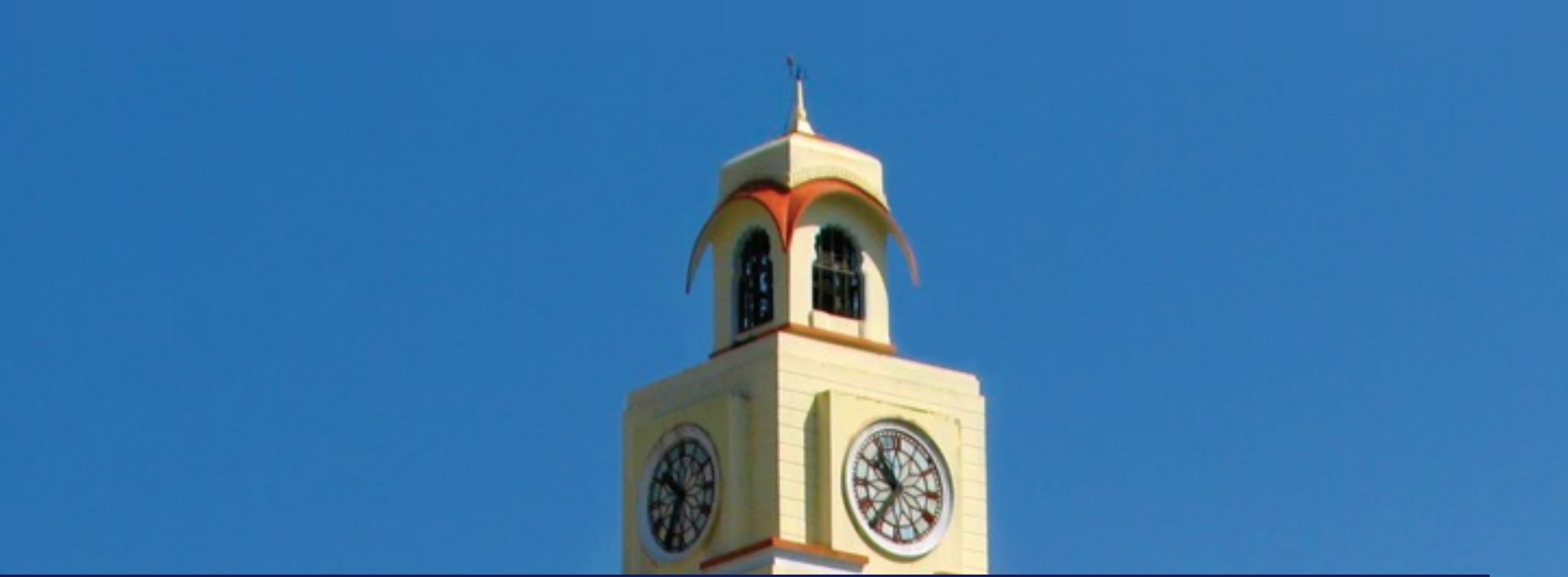

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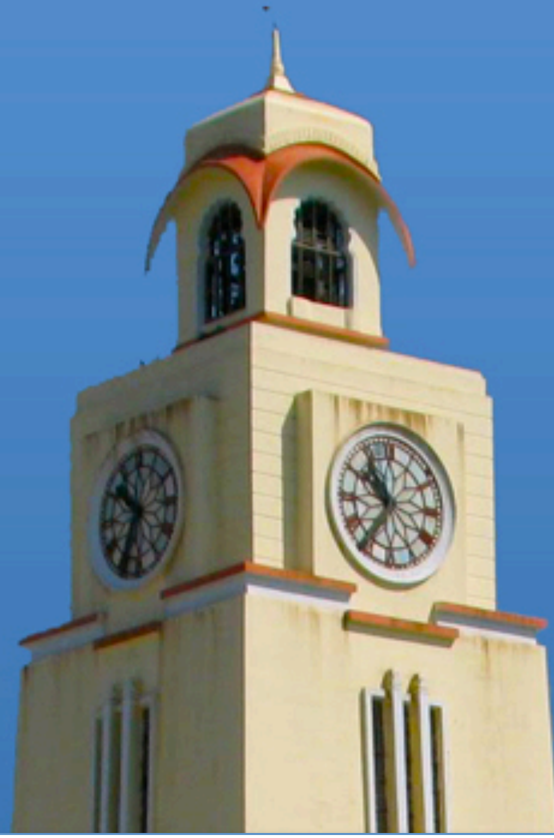
Mathematical Foundations for Data Science

