PANIMALAR ENGINEERING COLLEGE
MA 2262 PROBABILITY AND QUEUEING THEORY

UNIT I RANDOM VARIABLES AND STANDARD DISTRIBUTIONS

PART A

1. The CDF of a RV X is $F(x) = 1-(1+x)e^{-x}, x>0$. Find the pdf.
   Solution: $f(x) = F'(x) = (1+x)e^{-x} - e^{-x} = xe^{-x}, x>0$
2. If a random variable X takes the values 1,2,3,4 such that 2 P(X=1)=3P(X=2)=P(X=3)=5P(X=4). Find the probability distribution of X
   Solution:
   Assume P(X=3) = α By the given equation
   $P(X=1) = \frac{\alpha}{2}$ $P(X=2) = \frac{\alpha}{3}$ $P(X=4) = \frac{\alpha}{5}$
   For a probability distribution (and mass function) $\sum P(x) = 1$
   P(1)+P(2)+P(3)+P(4) =1
   $\frac{\alpha}{2} + \frac{\alpha}{3} + \alpha + \frac{\alpha}{5} = 1 \quad \Rightarrow \quad \frac{61}{30}\alpha = 1 \quad \Rightarrow \alpha = \frac{30}{61}$
   $P(X=1) = \frac{15}{61}$; $P(X=2) = \frac{10}{61}$; $P(X=3) = \frac{30}{61}$; $P(X=4) = \frac{6}{61}$
   The probability distribution is given by

X	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

3. Let X be a continuous random variable having the probability density function
   $$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & otherwise \end{cases}$$ Find the distribution function of x.
   Solution:
   $$F(x) = \int_1^x f(x)\,dx = \int_1^x \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_1^x = 1 - \frac{1}{x^2}$$
4. A random variable X has the probability density function *f(x)* given by
   $$f(x) = \begin{cases} cxe^{-x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$. Find the value of c and CDF of X.

Solution:

$$\int_0^\infty f(x)\,dx = 1$$

$$\int_0^\infty cxe^{-x}\,dx = 1$$

$$c\left[-xe^{-x} - e^{-x}\right]_0^\infty = 1$$

$$c(1) = 1$$

$$c = 1$$

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x cxe^{-x}\,dx$$

$$= \int_0^x xe^{-x}\,dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^x$$

$$= 1 - xe^{-x} - e^{-x}$$

5. A continuous random variable X has the probability density function *f(x)* given by $f(x) = ce^{-|x|}, -\infty < x < \infty$. Find the value of c and CDF of X.

Solution:

$$\int_{-\infty}^\infty f(x)\,dx = 1$$

$$\int_{-\infty}^\infty ce^{-|x|}\,dx = 1$$

$$2\int_0^\infty ce^{-|x|}\,dx = 1$$

$$2\int_0^\infty ce^{-x}\,dx = 1$$

$$2c\left[-e^{-x}\right]_0^\infty = 1$$

$$2c(1) = 1$$

$$c = \frac{1}{2}$$

$Case(i)\ x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\, dx$$

$$= c \int_{-\infty}^{x} e^{x}\, dx$$

$$= c\left[e^{x} \right]_{-\infty}^{x}$$

$$= \frac{1}{2} e^{x}$$

$Case(ii)\ x > 0$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{x} c\, e^{-|x|}\, dx$$

$$= c \int_{-\infty}^{0} e^{x}\, dx + c\int_{0}^{x} e^{-x} dx$$

$$= c\left[e^{x} \right]_{-\infty}^{0} + c\left[-e^{-x} \right]_{0}^{x}$$

$$= c - ce^{-x} + c$$

$$= c\left(2 - e^{-x} \right)$$

$$= \frac{1}{2}\left(2 - e^{-x} \right)$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x}, & x > 0 \\ \frac{1}{2}\left(2 - e^{-x} \right), & x < 0 \end{cases}$$

6. If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$.

   Find the probability that it will take on a value between 1 and 3. Also, find the probability that it will take on value greater than 0.5.

Solution:

$$P(1 < X < 3) = \int_1^3 f(x)\ dx = \int_1^3 2e^{-2x} dx = \left[-e^{-2x}\right]_1^3 = e^{-2} - e^{-6}$$

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\ dx = \int_{0.5}^{\infty} 2e^{-2x} dx = \left[-e^{-2x}\right]_{0.5}^{\infty} = e^{-1}$$

7. Is the function defined as follows a density function?

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

Solution:

$$\int_2^4 f(x)\ dx = \int_2^4 \frac{1}{18}(3+2x)\ dx = \left[\frac{(3+2x)^2}{72}\right]_2^4 = 1$$

Hence it is density function.

8. The cumulative distribution function (CDF) of a random variable X is $F(X) = 1-(1+x)e^{-x}$, $x > 0$. Find the probability density function of X.

Solution:

$$f(x) = F'(x)$$

$$= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right]$$

$$= xe^{-x}, \quad x > 0$$

9. The number of hardware failures of a computer system in a week of operations has the following probability mass function:

No of failures:	0	1	2	3	4	5	6
Probability :	0.18	0.28	0.25	0.18	0.06	0.04	0.01

Find the mean of the number of failures in a week.

Solution:

$$E(X) = \sum x\ P(x) = (0)(0.18) + (1)(0.28) + (2)(0.25) + (3)(0.18) + (4)(0.06) + (5)(0.04) + (6)(0.01)$$

$$= 1.92$$

10. Given the p.d.f of a continuous r.v X as follows: $f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$

Find the CDF of X.

Solution:

$$F(x) = \int_0^x f(x)\, dx = \int_0^x 6x(1 - x)\, dx = \int_0^x 6x - 6x^2\, dx = \left[3x^2 - 2x^3\right]_0^x = 3x^2 - 2x^3$$

11. A continuous random variable X has the probability function $f(x) = k(1+x),\ \ 2 \le x \le 5$. Find P(X<4). Solution:

$$\int_2^4 f(x)dx = 1 \qquad \Rightarrow k\int_2^5 (1+x)dx = 1$$

$$\Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$\Rightarrow k\frac{27}{2} = 1$$

$$\Rightarrow k = \frac{2}{27}$$

$$P(X < 4) = \int_2^4 f(x)\, dx = \frac{2}{27}\int_2^4 (1+x)\, dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4 = \frac{1}{25}(25-9) = \frac{16}{27}$$

12. Given the p.d.f of a continuous R.V X as follows:

$$.\ \ f(x) = \begin{cases} 12.5x - 1.25 & 0.1 \le x \le 0.5 \\ 0, & elsewhere \end{cases}$$

Find P(0.2 < X < 0.3)

Solution:

$$P(0.2 < X < 0.3) = \int_{0.2}^{0.3} (12.5x - 1.25)\, dx$$

$$= \left[12.5\frac{x^2}{2} - 1.25x\right]_{0.2}^{0.3}$$

$$= 1.25\left[5(0.3)^2 - 0.3 - 5(0.2)^2 + 0.2\right]$$

$$= 0.1875$$

13. Find the value of ‘c’ given the pdf of a random variable X as

$$f(x) = \begin{cases} \frac{c}{x^3},\ if\ 1 < x < \infty \\ 0,\ otherwise \end{cases}$$

Solution:

$$\int f(x)\,dx = 1 \quad \Rightarrow \quad \int_1^{\infty} \frac{c}{x^3}\,dx = 1$$

$$\Rightarrow c\left[-\frac{1}{2x^2}\right]_1^{\infty} = 1 \qquad \Rightarrow \frac{c}{2} = 1 \qquad \Rightarrow \qquad c = 2$$

14. Given the probability density function $f(x) = \dfrac{k}{1+x^2}, -\infty < x < \infty$, find k and C.D.F.

Solution:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$\Rightarrow k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\left[\left[\tan^{-1}\infty\right] - \left[\tan^{-1} - \infty\right]\right] = 1$$

$$\Rightarrow k\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

$$\Rightarrow k = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\left[\tan^{-1}\infty\right] - \left[\tan^{-1} - x\right]\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{1}{\pi}\cot^{-1} x$$

15. Find the value of (a) C and (b) mean of the following distribution

$$f(x) = \begin{cases} c\left(x - x^2\right), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Solution:

$$(a)\int_0^1 f(x)\,dx = 1 \qquad (b)E(X) = \int x\ f(x)\,dx$$

$$c\int_0^1 \left(x - x^2\right)dx = 1 \qquad = \int_0^1 x6\left(x - x^2\right)\ dx$$

$$c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \qquad = 6\int_0^1 \left(x^2 - x^3\right)\ dx$$

$$c\frac{1}{6} = 1 \qquad = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$c = 6 \qquad = \frac{1}{2}$$

16. Find the moment generating function where $f(x) = \begin{cases} 2/3,\ at\ x = 1 \\ 1/3,\ at\ x = 2 \\ 0,\ otherwise \end{cases}$

Solution:

$$M_X(t) = \sum e^{tx}\,p(x) = e^t\,p(1) + e^{2t}\,p(2) = e^t\ 2/3 + e^{2t}\ 1/3 = \frac{2e^t + e^{2t}}{3}$$

17. If the MGF of a continuous R.V X is given by $M_X(t) = \frac{3}{3-t}$. Find the mean and variance of X.

Solution:

$$M_X(t) = \frac{3}{3-t} = \frac{1}{1-\frac{t}{3}} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \ldots$$

$$E(X) = (coefficient\ \ of\ \ t)\ 1! = \frac{1}{3}\ \ is\ \ the\ \ \ mean$$

$$E(X^2) = \left(coefficient\ of\ t^2\right)2! = \frac{1}{9}2! = \frac{2}{9}$$

$$Variance\ = E(X^2) - E(X)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

18. If the MGF of a discrete R.V X is given by $M_X(t) = \frac{1}{81}\left(1 + 2e^t\right)^4$,find the distribution of X.

Solution:

$$M_X(t)=\frac{1}{81}\left(1+2e^t\right)^4$$

$$=\frac{1}{81}\left(1+4C_1\left(2e^t\right)+4C_2\left(2e^t\right)^2+4C_3\left(2e^t\right)^3+4C_4\left(2e^t\right)^4\right)$$

$$=\frac{1}{81}+\frac{8}{81}e^t+\frac{24}{81}e^{2t}+\frac{32}{81}e^{3t}+\frac{16}{81}e^{4t}$$

By definition of MGF ,

$$M_X(t)=\sum e^{tx}p(x)=p(0)+p(1)e^t+p(2)e^{2t}+p(3)e^{3t}+p(4)e^{4t}$$

On comparison with above expansion the probability distribution is

X	0	1	2	3	4
$p(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

19. Find the MGF of the R.V X whose p.d.f is $f(x)=\begin{cases}\frac{1}{10}, & 0<x<10\\ 0, & elsewhere\end{cases}$ .Hence its mean.

Solution:

$$M_X(t)=\int_0^{10}\frac{1}{10}e^{tx}\,dx$$

$$=\frac{1}{10}\left(\frac{e^{tx}}{t}\right)_0^{10}$$

$$=\frac{1}{10}\left(\frac{e^{10t}-1}{t}\right)$$

$$=\frac{1}{10t}\left(1+10t+\frac{100t^2}{2!}+\frac{1000t^3}{3!}+....-1\right)$$

$$=1+5t+\frac{1000}{31}t^2+.....$$

$$Mean\ =\ coefficient\ \ of\ \ \ t\ =5$$

20. If the range of x is the set $X=\{0,1,2,3,4\}$ and $P(X=x)=0.2$, determine the mean and variance of the random variable.

Solution:

$$Mean = E(X) = \sum xp(x) = \sum x(0.2) = 0.2\sum x = 0.2(0+1+2+3+4) = 2$$
$$E\left(X^2\right) = \sum x^2 p(x) = \sum x^2(0.2) = 0.2\sum x^2 = 0.2\left(0+1^2+2^2+3^2+4^2\right) = 6$$
$$Variance = E\left(X^2\right) - E(X)^2 = 6-4 = 2$$

21. Define MGF and write the formula to find mean and variance.

$$MGF = M_X(t) = E\left(e^{tX}\right) = \begin{cases} \sum e^{tx} p(x), & X \text{ discrete} \\ \int e^{tx} f(x)dx, & X \text{ continuous} \end{cases}$$

$Formula\ 1:$

$$E(X) = M_X{}'(0)$$
$$E\left(X^2\right) = M_X{}''(0)$$

$Formula\ 2:$

$$E(X) = 1! * coefficient\ of\ t\ in\ \exp ansion\ of\ MGF$$
$$E\left(X^2\right) = 2! * coefficient\ of\ t^2\ in\ \exp ansion\ of\ MGF$$
$$Mean = E(X)$$
$$Variance = E\left(X^2\right) - E(X)^2$$

22. Define central moments of a distribution.

Solution:

$$\mu_r = E\left[\left(X - \overline{X}\right)^r\right] = \begin{cases} \sum (x-\overline{x})^r\ p(x), & X \text{ is discrete} \\ \int (x-\overline{x})^r\ f(x)dx, & X \text{ is Continuous} \end{cases}$$

23. The mean and variance of the binomial distribution are 4 an d 3 respectively. Find P(X=0).

Solution:

$Mean = np = 4$ $\qquad$ $np = 4$

$Variance = npq = 3$ $\qquad$ $n\frac{1}{4} = 4$

$q = \frac{npq}{np} = \frac{3}{4}$ $\qquad$ $p = 1 - q = \frac{1}{4}$ $\qquad$ $n = 16$

$$P(X=0) = nC_0\, p^0 q^n = \left(\frac{3}{4}\right)^{16} = 0.01$$

24. The mean of a binomial distribution is 20 and standard deviation is 4. find the parameters of the distribution.

Solution:

$Mean = np = 20$ $\qquad$ $np = 20$

$Variance = npq = 4^2 = 16$ $\qquad$ $n\frac{1}{5} = 20$

$q = \frac{npq}{np} = \frac{16}{20} = \frac{4}{5}$ $\qquad$ $p = 1 - q = \frac{1}{5}$ $\qquad$ $n = 100$

25. Find the mean of Poisson distribution.

Solution:

$$Mean = E(X) = \sum xp(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda}\left[\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right] = e^{-\lambda}\lambda\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right]$$

$$= e^{-\lambda}\lambda e^{\lambda} = \lambda$$

26. It has been claimed that in 60 % of all solar heat installation the utility bill is Reduced by atleast one-third . Accordingly what are the probabilities that the utility bill will be reduced by atleast one-third in atleast four of five installation.

