2), we have $5x = 1 - 7 = -6 \Rightarrow x = -\frac{6}{5}$.

17. If the joint pdf of (X, Y) is given by $f_{XY}(x, y) = x + y$; $0 \le x, y \le 1$, find the pdf of $U = XY$.

Solution: Given $f_{XY}(x, y) = x + y$; $0 \le x, y \le 1$

Consider the auxiliary random variable $V = Y$

$$U = XY \qquad V = Y$$

$$u = xy \qquad v = y$$

$$u = xv \qquad y = v$$

$$x = \frac{u}{v}$$

$$\frac{\partial x}{\partial u} = \frac{1}{v}\ , \quad \frac{\partial y}{\partial u} = 0\ , \quad \frac{\partial x}{\partial v} = -\frac{u}{v^2}\ , \quad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} - 0 = \frac{1}{v} \Rightarrow |J| = \left|\frac{1}{v}\right| = \frac{1}{v}$$

Therefore, the joint density function of UV is given by

$$f_{UV}(u,v) = |J|\ f_{XY}(x,y) \qquad 0 \le x \le 1\ ,\ 0 \le y \le 1$$

$$= \frac{1}{v}(x+y) \qquad 0 \le \frac{u}{v} \le 1\ ,\ 0 \le v \le 1$$

$$= \frac{1}{v}\left(\frac{u}{v} + v\right) \qquad 0 \le u \le v\ ,\ 0 \le v \le 1$$

$$= \frac{u}{v^2} + 1 \qquad 0 \le u \le v \le 1$$

The pdf of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)dv = \int_u^1 \left(\frac{u}{v^2} + 1\right)dv = u\int_u^1 v^{-2}\,dv + \int_u^1 dv$$

$$= u\left[\frac{v^{-1}}{-1}\right]_u^1 + [v]_u^1 = -u\left(1 - \frac{1}{u}\right) + 1 - u = -u + 1 + 1 + -u$$

$$= 2 - 2u = 2(2-u),\ 0 \le u \le 1.$$

18. If the joint pdf of (X,Y) is given by $f_{XY}(x,y) = e^{-(x+y)}$; $x \ge 0$, $y \ge 0$, find the pdf of $U = \frac{X+Y}{2}$.

<u>**Solution**</u>**: Given** $f_{XY}(x,y) = e^{-(x+y)}$; $x \ge 0$, $y \ge 0$

Introduce the auxiliary random variable $V = Y$

$$U = \frac{X+Y}{2} \qquad V = Y$$

$$u = \frac{x+y}{2} \qquad v = y$$

$$2u = x + y \qquad y = v$$

$$2u = x + v$$

$$x = 2u - v$$

$$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2$$

$$|J| = |2| = 2$$

Therefore, the joint density function of UV is given by

$$f_{UV}(u,v) = |J|\ f_{XY}(x,y) \qquad x \ge 0\ ,\ y \ge 0$$

$$= 2e^{-(x+y)} \qquad 2u - v \ge 0\ ,\ v \ge 0$$

$$= 2e^{-(2u - v + v)} \qquad 2u \ge v\ ,\ v \ge 0$$

$$= 2e^{-2u} \qquad 0 \le v \le 2u$$

The pdf of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_0^{2u} 2e^{-2u}\,dv = 2e^{-2u}\int_0^{2u} dv = 2e^{-2u}\,[v]_0^{2u}$$
$$= 2e^{-2u}(2u-0) = 4ue^{-2u} \quad ,u \ge 0.$$

19. State and prove central limit theorem.

Statement:

If $X_1, X_2, X_3, \ldots\ldots, X_n, \ldots\ldots$ be a sequence of independent identically distributed random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2, i = 1, 2, \ldots\ldots$ and if $S_n = X_1 + X_2 + X_3 + \ldots\ldots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

Proof:

To prove that as $n \to \infty$, $Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ follows the standard normal distribution, we use uniqueness theorem.

"If two random variables have same probability distribution, then their moment generating function are identical".

We know that moment generating function of $Z = \dfrac{X-\mu}{\sigma} = e^{\frac{t^2}{2}}$. If we prove moment generating function of $Z_n = \dfrac{S_n - n\mu}{\sqrt{n}\,\sigma}$ is $e^{\frac{t^2}{2}}$, then Z_n follows standard normal distribution with mean 0 and variance 1.

Moment generating function of Z_n is,

$$M_{Z_n}(t) = E\left[e^{tZ_n}\right] = E\left[e^{t\left(\frac{S_n - n\mu}{\sqrt{n}\,\sigma}\right)}\right] = E\left[e^{t\left(\frac{(X_1+X_2+X_3+\ldots\ldots+X_N) - n\mu}{\sqrt{n}\,\sigma}\right)}\right]$$
$$= E\left[e^{t\left(\frac{X_1-\mu}{\sigma\sqrt{n}_1}\right)}.e^{t\left(\frac{X_2-\mu}{\sigma\sqrt{n}}\right)}\ldots\ldots\ldots\ldots\ldots e^{t\left(\frac{X_n-\mu}{\sigma\sqrt{n}}\right)}\right]$$

Because $S_n - n\mu = X_1 + X_2 + \ldots\ldots\ldots + X_n - \mu$ since $X_1, X_2, \ldots\ldots\ldots, X_n$ are independent and identically distributed.

$$M_{Z_n}(t) = E\left[e^{t\left(\frac{X-\mu}{\sigma\sqrt{N}}\right)}\right]E\left[e^{t\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)}\right]\ldots\ldots\ldots\ldots\ldots\ldots E\left[e^{t\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)}\right] = \left(E\left[e^{t\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)}\right]\right)^n$$

Consider $E\left[e^{t\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)}\right] = E\left[1 + \dfrac{t\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)}{1!} + \dfrac{t^2\left(\frac{X-\mu}{\sigma\sqrt{n}}\right)^2}{2!} + \ldots\ldots\ldots\ldots\right]$

$$= E[1] + \frac{t}{1!\sigma\sqrt{n}}E[X-\mu] + \frac{t^2}{2!\sigma^2 n}E\left[(X-\mu)^2\right] + \ldots\ldots\ldots$$

$$=1+\frac{t}{\sigma\sqrt{n}}\mu_1+\frac{t^2}{2\sigma^2 n}\mu_2+...........$$

$$=1+\frac{t}{\sigma\sqrt{n}}(0)+\frac{t^2}{2n\sigma^2}\sigma^2+.............$$

$$=1+\frac{t^2}{2n}+.................$$

Therefore, $M_{Z_n}(t)=\left[1+\frac{t^2}{2n}+..........\right]^n$

$$M_{Z_n}(t)=\left[1+\frac{t^2}{2n}+o\left(n^{-\frac{3}{2}}\right)\right]^n$$

$$M_{Z_n}(t)=\lim_{n\to\infty}\left[1+\frac{t^2}{2n}\right]^n$$

We know that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$

Comparing, we have $M_{Z_n}(t)=e^{\frac{t^2}{2}}$.

Therefore Z_n follows standard normal distribution with mean 0 and variance 1 and hence S_n follows normal distribution with mean $n\mu$ and variance $n\sigma^2$.

20. The lifetime of a certain brand of an electric bulb may be considered a RV with mean $1200h$ and standard deviation $250h$. Find the probability using central limit theorem that the average lifetime of 60 bulbs exceeds $1250h$.

Solution:

Let X_i represent the life of the bulb.

Given $E[X_i]=\mu=1200$ and $SD[X_i]=\sigma=250$

Let $\overline{X}$ denote the mean lifetime of 60 bulbs.

By central limit theorem, we have

$$\overline{X}\sim N\left(\mu,\frac{\sigma}{\sqrt{n}}\right)$$

$$\overline{X}\sim N\left(1200,\frac{250}{\sqrt{60}}\right)$$

$$P[\overline{X}>1250]=P\left[\frac{\overline{X}-1200}{\frac{250}{\sqrt{60}}}>\frac{1250-1200}{\frac{250}{\sqrt{60}}}\right]=P[Z>1.55]$$

$$=0.5-P[0<Z<1.55]=0.5-0.4392=0.0608.$$

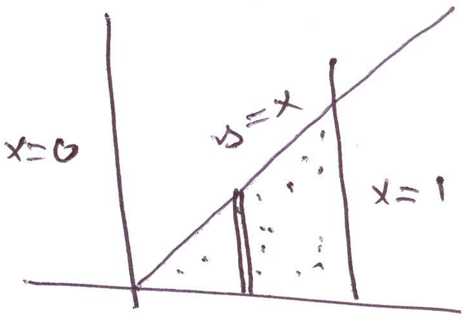
(1) The joint density function of RV X and Y is given by

$$f(x,y) = \begin{cases} 8xy & ; 0 < x < 1, \quad 0 < y < x \\ 0 & ; \text{otherwise} \end{cases}$$

Find the marginal density function of X and Y and find $P\left[Y < \frac{1}{8} / X < \frac{1}{2}\right]$, conditional density function y given x.

Soln

M D F of X



$$f(x) = \int f(x,y)\,dy$$

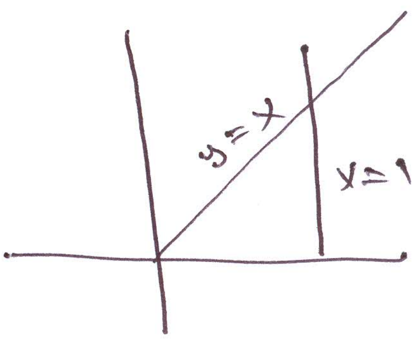
$$= \int_0^x 8xy\,dy$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x$$

$$= 8x\left[\frac{x^2}{2}\right] = 4x^3$$

$$f(x) = 4x^3 \qquad 0 \le x \le 1$$

M. D. f of y



$$f(y) = \int f(x,y)\,dx$$

$$= \int_0^1 8xy\,dx$$

$$= 8y\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{8y}{2}$$

$$= 4y$$

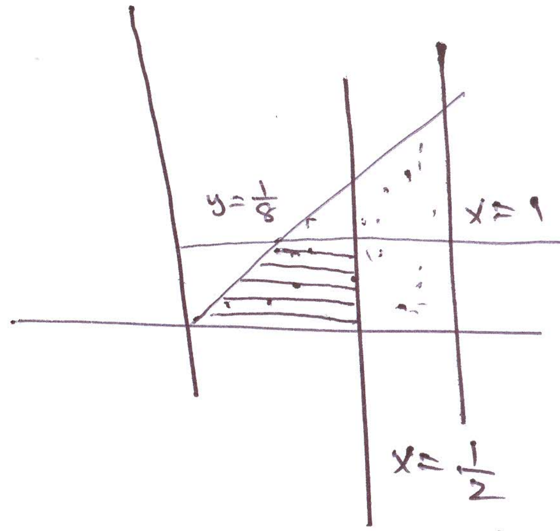
$$P\left[Y < \tfrac{1}{8} \,\middle|\, X < \tfrac{1}{2}\right]$$

$$= \frac{P\left[Y < \frac{1}{8},\ X < \frac{1}{2}\right]}{P\left[X < \frac{1}{2}\right]} \quad \text{—} \quad (1)$$

To find

$$P\left[\cancel{X} < \tfrac{1}{8},\ X < \tfrac{1}{2}\right]$$

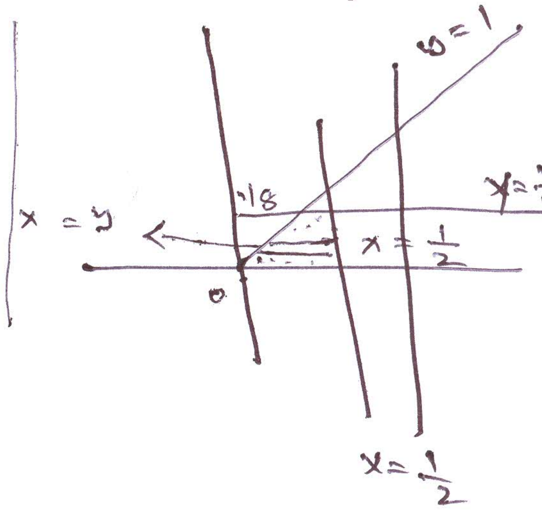
$$= \iint f(xy)\, dx\, dy$$



$$= \int_0^{1/8} \int_y^{1/2} 8xy \, dx\, dy$$

$$= \int_0^{1/8} 8y \left[\frac{x^2}{2}\right]_y^{1/2} dy$$

$$= \int_0^{1/8} \frac{8y}{2}\left[\frac{1}{4} - y^2\right] dy$$



$$= \int_0^{1/8} 4y\left[\frac{1}{4} - y^2\right] dy$$

$$= \int_0^{1/8} \frac{4y}{4} - 4y^3 \, dy$$

$$= \int_0^{1/8} y - 4y^3 \, dy$$

$$= \left[\frac{y^2}{2} - \frac{\cancel{4}y^4}{\cancel{4}}\right]_0^{1/8}$$

$$= \frac{1}{64\times 2} - \frac{1}{4096}$$

$$= \frac{32-1}{4096} = \frac{31}{4096}$$

$$P\left[x<\tfrac{1}{2}\right] = \int_0^{1/2} f(x).dx$$

$$= \int_0^{1/2} 4x^3.dx$$

$$= \left[\frac{4x^4}{4}\right]_0^{1/2}$$

$$= \frac{1}{16}$$

$$\therefore P\left[X<\tfrac{1}{8} \middle/ x<\tfrac{1}{2}\right]$$

$$= \frac{P\left[X<\tfrac{1}{8}, X<\tfrac{1}{2}\right]}{P\left[x<\tfrac{1}{2}\right]} = \frac{\frac{31}{4096}}{\frac{1}{16}}$$

$$= \frac{31\times16}{4096} = \frac{31}{256} \,//$$

Conditional density function of $f(y/x)$

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{8xy}{4x^3}$$

$$= \frac{2y}{x^2} \qquad 0<y<x$$

---

(2) The Two-Dimensional Random variable (x,y) has the joint density

$$f(xy) = \begin{cases} 8xy & : 0<x<y<1 \\ 0 & : 0 \text{ therwise} \end{cases}$$

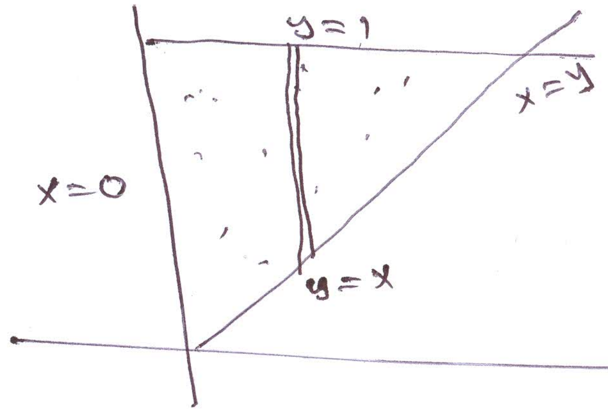
1. Find the marginal and conditional distribution (2) Find $P\left[x<\tfrac{1}{2} \cap y<\tfrac{1}{4}\right]$
(3) Are X and Y Independent.