MFDS Team



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Lecture No. 6

Agenda

- Motivation
- Matrix Norms
- Iterative Solution
    - Gauss Seidel Method
    - Gauss Jacobi Method
- Convergence Criteria
- Power Method

Motivation

Problem: Solve $7x = x + 18$. (Solution $x = 3$)

Iterative Procedure: $x_{i+1} = f(x_i)$, with x_1 given

$7x_{i+1} = x_i + 18$, $x_1 = 1$	$x_{i+1} = 7x_i - 18$, $x_1 = 1$
$x_2 = 2.7143$	$x_2 = -11$
$x_3 = 2.9592$	$x_3 = -95$
$x_4 = 2.9942$	$x_4 = -683$
$x_5 = 2.9992$	$x_5 = -4799$
$x_6 = 2.9999$	$x_6 = -33611$
Converges to $x = 3$	Diverges

a) Extension to Linear Systems?

b) Criteria for convergence (Similar to $\|r\| < 1$)

Matrix Norms

Some of the commonly used matrix norms are given below

Matrix norm corresponding to vector 1-norm is maximum absolute column sum

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

Matrix norm corresponding to vector ∞ - norm is maximum absolute row sum,

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$||\mathbf{A}||_2$ is the *Frobenius norm*

$$||\mathbf{A}||_2 = ||\mathbf{A}||_F = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} |a_{ij}|^2}$$

Gauss Seidel Method

- Given the system [A]{x}={B} and starting values $\{x\}^0$, Gauss-Seidel uses the first equation to solve for x_1, second for x_2, etc.

$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3 - \; .\,.\,.\,.\,. \; - a_{1n}x_n) / a_{11}$$
$$x_2 = (b_2 - a_{21}x_1 - a_{23}x_3 - \; .\,.\,.\,.\,. \; - a_{2n}x_n) / a_{22}$$
$$.\,.\,.\,.\,.\,.$$
$$x_n = (b_n - a_{n1}x_1 - a_{n2}x_2 - \; .\,.\,.\,.\,. \; - a_{(n-1)(n-1)}x_{n-1}) / a_{nn}$$

After the first iteration you get $\{x\}^1$. Use these values to start a new iteration. Repeat until the tolerance is satisfied as

$$|\varepsilon_{a,i}| = \left| \frac{x_i^k - x_i^{k-1}}{x_i^k} \right| 100\% < \varepsilon_s$$

for all the unknowns (i=1,…,n), where k and k-1 represent the present and previous iterations.

Iterative Methods for Systems

Consider the system $Ax = b$

Write $A = I + L + U$, where

- I is an identity matrix
- L is lower triangular matrix with diagonal entries 0
- U is upper triangular matrix with diagonal entries 0

$Ax = (I + L + U)x = b$

- $x^{k+1} = -Lx^k - Ux^k + b$ (Gauss Jacobi)
- $x^{k+1} = -(I + L)^{-1}Ux^k + (I + L)^{-1}b$ (Gauss Seidel)

Converges if $\|(I+L)^{-1}U\| < 1$

Gauss Seidel Method

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Solve the following system using the Gauss-Seidel Method

$$
\begin{aligned}
6x_1 - 2x_2 + x_3 &= 11 \\
-2x_1 + 7x_2 + 2x_3 &= 5 \\
x_1 + 2x_2 - 5x_3 &= -1
\end{aligned}
$$

starting with $x_1^0 = x_2^0 = x_3^0 = 0.0$

- Rearrange the equations

$$
\begin{aligned}
x_1 &= (11 + 2x_2 - x_3) / 6 \\
x_2 &= (5 + 2x_1 - 2x_3) / 7 \\
x_3 &= (1 + x_1 + 2x_2) / 5
\end{aligned}
$$

