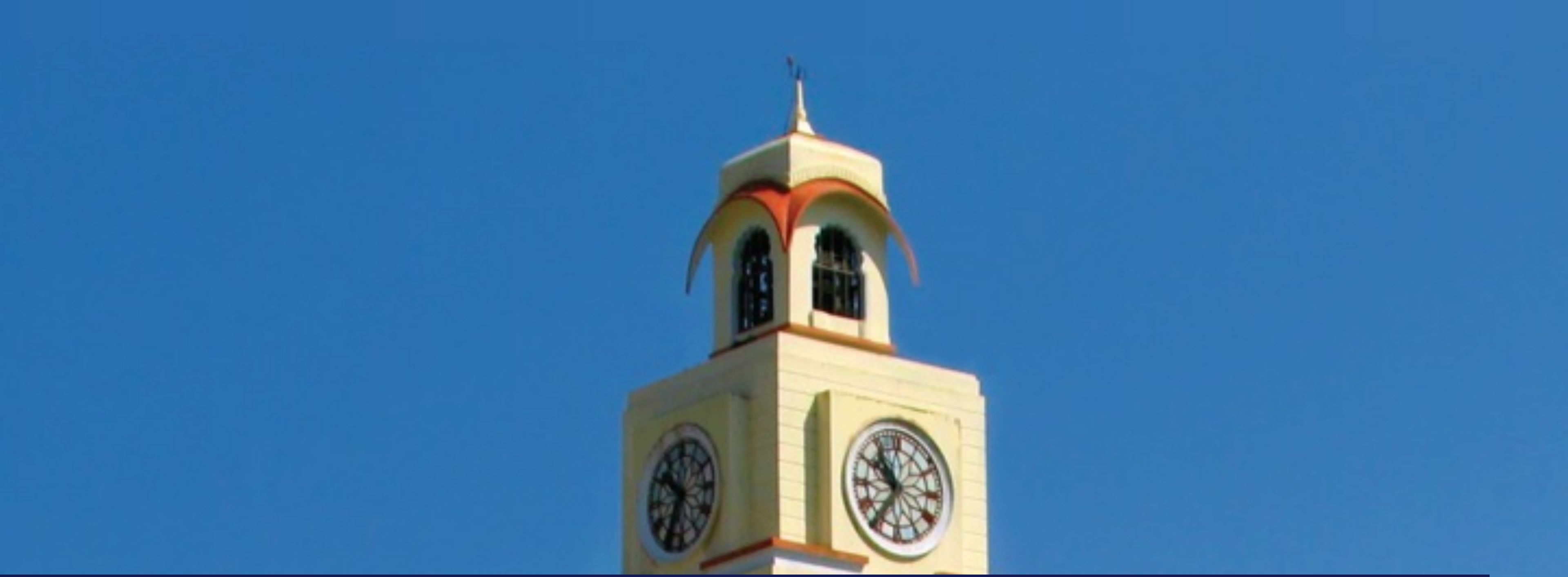

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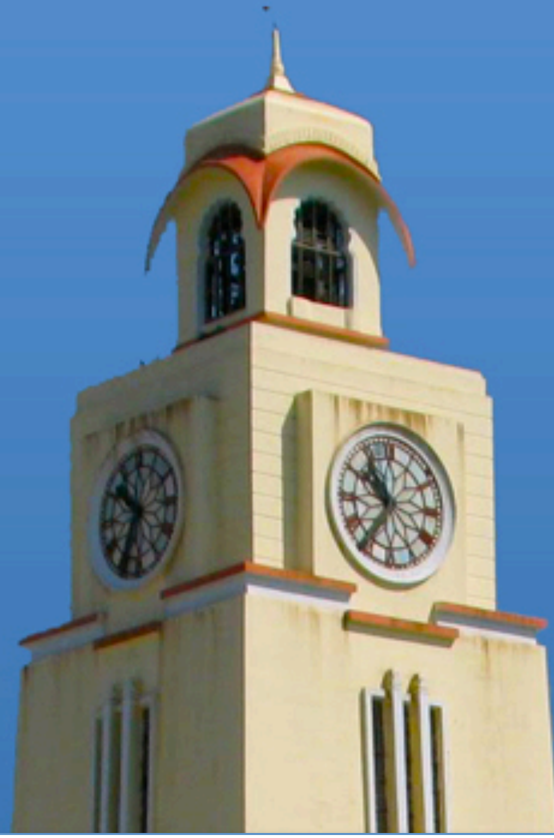