<u>Soln</u>

Given

$f(xy) = 8xy$

M.D.F of x



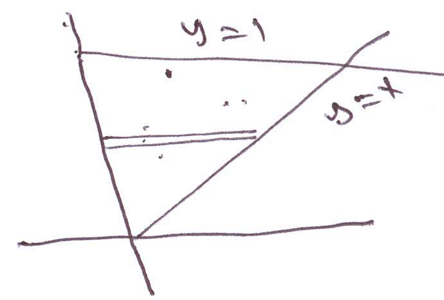
$f(x) = \int f(xy).dy$

$= \int_x^1 8xy.dy$

$= 8x \left[y^2/_2 \right]_x^1$

$= \frac{8x}{2} \left[1 - x^2 \right]$

$= 4x(1 - x^2) \; ; \quad 0 \le x \le 1$



$f(y) = \int f(xy).dx$

$= \int_0^y 8xy.dx$

$= 8y \left[\frac{x^2}{2} \right]_0^y$

$= \frac{8y}{2} \left[y^2 \right]$

$= 4y^3 \; ; \; 0 \le y \le 1$

conditional density function of x Given y

$$f(x/y) = \frac{f(xy)}{f(y)} = \frac{8xy}{4y^3}$$

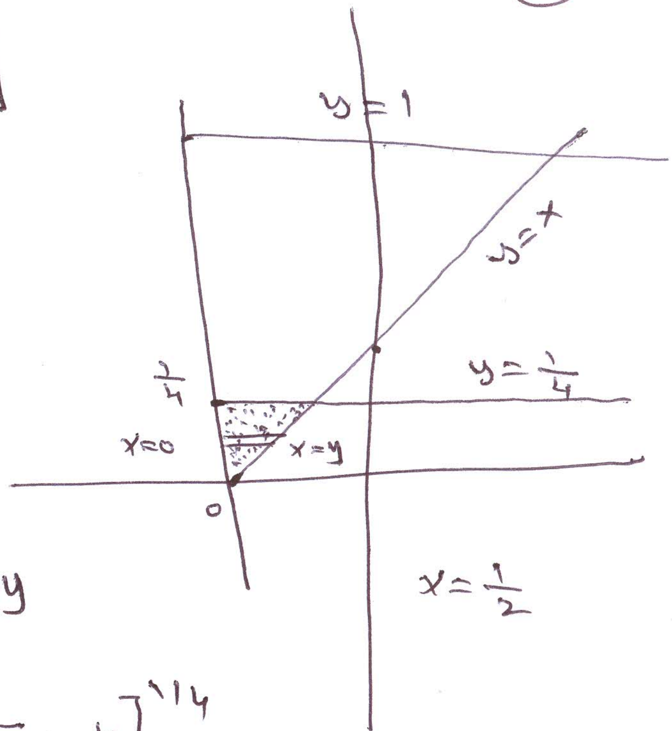
$$= \frac{8x}{4y^2} = \frac{2x}{y^2}$$

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{8xy}{4x(1-x^2)}$$

$$= \frac{8y}{4(1-x^2)} = \frac{2y}{1-x^2}$$

$P\left[X < \frac{1}{2} \cap Y < \frac{1}{4}\right]$

$$= \int_0^{1/4}\int_0^{y} 8xy\,dx\,dy$$



$$= \int_0^{1/4} 8y\left[\frac{x^2}{2}\right]_0^{y} dy$$

$$= \int_0^{1/4} \frac{8y}{2}\left[y^2 - 0\right] dy$$

$$= \int_0^{1/4} 4y^3\,dy = \left[\frac{\cancel{4}y^4}{\cancel{4}}\right]_0^{1/4}$$

$$= \left(\frac{1}{4}\right)^4 = \frac{1}{256}$$

(iii)

$$f(x).f(y) = 4x(1-x^2).4y^3$$

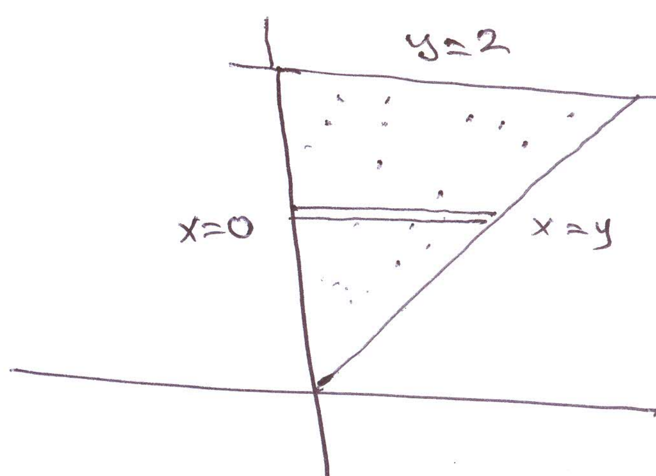
$$= 16xy^3(1-x^2)$$

$$\neq f(xy)$$

$\therefore$ x and y are not independent

---

(3) If the joint pdf of (x, y) is given by $f(xy) = K \quad 0 \le x < y \le 2$ find K and also marginal and conditional density function



Soln

$$\iint f(xy)\,dx\,dy = 1$$

$$\int_0^{2}\int_0^{y} K.dx\,dy = 1$$

⑥

$$K \int_0^2 [x]_0^y \, dy = 1$$

$$K \int_0^2 y \, dy = 1$$

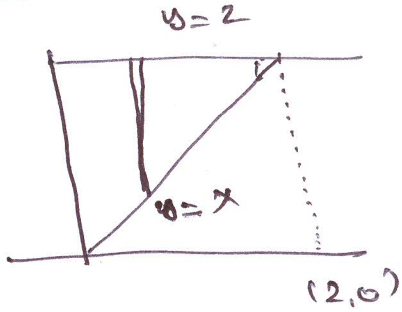
$$K \left[\frac{y^2}{2}\right]_0^2 = 1$$

$$K \left[\frac{4}{2}\right] = 1$$

$$2K = 1$$

$$\Rightarrow \underline{\underline{K = \frac{1}{2}}}$$

<u>M. D. f of x</u>



$$f(x) = \int f(xy).dy$$

$$f(x) = \int_x^2 \frac{1}{2}.dy$$

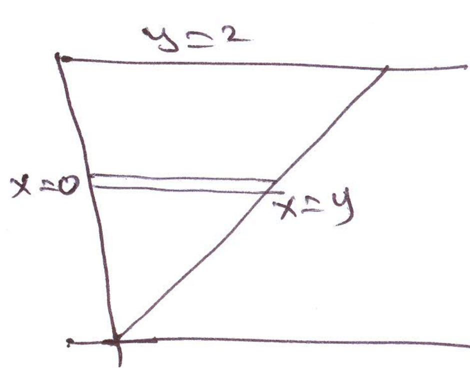
$$= 2 \, [y]_x^2$$

$$= \frac{1}{2}[2-x] = \cancel{\frac{1}{2}(2-x)}$$

$$= \underline{\underline{\cancel{4-2x}}} \qquad \cancel{0 \le x \le 2}$$

$$= \frac{1}{2}(2-x) \qquad 0 \le x \le 2$$

<u><u>M.D. f of y</u></u>



$$f(y) = \int f(xy).dx$$

$$= \int_0^y \frac{1}{2}.dx$$

$$= \frac{1}{2}[x]_0^y$$

$$= \frac{1}{2}y = \frac{y}{2} \; ; \; 0 \le y \le 2$$

conditional density function

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{\frac{1}{2}}{\frac{1}{2}(2-x)}$$

$$= \frac{1}{2-x} \qquad 0 < y < 2$$

$$f(x/y) = \frac{f(xy)}{f(y)} = \frac{\frac{1}{2}}{\frac{y}{2}} = \frac{1}{y} \qquad 0 < x < y$$

---

(4) A fair coin is tossed four times. Let X denote the number of heads obtained in the first two tosses, and Let Y denote the number of heads obtained in the last two tosses. Find the joint pmf of X and Y. Show that X and Y are independent random variables.

Soln The sample space and the value of X and Y are shown below.

S	X	Y
TTTT	0	0
TTTH	0	1
TTHT	0	1
TTHH	0	2
THTT	1	0
THTH	1	1
THHT	1	1
THHH	1	2
HTTT	1	0
HTTH	1	1
HTHT	1	1
HTHH	1	2
HHTT	2	0
HHTH	2	1
HHHT	2	1
HHHH	2	2

The joint pmf of X and Y is tabulated below

Y \ X	0	1	2	P(x)
0	1/16	2/16	1/16	4/16
1	2/16	4/16	2/16	8/16
2	1/16	2/16	1/16	4/16
P(y)	4/16	8/16	4/16	1

X	0	1	2
P(x)	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{4}{16}$

y	0	1	2
P(y)	4/16	8/16	4/16

$$P(X=0, y=0] = P[X=0] \cdot P[Y=0]$$

$$\frac{1}{16} = \frac{4}{16} \times \frac{4}{16} = \frac{16}{256} = \frac{1}{16}$$

Similarly

$$P[X=0, Y=1] = P[X=0] \cdot P[Y=1]$$

$$P[X=0, Y=2] = P[X=0] \; P[Y=2]$$

$$P[X=1 \; Y=0] = P[X=1] \; P[Y=0]$$

$$P[X=1 \; Y=1] = P[X=1] \cdot P[Y=1]$$

$$P[X=1, Y=2] = P[X=1] \cdot P[Y=2]$$

$$P[X=2, y=0] = P[X=2] \; P[Y=0]$$

$$P[X=2, y=1] = P[X=2] \cdot P[Y=1]$$

$$P[X=2 \cancel{P}[y=2] = P[X=2] \cdot P[Y=2]$$

for all value at $P(X=x, Y=y] = P[X=x] \cdot P[y=y]$

$\therefore$ X and Y are statistically independent.

Regression curves of the mean

(i) $E[Y/X=x]$ is called the Regression function of Y on X.

(ii) $E[X|Y=y]$ is called the Regression function of X on Y.

The Regression curve y on x is

$$\boxed{Y = E[Y|X=x]}$$

The Regression curve x on y is

$$\boxed{X = E[X|Y=y]}$$

Problem 1.

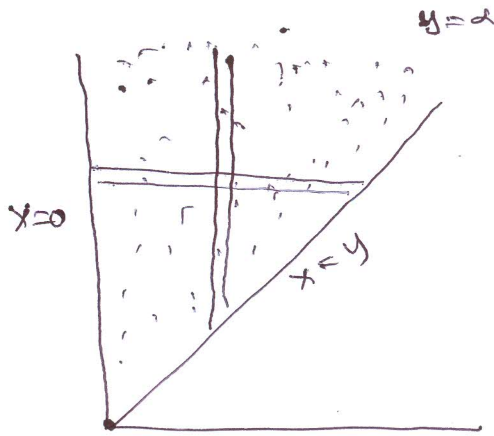
The joint p.d.f of a two dimensional Random variable (X,Y) is

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the regression curves of means.

Sol)

First To find the marginal p.d.f of x and y.



$$f(x) = \int f(xy)\,dy$$

$$= \int_x^{\infty} e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_x^{\infty}$$

$$= \left[0 - \left[\frac{e^{-x}}{-1}\right]\right]$$

$$f(x) = e^{-x}, \quad 0 \le x \le \infty$$

$$f(y) = \int_0^y e^{-y} \, dx$$

$$= \left[e^{-y} [x]_0^y \right]$$

$$= e^{-y} [y - 0] = y e^{-y}$$

$$f(y) = y e^{-y} \qquad 0 \le y \le \infty$$

conditional density function

$$f(y/x) = \frac{f(x\,y)}{f(x)} = \frac{e^{-y}}{e^{-x}}$$

$$= e^{x-y} \qquad 0 < x < y < \infty$$

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}}{y e^{-y}}$$

$$= \frac{1}{y} \qquad 0 < x < y < \infty$$

The Regression curve of y on x is given by

$$y = E[Y/x = x] \qquad \underline{0 < x < y < \infty}$$

$$= \int y \, f(y/x) \, dy$$

$$= \int_x^\infty y \, e^{x-y} \, dy$$

$$= e^x \int_x^\infty y e^{-y} \, dy$$

$$= e^x \left[\frac{y e^{-y}}{-1} - e^{-y} \right]_x^\infty$$

$$= e^x \left[0 - [-x e^{-x} - e^{-x}] \right]$$

$$= e^x . e^{-x} [x + 1]$$

$$\underline{y = x + 1}$$

$y = x+1$ is regression curve y on x

The Regression curve x on y is

$$x = E[X/y=y] \qquad \underline{0 \le x \le y \le d}$$

$$x = \int_0^y x . f(x/y) . dx$$

$$x = \int_0^y x . \frac{1}{y} . dx$$

$$= \frac{1}{y} \left[\frac{x^2}{2} \right]_0^y$$

$$= \frac{1}{y} \left[\frac{y^2}{2} \right]$$

$$x = \frac{y}{2}$$

$$2x = y$$

$$\underline{2x - y = 0}$$

$2x - y = 0$ is the regression curve of x on y

---

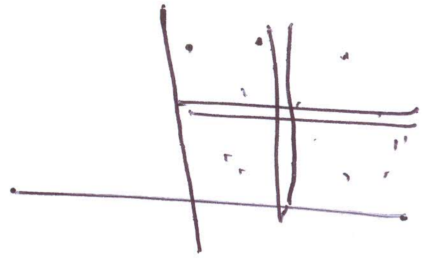
(2) The joint p.d.f of a two dimensional random variable is given by

$$f(x,y) = \begin{cases} x e^{-x(y+1)} & x \geq 0, \ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that the regression of y on x is not linear.

<u>Soln</u>

First find the marginal density function of x and y



$$f(x) = \int f(x,y)\,dy$$

$$= \int_0^{\infty} x\, e^{-xy} \cdot e^{-x} \cdot dy$$

$$= x e^{-x} \int_0^{\infty} e^{-xy} \cdot dy$$

$$= \cancel{x} e^{-x} \left[\frac{e^{-xy}}{-\cancel{x}} \right]_0^{\infty}$$

$$= e^{-x} [0 - 1]$$

$$f(x) = e^{-x}, \quad x \geq 0$$

$$f(y) = \int_0^{\infty} f(x,y) \cdot dx$$

$$= \int_0^{\infty} x\, e^{-xy} \cdot e^{-x} \cdot dx$$

$$= \int_0^{\infty} x\, e^{-x(y+1)} \cdot dx$$

$$= \left[x \frac{e^{-x(y+1)}}{-(y+1)} - \frac{e^{-x(y+1)}}{(y+1)^2} \right]_0^{\infty}$$

$$= \left[0 - \left(0 - \frac{1}{(y+1)^2} \right) \right]$$

$$f(y) = \frac{1}{(y+1)^2}, \quad y \geq 0$$

The conditional pdf ~~of~~ y on x is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{x\, e^{-x(y+1)}}{e^{-x}}$$

$$= x\, e^{-xy} \cdot e^{-x} \cdot e^{x}$$

$$f(y/x) = x\, e^{-xy} \qquad x, y \geq 0$$

The Regression curve Y on X is given by

$$y = E[Y/X=x]$$

$$= \int_0^\infty y\, f(y/x)\, dy$$

$$= \int_0^\infty y \cdot x e^{-xy}\, dy$$

$$= x \int_0^\infty y e^{-xy}\, dy$$

$$= x \left[\frac{y e^{-xy}}{-\cancel{x}} - \frac{e^{-xy}}{\cancel{x^2}} \right]_0^\infty$$

$$= x \left[(0-0) - \left(0 - \frac{1}{\cancel{x^2}}\right) \right]$$

$$y = x \left[\frac{1}{x^2} \right]$$

$$y = \frac{1}{x}$$

$$\underline{\underline{xy = 1}}$$

$\boxed{\because xy = c^2 \text{ is RH}}$

$\therefore xy = 1$ which is a rectangular hyperbola.

Hence the Regression of Y on X is not linear.

---

(3) The joint P.d.f of a Two dimensional Random variable is given by

$$f(x,y) = \frac{1}{3}(x+y) \quad 0 \le x \le 1, \quad 0 \le y \le 2$$

Find the two regression curves for mean.

