HOMEWORK - 3

Q.1

a] Let P be a real Square Matrix Satisfying $P = P^T$ and $P^2 = P$.

i) Can the Matrix P have Complex Eigenvalues? if so, Construct an Example, else, justify your answer.

ii) What are the Eigenvalues of P?

<u>Solution</u>:

We know that,

The Eigenvalues of a Symmetric Matrix are <u>Real</u>.

i) Given that,

P is a real Square Matrix, $P = P^T$

$\therefore$ P is Symmetric, always have real eigenvalues.

ii) To find Eigenvalues of P.

Let λ is Eigenvalue of P. with respect to eigenvector v.

$\therefore Pv = \lambda v$ Given $P^2 = P$

$$\boxed{P^2 v = Pv = \lambda v} \quad \text{—— (I)}$$

$Pv = \lambda v$

$$\begin{aligned} P^2 v &= P \cdot P v \\ &= P \lambda v \\ &= \lambda P v \\ &= \lambda \lambda v \end{aligned}$$

$$\boxed{P^2 v = \lambda^2 v} \quad \text{—— (II)}$$

From (I) & (II)

$$\lambda^2 v = \lambda v$$

$$(\lambda^2 - \lambda) v = 0$$

$$\lambda^2 - \lambda = 0, \quad v \neq 0$$

$$\lambda = 0, \quad \lambda = 1$$

$\therefore$ P has Real Eigenvalues.

b) Given that the following matrix

$$A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$$ where c and r

are arbitarary real numbers.

and $5.5 \leq r \leq 6.5$ and the fact that $\lambda_1 = 3$ is one of the Eigen-Value, is it possible to determine the other two Eigenvalues?

If so, compute them and give reasons for your Answer.

Solution: For Given Matrix A,

$\det(A) = 0$

$\Rightarrow$ One of the Eigenvalue is "Zero"

$\therefore \lambda_1 = 3$ (Given Eigenvalue)

$\therefore \lambda_2 = 0$ ($\det A = 0$)

$\therefore$ To find $\lambda_3 = ?$

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From Property,

Sum of Eigenvalues = Trace of Matrix

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 1 + 7$$

$$3 + 0 + \lambda_3 = 9$$

$$\boxed{\lambda_3 = 6}$$

Q2) The Fibonacci Sequence is
(Q.2) defined by $V_n = V_{n-1} + V_{n-2}$ for $n \geq 2$ with Starting Values $V_0 = 1$ and $V_1 = 1$. Observe that the Calculation of V_K requires the calculation of $V_2, V_3 \text{----} V_{K-1}$. To Avoid this, could this problem be written as an Eigenvalue Problem and Solved for V_n directly? If so, find the Explicit Formula for V_n.

Solution

The Fibonacci Sequence,

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ---

$$V_{n+1} = V_n + V_{n-1}$$

$$V_n = 1V_n + 0\,V_{n-1}$$

$$\begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix}$$

Consider

$$f_n = A f_{n-1}$$

i.e $f_1 = A f_0$

$f_2 = A f_1 = A^2 f_0$

$f_3 = \text{-----} = A^3 f_0$

$$\boxed{f_n = A^n f_0}$$

Calculate A^n, $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Consider

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

and,

Eigenvectors are

$$X_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-1+\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$$

We know that,

$$A^n = P D^n P^{-1}$$

$$A^n = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$$

$$f_n = A^n f_0$$

$$= A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right].$$

Q.3 Prove that, if A is a Square Matrix of size $n \times n$, then $A^k \to 0$ as $k \to \infty$ if and only if $|\lambda_i| < 1 \ \forall i$

Solution :

As $k \to \infty$, $A^k \to 0$

$$\iff |A|^k \to 0$$

$$\iff (\lambda_1 \lambda_2 \cdots \lambda_n)^k \to 0$$

$$\iff \lambda_1^k \lambda_2^k \cdots \lambda_n^k \to 0$$

$$\iff |\lambda_i| < 1 \quad \forall i$$

Q.4 Construct Example of Matrices for which the defect is positive, negative and zero wherever possible.

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Solution:

Given a square Matrix A,

Let $\mu_A(\lambda)$ denote Algebraic Multiplicity of Eigenvalue λ of Matrix A.

and

$\gamma_A(\lambda)$ denote the Geometric Multiplicity Means, the number of linearly Independent Eigenvectors corresponding to eigenvalue λ.

If $\gamma_A(\lambda) < \mu_A(\lambda)$ then, the Matrix is said to be Defective.

Ex consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

characteristic polynomial is $\lambda^2 - 2\lambda + 1 = 0$

$\therefore$ Eigenvalue $\underline{\lambda = 1}$ has Algebraic Multiplicity $\mu_A(\lambda) = \mu_A(1) = 2$.

To find eigenvector corresponding to $\lambda = 1$

Consider

$$(A - \lambda I)X = 0 \quad \text{for } \lambda = 1$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here, x_1 can takes any Arbitrary value and $x_2 = 0$

$\therefore$ eigenvector $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore \quad \gamma_A(\lambda) = \gamma_A(1) = 1$$

$$\Rightarrow \quad \gamma_A(\lambda) < \mu_A(\lambda)$$

$\therefore$ A is a defective Matrix.

Positive defect

eg) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\lambda^2 = 0$

$\Rightarrow \quad \lambda = 0$

Algebraic Multiplicity is 2.

But its Geometric Multiplicity is 1.

Hence, $\Delta = 2 - 1 = 1$.

Negative defect

Geometric Multiplicity $<$ Algebraic Multiplicity

& hence defect cannot be negative.

Zero defect

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$\Rightarrow \quad \lambda_1 = 5 \quad , \quad \lambda_2 = \lambda_3 = -3$

Algebraic Multiplicity for $\lambda = -3$ $= 2$

&

Geometric Multiplicity for $\lambda = -3$ $= 2$

$\therefore \; (\Delta)_{-3} = 2 - 2$

$$\boxed{(\Delta)_{-3} = 0}$$