DSECL ZC416, MFDS

Lecture No.16

Agenda

1) Introduction to Boolean Algebra

2) Boolean Expressions and Boolean Functions

3) Identities of Boolean Algebra and Duality

4) Functional Completeness

Introduction to Boolean Algebra

Boolean algebra has rules for working with elements from the set {0, 1} together with the following operators :

1. **+** (Boolean sum)
2. **.** (Boolean product)
3. $\bar{}$ (Boolean Complement)

These operators are defined by:

- *Boolean sum*: 1 + 1 = 1, 1 + 0 = 1, 0 + 1 = 1, 0 + 0 = 0
- *Boolean product*: 1 . 1 = 1, 1 .0 = 0, 0 .1 = 0, 0 . 0 = 0
- *complement*: $\bar{0}$ = 1, $\bar{1}$= 0

Example: Find the value of 1 . 0 + $\overline{(0+1)}$

Solution : 1 . 0 + $\overline{(0+1)}$ = 0 + $\bar{1}$

= 0 + 0

= 0

Boolean Expressions and Boolean Functions

Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \le i \le n\}$ is the set of all possible n-tuples of 0s and 1s.

The variable x is called a ***Boolean variable*** if it assumes values only from B, that is, if its only possible values are 0 and 1.

A function from B^n to B is called a ***Boolean function of degree n.***

Example: The function $F(x, y) = x.y + \bar{x}\,\bar{y}$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$ is a Boolean function of degree 2.

TABLE 1

x	y	$F(x, y)$
1	1	1
1	0	0
0	1	0
0	0	1

Boolean Expressions and Boolean Functions (*continued*)

Example: Find the values of the Boolean function represented by

$$F(x, y, z) = x.y + \bar{z}$$

Solution: We use a table with a row for each combination of values of x, y, and z to compute the values of $F(x,y,z)$.

TABLE 2

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Boolean Expressions and Boolean Functions (*continued*)

Definition: Boolean functions F and G of n variables are equal if and only if $F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$ whenever $b_1, b_2, \ldots, b_n$ belong to B. Two different Boolean expressions that represent the same function are *equivalent*.

Definition: The complement of the Boolean function F is the function $\bar{F}$, where $\bar{F}(x_1, x_2, \ldots, x_n) = \overline{F(x_1, x_2, \ldots, x_n)}$.

Definition: Let F and G be Boolean functions of degree n. The Boolean sum $F + G$ and the Boolean product FG are defined by

- $(F + G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)$
- $(FG)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n).G(x_1, x_2, \ldots, x_n)$

Boolean Functions

Example: How many different Boolean functions of degree *n* are there?

Solution: By the product rule for counting, there are 2^n different *n*-tuples of 0s and 1s. Because a Boolean function is an assignment of 0 or 1 to each of these different n-tuples, by the product rule there are 2^{2^n} different Boolean functions of degree *n*.

TABLE 4 The Number of Boolean Functions of Degree *n*.

Degree	*Number*
1	4
2	16
3	256
4	65,536
5	4,294,967,296
6	18,446,744,073,709,551,616

The example tells us that there are 16 different Boolean functions of degree two. We display these in Table 3.

TABLE 3 The 16 Boolean Functions of Degree Two.

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

Identities of Boolean Algebra

TABLE 5 Boolean Identities.

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Each identity can be proved using a table.

All identities in Table , except for the first and the last two come in pairs. Each element of the pair is the ***dual*** **of the other (obtained by switching Boolean sums and Boolean products and 0's and 1's.**

The Boolean identities correspond to the identities of propositional logic

Duality in Boolean Algebra

How to construct the dual of a given Boolean expression ?

- Interchange Boolean sum and Boolean Products and Interchange 1's and 0's
- Example : The dual of $\bar{x}.1 + (\bar{y} + z)$ is $(\bar{x} + 0).(\bar{y}.z)$
- The dual of a Boolean function F represented by a Boolean expression is the function represented by the dual of this expression. This dual function is usually denoted by F^d .

Construct an identity from absorption law by taking dual of
$x.(x + y) = x$

- By taking its dual we get $x + x.y = x$, which is the first absorption law as given in the Table.

Identities of Boolean Algebra

Example: Show that the distributive law $x.(y + z) = x.y + x.z$ is valid.

Solution: We show that both sides of this identity always take the same value by constructing this table.

