BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI
WORK INTEGRATED LEARNING PROGRAMMES

Mathematical Foundations for Data Science

Homework -10 Solution

1. Show that if A and B are sets, then $(A \cap B) \cup (A \cap \bar{B}) = A$.

Qs.20 **PROOF**

FIRST PART Let $x \in (A \cap B) \cup (A \cap \overline{B})$.

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in (A \cap B) \vee x \in (A \cap \overline{B})$$

Using the definition of the intersection, we know that x has to be in both sets:

$$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in \overline{B})$$

Using distributive law for propositions:

$$x \in A \wedge (x \in B \vee x \in \overline{B})$$

Use the definition of the complement:

$$x \in A \wedge (x \in B \vee \neg(x \in B))$$

Use negation law for propositions:

$$x \in A \wedge \mathbf{T}$$

Using the identity law for propositions:

$$x \in A$$

By the definition of a subset, we have then shown $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$.

SECOND PART Let $x \in A$

Using the identity law for propositions:

$$x \in A \wedge \mathbf{T}$$

Use the negation law for propositions:

$$x \in A \wedge (x \in B \vee \neg(x \in B))$$

Use the definition of the complement:

$$x \in A \wedge (x \in B \vee x \in \overline{B})$$

Using distributive law for propositions:

$$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in \overline{B})$$

Using the definition of the intersection, we know that x has to be in both sets:

$$(x \in A \cap B) \vee (x \in A \cap \overline{B})$$

Using the definition of the union, x is in the union when it is in one of the sets:

$$x \in (A \cap B) \cup (A \cap \overline{B})$$

Since $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$ and $A \subseteq (A \cap B) \cup (A \cap \overline{B})$, the two sets have to be the same: $(A \cap B) \cup (A \cap \overline{B}) = A$

2. Show that if A, B, and C are sets, then

Qs.44 $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$

Proof:

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup (B \cup C)| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - \{(A \cap B) \cup (A \cap C)\} \\
&= |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|\} \\
&= |A| + |B| + |C| - |A \cap B| - (B \cap C) - (C \cap A) + |A \cap B \cap C|
\end{aligned}$$

3.

Qs.45

Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find **a)** $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

SOLUTION

$$A_i = \{1,2,3,....,i\} = \{x \in \mathbb{N} | x > 0 \wedge x \leq i\}$$

(a) If $i \leq n$, then we note that A_i is a subset of A_n:

$$A_i \subset A_n$$

Let us take the union of all these sets A_i with $i \leq n$:

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} A_n$$

Use the idempotent law:

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} A_n = A_n$$

By the definition of the union, we also know that $A_n \subseteq \bigcup_{i=1}^{n} A_i$.

Since $\bigcup_{i=1}^{n} A_i \subseteq A_n$ and $A_n \subseteq \bigcup_{i=1}^{n} A_i$, the two sets then have to be equal:

$$\bigcup_{i=1}^{n} A_i = A_n$$

(b) If $i \geq 1$, then we note that A_1 is a subset of A_i:

$$A_1 \subset A_i$$

Let us take the intersections of all these sets A_i with $i \leq n$:

$$\bigcap_{i=1}^{n} A_1 \subseteq \bigcap_{i=1}^{n} A_i$$

Use the idempotent law:

$$A_1 = \bigcap_{i=1}^{n} A_1 \subseteq \bigcap_{i=1}^{n} A_i$$

By the definition of the intersection, we also know that $\bigcap_{i=1}^{n} A_i \subseteq A_1$.

Since $\bigcap_{i=1}^{n} A_i \subseteq A_1$ and $A_1 \subseteq \bigcap_{i=1}^{n} A_i$, the two sets then have to be equal:

$$\bigcap_{i=1}^{n} A_i = A_1$$

4.

Qs.46 Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find **a)** $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

SOLUTION

$$A_i = \{....,-2,-1,0,1,....,i\} = \{x \in \mathbb{Z} | x \leq i\}$$

(a) If $i \leq n$, then we note that A_i is a subset of A_n:

$$A_i \subset A_n$$

Let us take the union of all these sets A_i with $i \leq n$:

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} A_n$$

Use the idempotent law:

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} A_n = A_n$$

By the definition of the union, we also know that $A_n \subseteq \bigcup_{i=1}^{n} A_i$.

Since $\bigcup_{i=1}^{n} A_i \subseteq A_n$ and $A_n \subseteq \bigcup_{i=1}^{n} A_i$, the two sets then have to be equal:

(b) If $i \geq 1$, then we note that A_1 is a subset of A_i: $A_1 \subset A_i$

Let us take the intersections of all these sets A_i with $i \leq n$: $\bigcap_{i=1}^{n} A_1 \subseteq \bigcap_{i=1}^{n} A_i$

Use the idempotent law: $A_1 = \bigcap_{i=1}^{n} A_1 \subseteq \bigcap_{i=1}^{n} A_i$

By the definition of the intersection, we also know that $\bigcap_{i=1}^{n} A_i \subseteq A_1$.

Since $\bigcap_{i=1}^{n} A_i \subseteq A_1$ and $A_1 \subseteq \bigcap_{i=1}^{n} A_i$, the two sets then have to be equal:

$$\bigcap_{i=1}^{n} A_i = A_1$$

Since $\bigcup_{i=1}^{n} A_i \subseteq A_n$ and $A_n \subseteq \bigcup_{i=1}^{n} A_i$, the two sets then have to be equal: $\bigcup_{i=1}^{n} A_i = A_n$

5. Find $\bigcup_{i=1}^{\infty} A_i$, $\bigcap_{i=1}^{\infty} A_i$ for each of the following case

Qs.49
a. $A_i = \{-i, -i+1, -i+2, \ldots -1, 0, 1 \ldots, i-1, i\}$

b. $A_i = \{-i, i\}$

c. $A_i = [-i, i]$ that is a **Closed** interval on real line

d. $A_i = [i, \infty)$ that is a **semiopen** interval $i \leqslant x < \infty$.

Solution:

a. Observe the set structure, $A_1 = \{-1, 0, 1\}$,

as *i* increases the sets get larger and larger.

Thus, we have $A_2 \subseteq A_3 \subseteq A_4 \ldots\ldots.. \subseteq A_n \subseteq \cdots$

Therefore, $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$, and $\bigcap_{i=1}^{\infty} A_i = A_1$

b. Observe the set structure $A_1 = \{-1, 1\}, A_2 = \{-2, 2\}$ and so on. Thus every set is a subset of set of integers. Hence, $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z} - \{0\}$. Since each pair of sets is disjoint. Therefore, $\bigcap_{i=1}^{\infty} A_i = \phi$

c. Similar to above, the only difference is now its on real line. Thus,

$\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$ (set of reals), and $\bigcap_{i=1}^{\infty} A_i = [-1, 1]$.

d. Here, $A_1 = [1, \infty)$. As *i* increases the sets are getting smaller and smaller. Thus, $\bigcup_{i=1}^{\infty} A_i = A_1$.

As *i* increases every number gets excluded, so $\bigcap_{i=1}^{\infty} A_i = \phi$.