Soln

First find the marginal P.d.f of x and y

$$f(x) = \int f(xy).dy$$

$$= \int_0^2 \frac{1}{3}(x+y).dy$$

$$= \frac{1}{3}\left[xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{3}\left[2x + \frac{4}{2}\right]$$

$$= \frac{1}{3}\left[2x+2\right] = \frac{2}{3}(x+1) , 0 \le x \le 1$$

$$f(y) = \int f(xy).dx$$

$$= \int_0^1 \frac{1}{3}[x+y].dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + xy\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2} + y\right]$$

$$= \frac{1}{6}\left[1+2y\right] . \quad 0 \le y \le 2$$

conditional density function

$$f(x/y) = \frac{f(xy)}{f(y)} = \frac{\frac{1}{3}[x+y]}{\frac{1}{6}[1+2y]}$$

$$= \frac{1}{3} \times \frac{6}{1} \left[\frac{x+y}{1+2y}\right]$$

$$f(x/y) = \frac{2[x+y]}{1+2y}$$

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{\frac{1}{3}[x+y]}{\frac{2}{3}[x+1)}$$

$$f(y/x) = \frac{1}{2}\frac{[x+y]}{x+1}$$

The Regression curve y on x

$$y = E[y|x) = \int_0^{\infty} y\, f(y/x).dy$$

$$= \int_0^{2} y \, \frac{1}{2} \frac{(x+y)}{x+1} \, dy$$

$$= \frac{1}{2(x+1)} \int_0^{2} xy + y^2 . dy$$

$$= \frac{1}{2[x+1)} \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_0^2$$

$$= \frac{1}{2(x+1)} \left[2x + 8/3 \right]$$

$$= \frac{1}{2(x+1)} \frac{1}{3} \left[6x + 8 \right]$$

$$= \frac{1}{6(x+1)} 2[3x+4]$$

$$= \frac{2}{6} \frac{3x+4}{x+1}$$

$$y = \frac{1}{3} \frac{[3x+4]}{(x+1)}$$

The Regression curve x on y is

$$x = E[x|y=y]$$

$$= \int x \, f(x/y).dx$$

$$= \int_0^1 x \, \frac{2(x+y)}{1+2y}.dx$$

$$= \frac{2}{1+2y} \int_0^1 x^2 + xy . dx$$

$$= \frac{2}{1+2y}\left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_0^1$$

$$= \frac{2}{1+2y}\left[\frac{1}{3}+\frac{y}{2}\right]$$

$$= \frac{2}{1+2y}\left[\frac{2+3y}{6}\right] =$$

$$x = \frac{2(2+3x)}{6(1+2y)}$$

$$x = \frac{(2+3x)}{3(1+2y)}$$

---

(4) The joint p.d.f of (x,y) is given by

$$f(x,y) = \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}} \quad x \geq 0,\ y \geq 0$$

Obtain the Regression equation of y on x.

soln The Regression equation y on x is

$$y = E[y/x] = \int y\, f(y/x).dy.$$

$$f(y/x) = \frac{f(xy)}{f(x)}$$

First find the M.D.f of x

$$f(x) = \int_0^{\infty} f(xy).dy$$

$$= \int_0^{\infty} \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}}.dy$$

$$= \frac{1}{(1+x)^4}\int_0^{\infty} y.\, e^{\frac{-y}{1+x}}.dx$$

$$= \frac{1}{(1+x)^4}\left[y \cdot \frac{\frac{e^{-y}}{1+x}}{\frac{-1}{1+x}} - \frac{\frac{e^{-y}}{1+x}}{\frac{1}{(1+x)^2}} \right]_0^{\infty}$$

$$= \frac{1}{(1+x)^4}\left[0 - \left(0 - \frac{1}{\frac{1}{(1+x)^2}}\right) \right]$$

$$= \frac{1}{(1+x)^4}\left[(1+x)^2 \right]$$

$$= \frac{1}{(1+x)^2} \qquad 0 \leq x \leq \infty$$

conditional density function of $f(y/x)$

$$f(y/x) = \frac{f(xy)}{f(x)}$$

$$= \frac{\frac{y}{(1+x)^4} e^{-\frac{y}{1+x}}}{\frac{1}{(1+x)^2}}$$

$$= \frac{(1+x)^2 \, y \, e^{-\frac{y}{1+x}}}{(1+x)^4}$$

$$= \frac{y}{(1+x)^2} e^{-\frac{y}{1+x}}$$

y on x is

$$y = E[y/x] = \int_0^{\infty} y \, f(y/x) \, dy$$

$$= \int_{0}^{\infty} y \, f(y/x) \, dy$$

$$= \int_{0}^{\infty} y \cdot \frac{y}{(1+x)^2} e^{-\frac{y}{1+x}} \, dy$$

$$= \frac{1}{(1+x)^2} \int_{0}^{\infty} y^2 e^{-\frac{y}{1+x}} \, dy$$

$$= \frac{1}{(1+x)^2} \left[y^2 \frac{e^{-y}}{-\frac{1}{1+x}} - 2y \frac{e^{-\frac{y}{1+x}}}{\frac{1}{(1+x)^2}} + 2 \frac{e^{-\frac{y}{1+x}}}{-\frac{1}{(1+x)^3}} \right]_{0}^{\infty}$$

$$= \frac{1}{(1+x)^2} \left[(0 - 0 + 0) - \left(0 - 0 + 2 \frac{1}{-\frac{1}{(1+x)^3}} \right) \right]$$

$$= \frac{1}{(1+x)^2} \left[\frac{2}{\frac{1}{(1+x)^3}} \right]$$

$$y = \frac{2(1+x)^{\cancel{3}}}{\cancel{(1+x)^2}}$$

$$y = 2(1+x)$$

which is the Regression equation of Y on X.

---

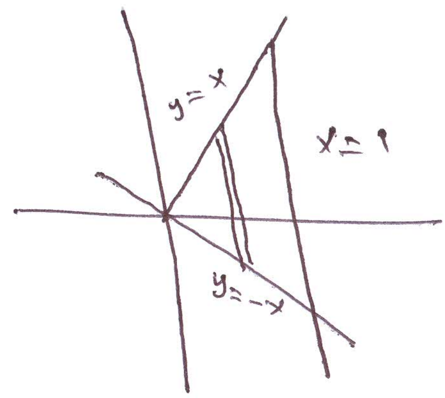
(5) The joint P.d.f of the Random variable (xy) is

$$f(xy) = \begin{cases} 1 & 0 < x < 1, \ |y| < x \\ 0 & \text{otherwise} \end{cases}$$

Show that the Regression curve Y on X is Linear. but regresion curve X on Y is not Linear

Soln

$$f(x,y) = \begin{cases} 1 & 0<x<1 \quad -x<y<x \\ 0 & \text{otherwise} \end{cases}$$

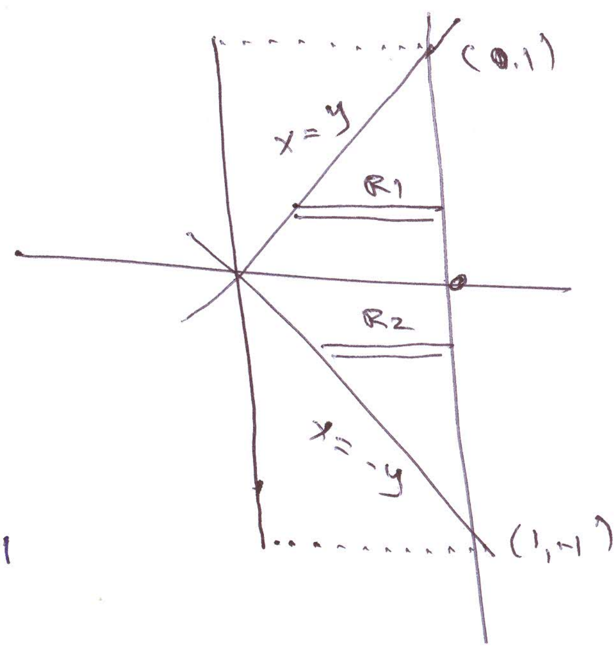


$$f(x) = \int f(x,y)\,dy$$

$$= \int_{-x}^{x} dy = 2\int_0^{x} dy$$

$$= 2[y]_0^{x} = 2x$$

$$f(x) = 2x \qquad 0 \le x \le 1$$



$$f(y) = \int f(x,y)\,dx$$

If $y>0$ (R_1)

$$f(y) = \int_y^1 dx = [x]_y^1$$

$$= \underline{\underline{1-y}}$$

$$f(y) = 1-y \qquad 0 \le y \le 1$$

If $y<0$ (R_2)

$$f(y) = \int_{-y}^{1} dx = [x]_{-y}^{1} = 1+y \qquad -1 \le y \le 0$$

$$\therefore f(y) = \begin{cases} 1-y & 0 \le y \le 1 \\ 1+y & -1 \le y \le 0 \end{cases}$$

Now $f(x/y) = \dfrac{f(x,y)}{f(y)} =$

$$f(x/y) = \begin{cases} \dfrac{1}{1-y} & 0<y<1, \quad y<x<1 \\ \dfrac{1}{1+y} & -1 \le y < 0 \quad -y \le x \le 1 \end{cases}$$

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{1}{2x} \qquad 0 \le x \le 1 \qquad |y| \le x$$

$$y = E[Y/X=x] = \int_{-x}^{x} y\, f(y/x).dy$$

$$= \int_{-x}^{x} y \frac{1}{2x} dy$$

$$= \cancel{y} \frac{1}{2x} \left[\frac{y^2}{2}\right]_{-x}^{x} = 0$$

$\therefore$

$$y = 0$$

The Regression curve y on x is $y=0$ which is a linear.

$$x = E[X/Y=y] = \cancel{\int y f(x/}$$ If $0 < y < 1$

$$= \int x\, f(x/y).dx$$

$$= \int_0^1 x \frac{1}{1-y} dx$$

$$= \frac{1}{1-y} \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2(1-y)} \qquad 0 < y < 1$$

$E[X/Y=y]$ If $-1 < y < 0$

$$x = E[X/Y=y] = \int x\, f(x/y).dx$$

$$= \int_{-1}^{0} x \frac{1}{1+y} .dx$$

$$= \frac{1}{1+y} \left[\frac{x^2}{2}\right]_{-1}^{0}$$

$$= \frac{1}{1+y}\left[0 - \frac{1}{2}\right] = \frac{1}{2(1+y)}$$

Regression curve x on y is

$$x = \begin{cases} \dfrac{1}{2(1-y)} & 0<y<1 \\ \dfrac{1}{2(1+y)} & -1<y<0 \end{cases}$$

which is not Linear

---

(6) For the joint p.d.f ~~f(x,y) = { 1/2 x~~

$$f(xy) = \begin{cases} \frac{1}{2} x^3 e^{-x(y+1)} & ; \; y \geq 0, \; x \geq 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

Find the regression curves for means.

Soln

first find the M.d.f of x and y

$$f(x) = \int f(xy).dy$$

$$= \int_0^\infty \frac{1}{2} x^3 . e^{-x(y+1)} .dy$$

$$= \frac{1}{2} x^3 \int_0^\infty e^{-x(y+1)} .dy$$

$$= \frac{1}{2} x^3 \left[\frac{e^{-x(y+1)}}{-x} \right]_0^\infty$$

$$= \frac{1}{2} x^3 \left\{ 0 - \frac{1}{-x} e^{-x[0+1]} \right]$$

$$= \frac{1}{2} x^3 \left[\frac{e^{-x}}{x} \right]$$

$$= \frac{1}{2} x^2 e^{-x} \quad ; \; x \geq 0$$

$$f(y) = \int f(xy).dx$$

$$f(y) = \int_0^\infty f(xy).dx$$

$$= \int_0^\infty \frac{1}{2} x^3 e^{-x(y+1)} .dx$$

$$= \frac{1}{2} \int_0^\infty x^3 e^{-x(y+1)} .dx$$

$$= \frac{1}{2} \left[x^3 \frac{e^{-x(y+1)}}{-(y+1)} - \frac{3x^2 e^{-x(y+1)}}{(y+1)^2} + 6x \frac{e^{-x(y+1)}}{-(y+1)^3} - \frac{6 e^{-x(y+1)}}{(y+1)^4} \right]_0^\infty$$

$$= \frac{1}{2} \left[0 - \left(0 - 0 + 0 - \frac{6}{(y+1)^4} \right] \right.$$

$$= \frac{6}{2} \frac{1}{(y+1)^4}$$

$$f(y) = \frac{3}{(y+1)^4} \qquad y \geq 0$$

conditional density function

$$f(x/y) = \frac{f(xy)}{f(y)} = \frac{\frac{1}{2} x^3 e^{-x(y+1)}}{\frac{3}{(y+1)^4}}$$

$$= \frac{1}{6} (y+1)^4 x^3 e^{-x(y+1)}$$

$$f(y/x) = \frac{f(xy)}{f(x)} = \frac{\cancel{\frac{1}{2} x^3} e^{-x(y+1)}}{\cancel{\frac{1}{2} x^3} e^{-x}}$$

$$= \frac{\cancel{d}\, x e^{-xy} . e^{-x}}{e^{-x}} = x e^{-xy}$$

$$f(y/x) = x e^{-xy}$$

Regression curve y on x

$$y = E[Y/X=x]$$

$$= \int y\, f(y/x)\, dy$$

$$= \int_0^{\alpha} y\, x\, e^{-xy}\, dy$$

$$= x \int_0^{\alpha} y\, e^{-xy}\, dy$$

$$= x \left[\frac{y\, e^{-xy}}{-x} - \frac{e^{-xy}}{x^2} \right]_0^{\alpha}$$

$$y = x\left[0 - \left(\frac{1}{x^2} \right) \right]$$

$$y = \frac{x}{x^2}$$

$$y = \frac{1}{x}$$

$$\boxed{xy = 1} \qquad \boxed{R.H.}$$

$$x = E[X/Y=y]$$

$$= \int x\, f(x/y)\, dx$$

$$= \int_0^{\alpha} x\, \frac{1}{6} (y+1)^4 x^3 \cdot e^{-x(y+1)}\, dx$$

$$= \frac{(y+1)^4}{6} \int_0^{\alpha} x^4\, e^{-x(y+1)}\, dx$$

$$= \cancel{\frac{(y+1)^4}{6} \left[x^4 \frac{e^{-x(y+1)}}{-(y+1)} - 4x^3 \right.}$$

$$= \frac{(y+1)^{\cancel{5}4}}{6}\left[x^4 \frac{e^{-x(y+1)}}{-(y+1)} - 4x^3 \frac{e^{-x(y+1)}}{(y+1)^2} + 12x^2 \frac{e^{-x(y+1)}}{-(y+1)^3} - \frac{24x\, e^{-x(y+1)}}{(y+1)^4} + 24 \frac{e^{-x(y+1)}}{-(y+1)^5}\right]_0^\infty$$

$$= \cancel{(y+1)}^{5}$$

$$= \frac{(y+1)^4}{6}\left[0 - \left(\frac{24}{-(y+1)^5}\right)\right]$$

$$x = \frac{(y+1)^{\cancel{4}}}{6(y+1)^{\cancel{5}}} \cdot 24$$

$$x = \frac{\cancel{24}}{\cancel{6}(y+1)}$$

$$x(y+1) = 4$$

Two function of Two Random variable:

If (X, Y) is a two-dimensional RV with joint p.d.f $f(x,y)$ and if $u = g(x,y)$ and $v = h(xy)$ are two Random variable, Then the joint pdf of (u,v) is given by

$$f(uv) = |J|\, f(xy)$$

$$\text{where } J = \frac{\partial(xy)}{\partial(uv)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(1) If X and Y each follows an exponential Distribution with parameter 1 and are Independent. Find the Pdf of $U = X - Y$.

Soln

If X and Y follows an exponential distribution with parameter 1 we have

$$f(x) = e^{-x}, \quad x \geq 0$$

$$f(y) = e^{-y}, \quad y \geq 0$$

Also X and Y are Independent given as

$$f(x,y) = f(x).f(y)$$

$$= e^{-x}.e^{-y}$$

$$= e^{-(x+y)}, \quad x \geq 0,\ y \geq 0$$

Let $U = X - y$ and $v = y$

$$x = u + y$$

$$x = u + v \qquad y = v$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = 1 \qquad \frac{\partial y}{\partial v} = 1$$

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint P.d.f of u and v is

$$f(u,v) = f(x,y)\,|J|$$

$$= e^{-(x+y)}\,1 \qquad \because\ x = u+v$$

$$= e^{-[u+v+v]} \qquad \because\ y = v$$

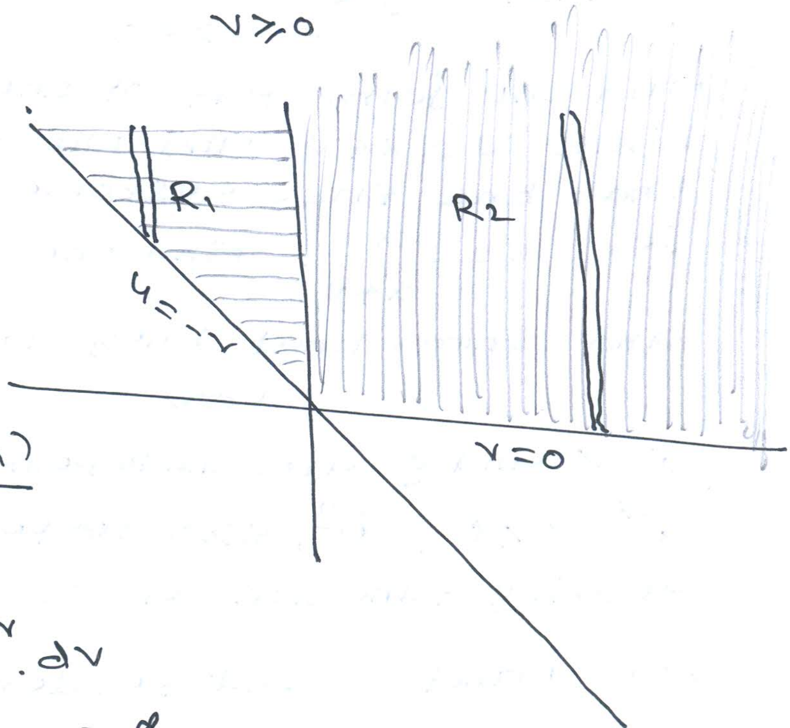
$$= e^{-(u+2v)}$$

<u>Range</u>

$x \geq 0$ $\qquad$ $y \geq 0$

$u+v \geq 0$ $\qquad$ $v \geq 0$

$u \geq -v$



P.d.f of u is

case(i) $\underline{u < 0 \quad (R_1)}$

$$f(u) = \int_{-u}^{\infty} f(u,v)\,dv$$

$$= \int_{-u}^{\infty} e^{-u} \cdot e^{-2v} \cdot dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{-u}^{\infty}$$

$$= e^{-u}\left[0 - \frac{e^{2u}}{-2}\right]$$

$$= \frac{e^{-u}\,e^{2u}}{-2} = \frac{e^{2u-u}}{2} = \frac{e^{u}}{2} \qquad u \leq 0$$

case(ii) $\underline{u > 0 \quad (R_2)}$

$$f(u) = \int_{0}^{\infty} e^{-u} \cdot e^{-2v} \cdot dv$$

$$= e^{-u}\left[\frac{e^{-2v}}{-2}\right]_{0}^{\infty} = e^{-u}\left[0 - \frac{1}{-2}\right] = \frac{e^{-u}}{2}$$

$\therefore$ Pdf of u is

$$f(u) = \begin{cases} \dfrac{e^{u}}{2} & u < 0 \\ \dfrac{e^{-u}}{2} & u \geq 0 \end{cases}$$

---

(2) The waiting times X and Y of two customers entering a bank at different times are assumed to be independent random variable with pdf

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Find the joint pdf of the sum of the waiting time $U = X+Y$ and the fraction of this time that the first customer spreads waiting (ie) $V = \dfrac{X}{X+Y}$. Find the marginal pdf's of u and v and show that they are independent.