Soln: Given n=5 , p=60 % =0.6 and q=1-p=0.4

$$p(x \geq 4) = p[x = 4] + p[x = 5]$$

$$= 5c_4 (0.6)^4 (0.4)^{5-4} + 5c_5 (0.6)^5 (0.4)^{5-5}$$

$$= 0.337$$

27. The no. of monthly breakdowns of a computer is a r.v. having poisson distbn with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Soln: $p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $\quad given\ \lambda = 1.8$

$$p(x = 1) = \frac{e^{-1.8}(1.8)^1}{1!} = 0.2975$$

28. In a company 5 % defective components are produced . What is the probability That atleast 5 components are to be examined in order to get 3 defectives.

Soln: To get 3 defectives ,3 or more components must be examined.

p=5 % =0.05 , q = 1- p=0.95 and k=success=3

$$p(X=x)=(x-1)c_{k-1}\, p^k\, q^{x-k}\ ,\ x=k,k+1,k+2,...$$

$$p(x\geq 5)=1-p(x<5)$$

$$=1-\left[p(x=3)+p(x=4)\right]$$

$$=1-\left[2c_2(0.05)^3(0.95)^0+3c_2(0.05)^3(0.95)^1\right]$$

$$1-0.00048=0.9995$$

29. . A discrete r.v X has mgf $M_x(t)=e^{2(e^t-1)}$ .Find E(x), var(x) , and p(x=0).

Soln: Given $M_x(t)=e^{2(e^t-1)}$

We know that mgf of poisson is $M_x(t)=e^{\lambda(e^t-1)}$

Therefore λ=2

In poisson E(x) = var(x) = λ

$\therefore\ Mean\, E(x)=\text{var}(x)=2$

$$p(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$$

$$\therefore\ p(X=0)=\frac{e^{-\lambda}\lambda^0}{0!}=e^{-\lambda}=e^{-2}=0.1353$$

30. Find the mean and variance of geometric distribution .

Soln: The pmf of Geometric distbn is given by

$$p(X=x)=p\,q^{x-1}\ ,x=1,2,3,.....$$

$$Mean\ E(x)=\sum x\,p(x)$$

$$=\sum_{x=1}^{\infty}x\,p\,q^{x-1}=p\sum_{x=1}^{\infty}x\,q^{x-1}$$

$$=p\left[1q^{1-1}+2q^{2-1}+3q^{3-1}+.....\right]$$

$$=p\left[1+2q\ +3q^2+.....\right]=p[1-q]^2$$

$$=p\,p^{-2}=p^{-1}=\frac{1}{p}$$

Mean $=\dfrac{1}{p}$

$$E\left[x^2\right]=\sum x^2\,p(x)$$

$$=\sum_{x=1}^{\infty}x^2\,p\,q^{x-1}$$

$$= \sum_{x=1}^{\infty} \left[x(x+1) - x\right] p\, q^{x-1}$$

$$= \sum_{x=1}^{\infty} x(x+1)\, p\, q^{x-1} - \sum_{x=1}^{\infty} x\, p\, q^{x-1}$$

$$= 1(1+1)\, p\, q^{1-1} + 2(2+1)\, pq^{2-1} + 3(3+1)\, pq^{3-1} + ... - \frac{1}{p}$$

$$= 2p + 2(3)\, pq^{1} + 3(4)\, pq^{2} + ... - \frac{1}{p}$$

$$= 2p\left[1 + 3q \;\; + 6q^{2} + ...\right] - \frac{1}{p}$$

$$= 2p\left[1 - q\right]^{-3} - \frac{1}{p} = 2\, p\, p^{-3} - \frac{1}{p}$$

$$= \frac{2}{p^{2}} - \frac{1}{p}$$

Variance $= E(x^{2}) - \left[E(x)\right]^{2}$

$$= \frac{2}{p^{2}} - \frac{1}{p} - \left(\frac{1}{p}\right)^{2} = \frac{2}{p^{2}} - \frac{1}{p} - \frac{1}{p^{2}}$$

$$= \frac{1}{p^{2}} - \frac{1}{p} = \frac{1-p}{p^{2}} = \frac{q}{p^{2}}$$

Variance $= \dfrac{q}{p^{2}}$

31. Find mgf of geometric distribution.

Soln: The pmf of geometric distribution is given by

$$Mgf \;\; M_{x}(t) = E(e^{tx}) = \sum e^{tx}\;\, p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx}\, p\, q^{x-1} = \sum_{x=1}^{\infty} e^{tx}\, p\;\, q^{x}\;\, q^{-1}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} \left(e^{t}\right)^{x}\, q^{x} = \frac{p}{q} \sum_{x=1}^{\infty} \left(qe^{t}\right)^{x}$$

$$= \frac{p}{q}\left[qe^{t} + \left(qe^{t}\right)^{2} + \left(qe^{t}\right)^{3} + ...\right]$$

$$= \frac{p}{q}\, qe^{t}\left[1 + qe^{t} + \left(qe^{t}\right)^{2} + ...\right]$$

$$= pe^{t}\left(1 - qe^{t}\right)^{-1} 1 - qe^{t}$$

$$\therefore \quad M_x(t) = 1 - qe^t$$

32. If on the average rain falls on 10 days in every 30 days, obtain the probability that rain will fall on atleast 3 days of a given week.

Solution:

$$\text{Success event is rain fall} \qquad p = \frac{10}{30} = \frac{1}{3} \qquad\qquad so\ q = \frac{2}{3}$$

$$A\ week\ is\ taken\ n = 7\ \ Applying\ \ binomial\ distribution$$

$$P(X \geq 3) = P(3) + P(4) + P(5) + P(6) + P(7) = 1 - P(0) - P(1) - P(2)$$

$$= 1 - 7C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^7 - 7C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^6 - 7C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^5$$

$$= 1 - 0.0585 - 0.2048 - 0.3072 = 0.4295$$

33. The probability of a successful optical alignment in the assembly of an optical data storage product is 0.8. Assume the trials are independent, what is the probability that the first successful alignment requires exactly four trials?

Solution:

First success in fourth trial, so let us apply Geometric distribution.

$p = 0.8$

$$P(X = 4) = q^{4-1}p = (0.2)^3 0.8 = 0.0064$$

34. If the probability is 0.10 that a certain kind of measuring device will show excessive drift, what is the probability that the fifth measuring device tested will be the first show excessive drift? Find its expected value also.

Solution:

First success(show excessive drift) in fifth trial. so let us apply Geometric distribution

$p = 0.1$

$$P(X = 5) = q^4 p = (0.9)^4 0.1 = 0.0656$$

$$E(X) = \frac{1}{p} = \frac{1}{0.1} = 10$$

35. One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core-size limitations. Find the probability that among a sample of 200 jobs there are no jobs that have to wait until weekends.

Solution:

Success event is waiting of a system for scheduling until weekends.

X number of systems waiting for scheduling until weekends.

p = 1 % = 0.01 n =200 Let us apply Poisson distribution

$$\lambda = np = 0.01 * 200 = 2$$

$$P(X = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-2} = 0.1353$$

36. If the probability is 0.40 that a child exposed to a certain contagious disease will catch it, what is the probability that the tenth child exposed to the disease will the third to catch it?

Solution:

Third success in tenth trial. so, let us apply negative binomial distribution

$$\text{Required probability} = \left(9C_2\, p^2 q^7\right)p = 36(0.4)^2(0.6)^7(0.4) = 0.0645$$

37. If X is uniformly distributed over (0,10) calculate the probability that (a) X>6 (b) 3<X<8

Solution:

$$f(x) = \frac{1}{10},\ 0 < x < 10$$

$$(a)\ P(X > 6) = \int_6^{10} \frac{1}{10}\,dx = \frac{1}{10}[x]_6^{10} = \frac{4}{10} = \frac{2}{5}$$

$$(b)\ P(3 < X < 8) = \int_3^{8} \frac{1}{10}\,dx = \frac{1}{10}[x]_3^{8} = \frac{5}{10} = \frac{1}{2}$$

38. Define exponential random variable and give an example.

Solution:

A continuous random variable X is said to follow exponential random variable

$f(x) = \lambda e^{-\lambda x},\ x > 0\ \ \lambda$ *being the parameter*

The time between failures of machines in a large factory is an exponential random variables.

39. Define a continuous random variable and give an example.

Solution:

A continuous random variable is a random variable whose values can not be counted and only measured and so it takes any value in an interval of $(-\infty, \infty)$

Height in cm of armymen in infantry is a random variable which is normally between $(170,190)$

40. Suppose that a bus arrives at a station every day between 10.00 am and 10.30 am at random. Let X be the arrival time, find the distribution function of X and sketch its graph.

Solution:

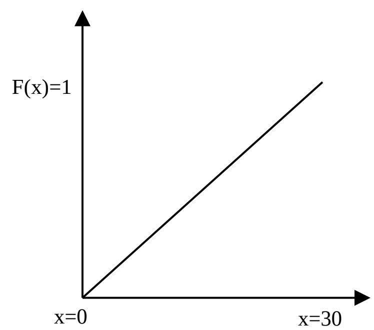
X is the uniform random variable and its density function is given by

$!0 am \text{ is treated as } 0 \min \text{ and } 10.30 am \text{ is treated as } 30 \min$

$$f(x) = \frac{1}{30-0} = \frac{1}{30},\ 0 \le x \le 30$$

$$F(x) = \int_0^x f(x)dx = \int_0^x \frac{1}{30}dx = \frac{x}{30}, 0 \le x \le 30$$

and its graph is



41. A fast food chain finds that the average time customers have to wait for service is 45 seconds. If the waiting time can be treated as an exponential random variable, what is the probability that a customer will have to wait more than 5 minutes given that already he waited for 2 minutes?

Solution:

$$\text{Average waiting time} = \frac{1}{\lambda} = 45 \sec = \frac{45}{60}\min = 0.75 \qquad \lambda = 1.33$$

$P(X > 5 | X > 2) = P(X > 3)$ *by memoryless property of exponential dist.*

$$= \int_3^\infty 1.33\, e^{-1.33x} dx = \left[-e^{-1.33x}\right]_3^\infty = e^{-1.33(3)} = 0.0185$$

42. Let X be a uniform random variable over [-1,1]. Find

(a) $P\left(|X| < \frac{1}{3}\right)$ (b) $P\left(|X| \ge \frac{3}{4}\right)$

Solution:

$$f(x) = \frac{1}{b-a} = \frac{1}{1-(-1)} = \frac{1}{2}, -1 < x < 1$$

$$P\left(|X| < \frac{1}{3}\right) = \int_{-\frac{1}{3}}^{\frac{1}{3}} f(x)dx = \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{2}dx = \frac{1}{2}[x]_{-1/3}^{1/3} = \frac{1}{3}$$

$$P\left(|X| \geq \frac{3}{4}\right) = 1 - P\left(|X| < \frac{3}{4}\right) = \int_{-\frac{3}{4}}^{\frac{3}{4}} f(x)dx = \int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{2}dx = \frac{1}{2}[x]_{-3/4}^{3/4} = \frac{3}{4}$$

43. Suppose X has an exponential distribution with mean equal to 10. Determine the value of 'x' such that P(X<x) = 0.95.

Solution:

$$\frac{1}{\lambda} = 10 \qquad \lambda = \frac{1}{10} = 0.1$$

$$P(X < x) = 0.95(given) \qquad \Rightarrow \qquad \int_0^x 0.1e^{-0.1x}\, dx = 0.95$$

$$\Rightarrow \left[-e^{-0.1x}\right]_0^x = 0.95 \qquad \Rightarrow \qquad 1 - e^{-0.1x} = 0.95$$

$$\Rightarrow e^{-0.1x} = 0.05 \qquad \Rightarrow \qquad -0.1x = \ln 0.05 = -3$$

$$x = 30$$

44. Show that for the uniform distribution $f(x) = \frac{1}{2a}$ $, -a < x < a$,the mgf about origin is $\frac{\sinh at}{at}$

Soln: Given $f(x) = \frac{1}{2a}$ $, -a < x < a$

MGF $M_x(t) = E\left[e^{tx}\right]$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-a}^{a} e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a} \int_{-a}^{a} e^{tx} dx = \frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$= \frac{1}{2at}\left[e^{at} - e^{-at}\right] = \frac{1}{2at} 2\sinh at = \frac{\sinh at}{at}$$

$$M_x(t) = \frac{\sinh at}{at}$$

45. Define exponential density function and find mean and variance of the same.

Soln: The density function of exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

$$Mean = E[x] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^{\infty} x\lambda\, e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x\, e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{-xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \lambda \left[(0-0) - \left(0 - \frac{1}{\lambda^2} \right) \right] = \lambda \left(\frac{1}{\lambda^2} \right) = \frac{1}{\lambda}$$

$$Mean = \frac{1}{\lambda}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$$

$$= \int_0^{\infty} x^2 \lambda\, e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x^2\, e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$$

$$= \lambda \left[(0-0-0) - \left(0 - 0 - \frac{2}{\lambda^3} \right) \right] = \lambda \left(\frac{2}{\lambda^3} \right) = \frac{2}{\lambda^2}$$

$$Variance = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

PART B

BAYES' THEOREM

1. In a bolt facotry machines A,B,C manufacture respectively 25,35 and 40 percent of the total production. Of their outputs 5,4 and 2 percent are defective bolts respectively. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manucatured by the machines B or C?
   Solution:
   A condition B (The drawn prodcut found to be defective) is given. Based on this condition we are asked to find the probability that it was manufactured by B Which is mutually exclusive with other events of produced by machine B,C.
   So apply Bayes' theorem. to apply this first identify the mutually exclusive events

