

Lecture 21

Theory of the Lasso II

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STAT 312/612

The Yale University logo, featuring the word "Yale" in a blue, serif typeface.

Class Notes

- Midterm II - Available now, due next Monday
- Problem Set 7 - Available now, due December 11th (grace period through the 16th)

LAST TIME

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Today's goal is to establish a bound on $\|\hat{\beta} - \beta\|_2^2$

The basic starting point from last time was the following decomposition, which had no assumptions beyond linearity of the true model:

$$\|X(\beta - b)\|_2^2 \leq 2\epsilon^t X(b - \beta) + \lambda \cdot (\|\beta\|_1 - \|b\|_1)$$

Where can think of this decomposition as the loss to be minimized, the empirical part, and the penalty term.

I then defined the set

$$\mathcal{A} = \{2\|\epsilon^t X\|_\infty \leq \lambda\}$$

And showed that for any $A > 1$ we have $\mathbb{P}\mathcal{A} \geq 1 - A^{-1}$ whenever

$$\lambda \geq A \cdot \sqrt{8 \log(2p) \sigma^2}.$$

Today we will motivate a stronger assumption on the model and use these two results to establish bounds on the prediction of β .

Also, it will be helpful to write the set \mathcal{A} as being parameterized by the value of λ_0 :

$$\mathcal{A}(\lambda_0) = \{2\|\epsilon^t X\|_\infty \leq \lambda_0\}$$

BOUNDS ON ESTIMATION ERROR

We already know that on $\mathcal{A}(\lambda_0)$ and with $\lambda > 2 \cdot \lambda_0$, we have:

$$\begin{aligned} \|X(\mathbf{b} - \beta)\|_2^2 + \lambda \cdot \|\mathbf{b}\|_1 &\leq 2\epsilon^t X(\mathbf{b} - \beta) + \lambda \cdot \|\beta\|_1 \\ &\leq \lambda_0 \|\mathbf{b} - \beta\|_1 + \lambda \cdot \|\beta\|_1 \end{aligned}$$

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Now, multiplying by two gives:

$$2\|X(\mathbf{b} - \beta)\|_2^2 + 2\lambda \cdot \|\mathbf{b}\|_1 \leq \lambda \|\mathbf{b} - \beta\|_1 + 2\lambda \cdot \|\beta\|_1$$

Recall that we defined the notation: $S = \{j : \beta_j \neq 0\}$, s is the size of the set S , and v_S is the vector v which has components not in S set to zero.

Notice that:

$$\begin{aligned}\|b\|_1 &= \|b_S\|_1 + \|b_{S^c}\|_1 \\ &\geq \|\beta\|_1 - \|b_S - \beta\|_1 + \|b_{S^c}\|_1\end{aligned}$$

Using the (reverse) triangle inequality and the fact that β_{S^c} is zero by definition.

Similarly, we have:

$$\|b - \beta\|_1 = \|b_S - \beta_S\|_1 + \|b_{S^c}\|_1$$

Where clearly β_S is redundant, but useful to keep the notation straight.

Plugging these in, we now get:

$$\begin{aligned} 2||X(\mathbf{b} - \boldsymbol{\beta})||_2^2 + 2\lambda \cdot ||\boldsymbol{\beta}_S||_1 - 2\lambda \cdot ||\mathbf{b}_S - \boldsymbol{\beta}_S||_1 + 2\lambda \cdot ||\mathbf{b}_{S^c}||_1 \\ \leq \lambda ||\mathbf{b} - \boldsymbol{\beta}||_1 + \lambda \cdot ||\mathbf{b}_S - \boldsymbol{\beta}_S||_1 + 2\lambda \cdot ||\mathbf{b}_{S^c}||_1 \end{aligned}$$

Plugging these in, we now get:

$$\begin{aligned} 2\|X(\mathbf{b} - \boldsymbol{\beta})\|_2^2 + 2\lambda \cdot \|\boldsymbol{\beta}_S\|_1 - 2\lambda \cdot \|\mathbf{b}_S - \boldsymbol{\beta}\|_1 + 2\lambda \cdot \|\mathbf{b}_{S^c}\|_1 \\ \leq \lambda \|\mathbf{b} - \boldsymbol{\beta}\|_1 + \lambda \cdot \|\mathbf{b}_S - \boldsymbol{\beta}_S\|_1 + 2\lambda \cdot \|\mathbf{b}_{S^c}\|_1 \end{aligned}$$

Which cancels out as:

$$2\|X(\mathbf{b} - \boldsymbol{\beta})\|_2^2 + \lambda \|\mathbf{b}_{S^c}\|_1 \leq 3 \cdot \lambda \cdot \|\mathbf{b}_S - \boldsymbol{\beta}_S\|_1$$

This result now actually gives two sub-results, as all three terms are positive and therefore each component of the left hand side is individually bounded by the right hand side.

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In particular, we have:

$$||b_{S^c}||_1 \leq 3 \cdot ||b_S - \beta_S||_1$$

Which implies that the amount of error in b can not be too highly concentrated on S^c .

The other sub-result gives:

$$2\|X(\mathbf{b} - \boldsymbol{\beta})\|_2^2 \leq 3\lambda \cdot \|\mathbf{b}_S - \boldsymbol{\beta}_S\|_1$$

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$$2||X(b - \beta)||_2^2 \leq 3\lambda \cdot ||b_S - \beta_S||_1$$

If σ_{min} is the minimum singular value of X , then the left hand side can be bounded below by:

$$2\sigma_{min}^2 ||b - \beta||_2^2 \leq 3\lambda \cdot ||b_S - \beta_S||_1$$

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Using the Cauchy-Schwarz inequality, this becomes:

$$\begin{aligned} 2\sigma_{min}^2 ||b - \beta||_2^2 &\leq 3\lambda \cdot \sqrt{s} ||b_S - \beta_S||_2 \\ ||b - \beta||_2 &\leq \frac{3\lambda\sqrt{s}}{2\sigma_{min}^2} \end{aligned}$$

Which gives a bound on the error of estimating β , which is exactly what we wanted to establish.