- First iteration

$$
\begin{aligned}
x_1^1 &= (11 + 2x_2^0 - x_3^0) / 6 = (11 + 0 - 0)/6 = 1.833 \\
x_2^1 &= (5 + 2x_1^1 - 2x_3^0) / 7 = (5 + 2*1.8333 - 0)/7 = 1.238 \\
x_3^1 &= (1 + x_1^1 + 2x_2^1) / 5 = (1 + 1.8333 + 2*1.2381)/5 = 1.062
\end{aligned}
$$

- Second iteration

$$
\begin{aligned}
x_1^2 &= (11 + 2x_2^1 - x_3^1) / 6 = (11 + 2*1.238 - 1.062)/6 = 2.069 \\
x_2^2 &= (5 + 2x_1^2 - 2x_3^1) / 7 = (5 + 2*2.069 - 2*1.062)/7 = 1.002 \\
x_3^2 &= (1 + x_1^2 + 2x_2^2) / 5 = (1 + 2.069 + 2*1.002)/5 = 1.015
\end{aligned}
$$

Gauss Seidel Method

Note: Always check the convergence using specified tolerance

	Zeroth	First	Second	Third	Fourth	Fifth
x_1	0.000	1.833	2.069	1.998	1.999	2.000
x_2	0.000	1.238	1.002	0.995	1.000	1.000
x_3	0.000	1.062	1.015	0.998	1.000	1.000

Gauss Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n
\end{gathered}
$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

Gauss Jacobi Method

- Gauss- Seidel always uses the newest available x values. Jacobi Method uses x values from the previous iteration.

Example 19: Repeat the previous example using the Jacobi method.

- Rearrange the equations

$$x_1 = (11 + 2x_2 - x_3) / 6$$
$$x_2 = (5 + 2x_1 - 2x_3) / 7$$
$$x_3 = (1 + x_1 + 2x_2) / 5$$

- First iteration (use x^0 values)

$$x_1^1 = (11 + 2x_2^0 - x_3^0) / 6 = (11 + 0 - 0)/6 = 1.833$$
$$x_2^1 = (5 + 2x_1^0 - 2x_3^0) / 7 = (5 + 0 - 0)/7 = 0.714$$
$$x_3^1 = (1 + x_1^0 + 2x_2^0) / 5 = (1 + 0 + 0)/5 = 0.200$$

- Second iteration (use x^1 values)

$$x_1^2 = (11 + 2x_2^1 - x_3^1) / 6 = (11 + 2*0.714 - 0.200)/6 = 2.038$$
$$x_2^2 = (5 + 2x_1^1 - 2x_3^1) / 7 = (5 + 2*1.833 - 2*0.200)/7 = 1.181$$
$$x_3^2 = (1 + x_1^1 + 2x_2^1) / 5 = (1 + 1.833 + 2*0.714)/5 = 0.852$$

Convergence of Jacobi Method

Show that for each of the following matrices **A**, the system **Ax** = **b** can be solved by Jacobi iteration with guaranteed convergence.

$$\text{(a)} \begin{bmatrix} 5 & -1 & 3 \\ 2 & -8 & 1 \\ -2 & 0 & 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} -2 & 0 & 4 \\ 2 & -8 & 1 \\ 5 & -1 & 3 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 4 & 2 & -2 \\ 0 & 4 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

(a) This matrix is diagonally dominated since $|5| > |-1| + |3|$, $|-8| > |2| + |1|$ and $|4| > |-2| + |0|$ are all true. From the above theorem we know that the Jacobi 'iteration matrix' **P** must have an ∞-norm that is strictly less than 1, but this is also easy to verify directly:

$$\mathbf{P} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$$

$$= -\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.125 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 \\ 2 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.2 & -0.6 \\ 0.25 & 0 & 0.125 \\ 0.5 & 0 & 0 \end{bmatrix}$$

Hence $\|\mathbf{P}\|_{\infty} = 0.8$, which as expected is < 1.

