Data Structures and Algorithms Design (DSECLZG519)

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SESSION 3-PLAN

Online Sessions(#)	List of Topic Title	Text/Ref Book/external resource
3	Analyzing Recursive Algorithms: Recurrence relations, Specifying runtime of recursive algorithms, Solving recurrence equations. Case Study: Analysing Algorithms	T1: 1.4

Analyzing Recursive Algorithms

- Recursive calls:-A procedure P calling itself-calls to P are for solving sub problems of smaller size.
- Recursive procedure call should always define a ***base case***.
- Base case-small enough that it can be solved directly without using recursion.
- A ***recurrence*** is an equation or inequality that describes a function in terms of its value on smaller inputs.
- ***Recurrence equation***: defines mathematical statements that the running time of a recursive algorithm must satisfy
- Recurrences can take many forms for example, a recursive algorithm might divide subproblems into unequal sizes, such as a 2/3-to -1/3 splits

Analyzing Recursive Algorithms

- Algorithm recursiveMax(A,n)

  // Input : An array A storing n>=1 integers

  //Output: The maximum element in A

  if n = 1 then

  return A[0]

  return max{ recursiveMax(A,n-1),A[n-1]}

Analyzing Recursive Algorithms

- Analysis of ***recursiveMax***
  - T(n)-Running time of algorithm on an input size n

$$T(n) = \begin{cases} 3 & \text{if } n=1 \\ T(n-1) + 7 & \text{otherwise} \end{cases}$$

```
Algorithm recursiveMax(A,n)
// Input : An array A storing  n>=1 integers
  //Output: The maximum element in A
 if  n = 1 then
 return A[0]
return max{ recursiveMax(A,n-1),A[n-1]}
```

Method of solving recurrences

- Iteration Method
- Substitution method
- Recursion-tree method
- Master method

Solving recurrences : Iterative Method

Analyzing Recursive Algorithms-Iterative method

- **General Plan-Iterative Method**
  - Identify the parameter to be considered based on the size of the input.
  - Identify the basic operation in the algorithm
  - Obtain the number of times the basic operation is executed.
  - Obtain an initial condition-base case
  - Obtain a recurrence relation
  - Solve the recurrence relation and obtain the order of growth and express using asymptotic notations.

Analyzing Recursive Algorithms

- Analysis of ***recursiveMax***
  - T(n)-Running time of algorithm on an input size n

$$T(n) = \begin{cases} 3 & \text{if } n=1 \\ T(n-1) + 7 & \text{otherwise} \end{cases}$$

```
Algorithm recursiveMax(A,n)
// Input : An array A storing  n>=1 ir
  //Output: The maximum element ir
 if  n = 1 then
 return A[0]
return max{ recursiveMax(A,n-1),A
```

Analyzing Recursive Algorithms-Example **1:-Factorial of a number**

– ***Algorithm fact(n)***

//Purpose: Computes factorial of n

//Input: A positive integer n

//Output: factorial of n

If(n=0)

return 1

return n*fact(n-1)

Analyzing Recursive Algorithms-Example **1:-Factorial of a number**

- **Analysis**
  - Parameter to be considered –n
  - Basic operation –Multiplication
  - T(n) = 0 if n=0

    1+T(n-1) Otherwise

Time taken to compute fact(n-1)

Time to multiply n*fact(n-1)

Analyzing Recursive Algorithms-Example **1:-Factorial of a number**

- – **Solve the recurrence**

  $T(n) = T(n\text{-}1) + 1$

  $[T(n\text{-}2) + 1] + 1 = \mathrm{T}(n\text{-}2) + 2$ substituted $T(n\text{-}2)$ for $T(n\text{-}1)$

  $[T(n\text{-}3) + 1] + 2 = T(n\text{-}3) + 3$ substituted $T(n\text{-}3)$ for $T(n\text{-}2)$

  .. a pattern evolves

  $\mathrm{T(n)} = 1+\mathrm{T(n\text{-}1)}$

  $=2+\mathrm{T(n\text{-}2)}$

  $=3+\mathrm{T(n\text{-}3)}$

  $=\ldots.$

  $=\mathrm{i}+\mathrm{T(n\text{-}i)}$

  When n=0 T(0)=0,No multiplications

  When i=n, T(n) $=\mathrm{n}+\mathrm{T(n\text{-}n)}$

  $=\mathrm{n}+0$

  $=\mathrm{n}$ $\underline{\mathbf{T(n)} \in \Theta(\mathbf{n})}$

Analyzing Recursive Algorithms- Example **2:-Tower of hanoi**

Step 1 − Move n-1 disks from **source** to **temp**

Step 2 − Move n^{th} disk from **source** to **dest**

Step 3 − Move n-1 disks from **temp** to **dest**

Algorithm Hanoi(n, source, dest, temp)

