Homework -4

Q1 Consider a matrix A and do the following.

i) Enter the matrix A in MATLAB / Octave using
A = [a11 a12 : : : a1n; a21 a22 : : : a2n; _ _ _ ; am1 am2 : : : amn];
for a _xed m and n which are small with m < n (say m = 2 and n = 3).

ii) Evaluate ATA and AAT and _nd their eigenvalues and eigenvectors. You could do it in MATLAB / Octave using the command [E; V] = eigs(B) for a given matrix B. Are the eigenvectors orthogonal? Are they orthonormal?

iii) Use the command [U S V] = svd(A) and compare the values of U and V with the eigenvector matrix obtained in step ii).

iv) Do you observe the decreasing order in which the singular values appear in S.

v) Repeat the above for the case m = n = 3 (say)

vi) Does the eigendecomposition of A in v) coincide with the SVD of A?

vii) Do you see some relationship between eigenvalues and singular values, in case of a square matrix?

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clc
% Homework -4
% Q.1 (i)
%created matrix of order 2 by 3
A=[1 2 3; 4 5 6]

% Q.1 (ii)
B=transpose(A)
%A*AT
P=A*B
%AT*A
Q=B*A
% Eigen values and vectors of A*AT
[EP, V1 ] = eigs(A*B)
% Eigen values and vectors of AT*A
[EQ, V2 ] = eigs(B*A)
% are the eigen vectors orthogonal E'=trans of E
%E1=E*E' % =I implies Matrix  E is ortho and hence vectors
% are the eigen vectors orthonormal
E1=EQ([1 2 3],[1])
E2=EQ([1 2 3],[2])
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E3=EQ([1 2 3],[3])
E1'*E1 %implies E1 is orthogonal
E2'*E2 %implies E2 is orthogonal
E3'*E3 %implies E3 is orthogonal
n1=norm(E1)
n2=norm(E2)
n3=norm(E3)% As norms are 1 the vectors are orthonormal

%Q.1 (iii), (iv)

[U S V ] = svd(B*A) %singular value decompostion
% U and V are coming same as eigen vector matrix (EQ) only order
of columns
% is different as we have decreasing order of singular values of
S



clc
% Homework -4 Part2
% Q.1 (V)
%created matrix of order 3 by 3
A=[1 2 3; 4 5 6;7 8 9]

% Q.1 (ii)of order 3 by 3
B=transpose(A)
%A*AT
P=A*B
%AT*A
Q=B*A
% Eigen values and vectors of A*AT
[EP, V1 ] = eigs(A*B)
% Eigen values and vectors of AT*A
[EQ, V2 ] = eigs(B*A)
% are the eigen vectors orthogonal E'=trans of E
%E1=E*E' % =I implies Matrix  E is ortho and hence vectors
% are the eigen vectors orthonormal
E1=EQ([1 2 3],[1])
E2=EQ([1 2 3],[2])
E3=EQ([1 2 3],[3])
E1'*E1 %implies E1 is orthogonal
E2'*E2 %implies E2 is orthogonal
E3'*E3 %implies E3 is orthogonal
n1=norm(E1)
n2=norm(E2)
n3=norm(E3)% As norms are 1 the vectors are orthonormal
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%Q.1 (iii), (iv)of order 3 by 3

[U S V ] = svd(B*A) %singular value decompostion
% U and V are coming same as eigen vector matrix (EQ) only order
of columns
% is different as we have decreasing order of singular values of
S

[E4, V4 ] = eigs(A)
[U5 S5 V5 ] = svd(A)

% Q.1 (vi)
% eigen decomposition of A in v) is different from the SVD of A?
%V4 not equal to S5 and E4 not equal to V5


% Q.1 (vii)
%There is a very special case in which the singular values of a
matrix are the same as the eigenvalues of a matrix.

%If A is a symmetric matrix, i.e. A=AT, then the singular values
of A are equal to the absolute values of the eigenvalues of A.
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Qus-2 The svd is derived from reduced svd. Refer to any web source and find out what reduced svd means and how svd is obtained from it.

Ans: Let $A \in R^{m\times n}$ with $m \geq n$ and $\text{rank}(A) = n$. Suppose the singular values of A are numbered in the descending order $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. If the sets $\{u_1, u_2, \ldots, u_n\}$, $\{v_1, v_2, \ldots, v_n\}$ are the sets of left and right singular vectors of A, respectively, then

$$Av_i = \sigma_i u_i$$

for each $i \in \{1, 2, \ldots, n\}$. This collection of vector equations can be expressed as a matrix equation,

$$AV = \widehat{U}\widehat{\Sigma}.$$

Here, $\widehat{\Sigma}$ is an $n \times n$ diagonal matrix with positive real entries σ_i (since A was assumed to have full rank n), $\widehat{U}$ is an $m \times n$ matrix with orthonormal columns u_i's, and V is an $n \times n$ matrix with orthonormal columns v_i's. Thus V is unitary, and we can multiply on the right by its inverse V^* to obtain

$$A = \widehat{U}\widehat{\Sigma}V^*. \tag{1}$$

This factorization of A is called a **reduced singular value decomposition**, or reduced SVD, of A.

However, this is not the standard way in which the idea of an SVD is usually formulated. The reason is as follows. The columns of $\widehat{U}$ are n orthonormal vectors in the m-dimensional space $\mathbb{C}^m$. Unless $m = n$, they do not form a basis of $\mathbb{C}^m$, nor is $\widehat{U}$ a unitary matrix. However, by adjoining an additional $m - n$ orthonormal columns, $\widehat{U}$ can be extended to a unitary matrix. Let us do this in an arbitrary fashion, and call the result U.

If $\widehat{U}$ is replaced by U in (1), then $\widehat{\Sigma}$ will have to change too. For the product to remain unaltered, the last $m - n$ columns of U should be multiplied by zero. Accordingly, let Σ be the $m \times n$ matrix consisting of $\widehat{\Sigma}$ in the upper $n \times n$ block together with $m - n$ rows of zeros below. We now have a new factorization, the **full SVD** of A

$$A = U\Sigma V^*.$$

Here U is $m \times m$ and unitary, V is $n \times n$ and unitary, and Σ is $m \times n$ and diagonal with positive real entries.