Convergence of Jacobi Method

$$(b) \begin{bmatrix} -2 & 0 & 4 \\ 2 & -8 & 1 \\ 5 & -1 & 3 \end{bmatrix}$$

(b) This matrix is not diagonally dominated since $|-2| > |0| + |4|$ is false; the third row also violates. However, the first and third equations of $\mathbf{Ax} = \mathbf{b}$ can be swapped to give a new system $\mathbf{A'x} = \mathbf{b'}$ with

$$\mathbf{A}' = \begin{bmatrix} 5 & -1 & 3 \\ 2 & -8 & 1 \\ -2 & 0 & 4 \end{bmatrix}$$

which is diagonally dominated; in fact it is the matrix of part (a). The reordered system is suitable for Jacobi iteration.

Convergence of Jacobi Method

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$$(c)\quad \begin{bmatrix} 4 & 2 & -2 \\ 0 & 4 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

Diagonal dominance is not necessary but sufficient condition

(c) This matrix is not diagonally dominated since $|4| > |2| + |-2|$ is false (we need strict satisfaction of the inequality in each row). In this case, we cannot we achieve diagonal dominance by suitably reordering the rows. We therefore press on to compute the Jacobi 'iteration matrix'

$$\mathbf{P} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})$$

$$= -\begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}\begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 0 & 0 & -0.5 \\ -0.25 & 0 & 0 \end{bmatrix}$$

and see what its norms look like. The row-sum norm $\|\mathbf{P}\|_\infty = 1$ Column sum norm $\|\mathbf{P}\|_1 = 1$. We are still looking for a matrix norm of **P** that is strictly less than 1. The Frobenius norm is our last chance, and happily we find

$$\|\mathbf{P}\|_{Fro} = \sqrt{(-0.5)^2 + (0.5)^2 + (-0.5)^2 + (-0.25)^2} = 0.901$$

Example of Divergence

Ex 1 Apply the Jacobi method to the system

$$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6, \end{aligned}$$

using the initial approximation $(x_1, x_2) = (0, 0)$, and show that the method diverges.

As usual, begin by rewriting the given system in the form

$$\begin{aligned} x_1 &= -4 + 5x_2 \\ x_2 &= -6 + 7x_1. \end{aligned}$$

Then the initial approximation $(0, 0)$ produces

$$\begin{aligned} x_1 &= -4 + 5(0) = -4 \\ x_2 &= -6 + 7(0) = -6 \end{aligned}$$

as the first approximation. Repeated iterations produce the sequence of approximations shown in Table 10.3.

TABLE 10.3

n	0	1	2	3	4	5	6	7
x_1	0	-4	-34	-174	-1244	-6124	$-42,874$	$-214,374$
x_2	0	-6	-34	-244	-1244	-8574	$-42,874$	$-300,124$

Example of Divergence

TABLE 10.3 JACOBI METHOD

n	0	1	2	3	4	5	6	7
x_1	0	−4	−34	−174	−1244	−6124	−42,874	−214,374
x_2	0	−6	−34	−244	−1244	−8574	−42,874	−300,124

For this particular system of linear equations you can determine that the actual solution is $x_1 = 1$ and $x_2 = 1$. So you can see from Table 10.3 that the approximations given by the Jacobi method become progressively *worse* instead of better, and you can conclude that the method diverges.

TABLE 10.4 GAUSS SEIDEL METHOD

n	0	1	2	3	4	5
x_1	0	−4	−174	−6124	−214,374	−7,503,124
x_2	0	−34	−1224	−42,874	−1,500,624	−52,521,874

Diagonally Dominant Matrix

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Definition of Strictly Diagonally Dominant Matrix Theorem

An $n \times n$ matrix A is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row. That is,

$$
\begin{aligned}
|a_{11}| &> |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
|a_{22}| &> |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\
&\vdots \\
|a_{nn}| &> |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|.
\end{aligned}
$$

Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

(a)
$$
\begin{aligned}
3x_1 - x_2 &= -4 \\
2x_1 + 5x_2 &= 2
\end{aligned}
$$

?