//Input: n :number of disks

//Output :All n disks on dest

If disk = 1

move disk from source to dest

Hanoi(n - 1, source, temp, dest) // Step 1

move nth disk from source to dest // Step 2

Hanoi(n - 1, temp, dest, source) // Step 3

Analyzing Recursive Algorithms- Example **2:-Tower of hanoi**

1.Problem size is n, the number of discs

2. The basic operation is moving a disc from rod to another

3. Base case $M(1) = 1$

4. Recursive relation for moving n discs

$$M(n) = M(n\text{-}1) + 1 + M(n\text{-}1) = 2M(n\text{-}1) + 1$$

Analyzing Recursive Algorithms- Example **2: Tower of hanoi**

Solve using backward substitution

$M(n) \quad = \; 2M(n\text{-}1) + 1$

$\quad = 2[2M(n\text{-}2) + 1] +1 = 2^2M(n\text{-}2) + 2+1$

$\quad =2^2[2M(n\text{-}3) +1] + 2+1$

$\quad = 2^3M(n\text{-}3) + 2^2 + 2 + 1$

$\quad \ldots$

$M(n) \quad = 2^iM(n\text{-}i) + \text{2^i-1+2^i-2+….+2^3+2^2+2^1+2^0}$

$M(n) \quad = 2^iM(n\text{-}i) + \; (\text{2^i-1})/(2\text{-}1)$ It’s a GP with a=1,r=2,n=i

$\quad = 2^iM(n\text{-}i) + \text{2^i-1}$

Analyzing Recursive Algorithms- Example **2:- Tower of hanoi**

- When i=n-1

$$M(n) = 2^{n-1} M(n-(n-1)) + 2^{n-1} - 1$$
$$= 2^{n-1} M(1) + 2^{n-1} - 1$$
$$= 2^{n-1} + 2^{n-1} - 1$$
$$= 2*2^{n-1} - 1$$
$$= 2*(2^{n} / 2) - 1$$
$$= 2^{n} - 1$$

$M(n) \in O(2^n)$

- Time complexity is exponential
- More computattions even for smaller value of n
- Doesnt necessarily mean algorithm is poor
- Nature of the problem itself is computationally expensive.

Analyzing Recursive Algorithms-Example **3:Exercise**

innovate achieve lead

ALGORITHM *BinRec(n)*

//Input: A positive decimal integer *n*

//Output: The number of binary digits in *n*'s binary representation

if *n* = 1 **return** 1

else return *BinRec(n/*2*)* + 1

The number of additions made in computing *BinRec(n/*2*)* is *T(n/*2*)*, plus one more addition is made by the algorithm to increase the returned value by 1. This leads to the recurrence

$$T(n) = \begin{cases} 0 & \text{if } n=1 \\ T(n/2)+1 & \text{otherwise} \end{cases}$$

Analyzing Recursive Algorithms-Example **3:Exercise**

Base condition $T(1)= 0$

$$T(n)=T(n/2)+1$$

- The standard approach to solving such a recurrence is to solve it only for $n = 2^k$
- ***Smoothness rule:*** the order of growth observed for $n = 2^k$ gives a correct answer about the order of growth for all values of n.

(S7)

$$T(2^{k-1}) = T(2^{(k-1)-1}) + 1$$
$$= T(2^{k-2}) + 1$$

Let $n = 2^k$

$$\therefore T(2^k) = T\left(\frac{2^k}{2}\right) + 1$$
$$= T(2^{k-1}) + 1$$
$$= T(2^{k-2}) + 1 + 1$$
$$= T(2^{k-3}) + 3$$
$$= \dots$$
$$= T(2^{k-i}) + i$$

Substitute the initial condition, $T(1) = 0$

to get $T(1)$, i shud be k

so that $T(2^{k-k}) = T(2^0) = T(1)$

$$= T(2^{k-k}) + k$$

$$T(2^k) = T(1) + k = \underline{\underline{k}}$$
$$= \underline{\underline{\log_2 n}}$$