(or)

If x and y are Independent Rvs with pdf's $e^{-x}, x \geq 0$, $e^{-y}, y \geq 0$ respectively. Find the density function of $U = \dfrac{x}{x+y}$ and $V = x+y$. Are u and v Independent.

Soln

Given $f(x) = e^{-x}.\ x \geq 0$

$f(y) = e^{-y} \quad y \geq 0$

x and y are Independent

$$f(xy) = f(x)\,f(y)$$
$$= e^{-x}\,e^{-y}$$
$$= e^{-(x+y)} \qquad x \geq 0,\ y \geq 0$$

Let $u = x+y \qquad v = \dfrac{x}{x+y}$

$$v = \frac{x}{u}$$

$$\Rightarrow x = uv$$

$$x+y=u$$
$$y=u-x$$
$$y=u-uv$$

$\therefore \quad x=uv \qquad\qquad y=u-uv$

$$\frac{\partial x}{\partial u}=v \qquad\qquad \frac{\partial y}{\partial u}=1-v$$

$$\frac{\partial x}{\partial v}=u \qquad\qquad \frac{\partial y}{\partial v}=-u$$

$$J=\frac{\partial(xy)}{\partial(uv)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix} v & u \\ 1-v & -u\end{vmatrix}$$

$$=-uv-u(1-v)$$
$$=-\cancel{uv}-u+\cancel{uv}$$
$$=-u$$

The joint p.d.f of (uv) is

$$f(uv)=f(xy)\,|J|$$
$$=e^{-(x+y)}\,|-u|$$
$$=u\,e^{-(uv+u-uv)}$$
$$=u\,e^{-u}$$

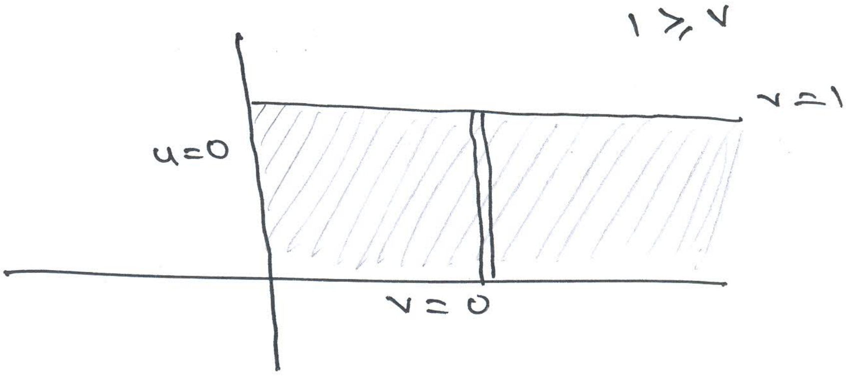
Range

$x\geq 0 \qquad\qquad y\geq 0$

$uv\geq 0 \qquad\qquad u-uv\geq 0$

$u\geq 0$ (or) $v\geq 0 \qquad\qquad u\geq uv$

$1\geq v$



The pdf of u is

$$f(u) = \int f(u,v).dv$$

$$= \int_0^1 u\, e^{-u}.dv$$

$$= u\, e^{-u} [v]_0^1$$

$$= u\, e^{-u} \qquad u \geq 0$$

The P.d.f of v is

$$f(v) = \int f(u,v).du$$

$$= \int_0^\infty u\, e^{-u}.du$$

$u = u$ $\quad dv = e^{-u}$

$u' = 1$ $\quad v = \frac{e^{-u}}{-1}$

$v_1 = \frac{e^{-u}}{1}$

$$= \left[\frac{u\, e^{-u}}{-1} - \frac{e^{-u}}{1}\right]_0^\infty$$

$$= (0) - (0 - 1)$$

$$= 1 \qquad 0 \leq v \leq 1$$

Test of u and v are Independent

$$f(u,v) = f(u).f(v)$$

$$= u\, e^{-u}.1$$

$$= f(u,v)$$

$\therefore$ Hence u and v are Independent.

---

(3) If x and y are Independent Rv's having the density $f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ and ~~$g(y) = \begin{cases} 3e^{-3y} \\ 0 \end{cases}$~~

$g(y) = \begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & y < 0 \end{cases}$. Find the density function of their sum $U = x + y$

(or)

The joint density of x_1 and x_2 is given by

$$f(x_1 x_2) = \begin{cases} 6\, e^{-3x_1 - 2x_2} & x_1 \geq 0 \quad x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = X_1 + X_2$

Soln

Assume $Y = u$

$u = x_1 + x_2 \qquad v = x_2$

$x_1 = u - x_2$

$x_1 = u - v \qquad x_2 = v$

$\frac{\partial x_1}{\partial u} = 1 \qquad \frac{\partial x_2}{\partial u} = 0$

$\frac{\partial x_1}{\partial v} = -1 \qquad \frac{\partial x_2}{\partial v} = 1$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The Joint density function of u and v is

$f(uv) = f(x_1 x_2)|J|$

$= 6\, e^{-3x_1 - 2x_2} \cdot 1$

$= 6\, e^{-3(u-v) - 2(v)}$

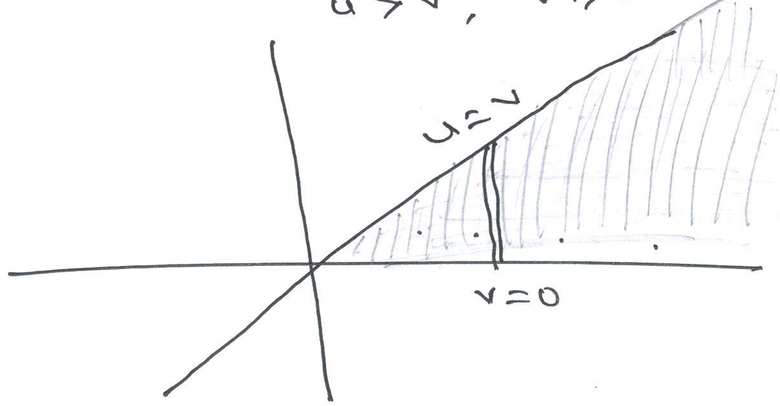
$= 6\, e^{-3u + 3v - 2v}$

$= 6\, e^{-3u + v}$

Range $\quad x_1 \geq 0 \qquad x_2 \geq 0$

$u - v \geq 0 \qquad v \geq 0$

$u > v, \quad v \geq 0$



The probability density function u is

$$f(u) = \int f(u,v)\,dv$$

$$= \int_0^u 6e^{-3u}.e^{v}.dv$$

$$= 6e^{-3u}\left[e^{v}\right]_0^u$$

$$= 6e^{-3u}\left[e^{u}-1\right]$$

$$= 6e^{-3u+u} - 6e^{-3u}$$

$$= 6e^{-2u} - 6e^{-3u}$$

$$\therefore f(u) = 6e^{-2u} - 6e^{-3u}, \quad u \geq 0$$

---

(4) The joint P.d.f of x and y is given by $f(x,y) = e^{-(x+y)}$, $x \geq 0$, $y \geq 0$ find the P.d.f of $U = \frac{x+y}{2}$.

Soln

Let $u = \frac{x+y}{2}$ $\quad v = y$

$$2u = x+y$$

$$\Rightarrow x = -y + 2u \qquad y = v$$

$$x = -v + 2u$$

$$x = 2u - v$$

$$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

The jpdf of ~~f(x,y)~~ f(u,v) is

$$f(u,v) = f(x,y)\,|J|$$

$$= e^{-(x+y)}.2 = 2e^{-(2u-v+v)}$$

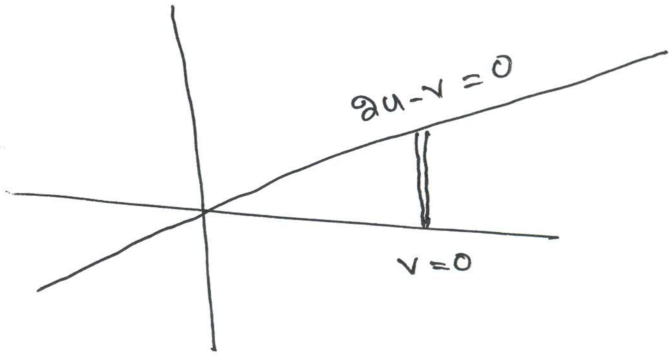
$$= 2e^{-2u}.$$

Range

$x \geq 0 \qquad y \geq 0$

$2u - v \geq 0 \qquad v \geq 0$

$2u \geq v$



The pdf of u is

$$f(u) = \int_0^{2u} f(u,v)\,dv$$

$$= \int_0^{2u} 2e^{-2u}\,.dv = 2e^{-2u}\left[v\right]_0^{2u}$$

$$= 2e^{-2u}\left[2u\right]$$

$$= 4u\,e^{-2u}, \quad u \geq 0$$

---

(5) The joint pdf of Random variable (X,Y) is given as $f(x,y) = \begin{cases} e^{-(x+y)}, & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$ Find the distribution of $\frac{1}{2}(X-Y)$

Soln

Let $u = \frac{1}{2}(x-y) \qquad y = v$

$2u = x - y$

$x = 2u + y \qquad y = v$

$x = 2u + v$

$\frac{\partial x}{\partial u} = 2 \qquad \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = 1 \qquad \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

The joint p.d.f of (u,v) is

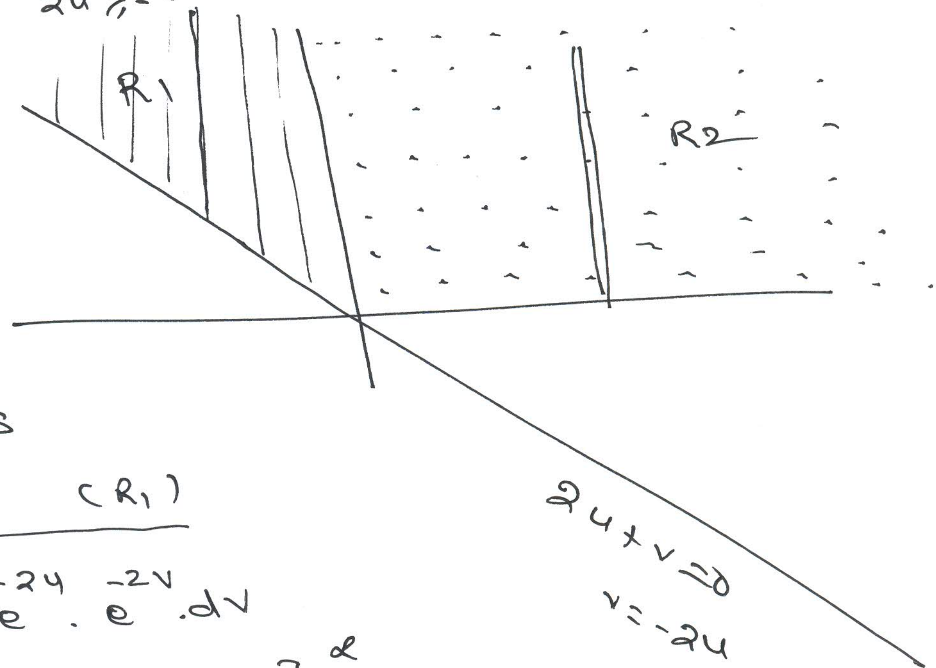
$$f(u,v) = f(x,y)\,|J|$$
$$= e^{-(x+y)}\,(2)$$
$$= 2\,e^{-[2u+v+v]}$$
$$= 2\,e^{-[2u+2v]}$$

Range

$x \geq 0$ $y \geq 0$

$2u+v \geq 0$ $v \geq 0$

$2u \geq -v$ $v \geq 0$



MDF of u is

case (i) $u < 0$ (R_1)

$$f(u) = \int_{-2u}^{\infty} 2\,e^{-2u} \cdot e^{-2v}\,dv$$
$$= 2\,e^{-2u}\left[\frac{e^{-2v}}{-2}\right]_{-2u}^{\infty}$$
$$= \cancel{2}\,e^{-2u}\left[0 - \frac{e^{4u}}{-\cancel{2}}\right]$$
$$= e^{-2u} \cdot e^{4u} = e^{2u} \quad u < 0$$

case (ii) $u > 0$ (R_2)

$$f(u) = \int_{0}^{\infty} 2\,e^{-2u} \cdot e^{-2v}\,dv = 2\,e^{-2u}\left[\frac{e^{-2v}}{-2}\right]_0^{\infty}$$
$$= 2\,e^{-2u}\left[0 - \frac{1}{-2}\right]$$
$$= e^{-2u} \quad u \geq 0$$

$\therefore$ Marginal density function of u is

$$f(u) = \begin{cases} e^{2u} & u < 0 \\ e^{-2u} & u \geq 0 \end{cases} \quad \text{(or)}$$

$$f(u) = e^{-2|u|} \qquad -\alpha \leq u \leq \alpha$$

---

(6) Suppose that X and Y are independent random variable having the pdf $f(x) = e^{-x}$ $x \geq 0$, $f(y) = e^{-y}$, $y \geq 0$ Find the pdf of Random variable $u = X/y$

Soln

If X and Y Independent Random variable

$$f(x,y) = f(x).f(y)$$
$$= e^{-x}.e^{-y}$$
$$= e^{-(x+y)} \qquad x \geq 0,\ y \geq 0$$

Let $u = \frac{x}{y}$ $\qquad v = y$

$$uy = x$$
$$x = uv \qquad \because y = v$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

The joint pdf of (u,v) is

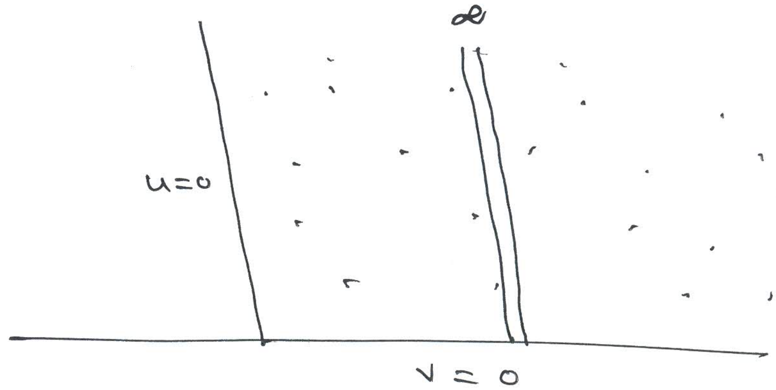
$$f(u,v) = f(x,y)\,|J|$$
$$= e^{-(x+y)}\,v$$
$$= v\,e^{-(uv+v)}$$

Range

$x \geq 0$ $\qquad y \geq 0$

$uv \geq 0$ $\qquad v \geq 0$

$u \geq 0,\ v \geq 0.$



The pdf of u is

$$f(u) = \int_0^u f(u,v)\,dv = \int_0^\infty v\, e^{-(uv+v)}\,dv$$

$$= \int_0^\infty v\, e^{-(u+1)v}\,dv$$

$u = v$ $\qquad dv = e^{-(u+1)v}$

$u' = 1$ $\qquad v = \dfrac{e^{-(u+1)v}}{-(u+1)}$

$\qquad v_1 = \dfrac{e^{-(u+1)v}}{(u+1)^2}$

$$= \left[v\,\frac{e^{-(u+1)v}}{-(u+1)} - \frac{e^{-(u+1)v}}{(u+1)^2} \right]_0^\infty$$

$$= \left[0 - \left(0 - \frac{1}{(u+1)^2} \right) \right]$$

$$f(u) = \frac{1}{(u+1)^2} \qquad u \geq 0$$

---

(7) If X and Y are Independent uniform (rectangular) variable in (0,1) find the density function of $U = XY$