(b)
$$
\begin{aligned}
4x_1 + 2x_2 - x_3 &= -1 \\
x_1 + 2x_3 &= -4 \\
3x_1 - 5x_2 + x_3 &= 3
\end{aligned}
$$

?

Diagonally Dominant Matrix

Ex 1

Begin by interchanging the two rows of the given system to obtain

$$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4. \end{aligned}$$

Note that the coefficient matrix of this system is strictly diagonally dominant. Then solve for x_1 and x_2 as follows.

$$\begin{aligned} x_1 &= \tfrac{6}{7} + \tfrac{1}{7}x_2 \\ x_2 &= \tfrac{4}{5} + \tfrac{1}{5}x_1 \end{aligned}$$

Using the initial approximation $(x_1, x_2) = (0, 0)$, you can obtain the sequence of approximations shown in Table 10.5.

TABLE 10.5

n	0	1	2	3	4	5
x_1	0.0000	0.8571	0.9959	0.9999	1.000	1.000
x_2	0.0000	0.9714	0.9992	1.000	1.000	1.000

So you can conclude that the solution is $x_1 = 1$ and $x_2 = 1$.

Diagonally Dominant Matrix

Do not conclude from Theorem(discussed in slide 17) that strict diagonal dominance is a necessary condition for convergence of the Gauss Jacobi or Gauss- Seidel methods. For instance, the coefficient matrix of the system

$$\begin{aligned} -4x_1 + 5x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

is not a strictly diagonally dominant matrix, and yet both methods converge to the solution $x_1 = 1$ and $x_2 = 1$ when you use an initial approximation of $(x_1, x_2) = (0,0)$.

Dominant Eigenvalue

Let $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ be eigen values of nxn matrix A

λ_1 is called **dominant eigen value** of A if

$$|\lambda_1| > |\lambda_i| \text{ for all } i = 2,3,\ldots\ldots n$$

The eigen vectors corresponding to λ_1 are called **dominant eigen vectors**

Note : Not every matrix has dominant eigen value

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = -1$) has no dominant eigenvalue. Similarly, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 1$) has no dominant eigenvalue.

Rayleigh's Quotient

Let A be an n x n real symmetric matrix

Let $x (\neq 0)$ be any real vector with n components

Let $y = Ax$, $m_0 = x^T x$, $m_1 = x^T y$, $m_2 = y^T y$

Rayleigh Quotient

$q = \frac{m_1}{m_0}$ is an approximation for an eigenvalue λ of A, and if we set $q = \lambda - \varepsilon$ so that ε is error of q, then

$$|\in| \leq \delta = \sqrt{\frac{m_2}{m_0} - q^2}$$

Power Method for approximating Eigenvalues

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_0 = 0.$$

For large values of n, polynomial equations like this one are difficult and time-consuming to solve. Moreover, numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors. In this section you will look at an alternative method for approximating eigenvalues. As presented here, the method can be used only to find the eigenvalue of A that is largest in absolute value—this eigenvalue is called the **dominant eigenvalue** of A. Although this restriction may seem severe, dominant eigenvalues are of primary interest in many physical applications.

$$\begin{aligned}
\mathbf{x}_1 &= A\mathbf{x}_0 \\
\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) &= A^2\mathbf{x}_0 \\
\mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) &= A^3\mathbf{x}_0 \\
&\vdots \\
\mathbf{x}_k = A\mathbf{x}_{k-1} = A(A^{k-1}\mathbf{x}_0) &= A^k\mathbf{x}_0.
\end{aligned}$$

Power Method

To find the largest eigenvalues of a nxn matrix A. Assume that A has dominant eigenvalue

1.STEP 1 : Choose a column vector $u_o = [1,1,.........1]^T$

2.STEP 2 : Compute $A * u_o$

3.STEP 3 : Normalize the resulting vector obtained in step 2 by dividing each component by the largest in magnitude

4.STEP 4 : Repeat steps 2 and 3 until the change in normalizing factor is negligible

CONCLUSION :