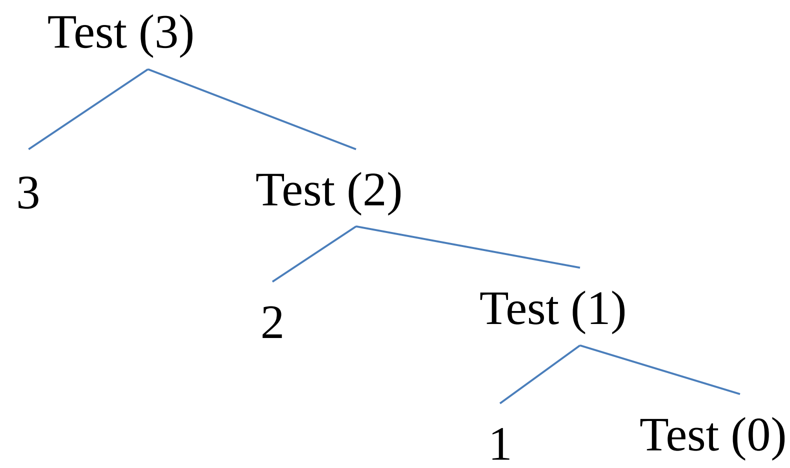
$n = 2^k$

$\underline{\underline{k = \log_2 n}}$

$$T(n) \in \Theta(\log n)$$

Solving recurrences : HW

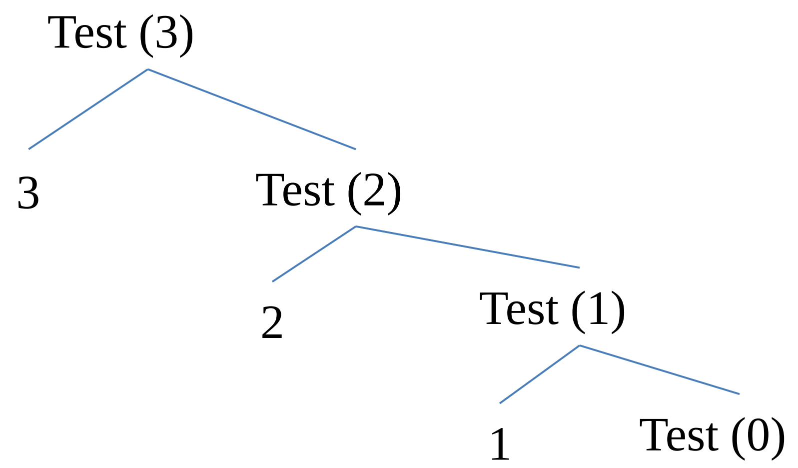
```
void test(int n)
{
        if(n>0)
        {
        printf(“%d”,n);
        test(n-1);
        }
}
```



Solving recurrences

- $$T(n) = \begin{cases} 0, & n = 0 \\ T(n-1) + 1 & n > 0 \end{cases}$$

Solve and discuss in Canvas



Solving recurrences : HW

```
Void Test (int n) ---------------
        {
        if(n>1)
        {
        for (i=1;i<n;i++)
        {
        stmt;
        }
        Test(n/2);
        Test(n/2);
        }
        }
```

$$T(n)=\begin{cases} 0, & n=1 \\ 2T(n/2)+n & n>1 \end{cases}$$

Solving recurrences : Substitution Method (Self reading)

Solving recurrences : Substitution Method Ref:Textbook R2

The most general method

- ***Guess*** the form of the solution.
- ***Use mathematical induction*** to find the constants and show that the solution works.

- ***We must be able to guess the form of the answer in order to apply it.***

Solving recurrences : Substitution Method Ref:Textbook R2

Solve T(n) = 2T(n/2) + n using substitution

– Guess $T(n) \leq cn \log n$ for some constant c

(that is, $T(n) = O(n \log n)$)

– Proof:

$T(n) \leq cn \log n$

$T(n) = 2T(n/2) + n$

$\leq 2(c\, n/2 \log n/2) + n$

$= cn \log n/2 + n$

$= cn \log n - cn \log 2 + n$

$= cn \log n - cn + n$

$= cn \log n - (cn - n)$ <=**cnlogn**

Solving recurrences : Substitution Method Ref:Textbook R2

- **Solve $T(n)=2T(\sqrt{n}) + \log n$**
  - Assume $n=2^m$, $m=\log n$
  - $T(2^m) = S(m)$
- **Show that the solution of $T(n)=T(n-1) + n$ is $O(n^2)$**
  - $T(n) <= cn^2$
  - $T(n)=T(n-1) + n$

$$<= c(n-1)^2 + n$$
$$<= c(n^2 - 2n+1) + n$$
$$<= cn^2 - 2cn + c + n$$
$$<= cn^2 - 2cn + c + n$$
$$<= cn^2 - (2cn - c - n)$$
$$<= cn^2 \quad \textbf{ie. } \mathbf{T(n) \text{ is } O(n^2)}$$

$$2cn - c - n >= 0$$
$$c(2n-1) >= n$$
$$c >= n/(2n-1)$$

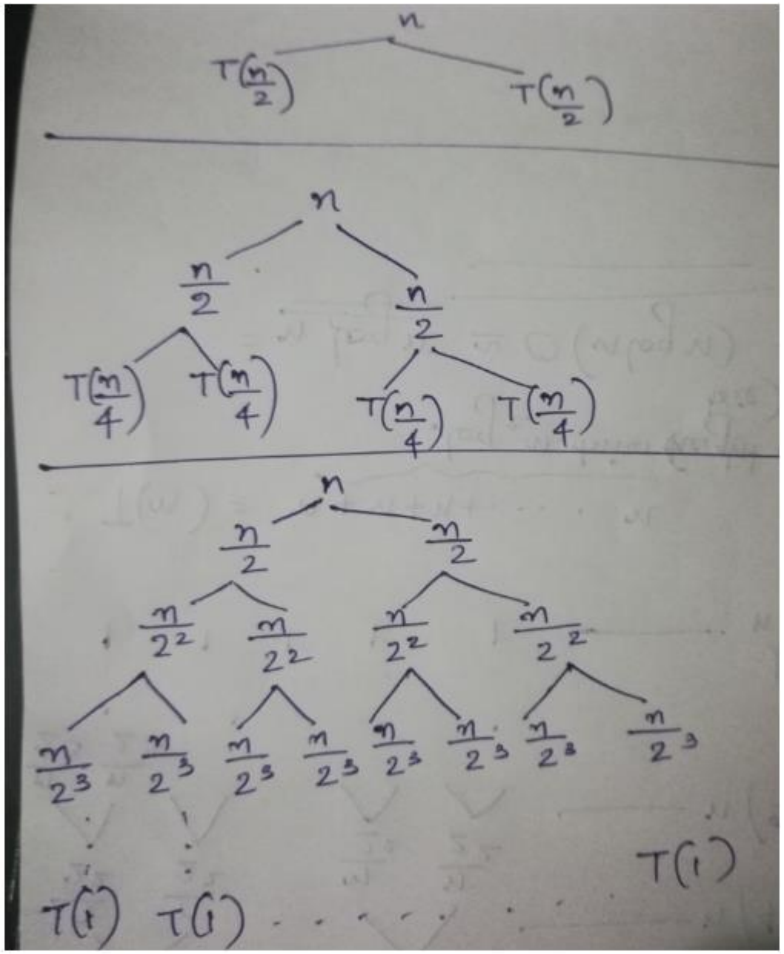
Solving recurrences : Recursion tree Method