Soln

Since X and Y are uniform variable Then pdf's are

$$f(x) = \frac{1}{1-0} = 1 \qquad 0 \leq x \leq 1$$

$\therefore$ Marginal density function of u is

$$f(u) = \begin{cases} e^{2u} & u < 0 \\ e^{-2u} & u \geq 0 \end{cases} \quad \text{(or)}$$

$$f(u) = e^{-2|u|} \quad -\alpha \leq u \leq \alpha$$

---

(6) Suppose that X and Y are independent random variable having the pdf $f(x) = e^{-x}$ $x \geq 0$, $f(y) = e^{-y}$, $y \geq 0$ Find the pdf of Random variable $u = X/y$

Soln

If X and Y Independent Random variable

$$f(x,y) = f(x).f(y)$$
$$= e^{-x}.e^{-y}$$
$$= e^{-(x+y)} \quad x \geq 0, y \geq 0$$

Let $u = \frac{x}{y}$ $\quad v = y$

$uy = x$

$x = uv$ $\quad \because y = v$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

The joint pdf of (u,v) is

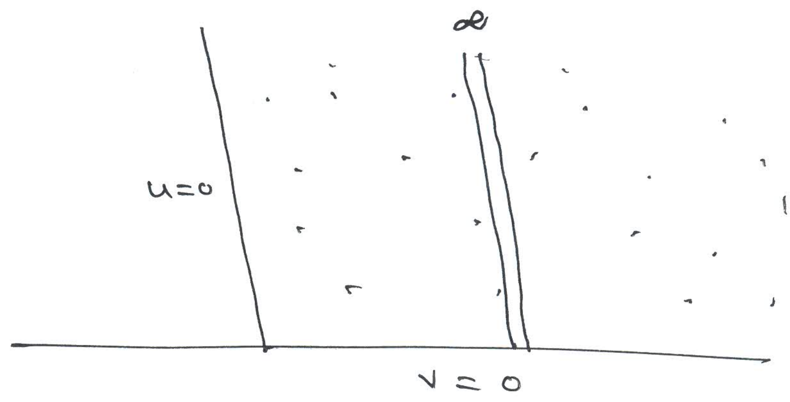
$$f(u,v) = f(x,y)\,|J|$$
$$= e^{-(x+y)}\,v$$
$$= v\,e^{-(uv+v)}$$

Range

$x \geq 0$ $\quad y \geq 0$

$uv \geq 0$ $\quad v \geq 0$

$u \geq 0, \; v \geq 0.$



The pdf of u is

$$f(u) = \int_0^u f(uv)\,dv = \int_0^\infty v\, e^{-(uv+v)}\,dv$$

$$= \int_0^\infty v\, e^{-(u+1)v}\,dv$$

$$u = v \qquad dv = e^{-(u+1)v}$$

$$u' = 1 \qquad v = \frac{e^{-(u+1)v}}{-(u+1)}$$

$$v_1 = \frac{e^{-(u+1)v}}{(u+1)^2}$$

$$= \left[v\,\frac{e^{-(u+1)v}}{-(u+1)} - \frac{e^{-(u+1)v}}{(u+1)^2} \right]_0^\infty$$

$$= \left[0 - \left(0 - \frac{1}{(u+1)^2} \right) \right]$$

$$f(u) = \frac{1}{(u+1)^2} \qquad u \geq 0$$

---

(7) If x and y are Independent uniform (rectangular) variable in (0,1) Find the density function of $U = XY$

<u>Soln</u>

Since x and y are uniform variable Then pdf's are

$$f(x) = \frac{1}{1-0} = 1 \qquad 0 \leq x \leq 1$$

$f(y) = \frac{1}{1-0} = 1 \qquad 0 \le y \le 1$

Since x and y are independent

$f(xy) = f(x)\,f(y)$

$= 1 \cdot 1 = 1 \qquad 0 \le x \le 1, \quad 0 \le y \le 1$

Let $u = xy \qquad v = y$

$x = \frac{u}{y}$

$x = \frac{u}{v} \qquad y = v$

$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = \frac{-u}{v^2} \qquad \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Joint pdf of u and v is

$f(uv) = f(xy)\,|J| = 1 \cdot \frac{1}{v} = \frac{1}{v}$

Range

$0 \le x \le 1 \qquad 0 \le y \le 1$

$0 \le \frac{u}{v} \le 1 \qquad 0 \le v \le 1$

$0 \le \frac{u}{v}, \quad \frac{u}{v} \le 1$

$0 \le u, \quad u \le v$

MDF of u is

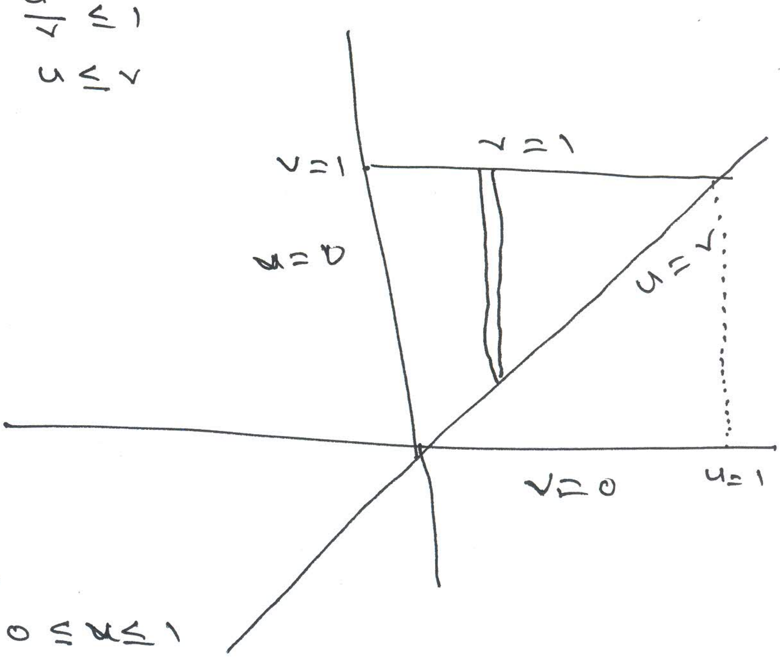
$f(u) = \int_u^1 f(uv)\,dv$

$= \int_u^1 \frac{dv}{v}$

$= [\log v]_u^1$

$= \log 1 - \log u$

$= -\log v \qquad 0 \le u \le 1$



(8) If X and Y are independent Random variable which are uniformly distributed over $(0,1)$ Find the distribution $u = x+y$ and $v = x-y$.

Soln

Given X and Y are uniformly Distributed in $(0,1)$

$$f(x) = \frac{1}{1-0} = 1 \qquad 0 \le x \le 1$$

$$f(y) = \frac{1}{1-0} = 1 \qquad 0 \le y \le 1$$

X and Y are Independent $\therefore$ Pdf of $f(xy)$ is

$$f(xy) = f(x).f(y)$$

$$= 1 \qquad 0 \le x \le 1, \quad 0 \le y \le 1$$

Let $u = x+y \qquad v = x-y$

$$\begin{array}{l} u = x+y \\ v = x-y \\ \hline u+v = 2x \end{array} \qquad \begin{array}{l} u = x+y \\ -v = x-y \\ \quad\; - \; + \\ \hline u-v = 2y \end{array}$$

$$x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}$$

$\therefore$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial u} = \frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$|J| = \frac{1}{2}$$

The jpd of u and v is

$$f(uv) = f(xy).|J| = 1.\frac{1}{2} = \frac{1}{2}$$

Range

$$0 \le x \le 1 \qquad\qquad 0 \le y \le 1$$

$$0 \le \frac{u+v}{2} \le 1 \qquad\qquad 0 \le \frac{u-v}{2} \le 1$$

$$\boxed{u+v = 0, \quad \frac{u+v}{2} = 1} \qquad \boxed{u-v = 0, \quad \frac{u-v}{2} = 1}$$

$$u = -v \qquad u+v = 2 \qquad\qquad u = v \qquad u-v = 2$$

(8) If X and Y are Independent Random variable which are uniformly distributed over $(0,1)$ Find the distribution $u = x+y$ and $v = x-y$.

Soln

Given X and Y are uniformly Distributed in $(0,1)$

$$f(x) = \frac{1}{1-0} = 1 \qquad 0 \le x \le 1$$

$$f(y) = \frac{1}{1-0} = 1 \qquad 0 \le y \le 1$$

X and Y are Independent $\therefore$ Pdf of $f(xy)$ is

$$f(xy) = f(x).f(y)$$
$$= 1 \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

Let $u = x+y \qquad v = x-y$

$$\begin{array}{r} u = x+y \\ v = x-y \\ \hline u+v = 2x \end{array} \qquad \begin{array}{r} u = x+y \\ -v = -x+y \\ \hline u-v = 2y \end{array}$$

$$x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}$$

$\therefore$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial u} = \frac{1}{2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$|J| = \frac{1}{2}$$

The jpd of u and v is

$$f(uv) = f(xy).|J| = 1.\frac{1}{2} = \frac{1}{2}$$

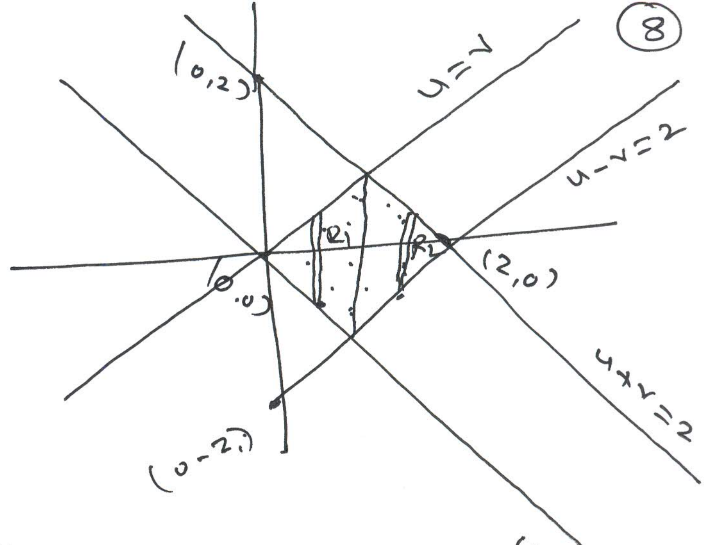
Range

$$0 \le x \le 1 \qquad\qquad 0 \le y \le 1$$

$$0 \le \frac{u+v}{2} \le 1 \qquad\qquad 0 \le \frac{u-v}{2} \le 1$$

$$\boxed{u+v = 0,\ \frac{u+v}{2} = 1} \qquad \boxed{u-v = 0,\ \frac{u-v}{2} = 1}$$

$$u = -v \qquad u+v = 2 \qquad\qquad u = v \qquad u-v = 2$$



The pdf of u is

limits are

when $0 \le u \le 1$ (R_1)

v varies from $-u$ to u

when $1 \le u \le 2$ (R_2)

v varies from $u-2$ to $2-u$

case (i) $0 \le u \le 1$ (R_1)

$$f(u) = \int_{-u}^{u} \frac{1}{2}\,du = \frac{1}{2}[v]_{-u}^{u}$$

$$= \frac{1}{2}[2u] = u \qquad 0 \le u \le 1$$

case (ii) $1 \le u \le 2$ (R_2)

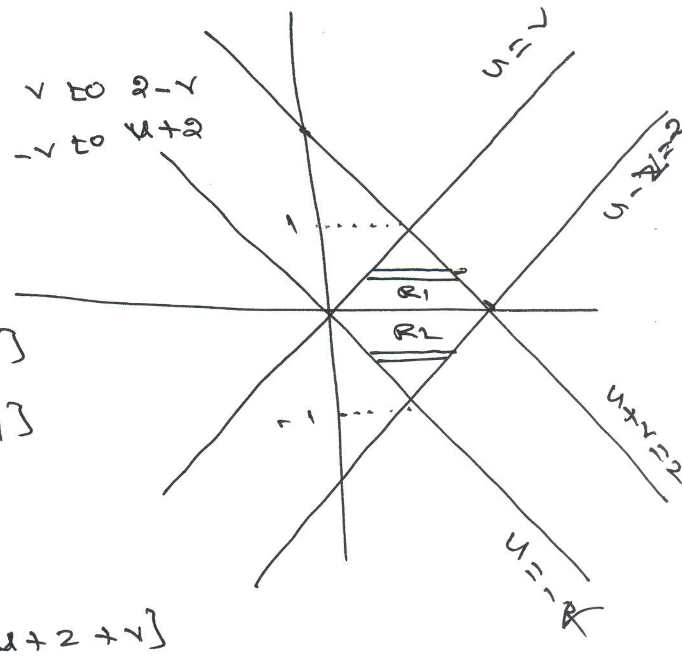
$$f(u) = \int_{u-2}^{2-u} \frac{1}{2}\,du = \frac{1}{2}[v]_{u-2}^{2-u} = \frac{1}{2}[2-u-u+2]$$

$$= \frac{1}{2}[4-2u]$$

$$= \frac{2}{2}[2-u]$$

$$f(u) = \begin{cases} u & 0 \le u \le 1 \\ 2-u & 1 \le u \le 2 \end{cases}$$

The pdf of v



when $0 \le v \le 1$, u varies from v to $2-v$

when $-1 \le v \le 0$ u varies from $-v$ to $v+2$

case (i) $0 \le v \le 1$ (R_1)

$$f(v) = \int_{v}^{2-v} \frac{1}{2}\,dv = \frac{1}{2}[2-v-v]$$

$$= \frac{1}{2}[2-2v]$$

$$= \underline{1-v}$$

case (ii) $-1 \le v \le 0$

$$f(v) = \int_{-v}^{v+2} \frac{1}{2}\,du = \frac{1}{2}[v+2+v]$$

$$= \frac{1}{2}[2+2v] = \frac{2}{2}[1+v] = \underline{1+v}$$

$$\boxed{f(v) = \begin{cases} 1-v & 0 \le v \le 1 \\ 1+v & -1 \le v < 0 \end{cases}}$$

(9) If x and y are independent rectangular variate on $(0,1)$ find the distribution of x/y

Soln

Since x and y are uniformly distributed in $(0,1)$

$$f(x) = \frac{1}{1-0} = 1 \qquad 0 \le x \le 1$$

$$f(y) = \frac{1}{1-0} = 1 \qquad 0 \le y \le 1$$

x and y are independent $\therefore$ pdf of $f(xy)$ is

$$f(xy) = f(x).f(y)$$

$$= 1.1 = 1 \qquad 0 \le x \le 1, \quad 0 \le y \le 1$$

Let $u = \frac{x}{y}$ $\qquad v = y$

$$x = uy$$

$$x = uv \qquad y = v$$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Joint p.d.f of u and v is

$$f(uv) = f(xy).|J| = 1.v = \underline{\underline{v}}$$

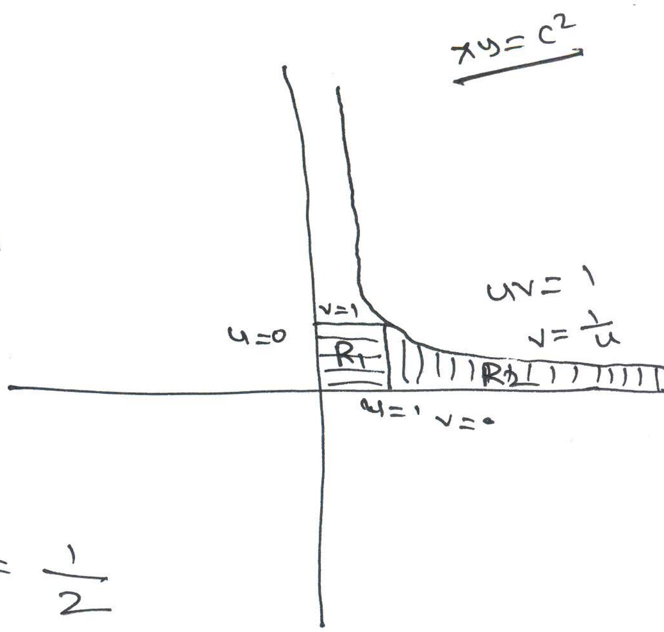
Range

$0 \le x \le 1 \qquad 0 \le y \le 1$

$0 \le uv \le 1 \qquad 0 \le v \le 1$

$u = 0, v = 0 \qquad v = 0 \quad v = 1$

$\underline{\underline{uv = 1}}$



Pdf of u

If $u < 1$

$$f(u) = \int_0^1 v\,du = \left[\frac{v^2}{2}\right]_0^1 = \frac{1}{2}$$

If $u > 1$

$$f(u) = \int_0^{1/u} v\,dv = \left[\frac{v^2}{2}\right]_0^{\frac{1}{u}} = \frac{1}{2}\frac{1}{u^2} - 0$$

$$= \frac{1}{2u^2}$$

$\therefore$

$$f(u) = \begin{cases} \frac{1}{2} & u < 1 \\ \frac{1}{2u^2} & u > 1 \end{cases}$$