   A_1 the product was manufactured by machine A
   A_2 the product was manufactured by machine B

A_3 the product was manufactured by machine C

Also, identify the event B(it is the condition given) which is occurring with each of A_i's
B the drawn product was found to be defective.
Question asked

$$P(\text{product was manufacutred machine B} \mid \text{product found to be defective}) = P(A_2 \mid B)$$

$$P(\text{product was manufacutred machine C} \mid \text{product found to be defective}) = P(A_3 \mid B)$$

$$P(\text{product was manufacutred machine B OR C} \mid \text{product found to be defective}) = P(A_2 \cup A_3 \mid B)$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$P(A_1) = 0.25 \qquad P(BA_1) = 0.05$$

$$P(A_2) = 0.35 \qquad P(B \mid A_2 = 0.04)$$

$$P(A_3) = 0.4 \qquad P(B \mid A_3) = 0.02$$

$$= \frac{0.35 * 0.04}{0.25 * 0.05 + 0.35 * 0.04 + 0.4 * 0.02} = \frac{0.14}{0.125 + 0.14 + 0.008}$$

$$= \frac{0.14}{0.273} = 0.5128$$

$$P(A_3 \mid B) = \frac{P(A_3)P(B \mid A_3)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$P(A_1) = 0.25 \qquad P(BA_1) = 0.05$$

$$P(A_2) = 0.35 \qquad P(B \mid A_2 = 0.04)$$

$$P(A_3) = 0.4 \qquad P(B \mid A_3) = 0.02$$

$$= \frac{0.4 * 0.02}{0.25 * 0.05 + 0.35 * 0.04 + 0.4 * 0.02} = \frac{0.008}{0.125 + 0.14 + 0.008}$$

$$= \frac{0.008}{0.273} = 0.0293$$

$$P(\text{product was manufacutred machine B OR C} \mid \text{product found to be defective}) = P(A_2 \cup A_3 \mid B)$$

$$= 0.5128 + 0.0293 = 0.5421$$

2. A certain disease is found to in one person in 5000. If a person does have the disease, in 92% of the cases the diagnostic procedure will show that he or she actually has it. If a person does not have the disease,the diagnostic procedure in one out of 500 cases gives a false result. What is the probability that a person with a postive test result has the disease?
   Solution:
   A condition B (The test result is positive) is given. Based on this condition we are asked to find the probability that the person actually has the disease.
   So apply Bayes' theorem. to apply this first identify the mutually exclusive events

A_1 the person actually has the disease
A_2 the person actually does not have the disease

Also, identify the event B(it is the condition given) which is occurring with each of A_i's
B the drawn test result is positive
Question asked $P(\text{person has the disease}|\text{test result is positive}) = P(A_1|B)$

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)}$$

$$P(A_1) = \frac{1}{5000} \qquad P(B|\ A_1) = 92\% = 0.92$$

$$P(A_2) = \frac{4999}{5000} \qquad P(B|\ A_2) = \frac{1}{500}$$

$$= \frac{\frac{1}{5000} * 0.92}{\frac{1}{5000} * 0.92 + \frac{4999}{5000} * \frac{1}{500}} = \frac{0.000184}{0.000184 + 0.002} = \frac{0.000184}{0.002184} = 0.084$$

3. An urn contains 10 white balls and 3 black balls. Another urn contains 3 white balls and 2 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then one ball is drawn at random from the later. What is the probability that it is a black ball?
Solution:
An information is given ie ball taken from the second urn is black.
Probability of this information is asked irrespective of the previous information (ie whether WW or BB or WB balls are transferred from first urn to second urn). So we need to consider all the cases and it is a total probability. in Bayes' theorem notation this is P(B) and denominator in the Bayes theorem formula.
$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$
where
A_1 2 white balls are (drawn) transferred from first urn (to second urn)
A_2 2 black balls are (drawn) transferred from first urn (to second urn)
A_3 1 white and 1 black balls are (drawn) transferred from first urn (to second urn)

B a balck ball is drawn from the second urn

$$P(A_1) = \frac{10C_2}{13C_2} = 0.5679 \qquad P(B|\ A_1) = \frac{2}{7} = 0.2857$$

$$P(A_2) = \frac{3C_2}{13C_2} = 0.038 \qquad P(B|A_3) = \frac{4}{7} = 0.5714$$

$$P(A_3) = \frac{10C_1 * 3C_1}{13C_2} = 0.3846 \qquad P(B|A_3) = \frac{3}{7} = 0.4286$$

$$\begin{aligned} P(B) &= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) \\ &= 0.5679 * 0.2857 + 0.038 * 0.5714 + 0.3846 * 0.4286 = 0.3488 \end{aligned}$$

RANDOM VARIABLES

1. If the probability density function of the random variable X is given by $f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & elsewhere \end{cases}$ (i) Find the rth moment (ii) Also evaluate E[(2x+1)2].

Solution:

$$(i) E\left(X^r\right) = \int x^r \ f(x) \ dx$$

$$= \int_0^1 x^r \ 2(1-x) \ dx$$

$$= 2\int_0^1 \left(x^r - x^{r+1}\right) dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= \frac{2}{(r+1)(r+2)}$$

$$(ii) E\left[(2X+1)^2\right] = E\left(4X^2 + 4X + 1\right)$$

$$= 4E\left(X^2\right) + 4E(X) + E(1)$$

$$= 4\frac{2}{(2+1)(2+2)} - 4\frac{2}{(1+1)(1+2)} + 1$$

$$= \frac{2}{3} - \frac{4}{3} + 1$$

$$= \frac{1}{3}$$

2. A random variable X has the following probability distribution

X:	-2	-1	0	1	2	3
P(x):	0.1	k	0.2	2k	0.3	3k

Find (i) the value of ‘k’

(ii) cumulative distribution of X

(iii) P(X<2) and P(-2<X<2)

(iv) Evaluate mean of X.

Solution:

(i)

$$\sum p(x) = 1$$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$0.6 + 6k = 1$$

$$6k = 0.4$$

$$k = \frac{1}{15}$$

Hence the probability distribution is

X:	-2	-1	0	1	2	3
P(x):	1/10	1/15	2/10	2/15	3/10	3/15

(ii) The cumulative distribution is

X:	-2	-1	0	1	2	3
P(x):	1/10	1/15	2/10	2/15	3/10	3/15
	3/30	2/30	6/30	4/30	9/30	6/30
F(x):	3/30	5/30	11/30	15/30	24/30	30/30

The same may be presented as

$$F(x)=\begin{cases} 0, & x<-2 \\ 3/30, & -2\le x<-1 \\ 5/30, & -1\le x<0 \\ 11/30, & 0\le x<1 \\ 15/30, & 1\le x<2 \\ 24/30, & 2\le x<3 \\ 1, & x\ge 3 \end{cases}$$

(iii) $p(X<2)=p(-2)+p(-1)+p(0)+p(1)=F(1)=15/30$

$p(-2<X<2)=p(-1)+p(0)+p(1)=12/30$

(iv) $E(X)=\sum xp(x)=(-2)\frac{3}{30}+(-1)\frac{2}{30}+(0)\frac{6}{30}+(1)\frac{4}{30}+(2)\frac{9}{30}+(3)\frac{6}{30}=\frac{32}{30}$

3. A random variable X has a uniform distribution over the interval (-3,3). Compute (i) $P(X=2)$ (ii) $P(|X-2|<2)$ (iii) Find 'k' such that $P(X>k)=\frac{1}{3}$.

Solution:

The probability density function of uniform distribution is given by

$$f(x)=\frac{1}{3-(-3)}=\frac{1}{6},-3<x<3$$

$$(i)P(X=2)=\int_2^2 f(x)dx=0$$

Aliter:

$$(i)P(X=2)=P(1.95<X<2.05)=\int_{1.95}^{2.05}\frac{1}{6}dx=\frac{2.05-1.95}{6}=\frac{0.1}{6}=0.017$$

$$(ii)P(|X-2|<2)=P(-2<X-2<2)$$
$$=P(0<X<4)$$
$$=\int_0^4 f(x)dx=\int_0^3 f(x)dx+\int_3^4 f(x)dx$$
$$=\int_0^3 \frac{1}{6}dx+\int_3^4 0dx=\frac{1}{2}$$

$$(iii)P(X>k)=\frac{1}{3}$$
$$\int_k^3 \frac{1}{6}dx=\frac{1}{3}$$
$$\frac{3-k}{6}=\frac{1}{3}$$
$$k=1$$

4. A continuous random variable has the probability density function $f(x)=kx^4,-1<x<0$. Find the value of 'k' and evaluate $P\left(X>-\frac{1}{2}\mid X<-\frac{1}{4}\right)$.

Solution:
Since it is a probability density function

$$\int f(x)dx=1$$

$$\int_{-1}^{0} kx^4dx=1$$
$$k\left[\frac{x^5}{5}\right]_{-1}^{0}=1$$
$$\frac{k}{5}=1$$
$$k=5$$

$$P\left(X > -\frac{1}{2} \mid X < -\frac{1}{4}\right) = \frac{P\left(X > -\frac{1}{2} \cap X < -\frac{1}{4}\right)}{P\left(X < -\frac{1}{4}\right)}$$

$$= \frac{P\left(-\frac{1}{2} < X < -\frac{1}{4}\right)}{P\left(X < -\frac{1}{4}\right)} = \frac{5\int_{-1/2}^{-1/4} x^4 dx}{5\int_{-1}^{-1/4} x^4 dx}$$

$$= \frac{5\left[\frac{x^5}{5}\right]_{-1/2}^{-1/4}}{5\left[\frac{x^5}{5}\right]_{-1}^{-1/4}} = \frac{\frac{1}{32} - \frac{1}{1024}}{1 - \frac{1}{1024}} = \frac{31}{1023}$$

5. If $p(x) = \begin{cases} xe^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (i) Show that p(x) is a p.d.f (ii) Find F(x)

Solution:

$$(i) \int f(x)dx = \int_0^\infty xe^{-x^2/2} dx$$

$$= \int_0^\infty e^{-t} dt \qquad Put\ t = \frac{x^2}{2}$$

$$= \left[-e^{-t}\right]_0^\infty = 1$$

$$(ii)\ F(x) = \int_0^x xe^{-x^2/2} dx$$

$$= \int_0^{\frac{x^2}{2}} e^{-t} dt \qquad Put\ t = \frac{x^2}{2}$$

$$= \left[-e^{-t}\right]_0^{\frac{x^2}{2}}$$

$$= 1 - e^{-x^2/2}$$

6. The sales of a convenience store on a randomly selected day are X thousand dollars, where X is a random variable with a distribution function of the following form:

$$F(x)=\begin{cases} 0, \ x<0 \\ \frac{x^2}{2}, \ 0\le x<1 \\ k\left(4x-x^2\right), 1\le x<2 \\ 1, x\ge 2 \end{cases}$$

Suppose that this convenience store's sales on any given day are less than \$2000

(i)Find the value of 'k'

(ii)Let A and B be the events that tomorrow the store's sales are between \$500 and \$ 1500 and over \$1500 respectively find P(A) and P(B) .

(iii) Are A and B independent?

Solution:

(i)Since f(x) is a probability density function

$$\int_0^2 f(x)dx=1$$

$$F(2)=1$$

$$k(4*2-4)=1$$

$$k=\frac{1}{4}$$

(ii) A: $0.5 \le X \le 1.5$

B: $X \ge 1.5$

$$P(A)=\int_{0.5}^{1.5} f(x)dx=F(1.5)-F(0.5)=\frac{1}{4}\left(4*1.5-1.5^2\right)-\frac{0.5^2}{2}=0.9375-0.125=0.8125$$

$$P(B)=\int_{1.0}^{2} f(x)dx=F(2)-F(1.0)=1-\frac{1}{4}\left(4*1-1.^2\right)=0.25$$

$$(iii)\ P(A\cap B)=P(1\le X\ \le 1.5)=F(1.5)-F(1)=0.9375-0.75=0.1875$$

$$P(A)P(B)=0.8125*0.25=0.20315$$

$$P(A\cap B)\ne P(A)P(B)$$

So, they are not independent.

7. If the density function of a continuous random variable X is given by

$$f(x)=\begin{cases} ax, & for\ 0\le x\le 1 \\ a, & for\ 1\le x\le 2 \\ 3a-ax, & for\ 2\le x\le 3 \\ 0, & elsewhere \end{cases}$$

(i)find the value of 'a'

(ii) find the c.d.f of X

(iii)If X_1,X_2,X_3 are 3 independent observations of X, what is the probability that exactly one of these is greater than 1.5?

