BITS Pilani
Pilani Campus

Mathematical Foundations for Data Science

MFDS Team

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DSECL ZC416, MFDS

Lecture No.1

Agenda

- Matrices
- Determinants
- Gauss Elimination
- Consistency of Linear Systems
- Gauss Jordan Method

Matrices

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A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets. For example,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

(1)

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

The numbers (or functions) are called **entries** or, less commonly, *elements* of the matrix.
The first matrix in (1) has two **rows**, which are the horizontal lines of entries.

Matrix – Notations

We shall denote matrices by capital boldface letters **A**, **B**, **C**, … , or by writing the general entry in brackets; thus **A** = $[a_{jk}]$, and so on.

By an ***m*** × ***n*** **matrix** (read *m by n matrix*) we mean a matrix with *m* rows and *n* columns—rows always come first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

(2)

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Vectors

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector.

We shall denote vectors by ***lowercase*** boldface letters **a**, **b**, … or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on.

Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{a} = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

A **column vector**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Equality of Matrices

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if (1) they have the same size and (2) the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.

Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

Algebra of Matrices

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$.

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}$.

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.

Matrix Multiplication

Multiplication of a Matrix by a Matrix

The **product C** = **AB** (in this order) of an $m \times n$ matrix **A** = $[a_{jk}]$ times an $r \times p$ matrix **B** = $[b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix **C** = $[c_{jk}]$ with entries

(1) $$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \qquad \begin{aligned} j &= 1, \cdots, m \\ k &= 1, \cdots, p. \end{aligned}$$

The condition $r = n$ means that the second factor, **B**, must have as many rows as the first factor has columns, namely n. A diagram of sizes that shows when matrix multiplication is possible is as follows:

$$\begin{array}{cccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{array}$$

Matrix Multiplication

EXAMPLE 1

Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined.

Matrix Multiplication

Matrix Multiplication Is *Not Commutative*, AB ≠ BA in General

This is illustrated by Example 1, where one of the two products is not even defined. But it also holds for square matrices. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

It is interesting that this also shows that $\mathbf{AB} = \mathbf{0}$ does ***not*** necessarily imply $\mathbf{BA} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

Transposition of Matrices & Vectors

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$(9) \qquad \mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

As a special case, transposition converts row vectors to column vectors and conversely.

Transposition of Matrices

Rules for transposition are

(10)

$$\text{(a)} \quad (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$$

$$\text{(b)} \quad (\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$$

$$\text{(c)} \quad (c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}}$$

$$\text{(d)} \quad (\mathbf{A}\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.$$

CAUTION! Note that in (10d) the transposed matrices are ***in reversed order.***

Special Matrices

- Symmetric: $a_{ij} = a_{ji}$ Eg: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$

- Skew Symmetric : $a_{ij} = - a_{ji}$ Eg: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

- Triangular: Upper Triangular → $a_{ij} = 0$ for all $i > j$

  Lower Triangular → $a_{ij} = 0$ for all $i < j$

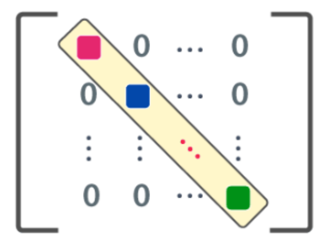
Upper triangular matrix: U

$$\begin{bmatrix} 1 & 1/2 & 3 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Lower triangular matrix: L

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

- Diagonal Matrix: $a_{ij} = 0$ for all $i \neq j$ Eg:



- Sparse Matrix: Many zeroes and few non-zero entities

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

Determinant

Determinant – A function from the set of n x n matrices to complex numbers

Example

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Properties of Determinants

Properties of Determinants:

1. det(AB) =det(A) * det (B)
2. det(A) nonzero implies there exists a matrix B such that AB=BA=I
3. Two Rows Equal➔ det = 0(Singular)
4. R_i and R_j swapped ➔ det gets a minus sign ($i \neq j$)
5. det(A) = det(A^T)
6. $R_i \leftarrow cR_j$ ➔det A $\leftarrow$c detA