Solving recurrences : Recursion Tree

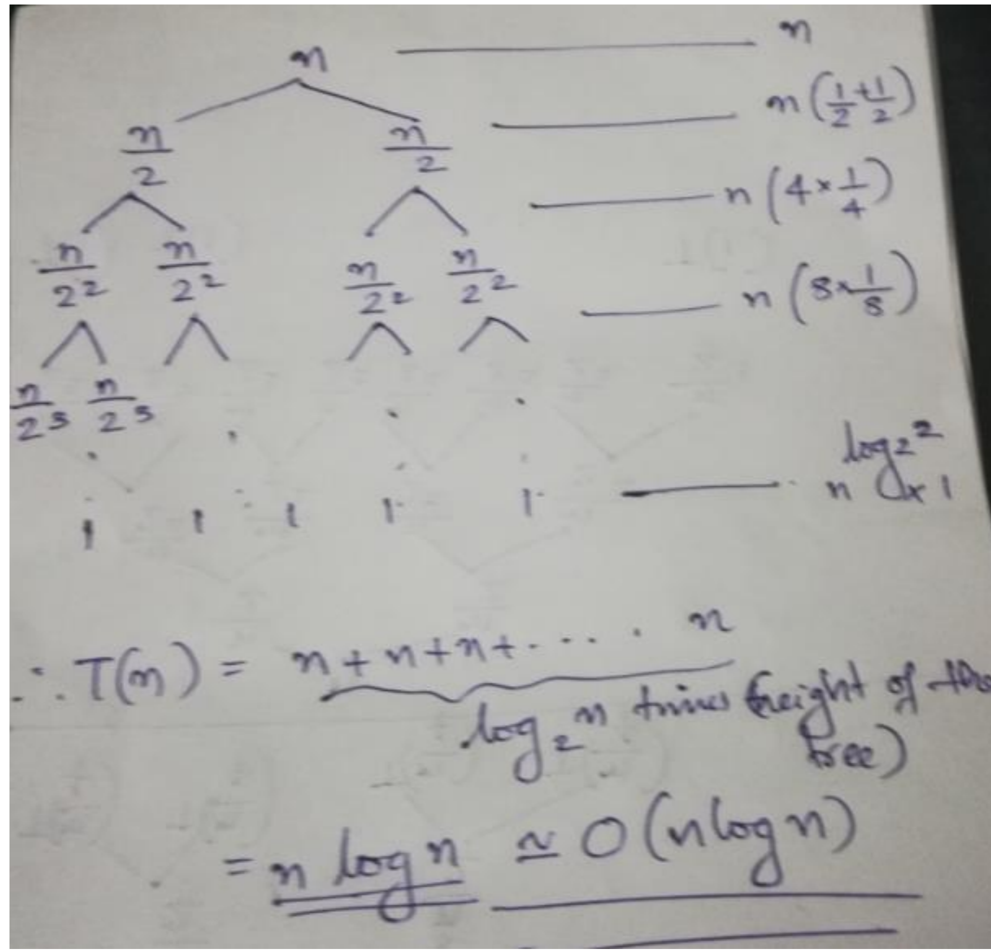
```
Void Test (int n) ---------------
        {
        if(n>1)
        {
                for (i=1;i<n;i++)
                {
                        stmt;
                }
                Test(n/2);        ----------
                Test(n/2);        ----------
        }
        }
```