TABLE 6 Verifying One of the Distributive Laws.

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Formal Definition of a Boolean Algebra

Definition: A *Boolean algebra* is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $^{-}$ such that for all x, y, and z in B:

$$x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$$

distributive laws

$$x \vee 0 = x$$
$$x \wedge 1 = x$$

identity laws

$$x \vee \bar{x} = 1$$
$$x \wedge \bar{x} = 0$$

complement laws

$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$

associative laws

$$x \vee y = y \vee x$$
$$x \wedge y = y \wedge x$$

commutative laws

The set of propositional variables with the operators $\wedge$ and $\vee$, elements **T** and **F**, and the negation operator $\neg$ is a Boolean algebra.

The set of subsets of a universal set with the operators $\cup$ and $\cap$, the empty set ($\emptyset$), universal set (U), and the set complementation operator ($^{-}$) is a Boolean algebra.

Sum-of-Products Expansion

Example: Find Boolean expressions that represent the functions (i) $F(x, y, z)$ and (ii) $G(x, y, z)$ in Table 1.

Solution:

(i) To represent F we need the one term $x\bar{y}z$ because this expression has the value 1 when $x = z = 1$ and $y = 0$.

(ii) To represent the function G, we use the sum $xy\bar{z} + \bar{x}y\bar{z}$ because this expression has the value 1 when $x = y = 1$ and $z = 0$, or $x = z = 0$ and $y = 1$.

The general principle is that each combination of values of the variables for which the function has the value 1 requires a term in the Boolean sum that is the Boolean product of the variables or their complements.

TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Sum-of-Products Expansion (*cont*)

- A ***literal*** is a Boolean variable or its complement.

- A ***minterm*** of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = \overline{x_i}$. Hence, a minterm is a product of n literals, with one literal for each variable.
  - *The minterm $y_1, y_2, \ldots, y_n$ has value has value 1 if and only if each x_i is 1.*
  - *This occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x_i}$.*

- *The sum of minterms that represents the function is called the **sum-of-products** expansion or **disjunctive normal form** of the Boolean function.*

Sum-of-Products Expansion (*cont*)

Example: Find the sum-of-products expansion for the function

$F(x,y,z) = (x + y)\,\bar{z}$.

Solution: We use two methods, first using a table and second using Boolean identities.

(i) Form the sum of the min term corresponding to each row of the table that has the value 1.

(ii) Including a term for each row of the table for which $F(x,y,z) = 1$ gives us

$$F(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}.$$

TABLE 2

x	y	z	$x+y$	$\bar{z}$	$(x+y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Sum-of-Products Expansion (*cont*)

(*ii*) We now use Boolean identities to find the disjunctive normal form of F(*x*,*y*,*z*):

$F(x,y,z)$	$= (x + y).\ \bar{z}$	
	$= x.\bar{z} + y\bar{z}$	distributive law
	$= x.1.\ \bar{z} + 1.y\bar{z}$	identity law
	$= x.(y + \bar{y}).\bar{z} + (x + \bar{x}).y\bar{z}$	unit property
	$= x.y.\bar{z} + x.\bar{y}.\bar{z} + x.y.\ \bar{z} + \bar{x}.y.\bar{z}$	distributive law
	$= x.y.\bar{z} + x.\bar{y}.\bar{z} + \bar{x}.y.\ \bar{z}$	idempotent law

Functional Completeness

Definition: Because every Boolean function can be represented using the Boolean operators ., +, and $^{-}$, we say that the set {., + , $^{-}$} is *functionally complete*.

- The set {., $^{-}$} is functionally complete since $x + y = \overline{\bar{x}\bar{y}}$.

- The set {+, $^{-}$} is functionally complete since $x.y = \overline{\bar{x} + \bar{y}}$.

NAND and NOR operations

The **NAND** operator, denoted by |, is defined by
1|1 = 0, and 1|0 = 0|1 = 0|0 = 1.

- The set consisting of just the one operator NAND {|} is functionally complete.
- $\bar{x} = x|x$
- $x.y = (x|y)|(x|y)$

The ***NOR*** operator, denoted by ↓, is defined by
0 ↓ 0 = 1, and 1 ↓ 0 = 0 ↓ 1 = 1 ↓ 1 = 0

- The set consisting of just the one operator nor {↓} is functionally complete.
- $\bar{x} = x \downarrow x$
- $x.y = (x \downarrow x) \downarrow (y \downarrow y)$