Orthogonal matrices: A such that $A^T = A^{-1}$ and det(A) = -1 or 1 $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Cramer's Rule

- $Ax = b$ for square matrices A
- System should be consistent
- $X_i = \dfrac{\det [\, a_1, a_2, \ldots\ldots\ldots a_{i-1}, b, a_{i+1}, \ldots\ldots a_n]}{\det (A)}$ where $\det(A) \neq 0$

Example: $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ $\det(A) = 1$, $b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$x_1 = \dfrac{\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}}{1}$ $\qquad$ $x_2 = \dfrac{\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}}{1}$

$x_1 = 1$ $\qquad$ $x_2 = 1$

Linear System, Coefficient Matrix, Augmented Matrix

A **linear system of m equations in n unknowns** $x_1, \ldots, x_n$ is a set of equations of the form

(1)

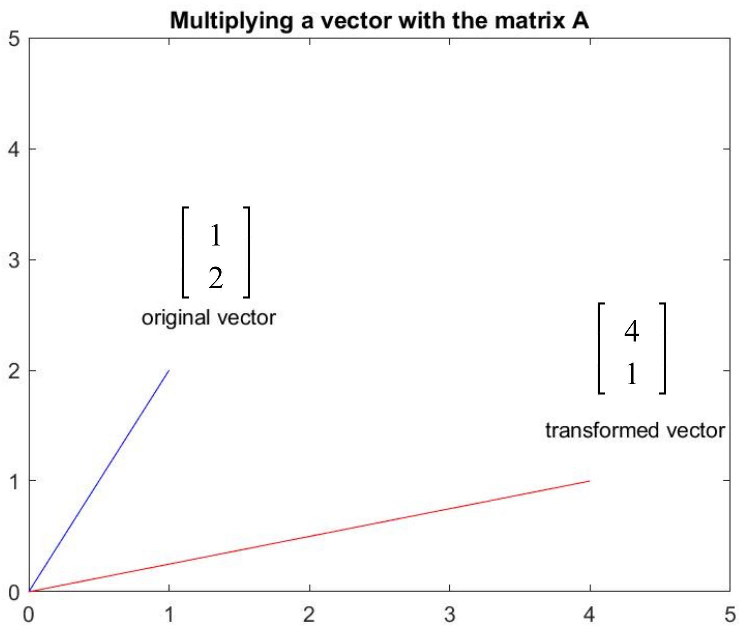
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line. $a_{11}, \ldots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \ldots, b_m$ on the right are also given numbers. If all the *bj* are zero, then (1) is called a **homogeneous system**. If at least one *bj* is not zero, then (1) is called a **nonhomogeneous system**.

Geometrical Interpretation

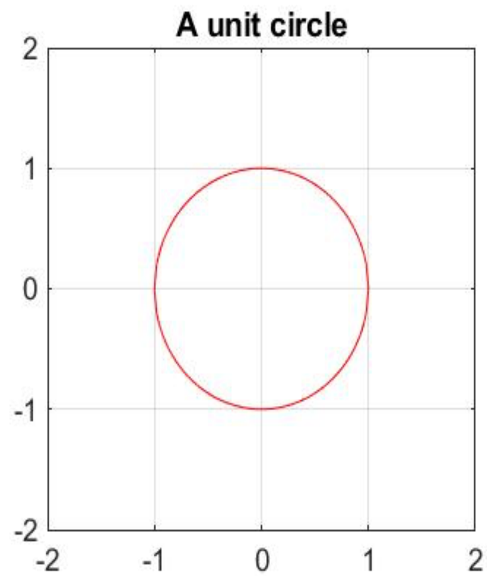
- Matrix Vector Product: Rotation and Stretching/ Contraction
- Multiplying by alpha produces Stretching $\alpha > 1$ and contraction for $\alpha < 1$
- Multiplication by an orthogonal matrix does only rotation
- In general a matrix is a combination of rotation and stretching

