Homework 1 (Solutions)

Question 1. *Let $A_{m\times n}$ be a given matrix with $m > n$. If the time taken to compute the determinant of a square matrix of size j is j^3, find upper bound on the*

a) total time taken to find the rank of A using determinants.

b) number of additions and multiplications required to determine the rank using the elimination procedure

Ans: a) Since $m > n$,

$$\text{rank}(A) \leq n.$$

Let $k \leq n$. According to the question, the time taken to compute the determinant of a $k \times k$ submatrix of A is k^3. Also, it is easy to see there are atmost $\binom{m}{k} \times \binom{n}{k}$ distinct $k \times k$ submatrices of A. Thus, time taken to compute the determinant of all $k \times k$ submatrices of A is

$$\binom{m}{k} \times \binom{n}{k} \times k^3.$$

Since the rank of A is the largest order of any non-zero minor in A, we have to compute the determinant of all $k \times k$ submatrices of A, where $1 \leq k \leq n$. Therefore, upper bound on the the total time taken to find the rank of A using determinants is

$$\sum_{k=1}^{n} \binom{m}{k} \times \binom{n}{k} \times k^3.$$

b) To make the entries below the digonal entry 0 in the j^{th} column, $(m-j) \times (n-j+1)$ number of multiplications are required. Thus,

$$\sum_{j=1}^{n-1} (m-j) \times (n-j+1)$$

is the upper bound on the number of multiplications. The number of additions is the same as the number of multiplications.

Question 2. *Let $A_{n\times n}$ be a given square matrix. Compute the number of multiplications and additions required to evaluate A^{28} using*

a) the naive method, $A^{28} = \underbrace{A \cdot A \cdots A}_{28 \text{ times}}$

b) A^2, $A^4 = A^2 \cdot A^2$, etc.

Ans: a) To compute the $(i,j)^{\text{th}}$ entry of the matrix $A \cdot A$, we need to multipy the i^{th} row of A with the j^{th} column of A. This operation requires n number of multiplications and $n-1$ number of additions. Thus, for all the n^2 entries of the matrix $A \cdot A$, $n^2 \times n$ multiplications and $n^2 \times (n-1)$ additions are performed.

Note that to compute A^{28}, 27 times matrix multiplication is performed. Thus, total number of multiplication and additions required to evaluate A^{28} is $27 \times n^3$ and $27 \times n^2(n-1)$, respectively.

b) The optimal way to compute A^{28} is by computing the following matrices:

$$A^2 = A \cdot A, \ A^4 = A^2 \cdot A^2, \ A^8 = A^4 \cdot A^4$$

$$A^{16} = A^8 \cdot A^8, \ A^{24} = A^{16} \cdot A^8, \text{ and } A^{28} = A^{24} \cdot A^4$$

A total of 6 matrix multiplication is needed to get A^{28} by using the above approach. Now, by using a), total number of multiplication and additions required to evaluate A^{28} is $6 \times n^3$ and $6 \times n^2(n-1)$, respectively.

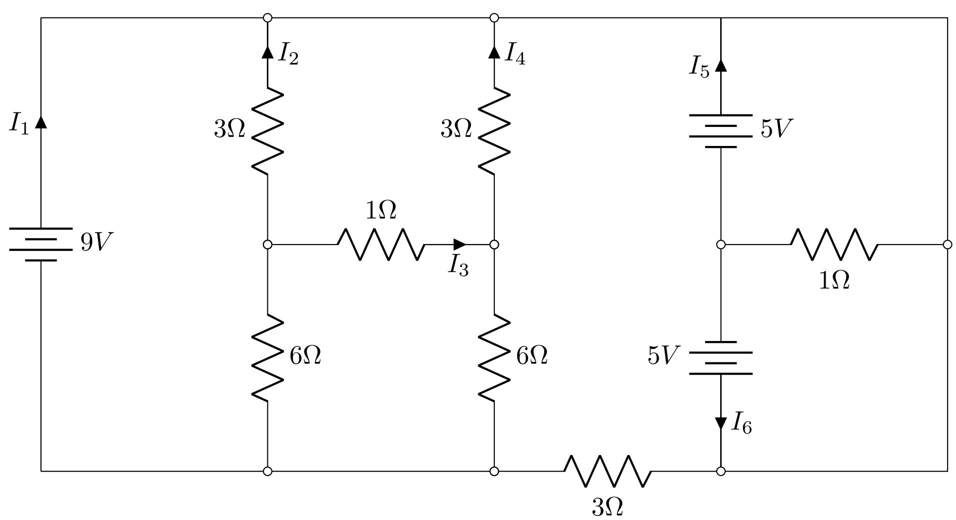
Question 3. *Modelling of electrical / traffic networks would lead to a linear system $Ax = b$. Refer to the text book / other resources and construct a network which has the following properties*

a) the number of equations are 6

b) A has rank 5

c) the system is consistent.

Ans: Consider the following circuit



There are 6 linear equations corresponding to the above circuit

$$3(I_1 - I_2) + 6(I_1 - I_3) = 9$$

$$3(I_2 - I_1) + 3(I_2 - I_4) + (I_2 - I_3) = 0$$
$$6(I_3 - I_1) + (I_3 - I_2) + 6(I_3 - I_4) = 0$$
$$3(I_4 - I_2) + 6(I_4 - I_3) + 3I_4 = 0$$
$$I_5 + I_6 = 5$$
$$I_5 + I_6 = 5$$

The corresponding simplified linear system is:

$$9I_1 - 3I_2 - 6I_3 = 9$$
$$-3I_1 + 7I_2 - I_3 - 3I_4 = 0$$
$$-6I_1 - I_2 + 13I_3 - 6I_4 = 0$$
$$-3I_2 - 6I_3 + 12I_4 = 0$$
$$I_5 + I_6 = 5$$
$$I_5 + I_6 = 5$$

Converting the given equations in augmented matrix form $[A \mid b]$, we have

$$\left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ -3 & -7 & -1 & -3 & 0 & 0 & 0 \\ -6 & -1 & 13 & -6 & 0 & 0 & 0 \\ 0 & -3 & -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

Applying the row reductions $R_2 \to R_2 + \dfrac{1}{3}R_1$, $R_3 \to R_3 + \dfrac{2}{3}R_1$, $R_3 \to R_3 + \dfrac{1}{2}R_2$, $R_4 \to R_4 + \dfrac{1}{2}R_2$, $R_4 \to R_4 + R_3$, and $R_6 \to R_6 - R_5$, we get

$$\left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} & 0 & 0 & \frac{15}{2} \\ 0 & 0 & 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\operatorname{rank}(A) = 5$. Also, for any $t \in \mathbb{R}$, the system has the following solution

$$I_1 = 5,\ I_2 = 4,\ I_3 = 4,\ I_4 = 3,\ I_5 = 5 - t,\ \text{and}\ I_6 = t.$$

Hence, the system is consistent.