Lecture 11
Logistic Regression

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Notes

- Problem Set #4 - Due in two weeks
- No class next Monday

Goals for today

- Logistic regression introduction
- Solving via least squares
- Running GLMs in R

Logistic regression introduction

Consider the case where $y_i \in \{0, 1\}$ for all values of i. If we write:

$$y = X\beta + \epsilon$$

Why does it not make sense for ϵ to be independent of X?

A natural extension of the classical linear regression to handel this case is, then:

$$\mathbb{E}(y|X) = g^{-1}(X\beta)$$

For some fixed and known function g, called the *link function*.

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If g is the identity how does this relate the linear case?

If y_i has a Bernoilli distribution, notice that this only has one unknown parameter $p_i = \mathbb{P}(y = 1)$. We can write the likelihood function as (just plug in the two possible values of y to see that this works:

$$L(y_i|p_i) = p_i^{y_i} \cdot (1 - p_i)^{1-y_i}$$

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Manipulating this a bit, we can write the likelihood as an exponential family:

$$\begin{aligned} L(y_i|p_i) &= (1 - p_i) \cdot \left(\frac{p_i}{1 - p_i} \right)^{y_i} \\ &= (1 - p_i) \cdot \exp\left(y_i \cdot \log\left(\frac{p_i}{1 - p_i} \right) \right) \end{aligned}$$

I won't derive the entire theory of exponential families today, but this form suggests that the 'canonical' parameter in the Bernoilli distribution is:

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&= \operatorname{logit}(p_i)
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Therefore, a natural choice is to say that η_i is a linear function of x_i:

$$\eta_i = x_i^t \beta$$

In other words, g is equal to the logit function.

Now, consider determining the mean of y_i given a regression vector β:

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\begin{aligned}
\log\left(\frac{p_i}{1-p_i}\right) &= x_i^t\beta \\
\frac{p_i}{1-p_i} &= e^{x_i^t\beta} \\
p_i &= (1-p_i)\cdot e^{x_i^t\beta} \\
\left(1+e^{x_i^t\beta}\right)p_i &= e^{x_i^t\beta} \\
p_i &= \frac{e^{x_i^t\beta}}{1+e^{x_i^t\beta}} \\
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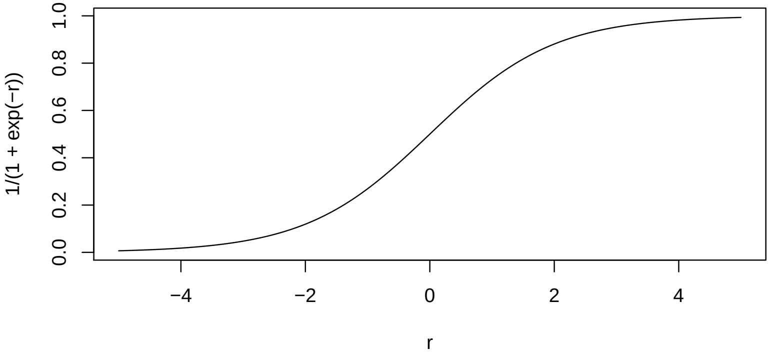
What does the relationship between $x^t\beta$ and p_i look like?



Note: we could use other link functions g, the logit is simply a popular choice given the theoretical connections to exponential families.

Parameter estimation

The parameters in logistic regression are generally fit using maximum likelihood estimation. The likelihood for the entire model, building on what we saw before, is given by:

$$L(y|X) = \prod_i p_i^{y_i} \cdot (1 - p_i)^{1-y_i}$$

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And therefore the log-likelihood is:

$$\begin{aligned} l(y|X) &= \sum_i \{y_i \cdot \log(p_i) + (1 - y_i) \cdot \log(1 - p_i)\} \\ &= \sum_i \left\{ y_i \beta^t x_i - \log\left(1 + e^{\beta^t x_i}\right) \right\} \end{aligned}$$

To find critical points of this, we set the first derivatives of the log-likelihood with respect to β to zero:

$$\frac{\partial}{\partial \beta} l(\beta) = \sum_i x_i \cdot (y_i - p_i(\beta)) = 0$$