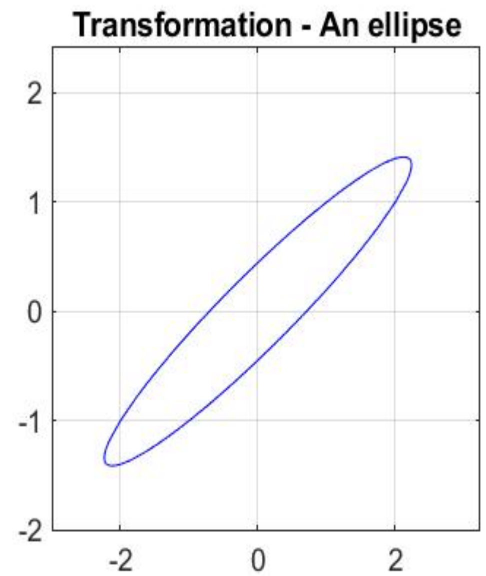
Geometrical Interpretation



$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$





Elementary Row Operations

Elementary Operations for Equations:

Interchange of two equations
Addition of a constant multiple of one equation to another equation
Multiplication of an equation by a ***nonzero*** *constant c*

Clearly, the interchange of two equations does not alter the solution set. Neither does their addition because we can undo it by a corresponding subtraction. Similarly for their multiplication, which we can undo by multiplying the new equation by $1/c$ (since $c \neq 0$), producing the original equation.

Elementary Row Operations

We now call a linear system S_1 **row-equivalent** to a linear system S_2 if S_1 can be obtained from S_2 by (finitely many!) row operations. This justifies Gauss elimination and establishes the following result.

Theorem 1

Row-Equivalent Systems

Row-equivalent linear systems have the same set of solutions.

Because of this theorem, systems having the same solution sets are often called *equivalent systems*. But note well that we are dealing with ***row operations***. No column operations on the augmented matrix are permitted in this context because they would generally alter the solution set.

Row Echelon Form(REF)

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At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix, and the system itself are called the **row echelon form**.

In it, rows of zeros, if present, are the last rows, and, in each nonzero row, the leftmost nonzero entry is farther to the right than in the previous row. For instance, in Example 5 the coefficient matrix and its augmented in row echelon form are

$$(8)\qquad \begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

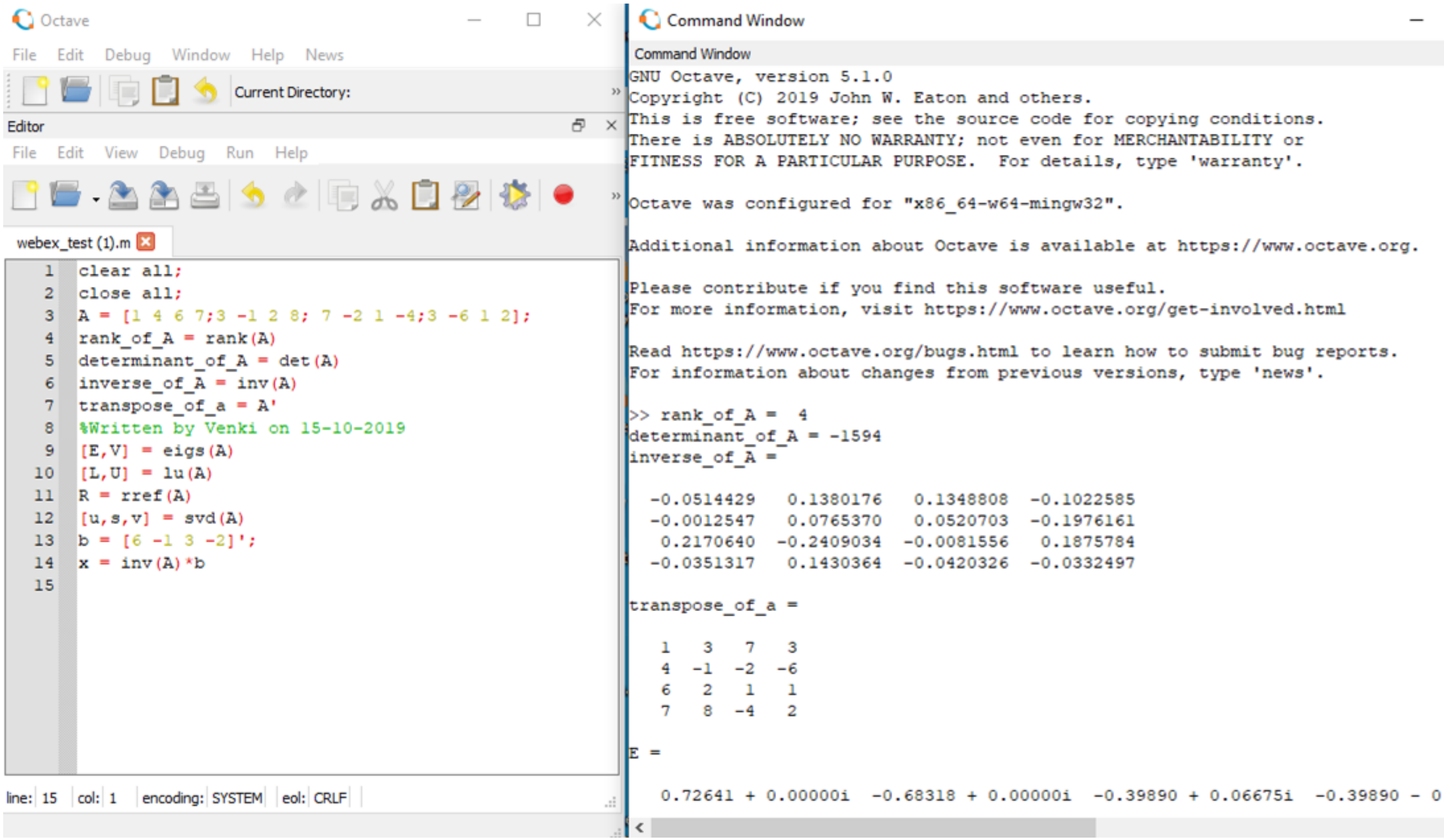
Note that we do not require that the leftmost nonzero entries be 1 since this would have no theoretic or numeric advantage.