$T(n)= 2T(n/2)+n$

Solution



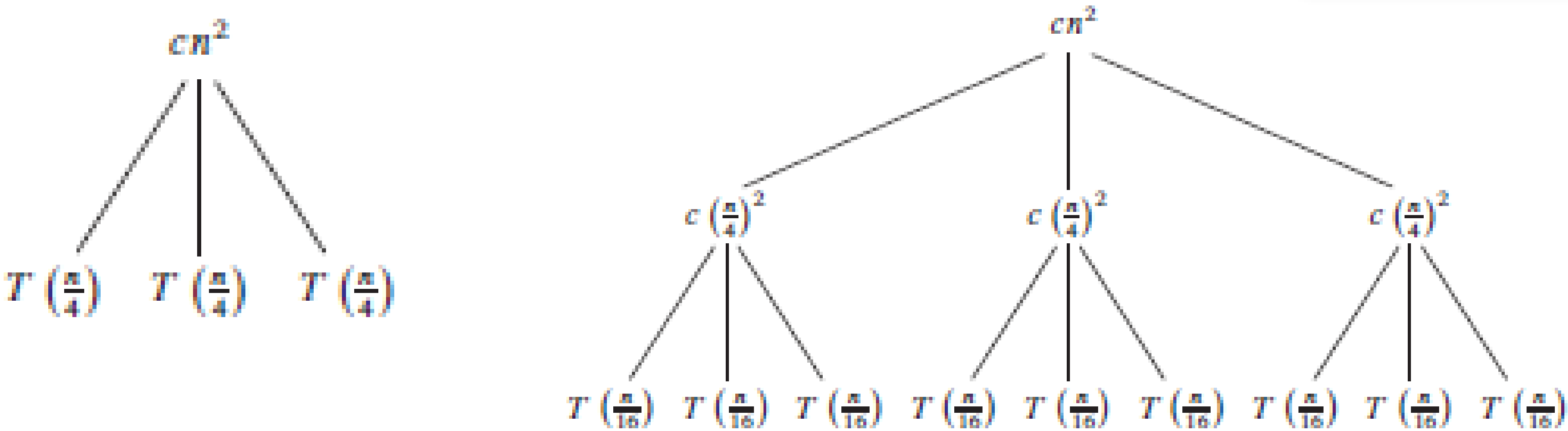
n
T(n/2) T(n/2)
n
n/2 n/2
T(n/4) T(n/4) T(n/4) T(n/4)
n
n/2 n/2
n/2^2 n/2^2 n/2^2 n/2^2
n/2^3 n/2^3 n/2^3 n/2^3 n/2^3 n/2^3 n/2^3 n/2^3
T(1) T(1) . . . . . T(1)



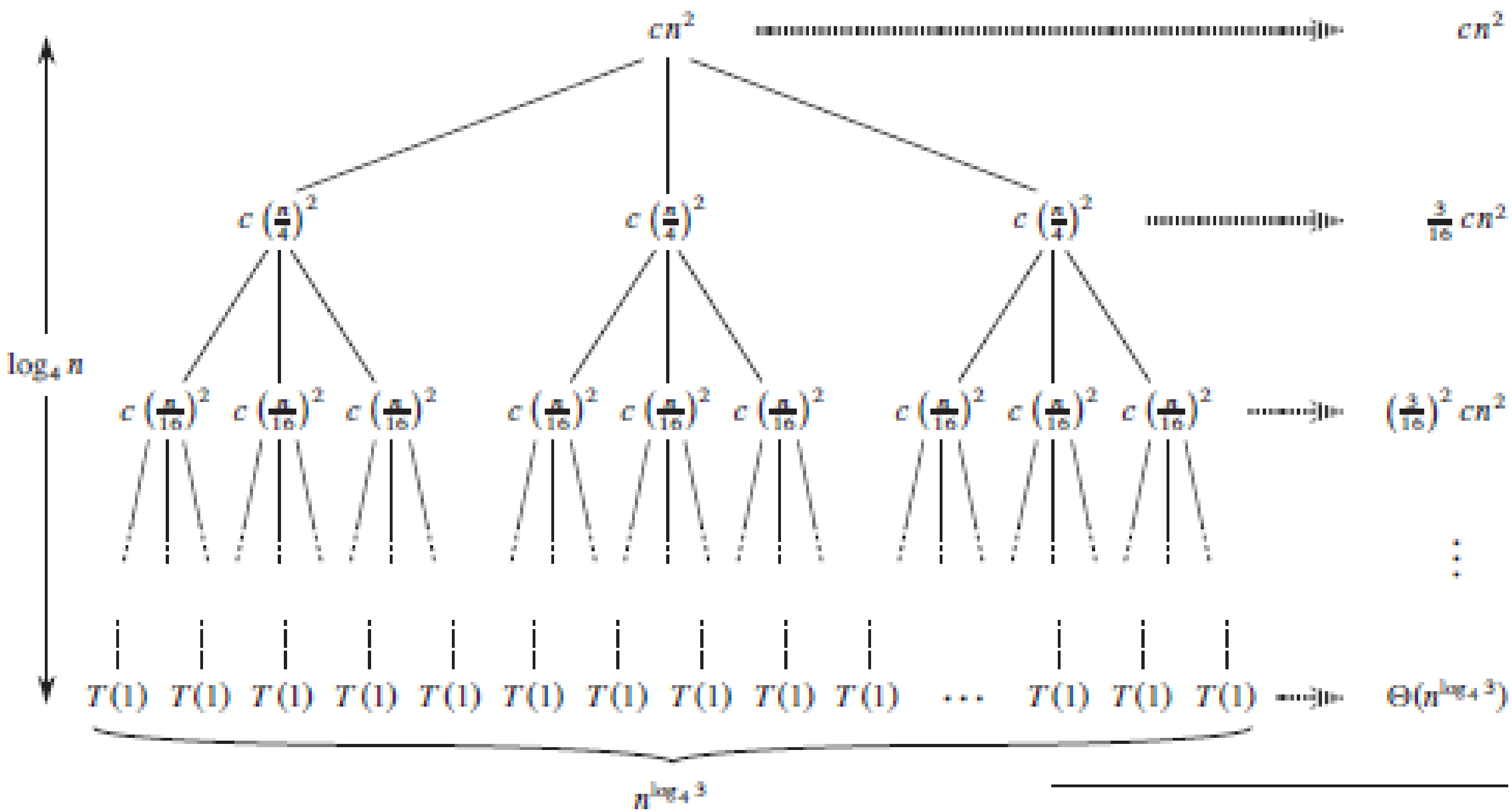
n
n/2 n/2
n/2^2 n/2^2 n/2^2 n/2^2
n/2^3 n/2^3
1 1 1 1 1
n
n(1/2 + 1/2)
n(4 × 1/4)
n(8 × 1/8)
n 2^(log_2 n) × 1
∴ T(n) = n + n + n + ... n
log_2 n times (height of the tree)
= n log n ≈ O(n log n)