(10) If the joint pdf of (x, y) is given by $f(x,y) = x + y \quad 0 \le x, y \le 1$ Find the pdf of $U = xy$

Soln Let ~~$U = x+y$~~ $\quad v = y \quad u = xy$

~~$x = u - y$~~ $\quad x = \frac{u}{y}$

$y = v \quad x = \frac{u}{v}$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{-u}{v^2} \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

The joint p.d.f of u and v is

~~$f(xy)$~~ $\quad f(uv) = f(xy)\,|J|$

$$= (x + y)\frac{1}{v} = \left[\frac{u}{v} + v\right]\frac{1}{v}$$

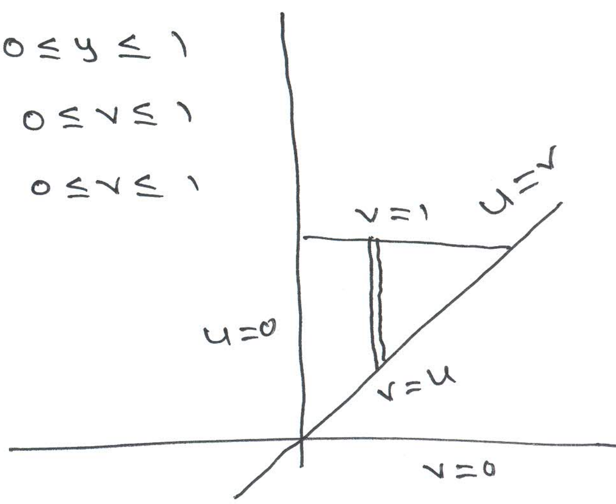
$$= \frac{u}{v^2} + 1$$

Range

$0 \le x \le 1 \qquad 0 \le y \le 1$

$0 \le \frac{u}{v} \le 1 \qquad 0 \le v \le 1$

$0 \le u \le v \qquad 0 \le v \le 1$



Pdf of u is,

$$f(u) = \int_u^1 f(u,v).dv$$

$$= \int_u^1 \frac{u}{v^2} + 1 . \, dv$$

$$= \int_u^1 u v^{-2} + 1 \, dv$$

$$= \left[\frac{u v^{-1}}{-1} + v \right]_u^1$$

$$= \left[\frac{u}{-v} + v \right]_u^1$$

$$= \cancel{\frac{u}{-u} + u}$$

$$= \left[-u + 1 \right) - \left(\frac{u}{-u} + u \right]$$

$$= -u + 1 + 1 - u$$

$$= 2 - 2u$$

$$= \underline{2(1-u)} \qquad 0 \le u \le 1$$

---

(11) If $f(x,y) = \begin{cases} x+y & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$ Find the density function of $U = x + y$

<u>Soln</u> Let $U = x + y$ $\qquad v = y$

$$x = u - y$$

$$x = u - v \qquad y = v$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Range

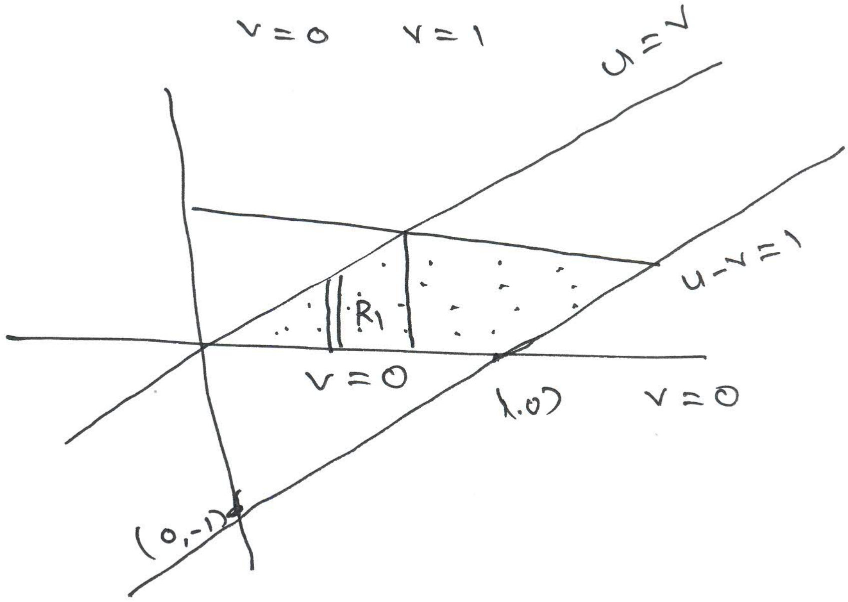
$0 \le x \le 1$ $\qquad$ $0 \le y \le 1$

$0 \le u-v \le 1$ $\qquad$ $0 \le v \le 1$

$v \le u \le 1+v$

$u = v$ $\qquad$ $u = 1+v$

$u - v = 1$



The pdf of u is $(u<1)$

case (i) $f(u) = \int_0^u u\,dv = u[v]_0^u = u[u] = u^2 \qquad 0 \le u \le 1$

case (ii) $1 \le u \le 2$ $\quad (R_2)$

$$f(u) = \int_{u-1}^{1} u\,dv = u[v]_{u-1}^{1} = u[1-u+1]$$
$$= u[2-u]$$
$$= 2u - u^2 \qquad 1 \le u \le 2$$

$$\therefore f(u) = \begin{cases} u^2 & 0 \le u \le 1 \\ 2u-u^2 & 1 \le u \le 2 \end{cases}$$

---

12 The joint P.d.f of a Two Dimensional Random variable (x,y) is given by

$$f(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of $U = \sqrt{x^2+y^2}$.

Soln Given

$$u = \sqrt{x^2+y^2} \qquad \text{Let } v = y$$

$$u^2 = x^2 + y^2$$

$$x^2 = u^2 - y^2$$

$$x^2 = u^2 - v^2 \qquad y = v$$

$$\boxed{\because x^2 = u^2 - v^2 \\ x = \sqrt{u^2-v^2}}$$

$$\cancel{2}x \frac{\partial x}{\partial u} = \cancel{2}u$$

$$\frac{\partial x}{\partial u} = \frac{u}{x} = \frac{u}{\sqrt{u^2-v^2}} \qquad \frac{\partial y}{\partial u} = 0$$

$$2x \frac{\partial x}{\partial v} = -\cancel{2}v \qquad \frac{\partial y}{\partial v} = 1$$

$$\frac{\partial x}{\partial v} = -\frac{v}{x} = \frac{-v}{\sqrt{u^2-v^2}}$$

$$J = \begin{vmatrix} \dfrac{u}{\sqrt{u^2-v^2}} & \dfrac{-v}{\sqrt{u^2-v^2}} \\ 0 & 1 \end{vmatrix} = \frac{u}{\sqrt{u^2-v^2}}$$

Now Joint pdf of (uv) is

$$f(uv) = f(xy).\,|J|$$

$$= 4xy\, e^{-(x^2+y^2)} . |J|$$

$$= 4 \cancel{\sqrt{u^2-v^2}} . v\, e^{-(u^2)} . \frac{u}{\cancel{\sqrt{u^2-v^2}}}$$

$$= 4uv\, e^{-u^2}$$

Range

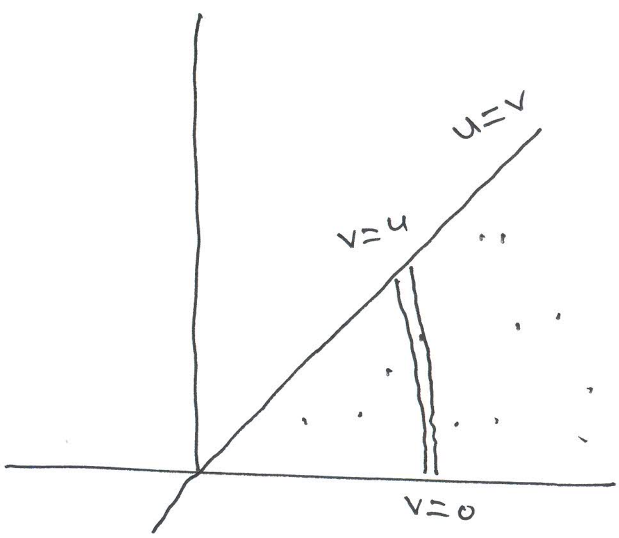
$x \geq 0 \qquad y \geq 0$

$\sqrt{u^2-v^2} \geq 0 \qquad v \geq 0$

$u^2 - v^2 \geq 0$

$u^2 \geq v^2$

$u \geq v \qquad v \geq 0$



The Probability density function of u is

$$f(u) = \int f(u,v)\,dv$$

$$= \int_0^u 4uv e^{-u^2}\,du$$

$$= 4u e^{-u^2} \int_0^u v\,du$$

$$= 4u e^{-u^2} \left[\frac{v^2}{2}\right]_0^u = 4u e^{-u^2}\left[\frac{u^2}{2}\right]$$

$$= \underline{\underline{2u^3 e^{-u^2}}} \qquad u \geq 0$$

---

13. Let X and Y are normally Distributed independent random variable with mean 0 and variance σ^2. Find the pdf of $R = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

Soln

If X and Y are normally Distributed with mean 0 and variance σ^2. The Pdf are

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \qquad -\infty \leq x \leq \infty$$

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2} \qquad -\infty \leq y \leq \infty$$

X and Y are Independent Then The jpdf of (X,Y) is

$$f(x,y) = f(x).f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2}$$

$$= \frac{1}{\sigma^2 2\pi} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

~~Put~~ $\cancel{x^2+y^2 = R^2}$

$$= \frac{\cancel{1}}{2\pi\sigma^2} e^{-\cancel{\frac{R^2}{2\sigma^2}}} \qquad . R \geq 0 \quad \cancel{0 \leq \theta \leq 2\pi}$$

$R = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1} y/x$

$x = R\cos\theta. \qquad y = R\sin\theta$

$$J = \frac{\partial(xy)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix}$$

$$= R\cos^2\theta + R\sin^2\theta$$

$$= R[\cos^2\theta + \sin^2\theta]$$

$$= \underline{\underline{R}}$$

The Joint pdf of $f(r,\theta)$ is

$$f(r,\theta) = f(xy).|J|$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}} R$$

$$= \frac{R}{2\pi\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \qquad R \geq 0, \; 0 \leq \theta \leq 2\pi$$

Marginal density function of R

$$f(R) = \int f(R\theta)\, d\theta$$

$$= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} R\, e^{-\frac{R^2}{2\sigma^2}} . d\theta$$

$$= \frac{1}{2\pi\sigma^2} R . e^{-R^2/2\sigma^2} [\theta]_0^{2\pi}$$

$$= \frac{1}{2\cancel{\pi}\sigma^2} R\, e^{-R^2/2\sigma^2} [2\cancel{\pi}]$$

$$= \underline{\underline{\frac{R}{\sigma^2} e^{-R^2/2\sigma^2} \; ; \; R \geq 0}}$$

Marginal density function of θ is

$$f(\theta) = \int f(r\theta).dr$$

$$= \frac{1}{2\pi\sigma^2} \int_0^{\infty} R . e^{-\frac{R^2}{2\sigma^2}} . dr$$

put $t = \frac{R^2}{2\sigma^2}$

$dt = \frac{2R}{2\sigma^2} dR$

$\sigma^2 dt = R dR$

$$f(\theta) = \frac{1}{2\pi\sigma^2} \int_0^{\infty} e^{-t} \cdot \sigma^2 dt$$

$$= \frac{1}{2\pi} \int_0^{\infty} e^{-t} dt = \frac{1}{2\pi} \left[\frac{e^{-t}}{-1} \right]_0^{\infty}$$

$$= \frac{1}{2\pi} [0+1]$$

$$= \frac{1}{2\pi}.$$

$f(\theta) = \frac{1}{2\pi}$; $0 \le \theta \le 2\pi$.

---

(14) If the joint p.d.f of the R.v's X and Y are given by

$$f(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the p.d.f of the R.V $U = \frac{X}{Y}$

Soln

Let $u = \frac{x}{y}$ $\quad v = y$

$x = uy$

$x = uv$ $\quad y = v$

$\frac{\partial x}{\partial u} = v$ $\quad \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = u$ $\quad \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Hence joint p.d.f of (u,v) is

$f(u,v) = f(x,y) |J| = 2v$

Range

$$0 \le x \le y \le 1$$

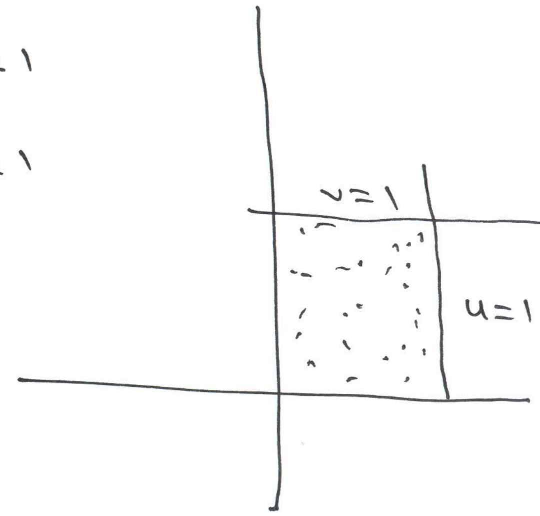
$$0 \le uv \le v \le 1$$

$0 \le uv \qquad uv \le v \qquad v \le 1$

$u \geq 0\,,\ v \geq 0\,, \qquad v \le 1$

$uv \le v$

$u \le 1$



P.d.f of u is

$$f(u) = \int f(u,v)\,dv$$

$$= \int_0^1 2v\,dv = \left[\frac{2\,v^2}{2}\right]_0^1 = 1 \qquad 0 \le u \le 1$$

$$\therefore f(u) = 1 \qquad 0 \le u \le 1$$

---

(15) The joint p.d.f of the two dimensional Random variable is

$$f(x,y) = \begin{cases} \frac{1}{2}\,x\,e^{-y}, & 0 < x < 2,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the p.d.f of $X+Y$.

