$$A = I + L + U$$

$$Ax = (I + L + U)\,x = b$$

$$x = b - Lx - Ux$$

Gauss Seidel

$$x^{k+1} = b - Lx^{k+1} - Ux^{k}$$

$$(I+L)\,x^{k+1} = b - Ux^{k}$$

multiplying by $(I+L)^{-1}$.

$$x^{k+1} = b(I+L)^{-1} - U(I+L)^{-1}x^{k}$$

$$\boxed{x^{k+1} = c\,x^{k} + b(I+L)^{-1}}$$

$$C = -(I+L)^{-1}U$$

Gauss Jacobi

$$x^{k+1} = b - (L+U)\,x^{k}$$

Multiplying by I^{-1}

$$x^{k+1} = I^{-1}b - I^{-1}(L+U)x^{k}$$

$$x^{k+1} = -I^{-1}(L+U)x^{k} + I^{-1}b$$

$$\boxed{x^{k+1} = C\,x^{k} + I^{-1}b}$$

where $C = -I^{-1}(L+U)$

Gauss Seidel & Gauss Jacobi iteration converges if and only if all eigenvalues of iteration matrix C have absolute value less then 1.

If the spectral radius of C (maximum of those absolute value) is SMALL then the convergence is RAPID

Convergence Condition $\|C\| < 1$

Matrix Norm

Pbm $5x_1 + x_2 + 2x_3 = 19$

$x_1 + 4x_2 - 2x_3 = -2$

$2x_1 + 3x_2 + 8x_3 = 39$

Eg for Gauss Seidel

$$x_1 + \frac{x_2}{5} + \frac{2}{5}x_3 = \frac{19}{5}$$

$$\frac{x_1}{4} + x_2 - \frac{1}{2}x_3 = \frac{-1}{2}$$

$$\frac{x_1}{4} + \frac{3}{8}x_2 + x_3 = \frac{39}{8}$$

$$\begin{bmatrix} 1 & 1/5 & 2/5 \\ 1/4 & 1 & -1/2 \\ 1/4 & 3/8 & 1 \end{bmatrix} = I + L + U = I + \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 0 \\ 1/4 & 3/8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$I + L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 0 \\ 1/4 & 3/8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 3/8 & 1 \end{bmatrix}$$

$$(I + L)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -5/32 & -3/8 & 1 \end{bmatrix}$$

$$C = -(I + L)^{-1}U = -\begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -5/32 & -\frac{3}{8} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -1/5 & -2/5 \\ 0 & 1/20 & 3/5 \\ 0 & 1/32 & -1/8 \end{bmatrix}$$

$$\|C\| = \sqrt{\frac{1}{25} + \frac{4}{25} + \frac{1}{400} + \frac{9}{25} + \frac{1}{32^2} + \frac{1}{64}}$$

$$= \sqrt{\frac{14}{25} + \frac{1}{400} + \frac{1}{32^2} + \frac{1}{64}}$$

$$\underline{\underline{\|C\| = 0.76098}} < 1.$$

<u>Gauss Jacobi</u> $\quad C = -I^{-1}(L+U)$

$$L+U = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 0 \\ 1/4 & 3/8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & 0 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/4 & 0 & -1/2 \\ 1/4 & 3/8 & 0 \end{bmatrix}$$

$$-I^{-1}(L+U) = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/5 & 2/5 \\ 1/4 & 0 & -1/2 \\ 1/4 & 3/8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/5 & -2/5 \\ -1/4 & 0 & 1/2 \\ -1/4 & -3/8 & 0 \end{bmatrix}$$

$$\|C\|_{Fro} = \sqrt{\frac{1}{25} + \frac{4}{25} + \frac{1}{16} + \frac{1}{4} + \frac{1}{16} + \frac{9}{64}}$$

$$= \sqrt{\frac{1}{5} + \frac{1}{8} + \frac{1}{4} + \frac{9}{64}}.$$

$$= \sqrt{0.715625} = \underline{\underline{0.845}} < 1.$$

Rayleigh Quotient

Let X be eigenvector of matrix $A_{n \times n}$. Real Symmetric matrix

$$Ax = \lambda X$$

Rayleigh Quotient given by $\boxed{\lambda_R = \frac{X^T A X}{X^T X}}$

where x is column vector
X^T is Row vector
A is square matrix

$y = Ax$

$m_0 = X^T X$ Quotient $\boxed{q = \frac{m_1}{m_0}.}$

$m_1 = X^T y$

$m_2 = y^T y$

Rayleigh Quotient is the same as the eigenvalue when X is an exact eigenvector.

$\boxed{q = \lambda - \epsilon}$ then Error of q is ϵ

$$|\epsilon| \leq \delta = \sqrt{\frac{m_2}{m_0} - q^2}.$$

δ → Bound for error ϵ of the approximation q of an eigenvalue of A.

Power Method Simplest method for computing one eigenvalue eigenvector pair

Repeatedly multiplies matrix times initial starting vector.