Solving recurrences : Recursion tree

- Solve $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$. ie. $T(n) = 3T(n/4) + cn^2$.



Solving recurrences : Recursion tree



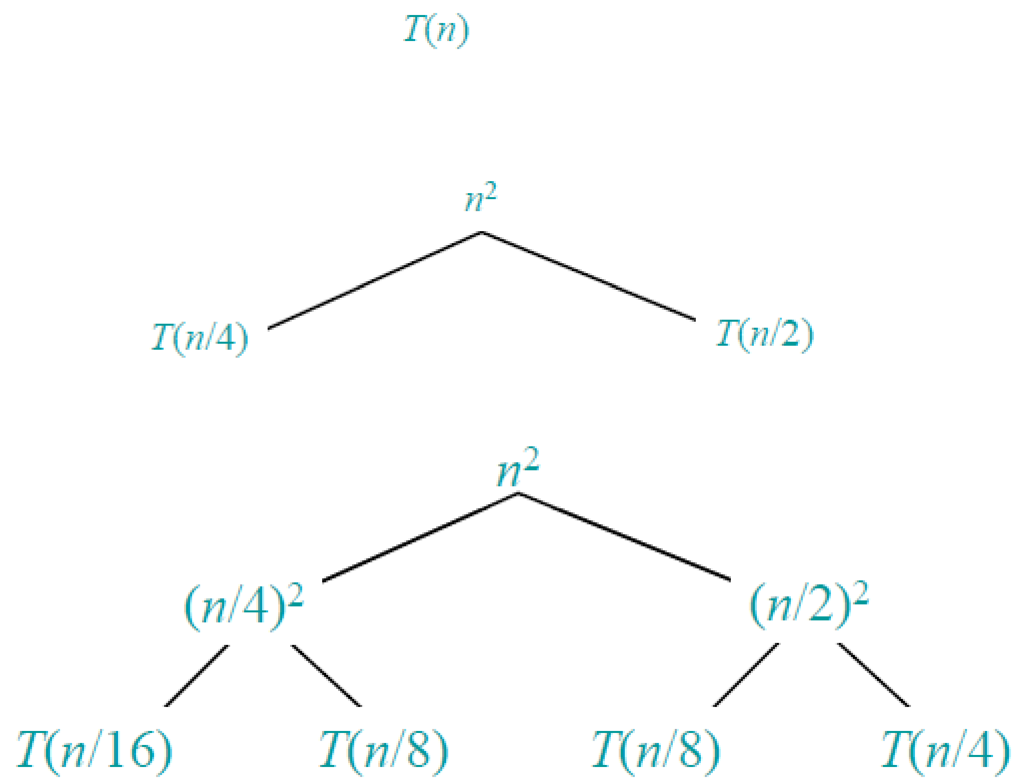
Solving recurrences : Recursion tree

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})
\end{aligned}
$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1 - (3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2) .
\end{aligned}
$$

