Lecture 07
Hypothesis Testing with Multivariate Regression

23 September 2015

Taylor B. Arnold
Yale Statistics
STAT 312/612

Yale

Goals for today

1. Galton heights data
2. Hypothesis tests

Linear models assumptions

Hypothesis tests

Consider testing the hypothesis $H_0 : beta_j = b$ for some particular b and j.

Hypothesis tests

Consider testing the hypothesis $H_0 : beta_j = b$ for some particular b and j.

Under assumptions I-V we have the following:

$$\widehat{\beta} - b \Big| X, H_0 \sim \mathcal{N}\left(0, \sigma^2 \left((X^t X)^{-1}_{jj}\right)\right)$$

Z-test

This suggests the following test statistic:

$$z = \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}}$$

With,

$$z | X, H_0 \sim \mathcal{N}(0, 1)$$

T-test

As in the simple linear linear regression case, we need to estimate σ^2 with s^2. This yields the following test statistic:

$$\begin{aligned} t &= \frac{\widehat{\beta} - b}{\sqrt{s^2 \left((X^t X)^{-1}_{jj} \right)}} \\ &= \frac{\widehat{\beta} - b}{\text{S.E.}(\widehat{\beta}_j)} \end{aligned}$$

T-test

The test statistic has a T-distribution with $(n-p)$ degrees of freedom

$$t|\,X, H_0 \sim t_{n-p}$$

T-test

The test statistic has a T-distribution with $(n-p)$ degrees of freedom

$$t|\, X, H_0 \sim t_{n-p}$$

This time around, we'll actually prove this.

Note that:

$$
\begin{aligned}
t &= \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\
&= \frac{z}{\sqrt{s^2/\sigma^2}} \\
&= \frac{z}{q/(n-p)}
\end{aligned}
$$

Note that:

$$
\begin{aligned}
t &= \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\
&= \frac{z}{\sqrt{s^2/\sigma^2}} \\
&= \frac{z}{q/(n-p)}
\end{aligned}
$$

Where $q = r^t r/\sigma^2$.

Note that:

$$
\begin{aligned}
t &= \frac{\widehat{\beta} - b}{\sqrt{\sigma^2 \left((X^t X)^{-1}_{jj} \right)}} \cdot \sqrt{\frac{\sigma^2}{s^2}} \\
&= \frac{z}{\sqrt{s^2/\sigma^2}} \\
&= \frac{z}{q/(n-p)}
\end{aligned}
$$

Where $q = r^t r/\sigma^2$.

We need to show that (1) $q|X \sim \chi^2_{n-p}$ and (2) $z \perp\!\!\!\perp q|X$.

Lemma If B is an idempotent matrix and $v \sim \mathcal{N}(0, \mathbb{I}_n)$, then $v^t B v \sim \chi^2_{\mathrm{rank(B)}}$.

Lemma If B is an idempotent matrix and $v \sim \mathcal{N}(0, \mathbb{I}_n)$, then $v^t B v \sim \chi^2_{\text{rank(B)}}$.

See `http://www2.econ.iastate.edu/classes/econ671/hallam/documents/QUAD_NORM.pdf` for a proof of this lemma.

(1) We know that $r^t r = \epsilon^t M \epsilon$, so:

$$\begin{aligned} q &= \frac{r^t r}{\sigma^2} \\ &= \frac{\epsilon^t}{\sigma} \cdot M \cdot \frac{\epsilon}{\sigma} \end{aligned}$$

(1) We know that $r^t r = \epsilon^t M \epsilon$, so:

$$\begin{aligned} q &= \frac{r^t r}{\sigma^2} \\ &= \frac{\epsilon^t}{\sigma} \cdot M \cdot \frac{\epsilon}{\sigma} \end{aligned}$$

From the lemma then $q \sim \chi^2_{\text{rank(M)}}$. Finally, noting that $rank(M) = tr(M) = n - p$ finishes the first part of the proof.