Soln

Let $u = x+y \qquad v = y$

$$x = u - y$$

$$x = u - v \qquad y = v$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

The jpdf of (u,v) is

$$f(u,v) = f(x,y)\,|J|$$

$$= \frac{1}{2} x e^{-y} \cdot 1$$

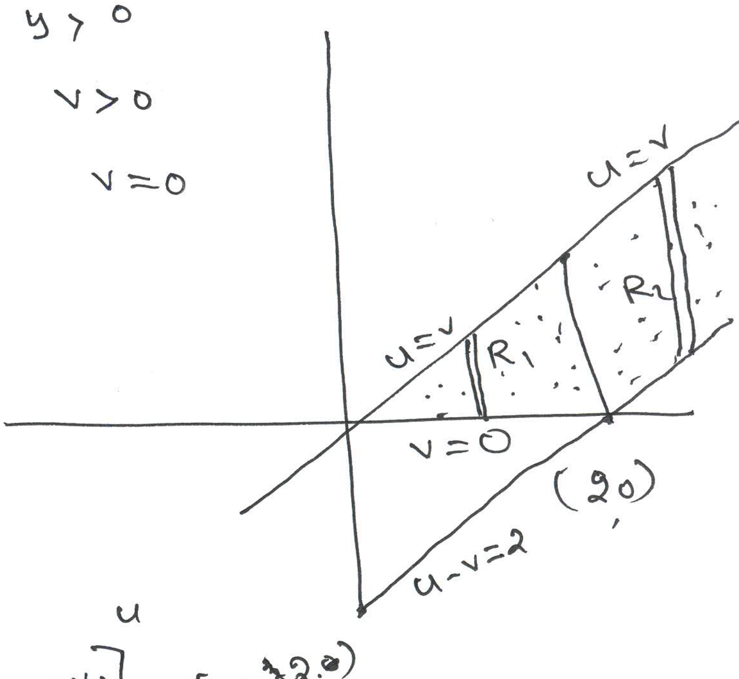
$$= \frac{1}{2}(u-v) e^{-v}.$$

Range

$0 < x < 2$ $\quad y > 0$

$0 < u-v < 2$ $\quad v > 0$

$\boxed{u-v=0 \quad u=v}$ $\quad u-v=2$ $\quad v=0$



Pdf of u

case (i) $u<2$ (R_1)

$$f(u) = \int_0^u \frac{1}{2}(u-v) e^{-v}\,dv$$

$$= \frac{1}{2}\left[(u-v)\frac{e^{-v}}{-1} - (-1)(e^{-v})\right]_0^u$$

$$= \frac{1}{2}\left[0 + e^{-u}\right] - \left[\frac{u}{-1} + 1\right]$$

$$= \frac{1}{2}\left[e^{-u} + u - 1\right] \quad u<2$$

case (ii) R_2 $\quad u>2$

$$f(u) = \int_{u-2}^{u} \frac{1}{2}(u-v) e^{-v}\,dv$$

$$= \frac{1}{2}\left[(u-v)\frac{e^{-v}}{-1} - (-1)(e^{-v})\right]_{u-2}^{u}$$

$$= \frac{1}{2}\left[0 + e^{-u}\right) - \left[(\cancel{u} - \cancel{u}+2)\frac{e^{-(u-2)}}{-1} + e^{-(u-2)}\right]$$

$$= \frac{1}{2}\left[e^{-u} + 2e^{-u+2} - e^{-u+2}\right]$$

$$= \frac{1}{2}\left[e^{-u} + e^{-u+2}\right]$$

$$\therefore f(u) = \begin{cases} \frac{1}{2}\left(e^{-u} + u - 1\right) & 0 \le u \le 2 \\ \frac{1}{2}\left[e^{-u} + e^{-u+2}\right] & 2 \le u < \infty \end{cases}$$

---

(16) The Random variable (x, y) as the joint pdf

$$f(x,y) = \begin{cases} 24xy, & x \geq 0, \; y \geq 0 \quad x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that $u = x+y$ $v = x/y$ are Independent

Soln

Let $u = x+y$ $\quad v = \frac{x}{y}$

$\Rightarrow v = x/y$

$x = vy$

$u = x+y$

$u = vy + y$

$u = y[v+1]$

$\Rightarrow y = \frac{u}{v+1}$

$x = vy$

$x = v\left[\frac{u}{v+1}\right]$

$x = \frac{uv}{v+1}$

$$\frac{\partial x}{\partial u} = \frac{v}{v+1}$$

$$\frac{\partial x}{\partial v} = \frac{(v+1)u - uv(1)}{(v+1)^2}$$

$$= \frac{\cancel{uv} + u - \cancel{uv}}{(v+1)^2}$$

$$= \frac{u}{(v+1)^2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v+1}$$

$$\frac{\partial y}{\partial v} = \frac{-u}{(v+1)^2}$$

$$J = \begin{vmatrix} \dfrac{v}{v+1} & \dfrac{u}{(v+1)^2} \\ \dfrac{1}{v+1} & -\dfrac{u}{(v+1)^2} \end{vmatrix}$$

$$= \frac{-uv}{(v+1)^3} - \frac{u}{(v+1)^3}$$

$$= \frac{-u}{(v+1)^{\cancel{3}\,2}}\left[\cancel{v+1}\right] = \frac{-u}{(v+1)^2}$$

$\therefore$ The joint pdf of (u,v) is given by

$$f(uv) = f(xy).|J| = 24xy\,|J|$$

$$= 24\,\frac{uv}{v+1}\,\frac{u}{v+1}\cdot\frac{u}{(v+1)^2}$$

$$= \frac{24vu^3}{(v+1)^4}$$

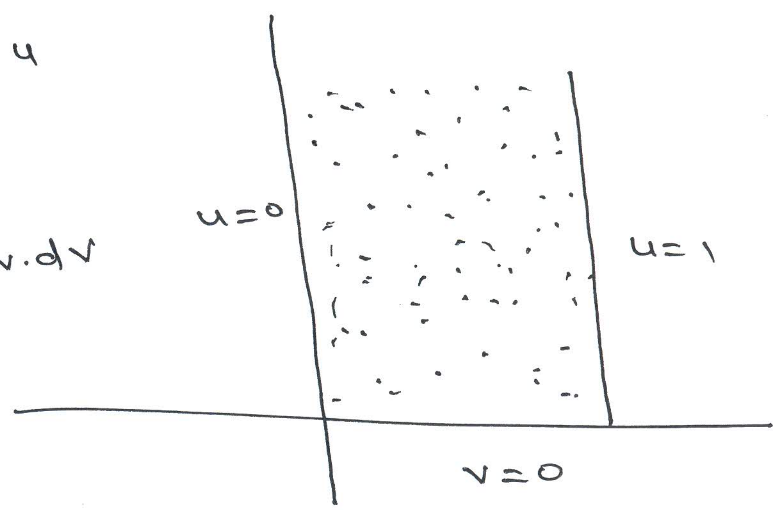
<u>Range</u>

$x \geq 0$	$y \geq 0$	$x+y \leq 1$
$\dfrac{uv}{v+1} \geq 0$	$\dfrac{u}{v+1} \geq 0$	$\dfrac{uv+u}{v+1} \leq 1$
$u \geq 0$	$v \geq 0$	$\dfrac{u\,[\cancel{v+1}]}{[\cancel{v+1}]} \leq 1$
$v \geq 0$		$u \leq 1$

The density function u

$$f(u) = \int f(uv).dv$$

$$= \int_0^\infty \frac{24u^3}{(v+1)^4}.v.dv$$



$$= 24u^3 \int_0^{\infty} \frac{v}{(1+v)^4} \, dv$$

Put $1+v = t \Rightarrow v = t-1$

$dv = dt$

$$\boxed{\begin{array}{l} v = 0 \Rightarrow t = 1 \\ v = \infty \Rightarrow t = \infty \end{array}}$$

$$= 24u^3 \int_1^{\infty} \frac{t-1}{t^4} \, dt$$

$$= 24u^3 \left[\int_1^{\infty} \frac{t}{t^4} \, dt - \int_1^{\infty} \frac{1}{t^4} \, dt \right]$$

$$= 24u^3 \left[\int_1^{\infty} t^{-3} \, dt - \int_1^{\infty} t^{-4} \, dt \right]$$

$$= 24u^3 \left\{ \left[\frac{t^{-2}}{-2} \right]_1^{\infty} - \left[\frac{t^{-3}}{-3} \right]_1^{\infty} \right\}$$

$$= 24u^3 \left[\frac{1}{-2t^2} \right]_1^{\infty} - \left[\frac{1}{-3t^3} \right]_1^{\infty}$$

$$= 24u^3 \left[0 - \frac{1}{-2} \right] - \left[0 - \frac{1}{-3} \right]$$

$$= 24u^3 \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$= 24u^3 \left[\frac{3-2}{6} \right] = \frac{24}{6} u^3$$

$$= \underline{\underline{4u^3}} \qquad 0 \le u \le 1$$

Marginal Density function v

$$f(v) = \int f(u,v) \, du$$

$$= \int_0^1 \frac{24u^3 v}{(1+v)^4} \, du = \frac{24v}{(1+v)^4} \int_0^1 u^3 \, du$$

$$= \frac{24v}{(1+v)^4} \left[\frac{u^4}{4} \right]_0^1 = \frac{6v}{(1+v)^4} \qquad 0 \le v \le \infty$$

To show u and v are independent

$$f(u).\,f(v) = 4u^3 . \frac{6v}{(1+v)^4}$$

$$= \frac{24u^3 v}{(1+v)^4}$$

$$= f(u,v)$$

Hence; u and v are Independent.

Central limit theorem:

Note: 1. Sum of the Random variable (S_n) follows Normal Distribution

$$S_n \sim N[n\mu, \sigma\sqrt{n}]$$

$$S_n \sim N[n\mu, \sqrt{\sigma^2 n}]$$

(2) Mean of the RV follows Normal

$$\bar{x} \sim N[\mu, \sigma/\sqrt{n}]$$

(3) Discrete of Random variable follows Normal $N[\mu, \sigma]$

---

(1) If $x_1, x_2 \dots x_n$ are Poisson variate with Parameter $\lambda = 2$. Use the CLT to estimate $P[120 \le S_n \le 160]$ where $S_n = x_1 + x_2 + \dots + x_n$ and $n = 75$.

Soln

By CLT sum of R.V S_n follows normal

$$S_n \sim N[n\mu, \sigma\sqrt{n}]$$

Given $n = 75$

$$\text{mean } \lambda = 2 = E(x)$$

$$\text{variance } \sigma^2 = \lambda = 2$$

$$SD = \sigma = \sqrt{2}$$

$$n\mu = 75 \times 2 = 150$$

$$\sigma\sqrt{n} = \sqrt{2}.\sqrt{75} = \sqrt{150} = 12.2474$$

Now

$$P[120 \le S_n \le 160] \qquad \therefore Z = \frac{S_n - 150}{12.2474}$$

$$P\left[\frac{120-150}{12.2474} \le Z \le \frac{160-150}{12.2474}\right]$$

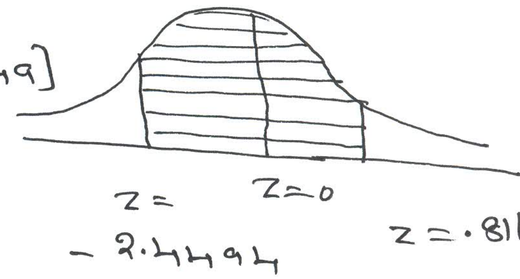
$$P[-2.4494 \le Z \le 0.81649]$$

$= P[-2.4494 \le Z \le 0] + P[0 \le Z \le 0.81649]$

$= P[0 \le Z \le 2.4494] + P[0 \le Z \le 0.81649]$

$= 0.49285 + 0.29289$

$= \underline{0.78574}$



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(2) Let $X_1, X_2 \dots X_{100}$ be Independed identically Distributed Random variable with $\mu = 2$. and $\sigma^2 = \frac{1}{4}$ Find. $P[192 \le X_1 + X_2 + \dots + X_{100} \le 210]$

Soln

By CLT sum of Random variable S_n follows Normal $S_n \sim N[n\mu, \sigma\sqrt{n}]$

Given $n = 100 \quad \sigma^2 = \frac{1}{4} \Rightarrow \sigma = \frac{1}{2}$

$\mu = 2$

$n\mu = 100 \times 2 = 200$

$\sigma\sqrt{n} = \frac{1}{2} \times \sqrt{100} = \frac{1}{2} 10 = 5$

$\therefore Z = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 200}{5}$

$P[192 \le S_n \le 210$

$= P\left[\frac{192 - 200}{5} \le Z \le \frac{210 - 200}{5}\right]$

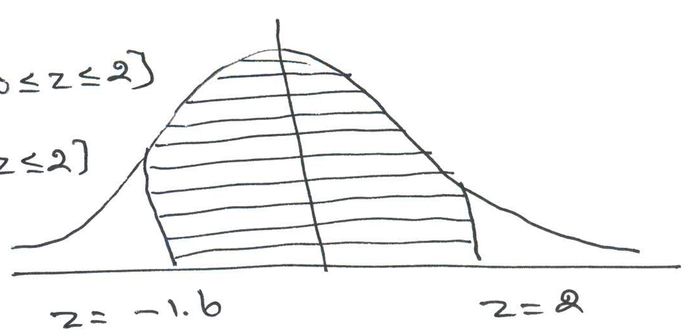
$= P[-1.6 \le Z \le \cancel{8}\,2]$

$= P[-1.6 \le Z \le 0] + P[0 \le Z \le 2]$

$= P[0 \le Z \le 1.6] + P[0 \le Z \le 2]$

$= 0.4452 + 0.4773$

$= \underline{\underline{0.92245}}$



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(3) The resistor r_1, r_2, r_3, and r_4 are independent Random variable and is uniform in the interval $(450, 550)$ using CLT. Find

~~REPZ~~ $P[1900 \le r_1 + r_2 + r_3 + r_4 \le 2100]$

Soln

x is uniform Distribution

Mean $E(x) = \frac{a+b}{2} = \frac{450+550}{2} = \frac{1000}{2} = 500$

variance $= \frac{(b-a)^2}{12} = \frac{(550-450)^2}{12} = \frac{(100)^2}{12} = 833.33$

$n = 4$

By CLT sum of Random variable S_n follows Normal. $S_n \sim N[n\mu, \sigma\sqrt{n}]$

$n\mu = 4 \times 500 = 2000$

$\sigma\sqrt{n} = \sqrt{833.33} \cdot \sqrt{4}$

$= 57.735$

$$z = \frac{S_n - 2000}{57.735}$$

$P[1900 \leq S_n \leq 2100]$

$$P\left[\frac{1900-2000}{57.735} \leq z \leq \frac{2100-2000}{57.735}\right]$$

$= P[-1.7320 \leq z \leq 1.7320]$

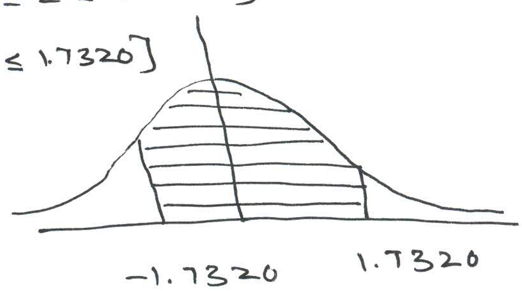
$= P[-1.7320 \leq z \leq 0] + P[0 \leq z \leq 1.7320]$

$= P[0 \leq z \leq 1.7320] + P[0 \leq z \leq 1.7320]$

$= 2 \times P[0 \leq z \leq 1.7320]$

$= 2 \times 0.45836$

$= \underline{\underline{0.91672}}$



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(4) If x_i, $i = 1, 2, 3 \ldots 50$ are Independent R.V's each having poisson distribution with Parameter $\lambda = 0.03$ and $S_n = x_1 + x_2 + \cdots x_n$. Evaluate $P[S_n \geq 3]$

Soln

By CLT S_n follows normal

$S_n \sim N[n\mu, \sigma\sqrt{n}]$

Given $n = 50$

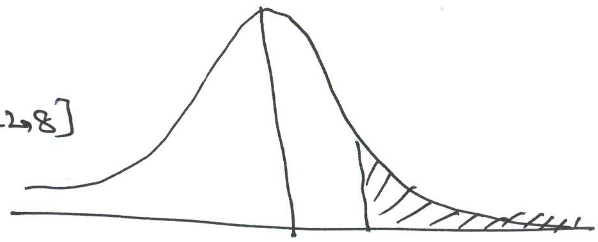
$\lambda = E(x) = \mu = 0.03$

variance $\sigma^2 = 0.03$

$$n\mu = 50 \times 0.03$$
$$= \underline{\underline{1.5}}$$
$$\sigma\sqrt{n} = \sqrt{0.03}\cdot\sqrt{50}$$
$$= \underline{\underline{1.2247}}$$

$$z = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 1.5}{1.2247}$$

$$P[S_n \geq 3] = P\left[z \geq \frac{3-1.5}{1.2247}\right]$$
$$= P[z \geq 1.2248]$$



$z = 1.2248$

$$= 0.5 - P[0 \leq z \leq 1.2248]$$
$$= 0.5 - 0.38967$$
$$= \underline{\underline{0.11033}}$$

---

(5) Suppose that orders at a restaurant are indentically independent RV with mean μ ₹ 8 and SD ₹ 2. Estimate

(i) The probability that the first 100 customers spend a total of more than ₹ 840. (ie) $P[X_1 + X_2 + \dots + X_{100} \geq 840]$

(ii) $P[780 \leq X_1 + X_2 + \dots + X_{100} \leq 820]$

Soln

By CLT sum of Random variable S_n follows Normal $S_n \sim N[n\mu, \sigma\sqrt{n}]$

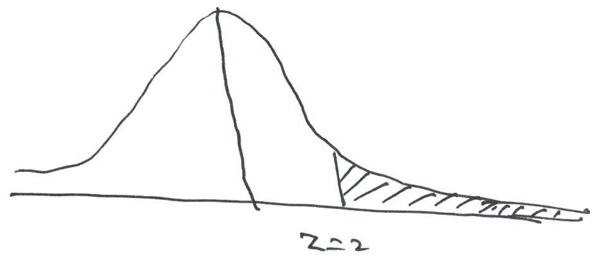
Given $n = 100$
$\mu = 8$
$\sigma = 2$

$$n\mu = 100 \times 8 = 800$$
$$\sigma\sqrt{n} = 2 \times \sqrt{100} = 2 \times 10 = 20$$
$$\therefore z = \frac{S_n - 800}{20}$$

i $P[S_n \geq 840]$

$P\left[z \geq \frac{840-800}{20}\right]$

$= P[z \geq 2]$



$= 0.5 - P[0 \leq z \leq 2]$

$= 0.5 - 0.4772$

$= \underline{0.0228}$

(ii) $P[780 \leq S_n \leq 820]$

$= P\left[\frac{780-800}{20} \leq z \leq \frac{820-800}{20}\right]$

$= P[-1 \leq z \leq 1]$

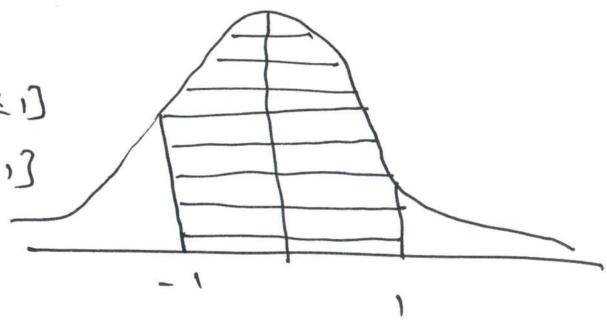
$= P[-1 \leq z \leq 0] + P[0 \leq z \leq 1]$

$= P[0 \leq z \leq 1] + P[0 \leq z \leq 1]$

$= 2 \times P[0 \leq z \leq 1]$

$= 2 \times 0.34134$

$= \underline{0.68268}$.