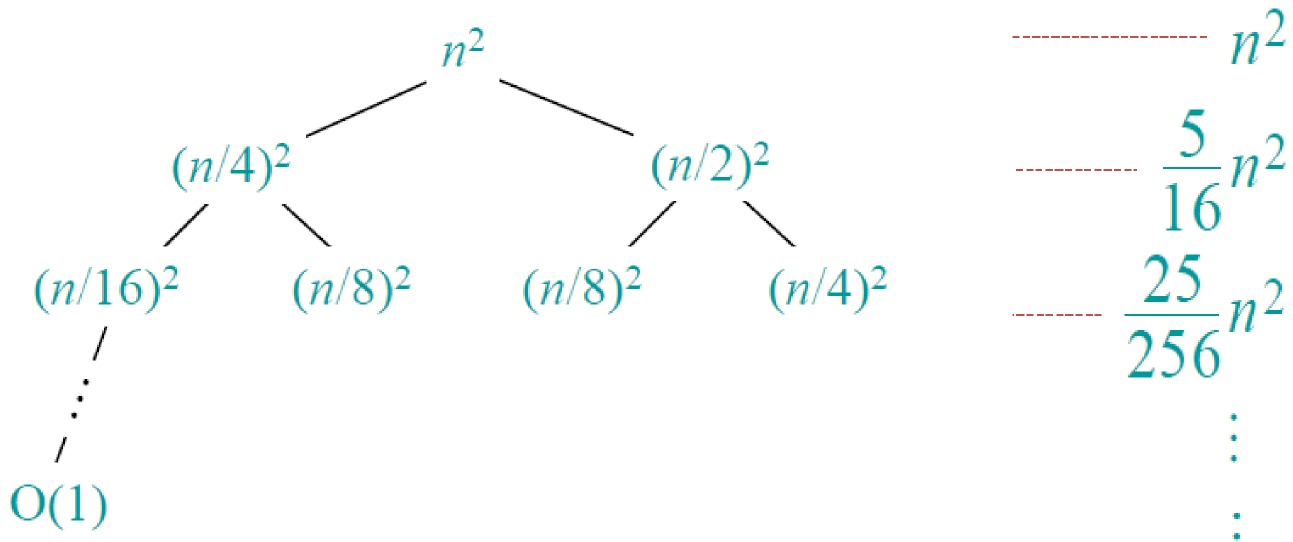
Solving recurrences : Recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Solving recurrences : Recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= O(n^2)$$ *geometric series*

Solving recurrences : Master Method

Solving recurrences: Master method Ref: Textbook R2

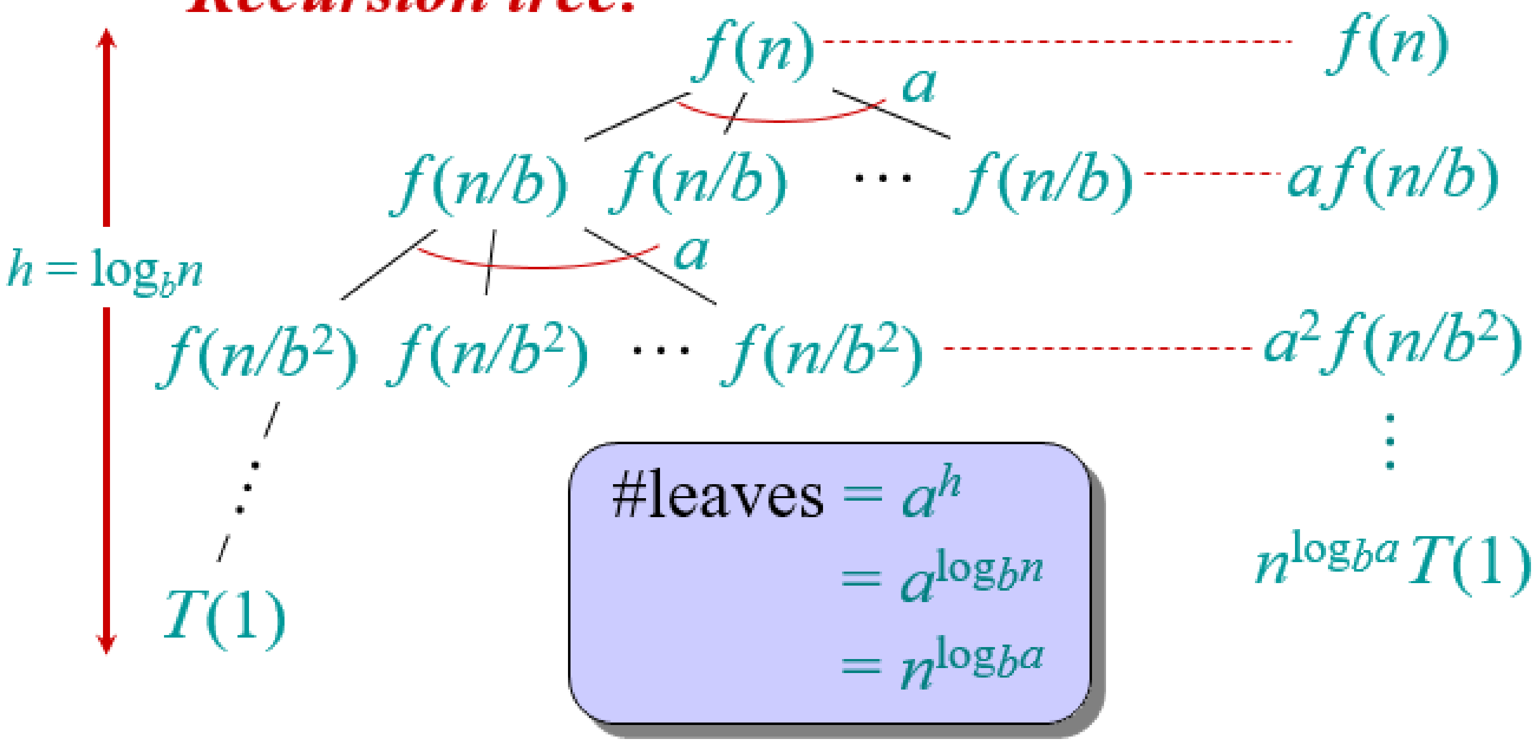
- The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