Solution:

(i) Since f(x) is a p.d.f $\int_0^3 f(x)\,dx = 1$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$$

$$\left[\frac{ax^2}{2}\right]_0^1 + [ax]_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\frac{a}{2} + 2a - a + 9a - \frac{9a}{2} - 6a + \frac{4a}{2} = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

Hence the p.d.f is

$$f(x) = \begin{cases} \frac{x}{2}, & \text{for } 0 \le x \le 1 \\ \frac{1}{2}, & \text{for } 1 \le x \le 2 \\ \frac{3-x}{2}, & \text{for } 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(ii) The c.d.f is

$$F(x) = \begin{cases} \int_0^x \frac{x}{2}\,dx, & \text{for } 0 \le x \le 1 \\ \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx, & \text{for } 1 \le x \le 2 \\ \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_1^x \frac{3-x}{2}\,dx, & \text{for } 2 \le x \le 3 \\ 1, & \text{for } x \ge 3 \end{cases}$$

$$F(x)=\begin{cases}\left[\dfrac{x^2}{4}\right]_0^x, & for\ 0\le x\le 1\\ \left[\dfrac{x^2}{4}\right]_0^1+\left[\dfrac{x}{2}\right]_1^x, & for\ 1\le x\le 2\\ \left[\dfrac{x^2}{4}\right]_0^1+\left[\dfrac{x}{2}\right]_1^x+\dfrac{1}{2}\left[3x-\dfrac{x^2}{2}\right]_1^x, & for\ 2\le x\le 3\\ 1, & for\ x\ge 3\end{cases}$$

$$F(x)=\begin{cases}\dfrac{x^2}{4}, & for\ 0\le x\le 1\\ \dfrac{1}{4}+\left[\dfrac{x-1}{2}\right], & for\ 1\le x\le 2\\ \dfrac{1}{4}+\dfrac{1}{2}+\dfrac{1}{2}\left[3x-\dfrac{x^2}{2}-3+\dfrac{1}{2}\right], & for\ 2\le x\le 3\\ 1, & for\ x\ge 3\end{cases}$$

$$F(x)=\begin{cases}\dfrac{x^2}{4}, & for\ 0\le x\le 1\\ \dfrac{2x-1}{4}, & for\ 1\le x\le 2\\ \dfrac{1}{2}\left[3x-\dfrac{x^2}{2}-2\right], & for\ 2\le x\le 3\\ 1, & for\ x\ge 3\end{cases}$$

(iii) $P(X\le 1.5)=F(1.5)=\dfrac{2(1.5)-1}{4}=\dfrac{1}{2}$

$$P\begin{pmatrix}exactly\ one\\ greater\ than\ 1.5\end{pmatrix}=P\left(\bar{A}\,BC\ \ or\ \ A\bar{B}C\ \ or\ \ AB\bar{C}\right)$$

$$=\left[\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]+\left[\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]+\left[\frac{1}{2}\frac{1}{2}\frac{1}{2}\right]$$

$$=\frac{3}{8}$$

8. If X has the distribution function

$$F(x)=\begin{cases}0, & x<1\\ \dfrac{1}{3}, & 1\le x<4\\ \dfrac{1}{2}, & 4\le x<6\\ \dfrac{5}{6}, & 6\le x<10\\ 1, & x\ge 10\end{cases}$$

Find

(i) the probability distribution of X

(ii)P(2<X<6)

(iii) Mean of X

(iv) Variance of X

Solution:

(i)Note that F(x) is constant in all cases mean that X is a discrete variable.The probability distribution of X is

X:	1	4	6	10
P(X):	$\frac{1}{3}$	$\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$	$\frac{5}{6}-\frac{1}{2}=\frac{1}{3}$	$1-\frac{5}{6}=\frac{1}{6}$

(ii) $P(2 < X < 6) = P(4) = \frac{1}{6}$

(iii)Mean of X

$$E(X) = \sum x\, p(x) = 1\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{1}{3}\right) + 10\left(\frac{1}{6}\right) = \frac{28}{6}$$

(iv)Variance of X = $E(X^2) - E(X)^2 = \frac{190}{6} - \left(\frac{28}{6}\right)^2 = \frac{190}{6} - \frac{784}{36} = \frac{356}{36}$

$$E(X^2) = \sum x^2\, p(x) = 1^2\left(\frac{1}{3}\right) + 4^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{3}\right) + 10^2\left(\frac{1}{6}\right) = \frac{190}{6}$$

9. Let X be a continuous random variable with p.d.f $f(x) = \frac{2}{x^2}$, $1 < x < 2$

   Find E(log X).

   Solution:

$$E(\log X) = \int_1^2 \log x\ f(x)\, dx = \int_1^2 \log x\ \frac{2}{x^2}\, dx = 2\left[\left(\log x\left(-\frac{1}{x}\right)\right)_1^2 - \int_1^2 \frac{1}{x}\left(-\frac{1}{x}\right)dx\right]$$

$$= 2\left[-\frac{1}{2}\log 2 + \left(-\frac{1}{x}\right)_1^2\right] = -\log 2 + \frac{1}{2}$$

10. The probability density function of the samples of speech wave forms is found to be decay exponentially, at the rate α, so the following p.d.f is proposed

    $f(x) = ce^{-|x|}$, $-\infty < x < \infty$. Find the constant ‘c’ and then find $P(|X| < \nu)$ and E(X).

    Solution:

$$\int_{-\infty}^{\infty} ce^{-|x|} dx = 1$$

$$2c\int_{0}^{\infty} e^{-|x|} dx = 1$$

$$2c\int_{0}^{\infty} e^{-x} dx = 1$$

$$2c\left[-e^{-x}\right]_0^{\infty} = 1 \qquad c = \frac{1}{2}$$

$$P(|X| < \nu) = \int_{-\nu}^{\nu} ce^{-|x|} dx$$

$$= 2c\int_{0}^{\nu} e^{-|x|} dx$$

$$= \int_{0}^{\nu} e^{-x} dx$$

$$= \left[-e^{-x}\right]_0^{\nu} = 1 - e^{-\nu}$$

$$E(X) = \int_{-\infty}^{\infty} x\ f(x)\ dx = \frac{1}{2}\int_{-\infty}^{\infty} x\ e^{-|x|} dx$$

$$= \int_{0}^{\infty} x e^{-|x|} dx = \int_{0}^{\infty} x e^{-x} dx = 1! = 1$$

11. If X is a continuous random variable with probability density function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ \frac{3}{2}(x-1)^2, & 1 \le x < 2 \\ 0, & otherwise \end{cases}$$ Find cumulative distribution function F(x).

Use it to find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$

Solution:

$$F(x)=\begin{cases}\int_0^x x\,dx, & 0\le x<1\\ \int_0^1 x\,dx+\int_1^x \frac{3}{2}(x-1)^2\,dx, & 1\le x<2\\ 1, & x\ge 2\end{cases}$$

$$=\begin{cases}\left[\frac{x^2}{2}\right]_0^x, & 0\le x<1\\ \left[\frac{x^2}{2}\right]_0^1+\frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_1^x, & 1\le x<2\\ 1, & x\ge 2\end{cases}$$

$$=\begin{cases}\frac{x^2}{2}, & 0\le x<1\\ \frac{1}{2}+\frac{1}{2}\left[(x-1)^3\right], & 1\le x<2\\ 1, & x\ge 2\end{cases}$$

$$P\left(\frac{3}{2}<X<\frac{5}{2}\right)=F\left(\frac{5}{2}\right)-F\left(\frac{3}{2}\right)=1-\left\{\frac{1}{2}+\frac{1}{2}\left[\left(\frac{3}{2}-1\right)^3\right]\right\}=\frac{7}{16}$$

12. The cumulative distribution function (cdf) of a random variable X is given by

$$F(x)=\begin{cases}0, & x<0\\ x^2, & 0\le x<\frac{1}{2}\\ 1-\frac{3}{25}(3-x)^2, & \frac{1}{2}\le x<3\\ 1, & x\ge 3\end{cases}$$

Find the pdf of X and evaluate $P(|X|\le 1)$ using both pdf and cdf.

Solution:

$$f(x)=F'(x)=\begin{cases}0, & x<0\\ 2x, & 0\le x<\frac{1}{2}\\ \frac{6}{25}(3-x)^2, & \frac{1}{2}\le x<3\\ 0, & x\ge 3\end{cases}$$

$$f(x)=\begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x)\ , & \frac{1}{2} \le x < 3 \\ 0, & elsewhere \end{cases}$$

$$(i)P(|X| \le 1) = \int_{-1}^{1} f(x)dx = \int_{-1}^{0} 0\,dx + \int_{0}^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)dx$$

$$= \left[x^2\right]_0^{\frac{1}{2}} + \left[\frac{6}{25}\frac{(3-x)^2}{-2}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} - \frac{3}{25}\left(4 - \frac{25}{4}\right) = \frac{1}{4} + \frac{27}{100} = \frac{52}{100} = \frac{13}{25}$$

$$(ii)P(|X| \le 1) = P(-1 < X < 1) = F(1) - F(-1) = 1 - \frac{3}{25}(3-1)^2 - 0 = \frac{13}{25}$$

13. Experience has shown that while walking in a certain park, the time X, (in mins), between seeing two people taking rest has a density function of the form

$$f(x)=\begin{cases} \lambda x e^{-x}, & x > 0 \\ 0, & otherwise \end{cases}$$

(i)Calculate the value of λ (ii)Find the distribution function of X

(iii) What is the probability that Jeff , who has just seen a person taking rest,will see another person taking rest in 2 to 5 minutes?

Solution:

$$(i)\int f(x)dx = 1$$

$$\int_0^\infty \lambda x e^{-x}\,dx = 1$$

$$\lambda \int_0^\infty x e^{-x}\,dx = 1$$

$$\lambda 1! = 1$$

$$\lambda = 1$$

$$(ii)F(x) = \int_0^x f(x)dx$$

$$= \int_0^x \lambda x e^{-x}\,dx$$

$$= \int_0^x x e^{-x}\,dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^x$$

$$= -xe^{-x} - e^{-x} + 1$$

$$= 1 - (1+x)e^{-x}$$

$$(iii) P(2 < X < 5) = F(5) - F(2)$$
$$= \left[1-(1+5)e^{-5}\right] - \left[1-(1+2)e^{-2}\right]$$
$$= 3e^{-2} - 6e^{-5}$$
$$= 0.3656$$

Moments and MGF

14. A continuous random variable $f(x) = kx^2 e^{-x},\ x > 0$. Find the rth moment of X, MGF. Hence find the mean and variance of X.

Solution:

$$\int_0^\infty kx^2 e^{-x} dx = 1$$
$$k\int_0^\infty x^2 e^{-x} dx = 1$$
$$k\ 2! \qquad = 1$$
$$k = \frac{1}{2}$$
$$MGF = M_X(t) = \int_0^\infty \frac{1}{2} x^2 e^{-x} e^{tx}\ dx$$
$$= \frac{1}{2}\int_0^\infty x^2 e^{-(1-t)x} dx$$
$$= \frac{1}{2}\left[x^2 \frac{e^{-(1-t)x}}{-(1-t)} - 2x\frac{e^{-(1-t)x}}{(1-t)^2} + 2\frac{e^{-(1-t)x}}{-(1-t)^3}\right]_0^\infty$$
$$= \frac{1}{(1-t)^3}$$

$$M_X(t) = \frac{1}{(1-t)^3} = (1-t)^{-3} = 1 + 3t + 6t^2 + 20t^3 + \ldots.$$

$$E(X^r) = \int_0^\infty x^r\ f(x)\ dx = \int_0^\infty x^r\ \frac{1}{2} x^2 e^{-x} = \frac{1}{2}\int_0^\infty x^{r+2} e^{-x} = \frac{(r+2)!}{2}$$

Mean = E(X) = coeff of t = 3

$E(X^2)$ = coeff of t^2 * 2 ! = 12

Variance = 12 – 9 = 3

15. If the moments of a random variable ‘X’ are defined by $E(X^r) = 0.6, \quad r = 1,2,3....$ Find the probability distribution of X.

Solution:

$$E(X^r) = (coefficient\ of\ t^r\ in\ M_x(t))\, r!$$

$$coefficient\ of\ t^r\ in\ M_x(t) = \frac{E(X^r)}{r!}$$

$$Note\ that\ E(X^0) = \sum x^0 p(x) = \sum p(x) = 1$$

$$Hence\ M_x(t) = \sum_{r=0,1,2,3...} \frac{E(X^r)}{r!} t^r = E(X^0) + \sum_{r=1,2,3...} \frac{E(X^r)}{r!} t^r$$

$$= 1 + \sum_{r=1,2,3..} \frac{0.6}{r!} t^r$$

$$= 1 + 0.6\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right]$$

$$= 0.4 + 0.6 + 0.6\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right]$$

$$= 0.4 + 0.6\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ...\right]$$

$$= 0.4 + 0.6e^t$$

$$Also, M_x(t) = \sum_x e^{tx}\ p(x) = p(0) + e^t p(1) + e^{2t} p(2) + e^{3t} p(3)$$

$$On\ comparison\ \ p(X=0) = 0.4,\ \ p(X=1) = 0.6\ \ and\ \ p(X \geq 2) = 0$$

16. A random variable X has density function given by $f(x) = \begin{cases} \frac{1}{k}, & for\ \ 0 < x < k \\ 0, & otherwise \end{cases}$

Find (i) MGF (ii) rth moment (iii) mean (iv) variance

Solution:

Note that it is p.d.f of uniform distribution.

$$(i) MGF = M_X(t) = \int e^{tx} f(x)dx$$

$$= \int_0^k e^{tx} \frac{1}{k} dx$$

$$= \frac{1}{k}\left[\frac{e^{tx}}{t}\right]_0^k$$

$$= \frac{1}{k}\left(\frac{e^{tk} - 1}{t}\right)$$

Expanding in powers of ‘t’ we have

$$MGF = M_X(t) = \frac{1}{k}\left(\frac{e^{tk}-1}{t}\right) = \frac{1}{kt}\left[\left(1+\frac{kt}{1!}+\frac{(kt)^2}{2!}+\frac{(kt)^3}{3!}+....\right)-1\right]$$

$$= \frac{1}{kt}\left[\frac{kt}{1!}+\frac{(kt)^2}{2!}+\frac{(kt)^3}{3!}+...\right]$$

$$= 1+\frac{kt}{2!}+\frac{(kt)^2}{3!}+\frac{(kt)^3}{4!}+...$$

Comparing the coefficients of 't','t^2','t^r' we have

$$(ii) E(X^r) = r!*coefficient\ of\ t^r\ in\ MGF = \frac{r!k^r}{(r+1)!} = \frac{k^r}{(r+1)}$$

$$(iii) Mean = E(X) = \frac{k}{2}$$

$$(iv) E(X^2) = \frac{k^2}{3} \qquad Variance = E(X^2) - E(X)^2 = \frac{k^2}{3} - \frac{k^2}{4} = \frac{k^2}{12}$$

17. If the probability density function of the random variable X is given by

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & elsewhere \end{cases}$$ (i) Find the rth moment (ii) Also evaluate E[(2x+1)2].