Row Echelon Form and Information from it

The original system of m equations in n unknowns has augmented matrix $[\mathbf{A} \mid \mathbf{b}]$. This is to be row reduced to matrix $[\mathbf{R} \mid \mathbf{f}]$.

The two systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Rx} = \mathbf{f}$ are equivalent: if either one has a solution, so does the other, and the solutions are identical.

Matlab/Octave Commands



Linear System

Matrix Form of the Linear System (1).
From the definition of matrix multiplication we see that the m equations of (1) may be written as a single vector equation

(2) $$\mathbf{Ax} = \mathbf{b}$$

where the **coefficient matrix A** = $[a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors.

Matrix Form of Linear System

Matrix Form of the Linear System (1). (continued)

We assume that the coefficients a_{jk} are not all zero, so that **A** is not a zero matrix. Note that **x** has n components, whereas **b** has m components. The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1). The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix **A** but came from vector **b**. Thus, we *augmented* the matrix **A**.

Gauss Elimination and Back Substitution

Triangular form:
Triangular means that all the nonzero entries of the corresponding coefficient matrix lie above the diagonal and form an upside-down 90° triangle. Then we can solve the system by **back substitution.**

Since a linear system is completely determined by its augmented matrix, ***Gauss elimination can be done by merely considering the matrices***.

(We do this again in the next example, emphasizing the matrices by writing them first and the equations behind them, just as a help in order not to lose track.)

Gauss Elimination

Solve the linear system

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= 0 \\ -x_1 + \quad x_2 - \quad x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 \qquad &= 80. \end{aligned}$$

Solution by Gauss Elimination.
This system could be solved rather quickly by noticing its particular form. But this is not the point. The point is that the Gauss elimination is systematic and will work in general, also for large systems. We apply it to our system and then do back substitution.

