BITS Pilani
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Mathematical Foundations for Data Science

MFDS Team

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DSECL ZC416, MFDS

Lecture No. 10

Slides are adapted version of slides from McGraw Hill Education

Agenda

- Venn diagram
- Set Operations
- Set Identities
- Cartesian Product
- Computer Representation of sets

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set.
- A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Roster Method

$S = \{a,b,c,d\}$

Order not important $S = \{a,b,c,d\} = \{b,c,a,d\}$

Each distinct object is either a member or not; listing more than once does not change the set. $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear. $S = \{a,b,c,d, \ldots\ldots,z\}$

Set of all vowels in the English alphabet: $V = \{\mathrm{a,e,i,o,u}\}$

Set of all odd positive integers less than 10: $O = \{1,3,5,7,9\}$

Set of all positive integers less than 100: $S = \{1,2,3,\ldots\ldots..,99\}$

Set of all integers less than 0: $S = \{\ldots., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$$S = \{x \mid P(x)\}$$

Example: $S = \{x \mid \text{Prime}(x)\}$

Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

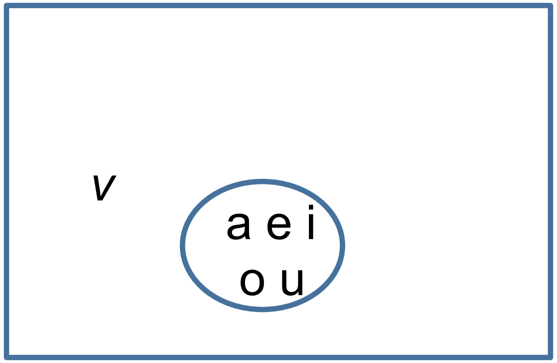
closed interval [a,b]

open interval (a,b)

Universal Set and Empty set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized $\varnothing$, but {} also used.

Venn Diagram



- The empty set is different from a set containing the empty set. $\varnothing \neq \{\varnothing\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if
$$\forall x (x \in A \leftrightarrow x \in B)$$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$
$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subset

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Subset

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

This is equivalent to

$A \subseteq B$ and $B \subseteq A$

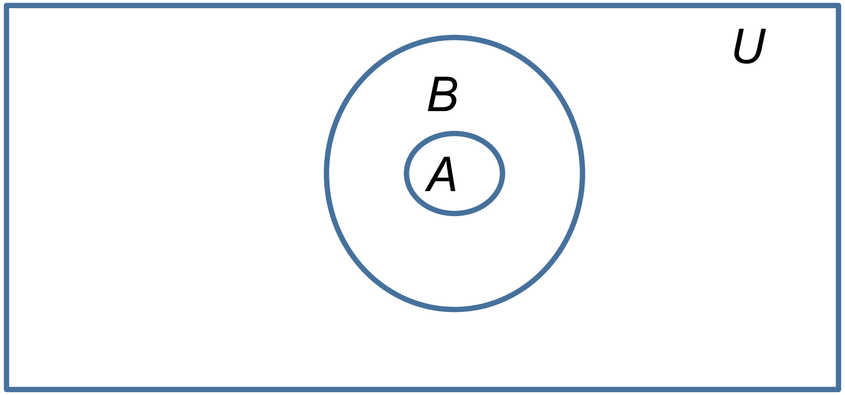
Proper subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $P(A)$, is called the *power set* of A.

Example: If $A = \{a,b\}$ then

$$P(A) = \{ø, \{a\},\{b\},\{a,b\}\}$$

If a set has n elements, then the cardinality of the power set is 2^n.

Tuples

The *ordered n-tuple* $(a_1,a_2,\ldots..,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i, for $i = 1, \ldots n$.

Example: What is $A \times B \times \mathrm{C}$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times \mathrm{C}$ **=** $\{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

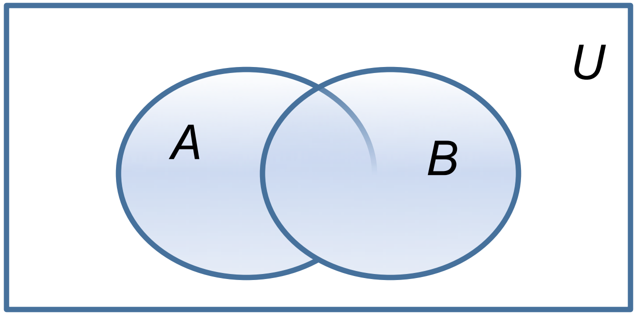
Set Operation - Union

Definition: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Set Operation - Intersection

Definition: The *intersection* of sets A and B, denoted by $A \cap B$, is

Note if the intersection is empty, then A and B are said to be *disjoint*.

Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

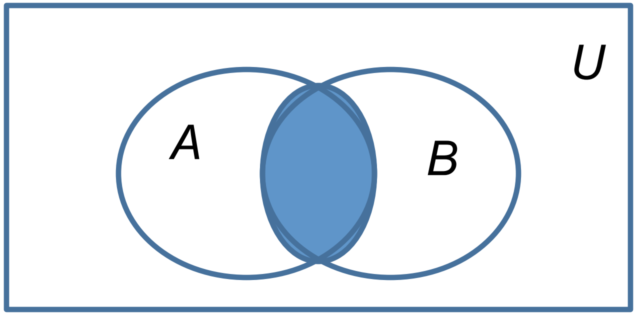
Solution: $\{3\}$

Example: What is?

