Lecture 14 PCR and Ridge Regression

04 November 2015

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- the final problem set, 7, is formally due the last day of classes (but we'll accept them through December 14th)

Goals for today

- ridge regression formulation and link to SVD
- principal component analysis
- applications to image data

Ridge regression

The ridge regression estimator is the solution to the following modified least squares optimization problem for some value of $\lambda > 0$.

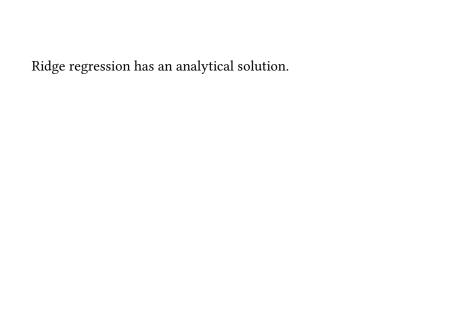
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$$\widehat{\beta}_{ridge} = \underset{b}{\arg\min} \left\{ ||y - Xb||_2^2 + \lambda ||b||_2^2 \right\}$$

The equation shrinks the coefficients towards zero, adding some bias but reducing the variance of the estimator.



Ridge regression has an analytical solution. To see this write the criterion as a matrix equation:

$$(y - Xb)^t(y - Xb) + \lambda b^t b = y^t y + b^t X^t Xb - 2y^t Xb + \lambda b^t b$$

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$$(y - Xb)^t(y - Xb) + \lambda b^t b = y^t y + b^t X^t Xb - 2y^t Xb + \lambda b^t b$$

And take its derivative:

$$rac{\partial}{\partial eta} \left(y^t y + b^t X^t X b - 2 y^t X b + \lambda b^t b
ight) = 2 X^t X b - 2 X^t y + 2 \lambda b$$

Setting this to zero yields

$$2X^{t}X\widehat{\beta} + 2\lambda\widehat{\beta} = 2X^{t}y$$
$$(X^{t}X + L_{t}\lambda)\widehat{\beta} = X^{t}y$$

$$(X^{t}X + I_{p}\lambda)\widehat{\beta} = X^{t}y$$

 $\widehat{\beta} = (X^{t}X + I_{p}\lambda)^{-1} \times X^{t}y$

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This is a useful analytical form, though as with least squares we would generally not invert the matrix directly but instead use a stable matrix decomposition.

Now consider the singular value decomposition $U\Sigma V^t$ of the matrix X. We can write the projection matrix P in terms of this as:

$$P = X(X^{t}X)^{-1}X^{t}$$

$$= U\Sigma V^{t}(V^{t}\Sigma^{2}V)^{-1}V\Sigma U^{t}$$

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Remember how important the projection matrix was? This is a very important result!

The analogue of the projection matrix for ridge regression is given by:

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Where P_0 is equal to the ordinary P.

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Where P_0 is equal to the ordinary P. As was the case last time, this matrix maps y into the predicted values \hat{y} .

$$X^{t}X + \lambda I_{p} = V\Sigma^{2}V^{t} + \lambda VV^{t}$$
$$= V(\Sigma^{2} + \lambda)V^{t}$$

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So what have we done to the

What is the decomposition of P_{λ} in terms of the singular value decomposition?

$$P_{\lambda} = X(X^{t}X + \lambda I_{p})^{-1}X^{t}$$

$$= U\Sigma V^{t}(V^{t}\Sigma^{2}V)^{-1}V\Sigma U^{t}$$

$$= U\Sigma(\Sigma^{2} + \lambda I_{p})^{-1}\Sigma U^{t}$$

$$= UDU^{t}$$

For the diagonal matrix *D*:

$$D = \operatorname{diag}\left(\frac{\sigma_1^2}{\sigma_1^2 + \lambda}, \dots, \frac{\sigma_p^2}{\sigma_p^2 + \lambda}\right) \tag{1}$$

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So we are shrinking the directions of the singular vectors, with more shrinkage on the smaller singular values.

What is the bias of the ridge regression?

$$Var(\widehat{\beta}|X) = (X^tX + I_p\lambda)^{-1} \times X^tVar(y|X)$$

 $= (X^t X + I_p \lambda)^{-1} \times X^t X \beta$

What is the bias of the ridge regression?

$$\mathbb{E}\widehat{\beta} = (X^t X + I_p \lambda)^{-1} \times X^t \mathbb{E} y$$
$$= (X^t X + I_p \lambda)^{-1} \times X^t X \beta$$