BITS Pilani
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Mathematical Foundations for Data Science

MFDS Team

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DSECL ZC416, MFDS

SECTION 2 AND 5 - WEBINAR 3

Topic: Eigenvalues and Eigenvectors

1. Let A be a real n x n real matrix. It is also known that for each row, the absolute value of diagonal entry is greater than sum of absolute values of non diagonal entries. Prove or disprove that A has only non-zero eigenvalues.

Hint:

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \qquad \forall i = 1, 2 \ldots n$$

Topic: Eigenvalues and Eigenvectors

Given the absolute value of diagonal entry is greater than sum of absolute values of non diagonal Entries

$$|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \quad \forall i=1,2...n ------------(1)$$

Let us assume A has zero eigenvalues. Let one of the eigenvalue $\lambda = 0$

by Gershgorin's theorem λ should satisfy

$$\lambda - \{a_{ii}\} \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \quad \forall i=1,2...n$$

$$\Rightarrow \{a_{ii}\} \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \quad \forall i=1,2...n$$

It will contradict to (1)
Hence A cannot have any zero eigenvalues.

Topic: Gauss Elimination

2. Apply Gauss elimination with and without partial pivoting and four digit arithmetic with rounding. Also compare the results with the exact solution $x_1 = 10.00$ and $x_2 = 1.000$

$$0.003x_1 + 59.14x_2 = 59.17$$

$$5.291x_1 - 6.130x_2 = 46.78$$

Hint: Partial Pivoting - rearrange the system for diagonally dominant

Topic: Gauss Elimination

Without Pivoting:

$$\begin{array}{c} \begin{pmatrix} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{pmatrix} \end{array} \begin{array}{l} R1 \\ R2 \end{array}$$

$$\begin{pmatrix} 0.003 & 59.14 & 59.17 \\ 0 & -104300 & -104400 \end{pmatrix} \begin{array}{l} R1 \rightarrow \textit{pivot row} \\ R2 - \dfrac{5.291}{0.003} R1 \end{array}$$

The system is reduced and gives the solution

$x_2 = 1.001$

$0.003 x_1 = 59.17 - 59.14 x_2$

Hence $x_1 = -10$

$x_1 = -10$ *and* $x_2 = 1.001$

With Partial Pivoting:

Interchange the system

$$\begin{pmatrix} 5.291 & -6.130 & 46.78 \\ 0.003 & 59.14 & 59.17 \end{pmatrix} \begin{array}{l} R1 \\ R2 \end{array}$$

$$\begin{pmatrix} 5.291 & -6.130 & 46.78 \\ 0 & 59.14 & 59.14 \end{pmatrix} \begin{array}{l} R1 \rightarrow \textit{pivot row} \\ R2 - \dfrac{0.003}{5.291} R1 \end{array}$$

The system is reduced and gives the solution

$x_2 = 1.000$

$x_1 = 10.00$

Topic: SVD

Q.3 Given a singular value decomposition of $A = U\Sigma V^T$, find the SVD of A^T.

How are the singular values of A and A^T related ?

Solution:

$$A = U \Sigma V^T$$

Apply Transpose on both sides

$$A^T = (U \Sigma V^T)^T$$

$$= (V^T)^T \Sigma^T U^T$$

$$= V \Sigma^T U^T$$

This is an SVD of A^T because U and V are orthogonal matrices and

Σ^T is an $n \times m$ diagonal matrix.

As Σ and Σ^T have the same non-zero diagonal entries

Hence A and A^T have the same non-zero singular values.

Topic: Norm

Q.4 For a vector $y \in R^m$ prove that $\|y\|_2 \le \sqrt{m}\,\|y\|_\infty$

Hint: $|y_i| \le \|y\|_\infty$

Solution:

Given a vector $y = (y_1, y_2, \ldots, y_m)$, we have the following definitions:

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$$\|y\|_\infty = \max\{|y_i|\} \text{ and } \|y\|_2 = \sqrt{\sum_{i=1}^{m} y_i^{\,2}}$$

$$\therefore \|y\|_2 = \sqrt{\sum_{i=1}^{m} |y_i|^2}$$

$$\le \sqrt{\sum_{i=1}^{m} \|y\|_\infty^{\,2}}$$

$$\le \sqrt{m\|y\|_\infty^{\,2}}$$

$$\le \sqrt{m}\,\|y\|_\infty$$

Topic: Power Method

Q.5 Under what conditions on the matrix A would we be able to compute the error δ in approximating the eigenvalue that is a largest in absolute value using the power method?

What happens when the initial starting vector X is an eigenvector of A?

Solution:

1. A must be Real Symmetric Matrix
2. When initial starting vector x is an eigen vector of A then $\delta = 0$

Topic: Gauss Jacobi

Q.6 Write a suitable numerical scheme for which Gauss Jacobi method for the following system converges

$3x_1 + 2x_2 + x_3 = 7$

$x_1 + 3x_2 + 2x_3 = 4$

$2x_1 + x_2 + 3x_3 = 7$

Solution: Given equations can be written as

$$x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{7}{3}$$

$$\frac{1}{3}x_1 + x_2 + \frac{2}{3}x_3 = \frac{4}{3}$$

$$\frac{2}{3}x_1 + \frac{1}{3}x_2 + x_3 = \frac{7}{3}$$

Topic: Gauss Jacobi

$$A=\begin{bmatrix}1 & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & 1 & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & 1\end{bmatrix}=I+L+U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0\\ \frac{1}{3}&0&0\\ \frac{2}{3}&\frac{1}{3}&0\end{bmatrix}+\begin{bmatrix}0&\frac{2}{3}&\frac{1}{3}\\0&0&\frac{2}{3}\\0&0&0\end{bmatrix}$$

For Jacobi method

$$C=-I^{-1}(L+U)=-(L+U)=\begin{bmatrix}0&\frac{-2}{3}&\frac{-1}{3}\\ \frac{-1}{3}&0&\frac{-2}{3}\\ \frac{-2}{3}&\frac{-1}{3}&0\end{bmatrix}$$

$$\|C\|_1=\text{Column sum norm }=\max_j\sum_{i=1}^{3}|a_{ij}|=1$$

$$\|C\|_\infty=\text{Row sum norm }=\max_i\sum_{j=1}^{3}|a_{ij}|=1$$

and Frobenius norm $\|C\|_2=\sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3}(a_{ij})^2}=\sqrt{\frac{15}{9}}=1.2$

$\|C\|\geq 1$ Given syetem of equation diverges.

Topic: Gauss Jacobi

Iteration	x_1	x_2	x_3
1	2.3333	1.3333	2.3333
2	0.6667	-1	0.3333
3	2.8889	0.8889	2.2222
4	1	-1.1111	0.1111
5	3.037	0.9259	2.037
6	1.037	-1.037	0
7	3.0247	0.9877	1.9877
8	1.0123	-1	-0.0123
9	3.0041	1.0041	1.9918
10	1	-0.9959	-0.0041
11	2.9986	1.0027	1.9986
12	0.9986	-0.9986	0
13	2.9991	1.0005	2.0005
14	0.9995	-1	0.0005
15	2.9998	0.9998	2.0003
16	1	-1.0002	0.0002
17	3.0001	0.9999	2.0001
18	1.0001	-1.0001	0

$\|C\| \geq 1$ Given syetem of equation diverges.