Home work - 5

Q3. How do you check if a matrix is positive semi-definite? Construct symmetric, positive semi-definite matrices and check their eigenvalues and eigenvectors. What do you observe?

Solution:
A matrix A is is positive definite, if the associated quadratic form has that property $x^T Ax > 0$ for $x \neq 0$; A quadratic form for which $x^T Ax \geq 0$ if $x \neq 0$ is called positive semi-definite

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$X^T AX = x^2 \text{ which is} > 0 \text{ for every } x \in R$$

$$\text{Therefore} \quad A \text{ is symmetric positive semi-definite}$$

Let $\text{Eigen values of } A:$

$$|A - \lambda I| = 0$$

$$(1 - \lambda)(0 - \lambda) = 0$$

$$\lambda = 0, 1$$

$$\text{Eigen values are non-negative}$$

Similarly

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{symmetric matrix}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$X^T AX = (x + y)^2 \text{ which is} > 0 \text{ for every } (x, y) \neq (0,0) \in R^2$$

$$\text{Therefore} \quad A \text{ is symmetric positive semi-definite}$$

$$\text{Eigen values of } A:$$

$$|A - \lambda I| = 0$$

$$(1 - \lambda)(1 - \lambda) - 1 = 0$$

$$\lambda = 0, 2$$

$$\text{Eigen values are non-negative}$$

Observation: Let A is symmetric matrix then $x^T Ax$ is positive semi-definite if and only if all eigenvalues of A are nonnegative

Proof:

- Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix
- Consider $f(x) = \mathbf{x}^T \mathbf{A} \mathbf{x}$, a *pure quadratic form*
- Eigenvalues of $\mathbf{A} : \lambda_1, \lambda_2, \ldots, \lambda_n$
- Orthonormal Eigenvectors of $\mathbf{A} : \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$
- $\mathbf{S} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$

$$\begin{aligned} \mathbf{x}^T \mathbf{A} \mathbf{x} &= \mathbf{x}^T \mathbf{S} \Lambda \mathbf{S}^T \mathbf{x} \\ &= \mathbf{y}^T \Lambda \mathbf{y} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 \end{aligned}$$

Therefore, $\lambda_i > 0 \quad \forall\, i \quad \Rightarrow \quad \mathbf{x}^T \mathbf{A} \mathbf{x} > 0$