Example – Gauss Elimination

Solution by Gauss Elimination. (continued)
As indicated, let us write the augmented matrix of the system first and then the system itself:

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1

Eliminate

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Equations

Pivot 1

Eliminate

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

Gauss Elimination

Solution by Gauss Elimination. ***(continued)***

Step 1. Elimination of x_1

Call the first row of **A** the **pivot row** and the first equation the **pivot equation**. Call the coefficient 1 of its x_1-term the **pivot** in this step. Use this equation to eliminate x_1 (get rid of x_1) in the other equations. For this, do:

Add 1 times the pivot equation to the second equation.

Add −20 times the pivot equation to the fourth equation.

This corresponds to **row operations** on the augmented matrix as indicated in BLUE behind the ***new matrix*** in (3). So the operations are performed on the ***preceding matrix***.

Gauss Elimination

Solution by Gauss Elimination. (continued)
Step 1. Elimination of x_1 **(continued)**
The result is

$$(3)\quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \qquad \begin{array}{l} \\ \text{Row 2} + \text{Row 1} \\ \\ \text{Row 4} - 20\ \text{Row 1} \end{array} \qquad \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 0 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80. \end{aligned}$$

Gauss Elimination

Solution by Gauss Elimination. (continued)

Step 2. Elimination of x_2

The first equation remains as it is. We want the new second equation to serve as the next pivot equation. But since it has no x_2-term (in fact, it is $0 = 0$), we must first change the order of the equations and the corresponding rows of the new matrix. We put $0 = 0$ at the end and move the third equation and the fourth equation one place up. This is called **partial pivoting** (as opposed to the rarely used *total pivoting*, in which the order of the unknowns is also changed).

Gauss Elimination

Solution by Gauss Elimination. (continued)
Step 2. Elimination of** x_2 **(continued)
It gives

Pivot 10

Eliminate 30

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot 10

Eliminate $30x_2$

$$\begin{aligned} x_1 - \quad x_2 + \quad x_3 &= \ 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80 \\ 0 &= \ 0. \end{aligned}$$

Gauss Elimination

Solution by Gauss Elimination. ***(continued)***

Step 2. Elimination of x_2 **(continued)**

To eliminate x_2, do:

Add –3 times the pivot equation to the third equation.

The result is

$$(4)\quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row 3 – 3 Row 2

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0. \end{aligned}$$

Gauss Elimination

Solution by Gauss Elimination. ***(continued)***

Back Substitution. Determination of $\boldsymbol{x_3, x_2, x_1}$ ***(in this order)***

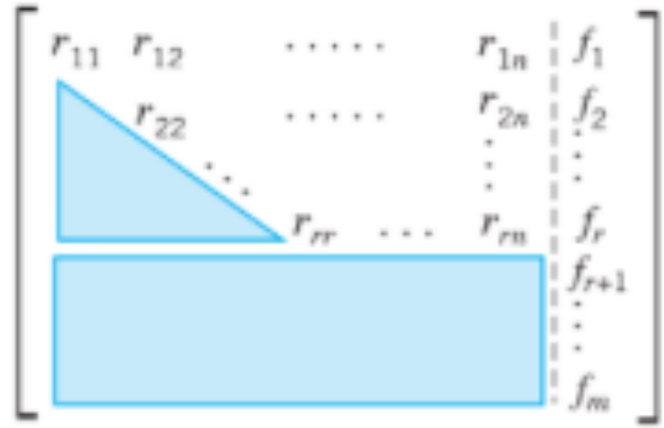
Working backward from the last to the first equation of this "triangular" system (4), we can now readily find x_3, then x_2, and then x_1:

$$\begin{aligned} -95x_3 &= -190 \\ 10x_2 + 25x_3 &= 90 \\ x_1 - x_2 + x_3 &= 0. \end{aligned}$$

This is the answer to our problem. The solution is unique.

Gauss Elimination

At the end of the Gauss elimination (before the back substitution), the row echelon form of the augmented matrix will be



- in upper triangular form
- having the first r rows non-zero
- Exactly n - r rows would be zero rows
- the rhs would also have the last r rows zero
- any one of the r last rows in non-zero would imply inconsistency
- complexity is O(n^3), where n is the number of rows
- facilitates the back substitution

Solution to System of Linear Equations

A linear system (1) is called **overdetermined** if it has more equations than unknowns, as in Example 2, **determined** if
$m = n$, as in Example 1, and **underdetermined** if it has fewer equations than unknowns.