Solution:

$$(i) E(X^r) = \int x^r\ f(x)\ dx$$

$$= \int_0^1 x^r\ 2(1-x)\ dx$$

$$= 2\int_0^1 \left(x^r - x^{r+1}\right) dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= \frac{2}{(r+1)(r+2)}$$

$$(ii) E\left[(2X+1)^2\right] = E\left(4X^2+4X+1\right)$$
$$= 4E\left(X^2\right)+4E(X)+E(1)$$
$$= 4\frac{2}{(2+1)(2+2)} - 4\frac{2}{(1+1)(1+2)} + 1$$
$$= \frac{2}{3} - \frac{4}{3} + 1$$
$$= \frac{1}{3}$$

18. Find the probability distribution of the total number of heads obtained in four tosses of a balanced coin. Hence obtain the MGF of X, mean and variance.

Solution:

The Sample space of tossing 4 coins is given by

HHHH HHHT HHTH HTHH THHH HHTT HTHT HTTH THHT TTHH
THTH HTTT THTT TTHT TTTH TTTT

Hence the probability distribution is given by

X	0	1	2	3	4
$p(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

In general, $p(X=x) = \frac{4Cx}{16}$ OR

X = number of heads in tossing 4 coins = 0,1,2,3,4

Probability of getting a head p = ½ q = ½

Hence the p.m.f is given by (using Binomial distribution)

$$p(X=x) = 4C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x} = 4C_x\left(\frac{1}{2}\right)^4 = \frac{4Cx}{16}$$

$$MGF = M_X(t) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{4} e^{tx}\frac{4Cx}{16}$$
$$= \frac{1}{16}\sum_{x=0}^{4} e^{tx}4C_x$$
$$= \frac{1}{16}\left[e^{t0}4C_0 + e^{t1}4C_1 + e^{t2}4C_2 + e^{t3}4C_3 + e^{t4}4C_4\right]$$
$$= \frac{1}{16}\left[4C_0e^{t0}1^4 + 4C_1e^t1^3 + 4C_2e^{2t}1^2 + 4C_3e^{3t}1 + 4C_4e^{4t}1^0\right]$$
$$= \frac{1}{16}\left[e^t+1\right]^4$$

Differentiating twice with respect to ‘t’

$$M_X{}'(t) = \frac{4}{16}\left[e^t+1\right]^3 e^t$$

$$M_X{}''(t) = \frac{12}{16}\left[e^t+1\right]^2 e^{2t} + \frac{4}{16}\left[e^t+1\right]^3 e^t$$

Put t = 0 above

$$E(X) = M_X{}'(0) = \frac{4}{16}\left[e^0+1\right]^3 e^0 = 2$$

$$E\left(X^2\right) = M_X{}''(0) = \frac{12}{16}\left[e^0+1\right]^2 e^0 + \frac{4}{16}\left[e^0+1\right]^3 e^0 = 5$$

$$Mean = E(X) = 2$$

$$Variance = E\left(X^2\right) - E(X)^2 = 5-4 = 1$$

19. The first four moments of a distribution about X = 4 are 1,4,10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu 3=0$ and $\mu 4=26$.

Solution:

$$Given\ \mu_1' = 1 \quad \mu_2' = 4 \quad \mu_3' = 10 \quad \mu_4' = 45$$

$$\mu_1 = 0$$

$$Mean = \mu_1' + X(about\ which\ moments\ are\ found) = 1+4 = 5$$

$$Variance\ \mu_2 = \mu_2' - \mu_1'^2 = 4-1 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 3\mu_1'\mu_1'^2 - \mu_1'^3 = 10-12+3-1 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 4\mu_1'\mu_1'^3 + \mu_1'^4 = 45-40+24-4+1 = 26$$

PROBABILITY DISTRIBUTIONS

20. In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law, for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

Solution:

The Success event here is finding a page with an error

X is the number of pages with error

Probability of success $p = \frac{390}{520} = \frac{39}{52}$

Experiment is repeated $n = 5$ pages

$\lambda = np = 5\frac{39}{52} = 3.75$

$\Pr obability\ that\ a\ page\ has\ no\ error = p(number\ of\ pages\ with\ error =)$

$$= p(X = 0)$$

$$= \frac{e^{-\lambda}\lambda^0}{0!} = e^{-3.75} = 0.02375$$

21. Find the MGF of a Poisson variable. Hence find its mean and variance. Also,Deduce that the sum of tow independent Poisson variables is a Poisson variate.

Solution:

The probability mass function of a Geometric variable is given by

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 1,2,3,...$

$$MGF = M_X(t) = \sum_{x=1}^{\infty} e^{tx}\ p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx}\ \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\left(\lambda e^t\right)^x}{x!}$$

$$= e^{-\lambda}\sum_{x=1}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\left(\lambda e^t\right)}{1!} + \frac{\left(\lambda e^t\right)^2}{2!} + \frac{\left(\lambda e^t\right)^3}{3!} + ...\right]$$

$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda\left(e^t - 1\right)} \qquad 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... = e^x$$

Differentiating twice with respect to ‘t’

$$M_X'(t) = e^{\lambda\left(e^t-1\right)}\lambda e^t$$

$$M_X''(t) = e^{\lambda\left(e^t-1\right)}\left(\lambda e^t\right)\lambda e^t + e^{\lambda\left(e^t-1\right)}\left(\lambda e^t\right)$$

Put t = 0 above

$$E(X) = M_X{}'(0) \qquad = e^{\lambda(e^0-1)}\lambda e^0 = \lambda$$
$$E(X^2) = M_X{}''(0) \qquad = e^{\lambda(e^0-1)}(\lambda e^0)\lambda e^0 + e^{\lambda(e^0-1)}(\lambda e^0) = \lambda^2 + \lambda$$
$$Mean = E(X) = \lambda$$
$$Variance = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Let X be a Poisson variate with parameter λ_1 and Y be another independent Poisson variate with parameter λ_2. Since X and Y are independent, MGF of their sum is Product of individual MGF's

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$
$$= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}$$ which is MGF of a Poisson variable with parameter $(\lambda_1+\lambda_2)$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

22. If X1 and X2 are independent random variables each having geometric distribution $q^x p,\ x = 0,1,2,3\ldots$ Show that the conditional distribution of X1 given X1 +X2 is uniform.

Solution:

$$p(X_1 = n \mid X_1 + X_2 = n+m) = \frac{p(X_1 = n \cap X_1 + X_2 = n+m)}{p(X_1 + X_2 = n+m)} \qquad p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$
$$= \frac{p(X_1 = n \cap X_2 = m)}{p(X_1 + X_2 = n+m)}$$
$$= \frac{p(X_1 = n)\ p(X_2 = m)}{p(X_1 + X_2 = n+m)} \quad X_1, X_2 \ \textit{are independent}$$
$$= \frac{q^n p\ q^m p}{q^{m+n} p} = p$$
$$p(X_1 = n \mid X_1 + X_2 = n+m) = p$$

Which is the probability mass function of Uniform distribution in discrete case.

23. Find the MGF of Geometric distribution and hence find its mean and variance.

Solution:

The probability mass function of a Geometric variable is given by

$$P(X = x) = q^{x-1}p, \quad x = 1,2,3,\ldots$$

$$MGF = M_X(t) = \sum_{x=1}^{\infty} e^{tx} \; p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \; q^{x-1} p$$

$$= e^t \; q^0 p + e^{t2} \; q^1 p + e^{t3} \; q^2 p + e^{t4} \; q^3 p + ...$$

$$= e^t \; q^0 p + e^{t2} \; q^1 p + e^{t3} \; q^2 p + e^{t4} \; q^3 p + ...$$

$$= pe^t \left[1 + qe^t + q^2 e^{2t} + q^3 e^{3t} + ...\right]$$

$$= pe^t \left[1 + qe^t + \left(qe^t\right)^2 + \left(qe^t\right)^3 + ...\right]$$

$$= \frac{pe^t}{1 - qe^t} \qquad 1 + x + x^2 + ... = \frac{1}{1-x}$$

Differentiating twice with respect to ‘t’

$$M_X{}'(t) = \frac{\left(1 - qe^t\right)pe^t - pe^t\left(-qe^t\right)}{\left(1 - qe^t\right)^2}$$

$$= \frac{pe^t}{\left(1 - qe^t\right)^2}$$

$$M_X{}''(t) = \frac{\left(1 - qe^t\right)^2 pe^{-t} - pe^t \, 2\left(1 - qe^t\right)\left(-qe^t\right)}{\left(1 - qe^t\right)^4}$$

Put t = 0 above

$$E(X) = M_X{}'(0) = \frac{pe^0}{\left(1 - qe^0\right)^2} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E\left(X^2\right) = M_X{}''(0) = \frac{\left(1 - qe^0\right)^2 pe^{-0} - pe^0 \, 2\left(1 - qe^0\right)\left(-qe^0\right)}{\left(1 - qe^0\right)^4}$$

$$= \frac{(1-q)^2 p - p2(1-q)(-q)}{(1-q)^4}$$

$$= \frac{p^3 + 2p^2 q}{p^4} \quad = \frac{1}{p} + 2\frac{q}{p^2}$$

$$Mean = E(X) = \frac{1}{p}$$

$$Variance = E\left(X^2\right) - E(X)^2 = \frac{1}{p} + 2\frac{q}{p^2} - \frac{1}{p^2} = \frac{p + 2q - 1}{p^2}$$

$$= \frac{p - 1 + 2q}{p^2} = \frac{2q - q}{p^2} = \frac{q}{p^2}$$

24. The probability of a component's failure is 0.05. Out of 14 components what is the probability that (i) atmost 3 will fail (ii) atleast 3 will fail?

Solution:

Component's failure is taken as success event.

So, probability of success = p= 0.05

Since the experiment is repeated n =14 times we shall apply Binomial or Poisson distribution to calculate the required probability.

X = number of failed components

Assuming Poisson distribution

$\lambda = np = 14 * 0.05 = 0.7$

$$(i) P(X \le 3) = P(0) + P(1) + p(2) + P(3)$$

$$= \frac{e^{-0.7}(0.7)^0}{0!} + \frac{e^{-0.7}(0.7)^1}{1!} + \frac{e^{-0.7}(0.7)^2}{2!} + \frac{e^{-0.7}(0.7)^3}{3!}$$

$$= e^{-0.7}\left[\frac{(0.7)^0}{0!} + \frac{(0.7)^1}{1!} + \frac{(0.7)^2}{2!} + \frac{(0.7)^3}{3!}\right]$$

$$= 0.4966[1 + 0.7 + 0.245 + 0.0572]$$

$$= 0.9943$$

$$(ii) P(X \ge 3) = P(3) + P(4) + P(5) + \ldots$$

$$= 1 - P(X < 3)$$

- $= 1 - [P(0) + P(1) + p(2)]$

$$= 1 - \left[\frac{e^{-0.7}(0.7)^0}{0!} + \frac{e^{-0.7}(0.7)^1}{1!} + \frac{e^{-0.7}(0.7)^2}{2!}\right]$$

$$= 1 - e^{-0.7}\left[\frac{(0.7)^0}{0!} + \frac{(0.7)^1}{1!} + \frac{(0.7)^2}{2!}\right]$$

$$= 1 - 0.4966[1 + 0.7 + 0.245]$$

$$= 0.0341$$

25. In each of the following case MGF of X is given. Determine the probability distribution of X and hence its mean.

$(i) M_X(t) = \left(\frac{1}{4}e^t + \frac{3}{4}\right)^7 \qquad (ii) M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$

Solution:

(i)The first MGF is of the form of Binomial distribution

$M_X(t) = \left(pe^t + q\right)^n$ *On comparison we have*

$p = \frac{1}{4} \quad q = \frac{3}{4} \quad n = 7$

So its probability distribution is $p(X = x) = 7C_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}$, $x = 0,1,2,3..,7$

So its mean $= np = \frac{7}{4}$

(ii) The second MGF is of the form of Gamma distribution

$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$

So, its probability mass function is $f(x) = \frac{\lambda^n}{\Gamma n} x^{n-1} e^{-\lambda x}, x > 0$

its mean $= \frac{n}{\lambda}$

26. VLSI chips essential to the running of a computer system, fail in accordance with Poisson at the rate of one chip in about 5 weeks. If there are two spare chips on hand, and if a new supply will arrive in 8 weeks, what is the probability that during the next 8 weeks the system will be down for a week or more owing to lack of chips.

Solution:

$$\text{average number of failures } \lambda = \frac{1}{5} = 0.2$$

$$\begin{aligned} P(\text{system will be down}) &= P(\text{more than 2 failures}) \\ &= P(X > 2) \\ &= 1 - P(X = 0,1,2) \\ &= 1 - \left[\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}\right] \\ &= 1 - e^{-0.2}\left[1 + \frac{0.2}{1!} + \frac{0.2^2}{2!}\right] \\ &= 1 - e^{-0.2}1.22 = 0.00115 \end{aligned}$$

27. Derive the mean and variance of Binomial distribution with parameters *n* and *p*.

Solution:

The probability mass function of Binomial distribution is given by

$P(X = x) = nC_x p^x q^{n-x}, \ n = 0,1,2...,n$

$$MGF = M_X(t) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x \left(pe^t\right)^x q^{n-x}$$

$$= \begin{cases} nC_0 \left(pe^t\right)^0 q^{n-0} + nC_1 \left(pe^t\right)^1 q^{n-1} + \\ nC_2 \left(pe^t\right)^2 q^{n-2} + ... + nC_n \left(pe^t\right)^n q^{n-n} \end{cases}$$

$$= \left(pe^t + q\right)^n, \qquad (a+b)^n = a^n + nC_1 a^{n-1} b + ... + b^n$$

Differentiating twice with respect to 't'

$$M_X{}'(t) = n\left(pe^t + q\right)^{n-1} pe^t$$

$$M_X{}''(t) = n(n-1)\left(pe^t + q\right)^{n-2} \left(pe^t\right)^2 + n\left(pe^t + q\right)^{n-1} pe^t$$

Put t = 0

$$M_X{}'(0) = E(X) = n\left(pe^0 + q\right)^{n-1} pe^0 = n(p+q)p = np$$

$$M_X{}''(0) = E\left(X^2\right) = n(n-1)\left(pe^0 + q\right)^{n-2} \left(pe^0\right)^2 + n\left(pe^0 + q\right)^{n-1} pe^0$$

$$= n(n-1)(p+q)^{n-2} (p)^2 + n(p+q)^{n-1} p$$

$$= n(n-1)(p)^2 + np$$

$$= (np)^2 - np^2 + np$$

$$Mean = E(X) = np$$

$$Variance = E\left(X^2\right) - E(X)^2 = (np)^2 - np^2 + np - (np)^2 = np(1-p) = npq$$

MISCELLANEOUS PROBLEMS

28. The density function of a random variable X is given by
$f(x) = kx(2-x), \ \ 0 \le x \le 2$. Find k, mean , variance and rth moment.

