

Lecture 07

Hypothesis Testing with Multivariate Regression

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Taylor B. Arnold
Yale Statistics
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Goals for today

1. Review of assumptions and properties of linear model
2. The multivariate T-test
3. Hypothesis test simulations
4. The multivariate F-test

REVIEW FROM LAST TIME

Classical linear model assumptions

I. Linearity $Y = X\beta + \epsilon$

II. Strict exogeneity $\mathbb{E}(\epsilon|X) = 0$

III. No multicollinearity $\mathbb{P}[\text{rank}(X) = p] = 1$

IV. Spherical errors $\mathbb{V}(\epsilon|X) = \sigma^2 \mathbb{I}_n$

V. Normality $\epsilon|X \sim \mathcal{N}(0, \sigma^2 \mathbb{I}_n)$

Finite sample properties

Under assumptions I-III:

$$(A) \mathbb{E}(\hat{\beta}|X) = \beta$$

Under assumptions I-IV:

$$(B) \mathbb{V}(\hat{\beta}|X) = \sigma^2(X^t X)^{-1}$$

(C) $\hat{\beta}$ is the best linear unbiased estimator (Gauss-Markov)

$$(D) \text{Cov}(\hat{\beta}, r|X) = 0$$

$$(E) \mathbb{E}(s^2|X) = \sigma^2$$

Under assumptions I-V:

(F) $\hat{\beta}$ achieves the Cramér–Rao lower bound

THE T-TEST

Hypothesis tests

Consider testing the hypothesis $H_0 : \beta_j = b_j$.

Under assumptions I-V we have the following:

$$\hat{\beta}_j - b_j \Big| X, H_0 \sim \mathcal{N}(0, \sigma^2 ((X^t X)_{jj}^{-1}))$$

Z-test

This suggests the following test statistic:

$$z = \frac{\hat{\beta}_j - b_j}{\sqrt{\sigma^2 ((X^t X)^{-1})_{jj}}}$$

With,

$$z|X, H_0 \sim \mathcal{N}(0, 1)$$

T-test

As in the simple linear regression case, we generally need to estimate σ^2 with s^2 .

This yields the following test statistic:

$$\begin{aligned} t &= \frac{\hat{\beta}_j - b_j}{\sqrt{s^2 ((X^t X)^{-1})_{jj}}} \\ &= \frac{\hat{\beta}_j - b_j}{\text{S.E.}(\hat{\beta}_j)} \end{aligned}$$

T-test

The test statistic has a T -distribution with $(n - p)$ degrees of freedom under the null hypothesis:

$$t|X, H_0 \sim t_{n-p}$$

This time around, we'll actually prove this.

To start, we re-write the test statistic as:

$$\begin{aligned} t &= \frac{\hat{\beta} - b}{\sqrt{\sigma^2 ((X^t X)^{-1})_{jj}}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\ &= \frac{z}{\sqrt{s^2/\sigma^2}} \\ &= \frac{z}{\sqrt{q/(n-p)}} \end{aligned}$$

Where $q = r^t r / \sigma^2$.

We need to show that (1) $q|X \sim \chi^2_{n-p}$ and (2) $z \perp\!\!\!\perp q|X$.

Lemma If B is a symmetric idempotent matrix and $u \sim \mathcal{N}(0, \mathbb{I}_n)$, then $u^t B u \sim \chi^2_{\text{tr}(B)}$.

Proof: The symmetric matrix B can be written as by its eigen-decomposition; for some orthonormal Q matrix and diagonal matrix Λ :

$$B = Q^t \Lambda Q$$

Notice that as B is idempotent:

$$\begin{aligned}(Q^t \Lambda Q) &= (Q^t \Lambda Q)(Q^t \Lambda Q) \\ &= Q^t \Lambda Q Q^t \Lambda Q \\ &= Q^t \Lambda^2 Q\end{aligned}$$

Since $Q^t = Q^{-1}$:

$$\Lambda = \Lambda^2$$

Therefore all the elements of Λ are 0 or 1.

By rotating the columns of Q , we can actually write:

$$\Lambda = \begin{pmatrix} \mathbb{I}_{\text{rank}(B)} & 0 \\ 0 & 0 \end{pmatrix}$$

Notice that this also shows that:

$$\begin{aligned} \text{tr}(B) &= \text{tr}(Q^t \Lambda Q) \\ &= \text{tr}(\Lambda Q Q^t) \\ &= \text{tr}(\Lambda) \\ &= \text{rank}(B) \end{aligned}$$

Now let $v = Qu$. We can see that the mean of v is zero:

$$\begin{aligned}\mathbb{E}v &= \mathbb{E}Qu \\ &= Q\mathbb{E}u \\ &= 0\end{aligned}$$

And the variance of v is:

$$\begin{aligned}\mathbb{E}vv^t &= Q\mathbb{E}(uu^t)Q \\ &= Q\mathbb{I}_nQ^t \\ &= \mathbb{I}_n\end{aligned}$$

Now we calculate the quadratic form $u^t B u$ by the transform of v .
 Substitute that $u = (Q)^{-1} v = Q^t v$:

$$\begin{aligned}
 u^t B u &= v^t Q B Q^t v \\
 &= v^t Q (Q^t \Lambda Q) Q^t v \\
 &= v^t Q Q^t \Lambda Q Q^t v \\
 &= v^t \Lambda v \\
 &= v^t \begin{pmatrix} \mathbb{I}_{\text{tr}(B)} & 0 \\ 0 & 0 \end{pmatrix} v \\
 &= \sum_{i=1}^{\text{tr}(B)} v_i^2 \\
 &\sim \chi_{\text{tr}(B)}^2
 \end{aligned}$$

Which completes the lemma.