- where $a \geq 1$, $b > 1$, and f is asymptotically positive.
- (f(n)>0 for n>=n0)

Idea of Master theorem

Recursion tree:



Solving recurrences: Master method
Ref: Textbook R2

Case 1:

If $f(n)=O(n^{\log_b a-\varepsilon})$, for some constant $\varepsilon > 0$, then $T(n)= \Theta(n^{\log_b a})$

f(n) grows polynomially slower than $n^{\log_b a}$

Case 2:

If $f(n)=\Theta(n^{\log_b a})$, then $T(n)= \Theta(n^{\log_b a} \log n)$

f(n) and $n^{\log_b a}$ grows at similar rates

Case 3:

*If $f(n)=\Omega(n^{\log_b a+\varepsilon})$ for some constant $\varepsilon > 0$, and if **$af(n/b)<=cf(n)$** for some constant $c < 1$ and all sufficiently large n, then $T(n)= \Theta(f(n))$*

f(n) grows polynomially faster than $n^{\log_b a}$

Solving recurrences: Master method
Ref: Textbook R2

Case 2 : (Generalisation):

If there is a constant k >= 0, such that f(n) is $\Theta(n^{\log_b a} \log^k n)$, then T (n) is $\Theta(n^{\log_b a} \log^{k+1} n)$

Example:

$T(n)= 2T(n/2) + n \log n$

$a=2,b=2$ $f(n)=n\log n$

$n^{\log_b a} = n$

f(n) is asymptotically larger than $n^{\log_b a}$. But it is not polynomially larger.

So no standard case of master theorem applies.

It belongs to case 2 general case.

$f(n) = \Theta(n^{\log_b a}) \log^k n) = \Theta(n^{\log_b a} \log^1 n)$

So $T(n)= \Theta(n \log^2 n)$

Solving recurrences: Master method

Example 1 : $T(n) = 2T(n/2) + n$

Sol:

Extract a=2, b=2 and f(n) =n

Determine $n^{\log_b a} = n^{\log_2 2} = n^1 = n$

Compare $n^{\log_b a} = n$

$f(n) = n$

Thus case 2: evenly distributed because

$$f(n) = \theta(n)$$

$$T(n) = \theta\,(n^{\log_b a} \log(n))$$

$$= \theta\,(n^1 \log(n))$$

$$= \theta\,(n\log n)$$

Solving recurrences: Master method

Example 2 : $T(n)=9T(n/3)+n$

$a= 9$ $b=3$ and $f(n) =n$

Determine $n^{\log_b a} = n^{\log_3 9} = n^2$

Compare: $n^{\log_b a} = n^2$

$f(n) = n$

Thus case1; (express $f(n)$ in terms of $n^{\log_b a}$) because $f(n)= O(n^{2-\varepsilon})$

$T(n)= \theta\ (n^{\log_b a}) = \theta(n^2)$

Solving recurrences: Master method

- **Example 3 :T(n) = 3T(n/4) + nlogn**

a= 3,

b=4,

f(n) = nlogn

Determine; $n^{\log_b a} = n^{\log_4 3}$ $\log_4 3 < 1$

Compare: $n^{\log_b a}$ and f(n)

$n^{\log_4 3} <= n\log n$ f(n) is asymptotically and polynomially larger

Thus case 3, but we have to check the reqularity condition!

The following should be true:

af(n/b) < = cf(n) where c<1

a(n/b) log (n/b) < = cf(n)

=> 3(n/4) log(n/4) <=c n log n

3/4nlog(n/4) <= c .n log n,

this is true for c=3/4 Hence.T(n) = $\theta\ (n\log(n))$

Master method Problems

- $T(n)=9T(n/3)+n$
- $T(n)=T(2n/3)+1$
- $T(n)=3T(n/4)+n\log n$
- $T(n)=2T(n/2)+n \lg n$
- $T(n)=8T(n/2)+\Theta(n^2)$

Case Study: Analyzing Algorithms

- **Computing the prefix averages of a sequence of numbers.**

  The ***i***-th prefix average of an array ***X*** is average of the first

  (***i*** + 1) elements of ***X*:**

  $\boldsymbol{A}[\boldsymbol{i}] = (\boldsymbol{X}[0] + \boldsymbol{X}[1] + \ldots + \boldsymbol{X}[\boldsymbol{i}])/(i+1)$

- **Applications**
- **Runtime analysis example:**

  Two algorithms for prefix averages

Case Study: Analyzing Algorithms

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm ***prefixAverages1***(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$A[i] \leftarrow s / (i + 1)$	n
return A	1

Algorithm ***prefixAverages1*** runs in $O(n^2)$ time

Case Study: Analyzing Algorithms

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\quad s \leftarrow s + X[i]$	n
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

Algorithm ***prefixAverages2*** runs in $O(n)$ time

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THANK YOU!