Furthermore, a system (1) is called **consistent** if it has at least one solution (thus, one solution or infinitely many solutions), but **inconsistent** if it has no solutions at all, as
$x_1 + x_2 = 1$, $x_1 + x_2 = 0$ in Example 1, Case (c).

Solution

The number of nonzero rows, r, in the row-reduced coefficient matrix **R** is called the **rank of R** and also the **rank of A**. Here is the method for determining whether **Ax** = **b** has solutions and what they are:

(a) No solution. If r is less than m (meaning that **R** actually has at least one row of all 0s) *and* at least one of the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ is not zero, then the system $\mathbf{Rx} = \mathbf{f}$ is inconsistent: No solution is possible. Therefore the system **Ax** = **b** is inconsistent as well.
See Example 4, where $r = 2 < m = 3$ and $f_{r+1} = f_3 = 12$.

Solution

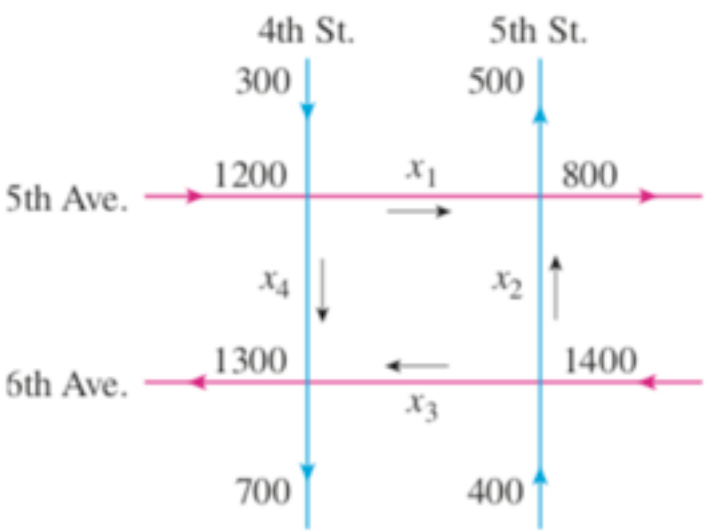
If the system is consistent (either $r = m$, or $r < m$ *and* all the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ are zero), then there are solutions.

(b) Unique solution. If the system is consistent and $r = n$, there is exactly one solution, which can be found by back substitution. See Example 2, where $r = n = 3$ and $m = 4$.

(c) Infinitely many solutions. To obtain any of these solutions, choose values of $x_{r+1}, \ldots, x_n$ arbitrarily. Then solve the rth equation for x_r (in terms of those arbitrary values), then the $(r - 1)$st equation for x_{r-1}, and so on up the line. See Example 3.

Traffic Flow Problem

Traffic Flow



$$x_1 + x_4 = 1500$$
$$x_1 + x_2 = 1300$$
$$x_2 + x_3 = 1800$$
$$x_3 + x_4 = 2000$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1300 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right]$$

After Elementary Row Operations

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Underdetermined System:
Three Linear equations and
Four Unknowns
Infinitely Many solutions

$$\begin{aligned} x_1 &= 1500 - t \\ x_2 &= -200 + t \\ x_3 &= 2000 - t \\ x_4 &= t \end{aligned}$$

Inverse

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} \quad \text{where } \det(A) \neq 0$$

Reiterate det(A) ≠ 0 → A is Non singular

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Gauss Jordan

$[A \mid I] \xrightarrow[\text{(ERO)}]{\text{Elementary Row Operations}} [I \mid A^{-1}]$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2 + 3 Row 1} \\ \text{Row 3 − Row 1} \end{array}$$

Gauss Jordan

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} -\text{Row } 1 \\ 0.5 \text{ Row } 2 \\ -0.2 \text{ Row } 3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Row 1 ->Row 1 + 2 Row 3

Row 2->Row 2 -3.5 Row 3

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \quad \text{Row } 1 + \text{Row } 2$$

Gauss Jordan

$$
= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]
$$

A^{-1}