(1) We know that $r^t r = \epsilon^t M \epsilon$, so:

$$\begin{aligned} q &= \frac{r^t r}{\sigma^2} \\ &= \frac{\epsilon^t}{\sigma} \cdot M \cdot \frac{\epsilon}{\sigma} \end{aligned}$$

Under Assumption V, $\frac{\epsilon}{\sigma} \sim \mathcal{N}(0, \mathbb{I}_n)$. Therefore from the lemma, then $q \sim \chi^2_{\text{tr}(M)}$; last time we showed the $\text{tr}(M) = n - p$, which finishes the first part of the proof.

(2) The random variables $\hat{\beta}$ and r are both linear combinations of ϵ , and are therefore jointly normally distributed. As they are uncorrelated (problem set 2), this implies that they are independent. Finally, this implies that $z = f(\hat{\beta})$ and $q = g(r)$ and themselves independent.

So, therefore, we have:

$$\begin{aligned}t &= \frac{\hat{\beta} - b}{\sqrt{\sigma^2 ((X^t X)^{-1})_{jj}}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\&= \frac{z}{q/(n-p)} \\&\sim t_{n-p}\end{aligned}$$

SIMULATION

F-TEST

F-test

Now consider the hypothesis test $H_0 : D\beta = d$ for a matrix D with k rows and rank k .

We'll form the following test statistic:

$$F = \frac{(D\hat{\beta} - d)^t [D(X^t X)^{-1} D^t]^{-1} (D\hat{\beta} - d) / k}{s^2}$$

And prove that it has an F-distribution with k and $n - p$ degrees of freedom.

We can re-write the test statistics as:

$$F = \frac{\textcolor{teal}{w}/k}{\textcolor{violet}{q}/(n-p)}$$

Where:

$$\begin{aligned}\textcolor{teal}{w} &= (D\hat{\beta} - d)^t [\sigma^2 D(X^t X)^{-1} D^t]^{-1} (D\hat{\beta} - d) \\ \textcolor{violet}{q} &= r^t r / \sigma^2\end{aligned}$$

We've already shown that $\textcolor{violet}{q}|X \sim \chi_{n-p}^2$, and can see that $\textcolor{violet}{q} \perp \textcolor{teal}{w}$ by the same argument as before. All that is left is to show that $\textcolor{teal}{w} \sim \chi_k^2$.

Let $v = D\hat{\beta} - d$. Under the null hypothesis $v = D(\hat{\beta} - \beta)$. Therefore:

$$\begin{aligned}\mathbb{V}(v|X) &= \mathbb{V}(D(\hat{\beta} - \beta)|X) \\ &= D\mathbb{V}(\hat{\beta} - \beta|X)D^t \\ &= \sigma^2 D(X^t X)^{-1} D^t\end{aligned}$$

Now, for the multivariate normally distributed v with zero mean, we have:

$$w = v^t \mathbb{V}(v|X)^{-1} v$$

Now, decompose $\mathbb{V}(v|X)^{-1} = \Sigma^{-1}$ as $Q^t \Lambda Q$. Notice that this implies:

$$\begin{aligned}\Sigma^{-1} &= Q^t \Lambda^{1/2} \Lambda^{1/2} Q \\ Q \Sigma^{-1} &= \Lambda^{1/2} \Lambda^{1/2} Q \\ \Lambda^{-1/2} Q \Sigma^{-1} &= \Lambda^{1/2} Q\end{aligned}$$

From here, we can write w as the inner product of a suitably defined vector u :

$$\begin{aligned}w &= v^t \Sigma^{-1} v \\&= v^t Q^t \Lambda^{1/2} \Lambda^{1/2} Q v \\&= u^t u\end{aligned}$$

With a mean of zero:

$$\begin{aligned}\mathbb{E}(u|X) &= \mathbb{E}(\Lambda^{1/2} Q v|X) \\&= \Lambda^{1/2} Q \mathbb{E}(v|X) \\&= 0\end{aligned}$$

And variance of

$$\begin{aligned}\mathbb{E}(uu^t|X) &= \mathbb{E}(\Lambda^{1/2}Qvv^tQ^t\Lambda^{1/2}|X) \\ &= \Lambda^{1/2}Q\mathbb{E}(vv^t|X)Q^t\Lambda^{1/2} \\ &= \Lambda^{1/2}Q\Sigma Q^t\Lambda^{1/2} \\ &= \Lambda^{-1/2}Q\Sigma^{-1}\Sigma Q^t\Lambda^{1/2} \\ &= \Lambda^{-1/2}\Lambda^{1/2} \\ &= \mathbb{I}_k\end{aligned}$$

And, finally:

$$w = uu^t \sim \chi_k^2$$

Which finishes the proof.

An alternative F-test

Now, consider the following estimator:

$$\tilde{\beta} = \arg \min_b ||y - Xb||_2^2, \quad \text{s.t.} \quad Db = d$$

We then define the restricted residuals $\tilde{r} = y - X\tilde{\beta}$.

An alternative F-test

An alternative expression for the F -test statistic is:

$$F = \frac{(\tilde{r}^t \tilde{r} - r^t r)/k}{r^t r / (n - p)}$$

Conceptually, it should make sense that this is large whenever the null hypothesis is false.

An alternative F-test

If we let SSR_U be the sum of squared residuals of the unrestricted model ($r^t r$) and SSR_R be the sum of squared residuals of the restricted model, then this can be re-written as:

$$F = \frac{(SSR_R - SSR_U)/k}{SSR_U/(n - p)}$$

This is the way it is written in the homework and in the Fumio Hayashi text. It is left for you to prove that this is equivalent to the other F test.

APPLICATION