•The normalizing factor is an approximate value of eigen value

•The final vector is the corresponding eigen vector

Power Method – Example

Find the dominant eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Step 1 : Choose column vector $u_o = [\,1,1,1]^T$

Step 2 :Multiply the matrix by the matrix [A] by $u_o = y_1$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 3 \\ -1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 5 \\ 3 \\ -1 \end{Bmatrix} = 5 \begin{Bmatrix} 1 \\ 0.6 \\ -0.2 \end{Bmatrix}$$

Step 3 :Normalize the resulting vector obtained in step 2 by dividing each component by the largest in magnitude

$u_1 = y_1 / 5 = [\,1, 0.6 , - 0.2\,]^T$ Normalizing factor $m_1 = 5$

Power Method - Example

Step 4 : $$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.6 \\ -0.2 \end{Bmatrix} = \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix} = 4.6 \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix}$$

$u_2 = y_2 / 4.6$ (normalizing factor m_2)

Power Method

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Now Repeating steps 2 and 3

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix} = \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix} = 4.2174 \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0183 \end{Bmatrix}$$

$u_3 = y_3 / 4.2174$
(normalizing factor m_3)

Power Method – Example

Continue the process $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0183 \end{Bmatrix} = \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix}$

$$\Rightarrow \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix} = 4.1134 \begin{Bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{Bmatrix}$$

$u_4 = y_4 / 4.1134$
(normalizing factor m_4)

m_1	m_2	m_3	m_4
5	4.6	4.2174	4.1134

Change in normalizing factor m_i's is now negligible

LARGEST Eigenvalue is $m_4 = 4.1134$
Corresponding Eigen vector $u_4 = [1,0, 0]^T$

Convergence of Power Method

If A is an $n \times n$ diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector $\mathbf{x}_0$ such that the sequence of vectors given by

$$A\mathbf{x}_0, \quad A^2\mathbf{x}_0, \quad A^3\mathbf{x}_0, \quad A^4\mathbf{x}_0, \quad \ldots, \quad A^k\mathbf{x}_0, \quad \ldots$$

approaches a multiple of the dominant eigenvector of A.

If the eigen values are ordered such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|,$$

then the power method will converge quickly if $|\lambda_2|/|\lambda_1|$ is small, and slowly if $|\lambda_2|/|\lambda_1|$ is close to 1.

Convergence of Power Method

(a) The matrix

$$A = \begin{bmatrix} 4 & 5 \\ 6 & 5 \end{bmatrix}$$

has eigenvalues of $\lambda_1 = 10$ and $\lambda_2 = -1$. So the ratio $|\lambda_2|/|\lambda_1|$ is 0.1. For this matrix, only four iterations are required to obtain successive approximations that agree when rounded to three significant digits. (See Table 10.7.)

TABLE 10.7

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\begin{bmatrix} 1.000 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} 0.818 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} 0.835 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} 0.833 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} 0.833 \\ 1.000 \end{bmatrix}$

Convergence of Power Method

(b) The matrix

$$A = \begin{bmatrix} -4 & 10 \\ 7 & 5 \end{bmatrix}$$

has eigenvalues of $\lambda_1 = 10$ and $\lambda_2 = -9$. For this matrix, the ratio $|\lambda_2|/|\lambda_1|$ is 0.9, and the power method does not produce successive approximations that agree to three significant digits until sixty-eight iterations have been performed, as shown in Table 10.8.

TABLE 10.8

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$		$\mathbf{x}_{66}$	$\mathbf{x}_{67}$	$\mathbf{x}_{68}$
$\begin{bmatrix}1.000\\1.000\end{bmatrix}$	$\begin{bmatrix}0.500\\1.000\end{bmatrix}$	$\begin{bmatrix}0.941\\1.000\end{bmatrix}$	$\begin{matrix}\cdots\\\cdots\end{matrix}$	$\begin{bmatrix}0.715\\1.000\end{bmatrix}$	$\begin{bmatrix}0.714\\1.000\end{bmatrix}$	$\begin{bmatrix}0.714\\1.000\end{bmatrix}$