Solution:

$$\int_0^2 f(x)\, dx = 1 \qquad \int_0^2 kx(2-x)\, dx = 1$$

$$k\int_0^2 \left(2x - x^2\right) dx = 1 \qquad k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left(4 - \frac{8}{3}\right) = 1 \qquad k = \frac{3}{4}$$

$$\mu_r{}' = \int_0^2 x^r \frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right)dx$$

$$= \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2 \qquad = \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right] \qquad = 6\left(2^r\right)\frac{1}{(r+2)(r+3)}$$

put r = 1,2 $\quad \mu_{1'} = \frac{12}{(3)(4)} = 1 \quad \mu_2{}' = \frac{24}{(4)(5)} = \frac{6}{5}$

Mean = 1 and variance = $\mu_{2'} - \mu_{1'^2} = \frac{6}{5} - 1 = \frac{1}{5}$

29. The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand:	1	2	3	4	5	6	7	8
Probability:	0.08	0.3k	0.19	0.24	$k2$	0.1	0.07	0.04

Determine the expected demand for watches. Also, compute the variance.

Solution:

$$\sum P(x) = 1$$

$$(0.08) + (0.3k) + (0.19) + (0.24) + (k^2) + (0.1) + (0.07) +)(0.04) = 1$$

$$k^2 + 0.3k - 0.28 = 0 \quad \Rightarrow k = 0.4$$

$$E(X) = \sum x\,P(x) = (1)(0.18) + (2)(0.12) + (3)(0.19) +$$
$$(4)(0.24) + (5)(0.16) + (6)(0.1) + (7)(0.07) + (8)(0.04)$$
$$= 4.02 \ \textit{is the mean}$$

$$E(X^2) = \sum x^2\,P(x) = (1)(0.18) + (4)(0.12) + (9)(0.19) +$$
$$(16)(0.24) + (25)(0.16) + (36)(0.1) + (49)(0.07) + (64)(0.04)$$
$$= 19.7$$

$$\textit{Variance} = E(X^2) - E(X)^2 = 19.07 - 4.02^2 = 3.54$$

30. The distribution of a random variable X is given by $F(X) = 1 - (1+x)e^{-x}$, $x > 0$ .Find the rth moment, mean and variance. Solution:

(i)

$$f(x) = F'(x)$$
$$= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right]$$
$$= xe^{-x}, \quad x > 0$$

(ii)

$$E\left(X^r\right) = \mu_r{}' = \int_0^\infty x^r \; f(x)\; dx$$
$$= \int_0^\infty x^r \; xe^{-x} \; dx$$
$$= \int_0^\infty x^{r+1} e^{-x} \; dx$$
$$= (r+1)!$$

(iii) $E(X) = \mu_1{}' = (1+1)! = 2$

(iv) $E\left(X^2\right) = \mu_2{}' = (2+1)! = 6$ Variance = $E\left(X^2\right) - E(X)^2$ =2

31. Suppose that the duration 'X' in minutes of long distance calls from your home, follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \; x > 0$. Find p(X > 5), p(3≤X≤6),mean and variance.

Solution:

(i) $p(X > 5) = \int_5^\infty f(x)\; dx = \int_5^\infty \frac{1}{5} e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}}\right]_5^\infty = e^{-1}$

(ii) $p(3 < X < 6) = \int_3^6 f(x)\; dx = \int_3^6 \frac{1}{5} e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}}\right]_3^6 = -e^{-1.2} + e^{-0.5}$

(iii)

$$E(X) = \int_0^\infty xf(x)\; dx = \int_0^\infty \frac{1}{5} xe^{-\frac{x}{5}} dx$$

$$= \frac{1}{5}\left[-xe^{-\frac{x}{5}}5 - e^{-\frac{x}{5}}25 \right]_0^{\infty}$$

$$= \frac{1}{5}(0+25) = 5$$

(iv) $E(X^2) = \int_0^{\infty} x^2 f(x)\, dx = \int_0^{\infty} \frac{1}{5} x^2 e^{-\frac{x}{5}} dx$

$$= \frac{1}{5}\left[-x^2 e^{-\frac{x}{5}}5 - 2xe^{-\frac{x}{5}}25 + 2e^{-\frac{x}{5}}125 \right]_0^{\infty}$$

$$= \frac{1}{5}(0+250) = 50$$

Variance = $E\left(X^2\right) - E(X)^2$ $= 50 - 25 = 25$

32. A random variable X has the following probability distribution.

X:	0	1	2	3	4	5	6	7
f(x):	0	k	$2k$	$2k$	$3k$	$k2$	$2k2$	$7k2+ k$

Find (i) the value of k (ii) $p(1.5 < X < 4.5 \mid X > 2)$ and

(iii) the smallest value of λ such that $p(X \le \lambda) > \frac{1}{2}$.

Solution:

$$\sum P(x) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow k = -1, \frac{1}{10}$$

$$k = \frac{1}{10} = 0.1$$

$$A = 1.5 < X < 4.5 = \{2,3,4\}$$

$$B = X > 2 = \{3,4,5,6,7\}$$

(ii) $A \cap B = \{3,4\}$

$$p(1.5 < X < 4.5 \mid X > 2) = p(A \mid B) = \frac{p(A \cap B)}{p(B)} = \frac{p(3,4)}{p(3,4,5,6,7)}$$

$$= \frac{2k+3k}{2k+3k+k^2+2k^2+7k^2+k} = \frac{5k}{10k^2+6k} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(iii)

X	p(X)	F(X)
0	0	0
2	2k = 0.2	0.3
3	2k = 0.2	0.5
4	3k = 0.3	0.8
5	k2=0.01	0.81
6	2k2 = 0.02	0.83
7	7k2+k = 0.17	1.00

From the table for X = 4,5,6,7 p(X) > $\frac{1}{2}$ and the smallest value is 4

Therefore $\lambda = 4$.

33. Find the MGF of triangular distribution whose density function is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & elsehwere \end{cases}$$ .Hence its mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\, dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

expanding the above in powers of t, we get

$$M_X(t) = \frac{e^{2t} - 2e^{t} + 1}{t^2} = \frac{1}{t^2}\left[\left(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \dots\right) - 2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + .\right) - 1\right]$$

$$= \frac{1}{t^2}\left(\frac{2t^2}{2!} + \frac{6t^3}{3!} + \frac{14t^4}{4!} + \dots\right)$$

$$= 1 + t + \frac{7t^2}{12} + \frac{t^3}{4} + \dots.$$

Mean = E(X) = (coefficient of t) 1! = 1

E(X2) = (coefficient of t2)2! = $\frac{7}{6}$

Variance = E(X2) - E(X)2 = $\frac{1}{6}$

34. Find the MGF of the RV X, whose pdf is given by $f(x) = \frac{1}{2} e^{-|x|}, \ -\infty < x < \infty$.

Hence its mean and variance.

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{tx} e^{x}\, dx + \int_{0}^{\infty} e^{tx} e^{-x}\, dx$$

$$= \int_{-\infty}^{0} e^{(t+1)x}\, dx + \int_{0}^{\infty} e^{-(1-t)x}\, dx$$

$$= \left[\frac{e^{(t+1)x}}{(t+1)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_{0}^{\infty}$$

$$M_X(t) = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \dots.$$

Mean = E(X) = (coefficient of t) 1! = 0

E(X2) = (coefficient of t2)2! =2

Variance = E(X2) - E(X)2 = 2

35. The p.m.f of a RV X, is given by $p(X=j)=\frac{1}{2^j}$, $j=1,2,3\ldots$ Find MGF, mean and variance. Solution:

$$M_X(t)=E\left(e^{tX}\right)=\sum e^{tx}p(x)$$

$$=\sum_{x=0}^{\infty}e^{tx}\frac{1}{2^x}$$

$$=\sum_{x=0}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$=\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\ldots.$$

$$=\frac{e^t}{2}\left(1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\ldots\right)$$

$$=\frac{e^t}{2}\frac{1}{1-\frac{e^t}{2}}=\frac{e^t}{2-e^t}$$

Differentiating twice with respect to t

$$M'_X(t)=\frac{\left(2-e^t\right)\left(e^t\right)-e^t\left(-e^t\right)}{\left(2-e^t\right)^2}=\frac{2e^t}{\left(2-e^t\right)^2}$$

$$M''_X(t)=\frac{\left(2-e^t\right)^2\left(2e^t\right)-2e^t2\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4}=\frac{4e^t+2e^{2t}}{\left(2-e^t\right)^3}$$

put t = 0 above $E(X)=M'_X(0)=2$

$E\left(X^2\right)=M''_X(0)=6$

$Variance=E\left(X^2\right)-E(X)^2=6-4=2$

36. Find MGF of the RV X, whose pdf is given by $f(x)=\lambda e^{-\lambda x}$, $x>0$ and hence find the first four central moments. Solution:

$$M_X(t) = E\left(e^{tX}\right) \quad = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{(\lambda - t)}$$

Expanding in powers of t

$$M_X(t) = \frac{\lambda}{(\lambda - t)} = \frac{1}{1 - \left(\frac{t}{\lambda}\right)} = 1 + \left(\frac{t}{\lambda}\right) + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \ldots$$

Taking the coefficient we get the raw moments about origin

$$E(X) = (coefficient \ \ of \ \ t\,)1! = \frac{1}{\lambda}$$

$$E\left(X^2\right) = \left(coefficient \ of \ t^2\right)2! = \frac{2}{\lambda^2}$$

$$E\left(X^3\right) = \left(coefficient \ of \ t^3\right)3! = \frac{6}{\lambda^3}$$

$$E\left(X^4\right) = \left(coefficient \ of \ t^4\right)4! = \frac{24}{\lambda^4}$$

and the central moments are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - 2C_1\mu_1'\mu_1' + \mu_1'^2$$

$$= \frac{2}{\lambda^2} - 2\frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3C_1\mu_2'\mu_1' + 3C_2\mu_1'\mu_1'^2 - \mu_1'^3$$

$$= \frac{6}{\lambda^3} - 3\frac{2}{\lambda^2}\frac{1}{\lambda} + 3\frac{1}{\lambda}\frac{1}{\lambda^2} - \frac{1}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' - 4C_1\mu_3'\mu_1' + 4C_2\mu_2'\mu_1'^2 - 4C_3\mu_1'^4 + \mu_1'^4$$

$$= \frac{24}{\lambda^4} - 4\frac{6}{\lambda^3}\frac{1}{\lambda} + 6\frac{2}{\lambda^2}\frac{1}{\lambda^2} - 4\frac{1}{\lambda^4} + \frac{1}{\lambda^4} = \frac{9}{\lambda^4}$$

37. If the MGF of a (discrete) RV X is $\dfrac{1}{5-4e^t}$ find the distribution of X and p (X = 5 or 6).

Solution:

$$M_X(t) = \frac{1}{5-4e^t} = \frac{1}{5\left(1-\frac{4e^t}{5}\right)}$$

$$= \frac{1}{5}\left[1+\left(\frac{4e^t}{5}\right)+\left(\frac{4e^t}{5}\right)^2+\left(\frac{4e^t}{5}\right)^3+\ldots\right]$$

By definition

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx}p(x)$$

$$= 1 + e^{t0}p(0) + e^{t}p(1) + e^{t2}p(2) + \ldots$$

On comparison

$$p(0) = \frac{1}{5} \qquad p(1) = \frac{4}{25} \qquad p(2) = \frac{16}{125} \qquad p(3) = \frac{64}{625}$$

In general $p(X=r)=\frac{1}{5}\left(\frac{4}{5}\right)^r, \quad r=0,1,2,3$

$$p(X=5 \ or \ 6) = p(X=5)+p(X=6)$$

$$=\frac{1}{5}\left(\frac{4}{5}\right)^5+\frac{1}{5}\left(\frac{4}{5}\right)^6$$

$$=\frac{1}{5}\left(\frac{4}{5}\right)^5\left(1+\frac{4}{5}\right)$$

$$=\frac{9}{25}\left(\frac{4}{5}\right)^5$$

38. If X has the probability density function $f(x)=k\,e^{-3x},\ x>0$

Find (i) k (ii) p(0.5 ≤X≤1) (iii) Mean of X.