(2) The random variables $\widehat{\beta}$ and r are both linear combinations of ϵ, and are therefore jouintly normally distributed. As they are uncorrelated (problem set 2), this implies that they are independent. Finally, this implies that $z = f(\widehat{\beta})$ and $t = g(r)$ and themselves independent.

F-test

Now consider the hypothesis test $H_0 : D\beta = d$ for a matrix D with k columns and rank k.

We'll form the following test statistic:

$$F = \frac{(D\widehat{\beta} - d)^t [D(X^t X)^{-1} D^t]^{-1} (D\widehat{\beta} - d)/k}{s^2}$$

And prove that is has an F-distribution with k and $n - p$ degrees of freedom.

We can re-write the test statistics as:

$$F = \frac{w/k}{q/(n-p)}$$

Where:

$$\begin{aligned} w &= (D\widehat{\beta} - d)^t [D(X^t X)^{-1} D^t]^{-1} (D\widehat{\beta} - d) \\ q &= r^t r / \sigma^2 \end{aligned}$$

We can re-write the test statistics as:

$$F = \frac{w/k}{q/(n-p)}$$

Where:

$$\begin{aligned} w &= (D\widehat{\beta} - d)^t [D(X^t X)^{-1} D^t]^{-1} (D\widehat{\beta} - d) \\ q &= r^t r/\sigma^2 \end{aligned}$$

We've already shown that $q|X \sim \chi^2_{n-p}$, and can see that $q \perp\!\!\!\perp w$ by the same argument as before. All that is left is to show that $w \sim \chi^2_k$.

Let $v = D\widehat{\beta} - d$. Under the null hypothesis $v = D(\widehat{\beta} - \beta)$. Therefore:

$$\mathbb{V}(v|X) \quad = \quad \mathbb{V}(D(\widehat{\beta} - \beta))$$

Let $v = D\widehat{\beta} - d$. Under the null hypothesis $v = D(\widehat{\beta} - \beta)$. Therefore:

$$\begin{aligned} \mathbb{V}(v|X) &= \mathbb{V}(D(\widehat{\beta} - \beta)) \\ &= D\mathbb{V}(\widehat{\beta} - \beta|X)D^t \end{aligned}$$

Let $v = D\widehat{\beta} - d$. Under the null hypothesis $v = D(\widehat{\beta} - \beta)$. Therefore:

$$
\begin{aligned}
\mathbb{V}(v|X) &= \mathbb{V}(D(\widehat{\beta} - \beta)) \\
&= D\mathbb{V}(\widehat{\beta} - \beta|X)D^t \\
&= \sigma^2 D(X^t X)^{-1} D^t
\end{aligned}
$$

Let $v = D\widehat{\beta} - d$. Under the null hypothesis $v = D(\widehat{\beta} - \beta)$. Therefore:

$$\begin{aligned}
\mathbb{V}(v|X) &= \mathbb{V}(D(\widehat{\beta} - \beta)) \\
&= D\mathbb{V}(\widehat{\beta} - \beta|X)D^t \\
&= \sigma^2 D(X^t X)^{-1} D^t
\end{aligned}$$

Now, for the multivariate normally distributed v with zero mean, we have:

$$w = v^t \mathbb{V}(v|X)^{-1} v$$

And therefore $w|X \sim \chi^2_k$

An alternative F-test

Now, consider the following estimator:

$$\widetilde{\beta} = \arg\min_b ||y - Xb||_2^2, \quad \text{s.t.} \quad Db = d$$

An alternative F-test

Now, consider the following estimator:

$$\widetilde{\beta} = \arg\min_{b} ||y - Xb||_2^2, \quad \text{s.t.} \quad Db = d$$

We then define the restricted residuals $\widetilde{r} = y - X\tilde{\beta}$.

An alternative F-test

An alternative expression for the F-test statistic is:

$$F = \frac{(\tilde{r}^t \tilde{r} - r^t r)/k}{r^t r/(n-p)}$$

Conceptually, it should make sense that this is large whenever the null hypothesis is false.