$\{1,2,3\} \cap \{4,5,6\}$?

Solution: $\varnothing$

Venn Diagram for $A \cap B$



Set Operation - Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

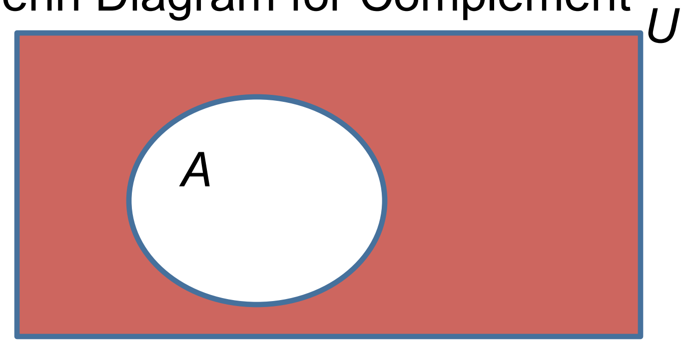
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

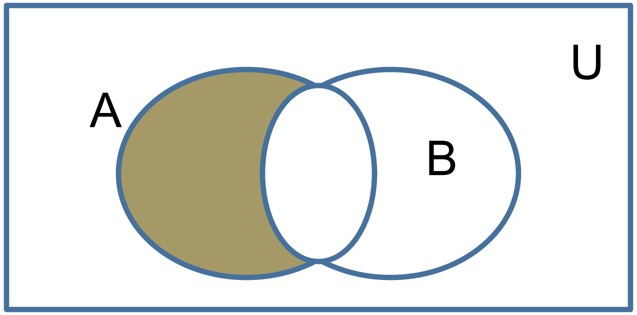
Venn Diagram for Complement



Set Operation - Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

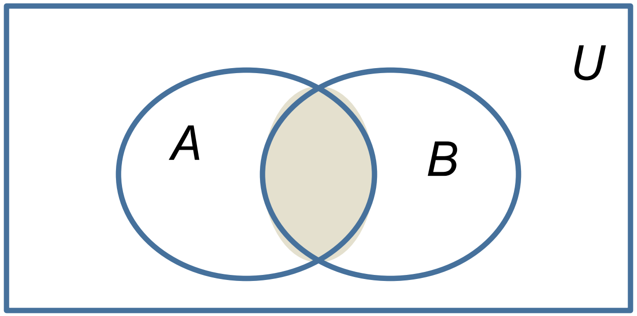


Venn Diagram for $A - B$

Cardinality of Union of two sets

Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Symmetric Difference

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

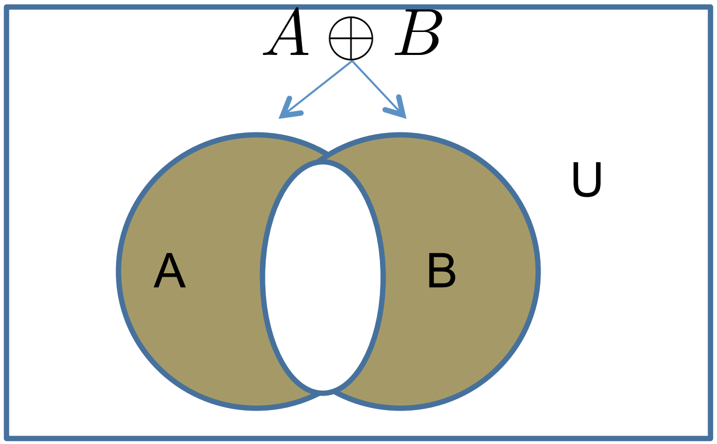
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is : $A \oplus B$

– **Solution**: {1,2,3,6,7,8}



Set Identities

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Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\overline{(\overline{A})} = A$$

Commutative laws

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Complement Law

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

De Morgan's Law

$$\boxed{\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}}$$

Absorption Law

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Proof of Second De Morgan Law

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These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x \mid x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x \mid \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x \mid \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x \mid \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x \mid x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x \mid x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x \mid x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Union and Intersection

Let A_1, A_2 ,…, A_n be an indexed collection of sets.
We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

For $i = 1,2,\ldots$, let $A_i = \{i, i+1, i+2, \ldots.\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Computer Representation of Sets

Represent a subset A of U with the bit string of length n, where the ith bit in the string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.

Example:

- Let $U = \{1,2,3,4,5,6,7,8,9,10\}$, and the ordering of elements of U has the elements in increasing order; that is $a_i = i$.

  What bit string represents the subset of all odd integers in U?

  Solution: 10 1010 1010

  What bit string represents the subset of all even integers in U?

  Solution: 01 010 10101

  What bit string represents the subset of all integers not exceeding 5 in U?

  Solution: 11 1110 0000

  What bit string represents the complement of the set $\{1,3,5,7,9\}$?

  Solution: 01 0101 0101

Set Operations

- The bit string for the union is the bitwise *OR* of the bit string for the two sets. The bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.
- Example:
  - The bit strings for the sets {1,2,3,4,5} and {1,3,5,7,9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

    Solution:

    Union:

    $11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010$, {1,2,3,4,5,7,9}

    Intersection:

    $11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000$, {1,3,5}