Solution:

$$(i)\int_0^\infty ke^{-3x}dx=1$$

$$k\left[-\frac{e^{-3x}}{3}\right]_0^\infty=1$$

$$k\frac{1}{3}=1$$

$$k=3$$

$$(ii)\,p(0.5\le X\le 1) = \int_{0.5}^{1} f(x)\ dx$$

$$= \int_{0.5}^{1} 3e^{-3x}\ dx$$

$$= 3\left[\frac{-e^{-3x}}{3}\right]_{0.5}^{1}$$

$$= -e^{-3}+e^{-1.5}$$

$$(iii) Mean = E(X) = \int_0^\infty x\, f(x)\, dx$$

$$= \int_0^\infty 3x\, e^{-3x}\, dx$$

$$= 3\left[-x\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9} \right]_0^\infty$$

$$= 3\left(\frac{1}{9}\right) = \frac{1}{3}$$

39. If a RV X has the pdf $f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & otherwise \end{cases}$.

Obtain (i) p(X <1) (ii) p($|X|$ >1) (iii) p(2X+3 > 5)

(iv) p ($|X|$ < 0.5 | X < 1

Solution:

(i) p(X < 1) = $\int_{-2}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{1} = \frac{3}{4}$

(ii) p($|X| \le 1$) = $\int_{-1}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{1} = \frac{1}{2}$

Hence p($|X|$ >1)= 1 - p($|X| \le 1$) = $\frac{1}{2}$

(iii) p(2X+3 > 5) = p(X > 1) = 1- p(X $\le$1) = 1- $\frac{3}{4} = \frac{1}{4}$

(iv) p ($|X|$ < 0.5 | X < 1) = $\frac{p(|X| < 0.5 \cap X < 1)}{p(X < 1)}$

$$= \frac{p([-0.5 < X < 0.5] \cap X < 1)}{p(X < 1)}$$

$$= \frac{p([-0.5 < X < 0.5]\ )}{p(X < 1)}$$

$$= \frac{\int_{-1}^{1} \frac{1}{4} dx}{\frac{3}{4}} = \frac{[x]_{-0.5}^{0.5}}{3} = \frac{1}{3}$$

40. If X has the distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x > 10 \end{cases}$$

(1) Probability distribution of X (2) p(2<X<6) (3) Mean (4) variance

Solution:

(1)As there is no x terms in the distribution function given is a discrete random variable. Hence the probability distribution is given by

X	1	4	6	10
$p(X)$	$\frac{1}{3}$	$\frac{1}{2}-\frac{1}{3}$	$\frac{5}{6}-\frac{1}{2}$	$1-\frac{5}{6}$
	$\frac{1}{3}$	$=\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(2) p(2<X<6) = p(4) = $\frac{1}{6}$

(3) Mean = E(X) = $\sum x \ p(x) = (1)\left(\frac{1}{3}\right)+(4)\left(\frac{1}{6}\right)+(6)\left(\frac{1}{3}\right)+(10)\left(\frac{1}{6}\right) = \frac{14}{3}$

(4) E(X2) = $\sum x^2 \ p(x) = (1)\left(\frac{1}{3}\right)+(16)\left(\frac{1}{6}\right)+(36)\left(\frac{1}{3}\right)+(100)\left(\frac{1}{6}\right) = \frac{95}{3}$

Variance = E(X2) – E(X)2 = $\frac{95}{3}$ - $\frac{196}{9}$ = $\frac{89}{9}$

41. A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0 & : \quad x \le 1 \\ k(1-x)^4 & : \quad 1 < x \le 3 \\ 0 & : \quad x > 3 \end{cases}$$

Find *k*, the probability density function *f(x)* and *P(X <2)*.

Solution:

Since it is a distribution function

$$F(\infty) = F(3) = 1$$

$$k(3-1)^4 = 1$$

$$k = \frac{1}{16}$$

The density function is $f(x) = F'(x) = \frac{1}{16}4(1-x)^3 = \frac{(1-x)^3}{4}, \ 1 \le x \le 3$

p(X < 2) = F(2) = $\frac{1}{16}(2-1)^4 = \frac{1}{16}$

42. If the cumulative distribution function of a R.V X is given by

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & x > 2 \\ 0, & x \le 2 \end{cases}$$ find (i) P(X < 3) (ii) P(4 < X < 5) (iii) P (X ≥3).

Solution:

(i) P(X < 3) = F(3) = $1 - \dfrac{4}{3^2} = \dfrac{5}{9}$

(ii) P(4 < X < 5) = F(5) - F(4) = $\left(1 - \dfrac{4}{5^2}\right) - \left(1 - \dfrac{4}{4^2}\right) = \dfrac{21}{25} - \dfrac{3}{4} = \dfrac{9}{100}$

(iii) P (X ≥3) = 1- F(3) = 1 - $\left(1 - \dfrac{4}{3^2}\right)$ = 1- $\dfrac{5}{9} = \dfrac{4}{9}$

43. Find the recurrence relation for the moments of the Binomial distribution.

Soln: The k^{th} order central moment is given by

$$\mu_k = E\left[\left(X - \overline{X}\right)^k\right] = E\left[\left(X - np\right)^k\right]$$

$$= \sum_{x=0}^{n} (x - np)^k \, p(x)$$

$$= \sum_{x=0}^{n} (x - np)^k \, nc_x \, p^x q^{n-x}$$

$$\mu_k = \sum_{x=0}^{n} nc_x \left[(x - np)^k \, p^x q^{n-x}\right] - - - - - - - - - - - - - - (1)$$

Differentiating (1) w.r.to p,we have

$$\frac{d\mu_k}{dp} = \sum_{x=0}^{n} n c_x \left[k(x-n p)^{k-1}(-n) p^x q^{n-x} + (x-n p)^k \left(x.p^{x-1}q^{n-x} + p^x (n-x) q^{n-x-1}(-1)\right)\right]$$

$$= -nk\sum_{x=0}^{n} n c_x (x-n p)^{k-1} p^x q^{n-x} + \sum_{x=0}^{n} n c_x (x-n p)^k p^{x-1} q^{n-x-1} \left[xq-(n-x)p\right]$$

$$= -nk\sum_{x=0}^{n} n c_x (x-n p)^{k-1} p^x q^{n-x} + \sum_{x=0}^{n} n c_x (x-n p)^k \frac{p^x}{p}\frac{q^{n-x}}{q}\left[x(p+q)-np\right]$$

$$= -nk\mu_{k-1} + \sum_{x=0}^{n} n c_x (x-n p)^k \frac{p^x}{p}\frac{q^{n-x}}{q}\left[x-np\right]$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\sum_{x=0}^{n} n c_x (x-n p)^{k+1} p^x q^{n-x}$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\mu_{k+1} \qquad , \ by\,(1)$$

$$\Rightarrow \mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\mu_{k-1}\right]$$

This is the recurrence relation for the moments of the Binomial distribution

44. Prove that poisson distribution is the limiting case of Binomial distribution.

(or)

Poisson distribution is a limiting case of Binomial distribution under the following conditions

(i) n , the no.of trials is indefinitely large , i.e, $n \to \infty$

(ii) p, the constant probability of success in each trial is very small ,i.e $p \to 0$

(iii) $np = \lambda$ is infinite or $p = \dfrac{\lambda}{n}$ and $q = 1 - \dfrac{\lambda}{n}$, λ is positive real

Soln: If Xis binomial r.v with parameter n & p ,then

$$p(X = x) = nc_x p^x q^{n-x},\ x = 0,1,2,....n$$

$$= \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))(n-x)!}{(n-x)!x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{n^x\,x!} n.n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right).....n\left(1-\frac{x-1}{n}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit as $n \to \infty$ on both sides

$$\lim_{n\to\infty} p(X = x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right) \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}(1.1....1).\left(e^{-\lambda}\right).1 = \frac{e^{-\lambda}\lambda^x}{x!} \quad ,\ x = 0,1,2,.....$$

$\therefore\ p(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,2,.....$ and it is poisson distbn.

Hence the proof.

45. Find the recurrence relation for the central moments of the poisson distbn. and hence find the first three central moments .

Soln: The k^{th} order central moment μ_k is given by

$$\mu_k = E\left(X - \overline{X}\right)^k = E\left(X - \lambda\right)^k$$

$$= \sum_{x=0}^{\infty} \left(x-\lambda\right)^k p(x)$$

$$\therefore\quad \mu_k = \sum_{x=0}^{\infty} \left(x-\lambda\right)^k \frac{e^{-\lambda}.\lambda^x}{x!} \text{ -----------(1)}$$

Diff. (1) w.r.to λ we have

$$\frac{d\mu_k}{d\lambda} = \sum_{x=0}^{\infty}\left[k(x-\lambda)^{k-1}(-1)\frac{e^{-\lambda}.\lambda^x}{x!} + \frac{(x-\lambda)^{k-1}}{x!}\left[e^{-\lambda}.x\lambda^{x-1} + (-e^{-\lambda}).\lambda^x\right]\right]$$

$$= -k\sum_{x=0}^{\infty}(x-\lambda)^{k-1}\frac{e^{-\lambda}.\lambda^x}{x!} + \sum_{x=0}^{\infty}(x-\lambda)^k\frac{e^{-\lambda}.\lambda^{x-1}}{x!}(x-\lambda)$$

$$= -k\mu_{k-1} + \sum_{x=0}^{\infty}(x-\lambda)^{k+1}\frac{e^{-\lambda}.\lambda^x}{x!\lambda} \qquad by(1)$$

$$= -k\mu_{k-1} + \frac{1}{\lambda}\mu_{k+1}$$

$$\therefore \mu_{k+1} = \lambda\left[\frac{d\mu_k}{d\lambda} + k\mu_{k-1}\right] \text{---------------}(2)$$

which is the recurrence formula for the central moments of the poisson distbn.

since $\mu_0 = 1$ *and* $\mu_1 = 0$

put k= 1 in (2)

$$\mu_{1+1} = \lambda\left[\frac{d}{d\lambda}\mu_1 + 1\mu_{1-1}\right]$$

$$= \lambda\left[\frac{d}{d\lambda}(0) + 1\mu_0\right]$$

$$\mu_2 = \lambda$$

put k= 2 in (2)

$$\mu_{2+1} = \lambda\left[\frac{d}{d\lambda}\mu_2 + 2\mu_{2-1}\right]$$

$$= \lambda\left[\frac{d}{d\lambda}(\lambda) + 2\mu_1\right]$$

$$\mu_3 = \lambda(1+0)$$

$$= \lambda$$

46. Prove that the sum of two independent poisson variates is a poisson variate, while the difference is not a poisson variate.

Soln: Let X_1 and X_2 be independent r.v.s that follow poisson distbn. with Parameters λ_1 and λ_2 respectively.

Let $X = X_1 + X_2$

$$p(X=n)=p(X_1+X_2=n)$$

$$=\sum_{r=0}^{n} p[X_1=r].p[X_2=n-r] \quad \text{since } X_1 \,\&\, X_2 \text{ are independent}$$

$$=\sum_{r=0}^{n} \frac{e^{-\lambda_1}.\lambda_1^{\ r}}{r!}.\frac{e^{-\lambda_2}.\lambda_2^{\ n-r}}{(n-r)!}$$

$$=e^{-\lambda_1}\,e^{-\lambda_2}\ \sum_{r=0}^{n} \frac{.\lambda_1^{\ r}}{r!}.\frac{1}{n!}\frac{n!.}{(n-r)!}\lambda_2^{\ n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.\frac{n!.}{r!\,(n-r)!}\lambda_1^{\ r}\lambda_2^{\ n-r}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.nc_r\lambda_1^{\ r}\lambda_2^{\ n-r} \quad =\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

This is poisson with parameter ($\lambda_1+\lambda_2$)

(ii) Difference is not poisson

Let $X=X_1-X_2$

$$E(X)=E[X_1-X_2]$$
$$=E(X_1)-E(X_2)$$
$$=\lambda_1-\lambda_2$$
$$E(X^2)=E\left[(X_1-X_2)^2\right]$$
$$=E\left[X_1^{\ 2}+X_1^{\ 2}-2X_1X_2\right]$$
$$=E\left[X_1^{\ 2}\right]+E\left[X_2^{\ 2}\right]-2E[X_1]E[X_2]$$
$$=(\lambda_1^{\ 2}+\lambda_1)+(\lambda_2^{\ 2}+\lambda_2)-2(\lambda_1\lambda_2)$$
$$=(\lambda_1-\lambda_2)^2+(\lambda_1+\lambda_2)$$
$$\neq(\lambda_1-\lambda_2)^2+(\lambda_1-\lambda_2)$$

It is not poisson.

47. If X and Y are two independent poisson variates , show that the conditional distbn. of X, given the value of X+Y is Binomial.

Soln: Let X and Y follow poisson with parameters λ_1 and λ_2 respectively.

$$p[X = r / X + Y = n] = \frac{p[X = r \text{ and } X + Y = n]}{p[X + Y = n]}$$

$$= \frac{p[X = r].p[X + Y = n]}{p[X + Y = n]} \quad \textit{by independent}$$

$$= \frac{\dfrac{e^{-\lambda_1}.\lambda_1^{\,r}}{r!} \cdot \dfrac{e^{-\lambda_2}.\lambda_2^{\,n-r}}{(n-r)!}}{\dfrac{e^{-(\lambda_1+\lambda_2)}.(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \frac{e^{-\lambda_1}.\lambda_1^{\,r}.e^{-\lambda_2}.\lambda_2^{\,n-r}}{e^{-\lambda 1} e^{-\lambda_2}..(\lambda_1+\lambda_2)^n}$$

$$= n c_r \frac{\lambda_1^{\,r}.\lambda_2^{\,n-r}}{(\lambda_1+\lambda_2)^r.(\lambda_1+\lambda_2)^{n-r}}$$

$$= n c_r \left[\frac{\lambda_1}{\lambda_1+\lambda_2}\right]^r \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^{n-r} = n c_r \, p^r \, q^{n-r} \quad .$$

$$\text{where} \quad p = \frac{\lambda_1}{\lambda_1+\lambda_2} \;\; and \;\; q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

This is binomial distbn.

48. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast ,exactly,atmost 2 defectives in a consignment of 1000 packets using poisson.

Soln: Give n = 20 , p = 0.05 , N = 1000

Mean $\lambda = n\,p = 1$

Let X denote the no. of defectives.

$$p[X = x] = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1}.1^x}{x!} = \frac{e^{-1}.}{x!} \quad x = 0,1,2,...$$

$$p[x \ge 2] = 1 - p[x < 2]$$

$$= 1 - [p(x = 0) + p(x = 1)]$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!}\right] = 1 - 2e^{-1} = 0.2642$$

Therefore , out of 1000 packets , the no. of packets containing atleast 2 defectives

$$= N.p[x \ge 2] = 1000 * 0.2642 \cong 264 \;\; \textit{packets}$$

(ii) $p[x = 2] = \dfrac{e^{-1}}{2!} = 0.18395$

Out of 1000 packets,=N*p[x=2]=184 packets

(iii)

$$p[x \le 2] = p[x=0] + p[x=1] + p[x=2]$$
$$= \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-1}}{2!} = 0.91975$$

For 1000 packets = 1000*0.91975=920 packets approximately.