(6) In a particular circuit 20 resistor are connected in series. The mean and variance of the resistance of each resistor is 5 and 0.2 respectively. what is the probability that total resistance of the circuit will exceed 98. Assuming independence.?

Soln

Sum of R.v's $S_n \sim N[n\mu, \sigma\sqrt{n})$

$n = 20 \quad \mu = 5. \quad \sigma^2 = 0.2$

$n\mu = 20 \times 5 = 100$

$\sigma\sqrt{n} = \sqrt{0.2}.\sqrt{20} = 2//$

$z = \frac{S_n - 100}{2}$

now $P[S_n \geq 98]$

$= P[z \geq \frac{98-100}{2}]$

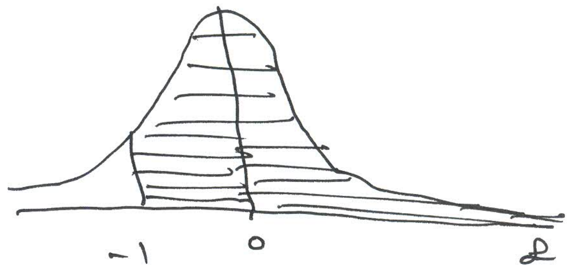
$= P[z \geq -1]$

$= 0.5 + P[-1 \leq z \leq 0]$

$= 0.5 + P[0 \leq z \leq 1]$

$= 0.5 + 0.34134$

$= \underline{\underline{0.84134}}$



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(7) A Large freight elevator can Transport a maximum of 9800 Pounds. Suppose a load of cargo containing 49 boxes must be Transported via the elevator. Experience has shown that the weight of boxes of this type of cargo follows a distribution with mean $\mu = 205$ pounds and standard deviation $\sigma = 15$ pounds. Based on this information, what is the probability that all 49 boxes can be safely loaded onto the freight elevator and transported.

<u>soln</u>

$n = 49 \quad \mu = 205 \quad \sigma = 15$

$S_n \sim N[n\mu, \sigma\sqrt{n})$

$n\mu = 49 \times 205 = 10045$

$\sigma\sqrt{n} = 15 \times \sqrt{49} = 15 \times 7 = 105$

$Z = \frac{S_n - 10045}{105}$

The elevator can Transport upto 9800 Pounds.

∴ These 49 boxes will be safely Transported if they weigh in total Less than 9800 Pounds.

$P[S_n < 9800]$

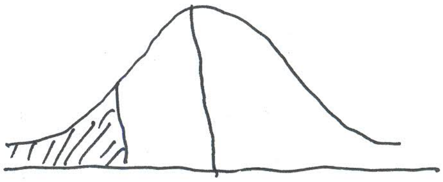
$= P[z < \frac{9800 - 10045}{105}]$

$= P[z < -2.333]$

$= 0.5 - P[0 \le z \le 2.33]$

$= 0.5 - 0.49018$

$= \underline{\underline{0.00982}}$



-2.332

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(8) From the past experience, it is known that the number of tickets purchased by a student standing in the line at the ticket window for the football match of ULCA against USC follows a distribution that has mean $\mu = 2.4$ and standard deviation $\sigma = 2.0$ suppose that few hours before the start of one of these matches there are 100 eager student standing in line to purchase tickets. If only 250 Tickets remain, what is the probability that all 100 students will be able to purchase the tickets the ~~dis~~ desire?

Soln

sum of the RV's follows normal $S_n \sim N[n\mu, \sigma\sqrt{n})$

$n = 100 \quad \sigma = 2 \quad \mu = 2.4$

$n\mu = \cancel{100 \times 2 = 200} = 100 \times 2.4 = 240$

$\sigma\sqrt{n} = 2 \times \sqrt{100} = 20$

$$z = \frac{S_n - 200}{20}$$

There are 250 Tickets available, so the 100 students will be able to purchase the Tickets they want if all together ask for less than 250 tickets.

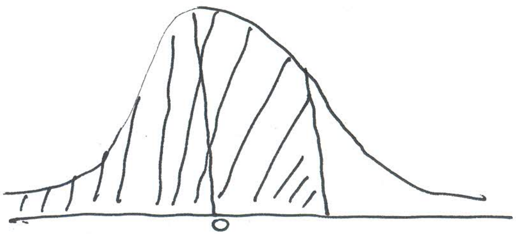
$$P[S_n \le 250] = P\left[z \le \frac{250-240}{20}\right] = P[z < 0.5)$$

$= 0.5 + P[0 \le z \le 0.5]$

$= 0.5 + \cancel{0.49379}$ 0.19146

$= \cancel{0.99379}$

$= \underline{\underline{0.69146}}$



0

$z = 2.5$

(9) The Life time of a certain brand of a Tube light may be considered as a Random variable with mean 1200 h and standard deviation 250h. Find the Probability, using CLT, the average life time of 60 light exceeds 1250 h.

Soln

by CLT $\bar{x}$ follows a normal distribution with mean μ and SD $\sigma/\sqrt{n}$ $\quad \bar{x} \sim N[\mu, \sigma/\sqrt{n}]$

$n = 60 \quad \mu = 1200 \quad SD = 250$

$$z = \frac{\bar{x} - 1200}{250/\sqrt{60}}$$

Probability of average life time of 60 lights exceeds 1250 h

$$P[\bar{x} > 1250] = P\left[z > \frac{1250 - 1200}{250/\sqrt{60}}\right]$$

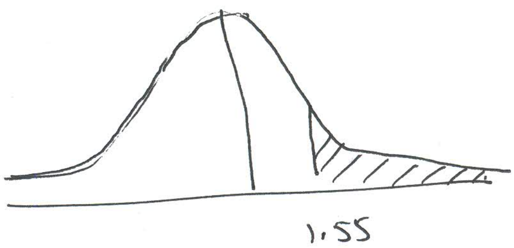
$$= P\left[z > \frac{50 \times \sqrt{60}}{250}\right]$$

$$= P[z > 1.55]$$

$= 0.5 - P[0 \le z \le 1.55]$

$= 0.5 - 0.43943$

$= \underline{0.06057}$



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(10) A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using CLT with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4.

Soln

Given $n = 100, \quad \mu = 60 \quad \sigma^2 = 400$

by CLT $\underset{(\text{mean})}{\bar{x}}$ follows normal $\bar{x} \sim (\mu, \sigma/\sqrt{n})$

$$z = \frac{\bar{x}-60}{\frac{20}{\sqrt{100}}} = \frac{\bar{x}-60}{\frac{20}{10}} = \frac{\bar{x}-60}{2}$$

To Find sample will not differ from $\mu = 60$ by more than 4 .lies $|\bar{x}-60| \le 4$

$$P[\,|\bar{x}-60| \le 4]$$

$$= P[-4 \le (\bar{x}-60) \le 4]$$

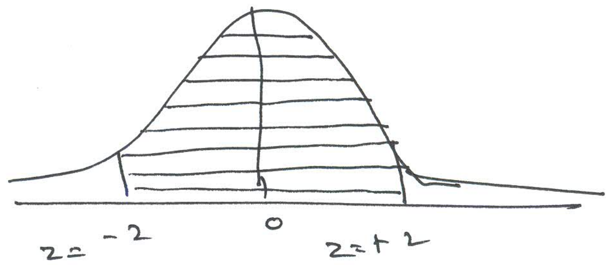
$$= P[-4+60 \le \bar{x} \le 4+60]$$

$$= P[56 \le \bar{x} \le 64]$$

$$= P\left[\frac{56-60}{2} \le z \le \frac{64-60}{2}\right]$$

$$= P[-2 \le z \le 2]$$

$$= P[-2 \le z \le 0] + P[0 \le z \le 2]$$



$$= P[0 \le z \le 2] + P[0 \le z \le 2]$$

$$= 2 \times P[0 \le z \le 2]$$

$$= 2 \times 0.4773 = \underline{\underline{0.9546}}$$

---

(11) A distribution with unknown mean μ has variance equal to 1.5. Use central limit theorem to determine how large a sample should be taken from the distribution in order that the Probability will be atleast 0.95 that the sample mean will be within 0.5 of the Population mean.

Soln:

$E(x) = \text{mean} = \mu$

$\text{var}(x) = \sigma^2 = 1.5$

Let n be the sample size.

by CLT Mean of RV follows Normal $\bar{x} \sim N[\mu, \sigma/\sqrt{n}]$

Given

The probability will be atleast 0.95 that the sample mean will be within 0.5 of the population mean.

$$P[\,|x-\mu| \le 0.5] \geq 0.95$$

$$z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-\mu}{\frac{\sqrt{1.5}}{\sqrt{n}}} = \frac{(x-\mu)\sqrt{n}}{1.2247}$$

To find 'n' such that

$$P[\mu-0.5 < \bar{x} < \mu+0.5] \geq 0.95$$

$$P[-0.5 < \bar{x}-\mu < 0.5] \geq 0.95$$

$$P[\,|\bar{x}-\mu| \le 0.5] \geq 0.95$$

$$P\left[\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} < \frac{0.5}{\frac{1.2248}{\sqrt{n}}}\right] \geq 0.95$$

$$P[\,|z| < 0.4082\sqrt{n}] \geq 0.95$$

where z is the standard normal variate

The least value of n is obtained from

$$P[\,|z| \le 0.4082\sqrt{n}] = 0.95$$

$$P[-0.4082\sqrt{n} \le z \le 0.4082\sqrt{n}] = 0.95$$

(ie) $2P[0 \le z \le 0.4082] = 0.95$

$$\Rightarrow P[0 \le z \le 0.4082] = \frac{0.95}{2}$$

$$P[0 \le z \le 0.4082] = 0.4750$$

$$0.4082\sqrt{n} = 1.96 \text{ (from table)}$$

$$\sqrt{n} = \frac{1.96}{0.4082}$$

$$\sqrt{n} = 4.8016$$

$$\sqrt{n} = 23.05$$

∴ The size of the sample atleast 24

12

A coin is tossed 10 times. what is the Probability of getting 3 or 4 or 5 heads. use central limit theorem.

Sol

$n = 10$ In coin $p = \frac{1}{2}$ $q = \frac{1}{2}$

mean $\mu = np = 10 \times \frac{1}{2} = 5$

variance $npq = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

$S.D = \sqrt{var} = \sqrt{2.5} = \underline{\underline{1.58}}$

In Discrete Random variable follows normal Then D $\sim N[\mu, \sigma]$ (ie) $z = \frac{\bar{x} - \mu}{\sigma}$

$$z = \frac{x - 5}{1.58}$$

To approximate the discrete probability distribution to continuous probability Distribution to add 0.5 to upper bound and subtract 0.5 from the lower bound.

$$P[3 \le x \le 5]$$

$$P[3 - 0.5 \le x \le 5 + 0.5]$$

$$= P[2.5 \le x \le 5.5]$$

$$= P\left[\frac{2.5 - 5}{1.58} \le z \le \frac{5.5 - 5}{1.58}\right]$$

$$= P[-1.58 \le z \le 0.32]$$

$$= P[-1.58 \le z \le 0] + P[0 \le z \le 0.32]$$

$$= P[0 \le z \le 1.58] + P[0 \le z \le 0.32]$$

$$= 0.4429 + 0.1255$$

$$= \underline{\underline{0.5684}}$$

(13) A coin is tossed 300 times found the Probability that heads will appear more than 140 times and Less than 150 times.

soln Given $n = 300$

$$p = \frac{1}{2}, \quad q = \frac{1}{2}$$

$$\text{mean } \mu = np = \frac{300}{2} = 150$$

$$\text{variance } npq = 300 \times \frac{1}{2} \times \frac{1}{2} = 75$$

$$\sigma = \sqrt{npq} = \sqrt{75} = \underline{8.660}$$

If the discrete Rv's follows normal D ~ N[μ.σ]

$$z = \frac{x - \mu}{\sigma} = \cancel{x - 75} \quad \frac{x - 150}{8.660}.$$

To approximate the discrete probability Distribution to continuous probability Distribution add 0.5 to the upper bound and 0.5 from the Lower bound.

$$P[140 \le x \le 150]$$

$$= P[140 - 0.5 \le x \le 150 + 0.5]$$

$$= P[139.5 \le x \le 150.5]$$

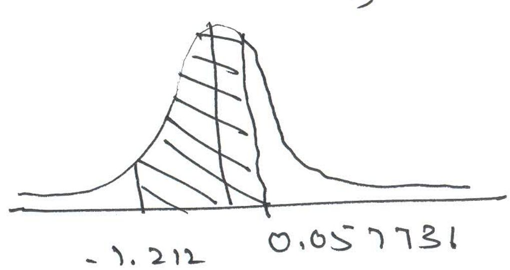
$$= P\left[\frac{139.5 - 150}{8.660} \le z \le \frac{150.5 - 150}{8.660}\right]$$

$$= P[-1.21247 \le z \le 0.057736]$$

$$= P[0 \le z \le 1.21247] + P[0 \le z \le 0.057736]$$

$$= 0.38733 + 0.02302$$

$$= \underline{0.41035}$$



(14) If a can of paint covers on the average 513.3 square feet with a standard Deviation of 31.5 square feet, what is the probability that the mean area covered by a sample of 40 of these cans will be any where from 510.0 to 520.0 square feet?

Soln

Given $\mu = 513.3$ $\sigma = 31.5$ and $n = 40$

By CLT mean of the RV follows normal $\bar{X} \sim N[\mu, \sigma/\sqrt{n})$

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 513.3}{\frac{31.5}{\sqrt{40}}} = \frac{\bar{X} - 513.3}{4.980587}$$

To find

$$P[510.0 \le \bar{x} \le 520.0]$$

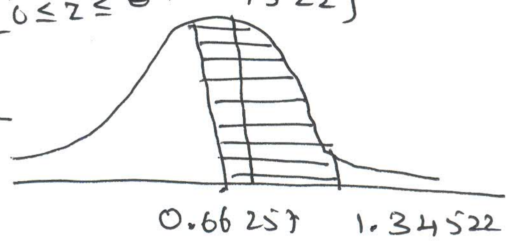
$$= P\left[\frac{510 - 513.3}{4.980587} \le z \le \frac{520 - 513.3}{4.980587}\right]$$

$$= P[-0.66257 \le z \le 1.34522]$$

$$= P[0 \le z \le 0.66257] + P[0 \le z \le 1.34522]$$

$$= 0.2462 + 0.41072$$

$$= \underline{\underline{0.65692}}$$



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(14) If 20 random number are selected independently in the interval (0,1) what is the approximate Probability that the sum of these numbers is atleast 8?

Soln

Given $n = 20$

uniform mean $= \frac{\cancel{a}+b}{2} = \frac{1+0}{2} = \frac{1}{2}$

variance $= \frac{b-a}{12} = \frac{1}{12}$

$$SD = \sqrt{\frac{1}{12}} = 0.288675$$

Sum of the RV follows normal $S_n \sim N(n\mu, \sigma\sqrt{n})$

$$z = \frac{S_n - n\mu}{\cancel{\sigma/\sqrt{n}}\ \sigma\sqrt{n}} = \frac{S_n - 10}{0.28865\sqrt{40}}$$

$$= \frac{S_n - 10}{1.29099}$$

To find $\underline{P[S_n \ge 8]}$

$P[S_n \geq 8]$

$= P\left[Z \geq \frac{8-10}{1.290}\right]$

$= P[Z \geq -1.54919]$

$= 0.5 + P[0 \leq Z \leq 1.54919]$

$= 0.5 + 0.43933$

$= \underline{\underline{0.93934}}$