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We can get around this problem by defining a modified version of the minimum eigenvector (or squared singular value) by only considering $b - \beta$ such that:

$$\|b_{Sc}\|_1 \leq 3 \cdot \|b_S - \beta_S\|_1$$

The (minimum) restricted eigenvalue ϕ_S on the set S is defined as:

$$\phi_S = \arg \min_{v \in \mathcal{V}_S} \frac{\|Xb\|_2}{\|b\|_2}$$

Where:

$$\mathcal{V}_S = \{v \in \mathbb{R}^p \text{ s.t. } \|v_{S^c}\|_1 \leq 3 \cdot \|v_S\|_1\}$$

Because we do not know S , it is impossible to calculate ϕ_S in practice. In theoretical work, often one considers **the** restricted eigenvalue ϕ defined as the smallest ϕ_S for all sets S with size bounded by some predefined s_0 .

Now, we can bound the following using our prior result:

$$\begin{aligned} 2||X(\mathbf{b} - \beta)||_2^2 + \lambda \cdot ||\mathbf{b} - \beta||_1 \\ &= 2||X(\mathbf{b} - \beta)||_2^2 + \lambda \cdot ||\mathbf{b}_S - \beta_S||_1 + \lambda \cdot ||\mathbf{b}_{S^c}||_1 \\ &= 4\lambda \cdot ||\mathbf{b}_S - \beta_S||_1 \end{aligned}$$

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Using Cauchy-Schwarz again, we can change the ℓ_1 -norm to an ℓ_2 -norm at the cost of a factor of \sqrt{s} :

$$2||X(\mathbf{b} - \beta)||_2^2 + \lambda \cdot ||\mathbf{b} - \beta||_1 \leq 4\lambda \cdot \sqrt{s} \cdot ||\mathbf{b}_S - \beta_S||_2$$

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Finally, we now use the restricted eigenvalue ϕ to convert from β space to $X\beta$ space:

$$2||X(\mathbf{b} - \beta)||_2^2 + \lambda \cdot ||\mathbf{b} - \beta||_1 \leq 4\lambda \cdot \sqrt{s} \cdot ||X(\mathbf{b}_S - \beta_S)||_2 / \phi$$

I am now going to use an inequality trick that is often useful in theoretical statistics derivations. For any u and v , notice that $4uv \leq u^2 + 4v^2$.

For a proof, notice that it is trivially true at zero and negative values of u and v . Then look at the derivatives and notice that the right hand side grows faster than the left hand side in the directions of both u and v .

Setting $u = \|X(b_S - \beta_S)\|_2$, we then have:

$$\begin{aligned} 2\|X(b - \beta)\|_2^2 + \lambda \cdot \|b - \beta\|_1 &\leq \|X(b_S - \beta_S)\|_2 + 4\lambda^2 \cdot s \cdot / \phi^2 \\ &\leq \|X(b - \beta)\|_2^2 + 4\lambda^2 \cdot s \cdot / \phi^2 \end{aligned}$$

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And when canceling one factor of $\|X(b - \beta)\|_2$:

$$\|X(b - \beta)\|_2^2 + \lambda \cdot \|b - \beta\|_1 \leq 4\lambda^2 \cdot s \cdot / \phi^2$$

Which holds on the entire set $\mathcal{A}(\lambda_0)$.

This establishes two simultaneous bounds:

$$\begin{aligned}\|X(\mathbf{b} - \beta)\|_2^2 &\leq 4\lambda^2 \cdot s \cdot / \phi^2 \\ \|\mathbf{b} - \beta\|_1 &\leq 4\lambda \cdot s \cdot / \phi^2\end{aligned}$$

Though the first is slightly less satisfying than our result in last class as it relies on ϕ^2 , though it no longer requires the norm of β .

ASYMPTOTIC ANALYSIS

As before, we can convert a more natural re-scaled problem by dividing all of the λ parameters by \sqrt{n}

Also, remember that for some $A > 1$, we have $\mathbb{P}\mathcal{A}(\lambda_0) \geq 1 - A^{-1}$ for all $\lambda > A \cdot \sqrt{16n^{-1} \log(2p)\sigma^2}$.

Therefore, we have:

$$\begin{aligned}\|b - \beta\|_1 &\leq 4\lambda \cdot s \cdot \phi^2 \\ &\leq 8 \cdot A\sigma^2 / \phi^2 \cdot \frac{s_n^2 \log(2p_n)}{n}\end{aligned}$$

Which is the same result as from the Bickel, Ritov, Tsybakov paper.

To establish consistency of the estimator under constant noise and restricted eigenvalues ϕ^2 , we need the following limit to go to zero:

$$\lim_{n \rightarrow \infty} \frac{s_n^2 \log(2p_n)}{n} = 0$$

Which can happen with a number of different scalings, such as a constant number of non-zero terms but an exponential number of non-zero terms. Or, s_n growing like $n^{1/3}$ and p_n growing linearly with s_n .

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5. I have always been skeptical of the asymptotic results for the same reason; ϕ likely depends on n , p_n and s_n in complex ways that are not accounted for

For our next (and last) week we will:

1. use the lasso to encode more complex forms of linear sparsity (e.g., outlier detection and the fused lasso)
2. give an alternative approach to solving for the lasso solution at a particular value of λ