49. The atoms of radio active element are randomly disintegrating. If every gram of this element , on average, emits 3.9 alpha particles per second, what is the probability during
the next second the no. of alpha particles emitted from 1 gram is
(i) atmost 6 (ii) atleast 2 (iii) atleast 3 and atmost 6 ?

Soln: Given $\lambda = 3.9$

Let X denote the no. of alpha particles emitted

$$(i)\ p(x \le 6) = p(x=0) + p(x=1) + p(x=2) + \ldots\ldots + p(x=6)$$
$$= \frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!} + \frac{e^{-3.9}(3.9)^2}{2!} + \ldots\ldots + \frac{e^{-3.9}(3.9)^6}{6!}$$
$$= 0.898$$

$$(ii)\quad p(x \ge 2) = 1 - p(x < 2)$$
$$= 1 - [p(x=0) + p(x=1)]$$
$$= 1 - \left[\frac{e^{-3.9}(3.9)^0}{0!} + \frac{e^{-3.9}(3.9)^1}{1!}\right]$$
$$= 0.901$$

$$(iii)\quad p(3 \le x \le 6) = p(x=3) + p(x=4) + p(x=5) + p(x=6)$$
$$= \frac{e^{-3.9}(3.9)^3}{3!} + \frac{e^{-3.9}(3.9)^4}{4!} + \frac{e^{-3.9}(3.9)^5}{5!} + \frac{e^{-3.9}(3.9)^6}{6!}$$
$$= 0.645$$

50. Derive the Negative binomial distbn X~NB(k , p) where X is the no. of failures preceeding the k th success in a sequence of Bernoulli trials and p = probability of success. Obtain the mgf , and hence mean and variance.

Soln: If repeated independent trials can result in a success with probability p,
And a failure with probability q=1-p,
Then the probability distbn of the r.v.X , the no. of failures preceeding the k th success is given by

$$p[X = x] = (x+k-1)c_{k-1}\ p^k\ q^x \quad , \ x = 0,1,2,\ldots..$$

Moment generating function(mgf)

$$M_x(t) = E\left[e^{tx}\right] = \sum e^{tx}\ p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx}\,(x+k-1)\,c_{k-1}\ p^k\ q^x$$

$$= p^k \sum_{x=0}^{\infty} (x+k-1)\,c_{k-1}\left(qe^t\right)^x$$

$$= p^k\left[(k-1)\,c_{k-1}\left(qe^t\right)^0 + (1+k-1)\,c_{k-1}\left(qe^t\right)^1 + (2+k-1)\,c_{k-1}\left(qe^t\right)^2 + ...\right]$$

$$= p^k\left[1 + k\,c_{k-1}\left(qe^t\right)^1 + (k+1)\,c_{k-1}\left(qe^t\right)^2 + ...\right]$$

$$= p^k\left[1 + \frac{k!}{(k-k+1)!(k-1)!}\left(qe^t\right) + \frac{(k+1)!}{(k+1-k+1)!}\left(qe^t\right)^2 + ...\right]$$

$$= p^k\left[1 + \frac{k.(k-1)!}{1\,!(k-1)!}\left(qe^t\right) + \frac{(k+1)!}{2!(k-1)!}\left(qe^t\right)^2 + ...\right]$$

$$= p^k\left[1 + \frac{k.}{1\,!!}\left(qe^t\right) + \frac{k.(k+1)}{2!!}\left(qe^t\right)^2 + ...\right]$$

$$= p^k\left[1 - qe^t\right]^{-k}$$

$$M_x(t) = p^k\left[1 - qe^t\right]^{-k}$$

$$Mean = \mu_1{}' = \left[\frac{d}{dt}\left[M_x(t)\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\,p^k\left[1 - qe^t\right]^{-k}\right]_{t=0}$$

$$= p^k\left[-k\left[1 - qe^t\right]^{-k-1}(-qe^t)\right]_{t=0}$$

$$= p^k\left[kq\left[1-q\right]^{-k-1}\right] = p^k\ k\,q\ p^{-k-1} = \frac{q\,k}{p}$$

Mean = $\dfrac{q\,k}{p}$

Variance

$$Var(x) = \mu_2{}' - \left(\mu_1{}'\right)^2$$

$$\mu_2' = \left[\frac{d^2}{dt^2}[M_x(t)]\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[-k\left[1-qe^t\right]^{-k-1}(-qe^t)\right]\right\}_{t=0}$$

$$= \left\{k\,q\,p^k\left[e^t(-k-1)\left[1-qe^t\right]^{-k-2}(-qe^t) + e^t\left[1-qe^t\right]^{-k-1}\right]\right\}_{t=0}$$

$$= k\,q\,p^k\left[(k+1)\,q\,p^{-k-2} + p^{-k-1}\right]$$

$$= k(k+1)q^2 p^{-2} + k\,q\,p^{-1}$$

$$\frac{k^2q^2}{p^2} + \frac{kq^2}{p^2} + \frac{kq}{p}$$

$$\therefore\ Var(x) = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{k^2q^2}{p^2} + \frac{kq^2}{p^2} + \frac{kq}{p} - \left(\frac{q\,k}{p}\right)^2$$

$$= \frac{kq}{p^2}(p+q) = \frac{kq}{p^2}$$

$$Variance = \frac{kq}{p^2}$$

51. Establish the memoryless property of geometric distbn.

Soln: If X is a discrete r.v. following a geometric distbn.

$$\therefore\ p(X = x) = pq^{x-1} \quad , \ x = 1,2,.....$$

$$p(x > k) = \sum_{x=k+1}^{\infty} pq^{x-1}$$

$$= p\left[q^k + q^{k+1} + q^{k+2} + ...\right]$$

$$= p\,q^k\left[1 + q + q^2 + ...\right] = p\,q^k(1-q)^{-1}$$

$$= p\,q^k\,p^{-1} = q^k$$

Now

$$p\left[x > m+n/x > m\right] = \frac{p\left[x > m+n \ and \ x > m\right]}{p\left[x > m\right]}$$

$$= \frac{p\left[x > m+n\right]}{p\left[x > m\right]} = \frac{q^{m+n}}{q^m} = q^n = p\left[x > n\right]$$

$$\therefore\ p\left[x > m+n/x > m\right] = p\left[x > n\right]$$

52. If X_1, X_2 be independent r.v. each having geometric distbn $pq^k, k = 0,1,2,....$ Show that th conditional distribution of X_1 given $X_1 + X_2$ is Uniform distbn.

Soln:

$$p[X_1 = r / X_1 + X_2 = n] = \frac{p[X_1 = r \text{ and } X_1 + X_2 = n]}{p[X_1 + X_2 = n]}$$

$$= \frac{p[X_1 = r \quad and \quad X_2 = n - r]}{\sum_{s=0}^{n} p[X_1 = s \quad and \quad X_2 = n - s]}$$

$$= \frac{p[X_1 = r].p[X_2 = n - r]}{\sum_{s=0}^{n} p[X_1 = s].p[X_2 = n - s]} \qquad by \ independent$$

$$= \frac{q^r \ p \, q^{n-r} \ p}{\sum_{s=0}^{n} q^s \ p \, q^{n-s} \ p} = \frac{q^n}{\sum_{s=0}^{n} q^n} = \frac{1}{n+1}, \ r = 0,1,2,.....n$$

$$(i.e) \ \ p[X_1 = r / X_1 + X_2 = n] = \frac{1}{n+1} \ , \ this \ is \ uniform \ distbn$$

53. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability

That the target is shot on any one shot is 0.7.

(i) What is the probability that the target would be hit in 10 th attempt?

(ii) What is the probability that it takes him less than 4 shots?

(iii)What is the probability that it takes him an even no. of shots?

(iv) What is the average no. of shots needed to hit the target?

Soln: Let X denote the no. of shots needed to hit the target and X follows geometric distribution with pmf $p[X = x] = p\,q^{x-1}$, $x = 1,2,....$

Given p=0.7 , and q=1-p=0.3

(i) $p[x = 10] = (0.7)(0.3)^{10-1} = 0.0000138$

(ii)

$$p[x < 4] = p(x = 1) + p(x = 2) + p(x = 3)$$
$$= (0.7)(0.3)^{1-1} + (0.7)(0.3)^{2-1} + (0.7)(0.3)^{3-1}$$
$$= 0.973$$

(iii)

$$p[x \text{ is an even number}] = p(x = 2) + p(x = 4) + ...$$
$$= (0.7)(0.3)^{2-1} + (0.7)(0.3)^{4-1} + ...$$
$$= (0.7)(0.3)[1 + (0.3)^2 + (0.3)^4 ....]$$

$$= 0.21\left[1 + \left((0.3)^2\right) + \left((0.3)^2\right)^2 + ...\right]$$

$$= 0.21\left[1 - (0.3)^2\right]^{-1} = (0.21)(0.91)^{-1}$$

$$= \frac{0.21}{0.91} = 0.231$$

(iv) Average no. of shots $= E(X) = \frac{1}{p} = \frac{1}{0.7} = 1.4286$

54. The number of personel computer (pc) sold daily at a computer world is uniformly distributed with a minimum of 2000 pc and a maximum of 5000 pc. Find

(1) The probability that daily sales will fall between 2500 and 3000 pc

(2)What is the probability that the computer world will sell atleast 4000 pc's?

(3) What is the probability that the computer world will sell exactly 2500 pc's?

Soln: Let X~U(a , b) , then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , a < x < b$$

$$= \frac{1}{5000-2000} \quad , \quad 2000 < x < 5000$$

$$= \frac{1}{3000} \quad , \quad 2000 < x < 5000$$

(1)

$$p[2500 < x < 3000] = \int_{2500}^{3000} f(x)\,dx$$

$$= \int_{2500}^{3000} \frac{1}{3000}\,dx = \frac{1}{3000}\left[x\right]_{2500}^{3000}$$

$$= \frac{1}{3000}[3000 - 2500] = 0.166$$

(2)

$$p[x \geq 4000] = \int_{4000}^{5000} f(x)\,dx$$

$$= \int_{4000}^{5000} \frac{1}{3000}\,dx = \frac{1}{3000}\left[x\right]_{4000}^{5000}$$

$$= \frac{1}{3000}[5000 - 4000] = 0.333$$

(3) $p[x = 2500] = 0$, (i.e) it is particular point ,the value is zero.

55. Starting at 5.00 am every half an hour there is a flight from San Fransisco airport to Losangles .Suppose that none of three planes is completely sold out and that they always have room for passengers . A person who wants to fly to Losangles arrive at a random time between 8.45 am and 9.45 am. Find the probability that she waits

(a) Atmost 10 min (b) atleast 15 min

Soln: Let X be the uniform r.v. over the interval (0,60)

Then the pdf is given by

$$f(x) = \frac{1}{b-a} \ , a < x < b$$

$$= \frac{1}{60} \quad , \quad 0 < x < 60$$

(a) The passengers will have to wait less than 10 min. if she arrives at the airport

$$= p(5 < x < 15) + p(35 < x < 45)$$

$$= \int_{5}^{15} \frac{1}{60} dx + \int_{35}^{45} \frac{1}{60} dx$$

$$= \frac{1}{60}\left[x\right]_{5}^{15} + \frac{1}{60}\left[x\right]_{35}^{45}$$

$$= \frac{1}{3}$$

(b)The probability that she has to wait atleast 15 min.

$$= p(15 < x < 30) + p(45 < x < 60)$$

$$= \int_{15}^{30} \frac{1}{60} dx + \int_{45}^{60} \frac{1}{60} dx$$

$$= \frac{1}{60}\left[x\right]_{15}^{30} + \frac{1}{60}\left[x\right]_{45}^{60}$$

$$= \frac{1}{2}$$

56. Establish the memoryless property of exponential distbn.

Soln: If X is exponentially distributed , then

$$p\left[x > s+t/x > s\right] = p\left[x > t\right] \ \ for\,any\,s,t > 0$$

The pdf of exponential distbn is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , \ otherwise \end{cases}$$

$$p(x>k)=\int_k^\infty f(x)\,dx$$

$$=\int_k^\infty \lambda e^{-\lambda x}\,dx=\lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty$$

$$=-\left[0-e^{-\lambda k}\right]=e^{-\lambda x}\;---------(1)$$

$$p\left[x>s+t/x>s\right]=\frac{p\left[x>s+t\ \ and\ \ x>s\right]}{p\left[x>s\right]}$$

$$=\frac{p\left[x>s+t\ \right]}{p\left[x>s\right]}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}=e^{-\lambda t}=p\left[x>t\right]$$

$$\therefore\quad p\left[x>s+t/x>s\right]=p\left[x>t\right]\ \ for\,any\,s,t>0$$

57. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda=\dfrac{1}{2}$.

(a)What is the probability that the repair time exceeds 2 hrs ?

(b)What is the conditional probability that a repair takes atleast 11 hrs given that Its direction exceeds 8 hrs ?

Soln: If X represents the time to repair the machine, the density function Of X is given by

$$f(x)=\lambda e^{-\lambda x}\qquad,\ x\geq 0$$

$$=\frac{1}{2}e^{-x/2}\qquad x\geq 0$$

(a)

$$p(x>2)=\int_2^\infty f(x)\,dx=\int_2^\infty \lambda e^{-\lambda x}\,dx$$

$$=\int_2^\infty \frac{1}{2}e^{\frac{-x}{2}}\,dx=\frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^\infty$$

$$=-\left[0-e^{-1}\right]=0.3679$$

$$p[x \geq 11/x > 8] = p[x > 3]$$

$$= \int_3^\infty f(x)\,dx = \int_3^\infty \lambda\, e^{-\lambda x}\,dx$$

$$= \int_3^\infty \frac{1}{2} e^{\frac{-x}{2}}\,dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_3^\infty$$

$$= -\left[0 - e^{-\frac{3}{2}}\right] = e^{-\frac{3}{2}} = 0.2231$$