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Mathematical Foundations for Data Science

MFDS Team

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DSECL ZC416, MFDS

Lecture No. 7

Agenda

- Reddy Mikks Problem
  - Formulation and graphical method
  - TORA
- Urban Renewal Model
  - Formulation and feasibility
- Currency Arbitrage Model
  - Formulation and applicability

Problem 1: Reddy Mikks Company

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$\boxed{\begin{aligned} 6x_1 + 4x_2 &\le 24 \\ x_1 + 2x_2 &\le 6 \end{aligned}} \quad \begin{aligned} &(\text{Raw material } M1) \\ &(\text{Raw material } M2) \end{aligned}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$\boxed{x_2 - x_1 \le 1} \quad (\text{Market limit})$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$\boxed{x_2 \le 2} \; (\text{Demand limit})$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \ge 0$, $x_2 \ge 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\le 24 &\quad (1) \\ x_1 + 2x_2 &\le 6 &\quad (2) \\ -x_1 + x_2 &\le 1 &\quad (3) \\ x_2 &\le 2 &\quad (4) \\ x_1, x_2 &\ge 0 &\quad (5) \end{aligned}$$

Problem 2: Urban Renewal Model

Example 2.3-1 (Urban Renewal Model)

The city of Erstville is faced with a severe budget shortage. Seeking a long-term solution, the city council votes to improve the tax base by condemning an inner-city housing area and replacing it with a modern development.

The project involves two phases: (1) demolishing substandard houses to provide land for the new development, and (2) building the new development. The following is a summary of the situation.

1. As many as 300 substandard houses can be demolished. Each house occupies a .25-acre lot. The cost of demolishing a condemned house is $2000.
2. Lot sizes for new single-, double-, triple-, and quadruple-family homes (units) are .18, .28, .4, and .5 acre, respectively. Streets, open space, and utility easements account for 15% of available acreage.
3. In the new development the triple and quadruple units account for at least 25% of the total. Single units must be at least 20% of all units and double units at least 10%.
4. The tax levied per unit for single, double, triple, and quadruple units is $1,000, $1,900, $2,700, and $3,400, respectively.
5. The construction cost per unit for single-, double-, triple-, and quadruple- family homes is $50,000, $70,000, $130,000, and $160,000, respectively. Financing through a local bank can amount to a maximum of $15 million.

How many units of each type should be constructed to maximize tax collection?

Problem 2: Urban Renewal Model

Step 1: Define Variables of the problem

x_1 = Number of units of single-family homes

x_2 = Number of units of double-family homes

x_3 = Number of units of triple-family homes

x_4 = Number of units of quadruple-family homes

x_5 = Number of old homes to be demolished

Step 2: Define Objective that we need to optimize

The objective is to maximize total tax collection from all four types of homes—that is,

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

Problem 2: Urban Renewal Model

Lot sizes for new single-, double-, triple-, and quadruple-family homes (units) are .18, .28, .4, and .5 acre, respectively. Streets, open space, and utility easements account for 15% of available acreage.

The first constraint of the problem deals with land availability.

$$\boxed{\begin{pmatrix}\text{Acreage used for new}\\ \text{home construction}\end{pmatrix} \leq \begin{pmatrix}\text{Net available}\\ \text{acreage}\end{pmatrix}}$$

From the data of the problem we have

$$\text{Acreage needed for new homes} = .18x_1 + .28x_2 + .4x_3 + .5x_4$$

To determine the available acreage, each demolished home occupies a .25-acre lot, thus netting $.25x_5$ acres. Allowing for 15% open space, streets, and easements, the net acreage available is $.85(.25x_5) = .2125x_5$. The resulting constraint is

$$\boxed{.18x_1 + .28x_2 + .4x_3 + .5x_4 \leq .2125x_5}$$

or

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

Problem 2: Urban Renewal Model

As many as 300 substandard houses can be demolished. $\boxed{x_5 \leq 300}$

In the new development the triple and quadruple units account for at least 25% of the total. Single units must be at least 20% of all units and double units at least 10%.

(Number of single units) ≥ (20% of all units)

(Number of double units) ≥ (10% of all units)

(Number of triple and quadruple units) ≥ (25% of all units)

These constraints translate mathematically to

$$x_1 \geq .2(x_1 + x_2 + x_3 + x_4)$$

$$x_2 \geq .1(x_1 + x_2 + x_3 + x_4)$$

$$x_3 + x_4 \geq .25(x_1 + x_2 + x_3 + x_4)$$

Problem 2: Urban Renewal Model

The only remaining constraint deals with keeping the demolishition/construction cost within the allowable budget—that is,

$$(\text{Construction and demolition cost}) \leq (\text{Available budget})$$

Expressing all the costs in thousands of dollars, we get

$$\boxed{(50x_1 + 70x_2 + 130x_3 + 160x_4) + 2x_5 \leq 15000}$$

Problem 2: Urban Renewal Model

The complete model thus becomes

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

subject to

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \le 0$$

$$x_5 \le 300$$

$$-.8x_1 + .2x_2 + .2x_3 + .2x_4 \le 0$$

$$.1x_1 - .9x_2 + .1x_3 + .1x_4 \le 0$$

$$.25x_1 + .25x_2 - .75x_3 - .75x_4 \le 0$$

$$50x_1 + 70x_2 + 130x_3 + 160x_4 + 2x_5 \le 15000$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Problem 2: Urban Renewal Model

Total tax collection $= z =$ \$343,965
Number of single homes $= x_1 = 35.83 \simeq 36$ units
Number of double homes $= x_2 = 98.53 \simeq 99$ units
Number of triple homes $= x_3 = 44.79 \simeq 45$ units
Number of quadruple homes $= x_4 = 0$ units
Number of homes demolished $= x_5 = 244.49 \simeq 245$ units

However, the feasible solution is
$x_1 = 36$, $x_2 = 98$, $x_3 = 45$, $x_4 = 0$ and $x_5 = 245$

Problem 3: Currency Arbitrage Model

Example 2.3-2 (Currency Arbitrage Model)

Suppose that a company has a total of 5 million dollars that can be exchanged for euros (€), British pounds (£), yen (¥), and Kuwaiti dinars (KD). Currency dealers set the following limits on the amount of any single transaction: 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. The table below provides typical spot exchange rates. The bottom diagonal rates are the reciprocal of the top diagonal rates. For example, rate(€ → \$) = 1/rate(\$ → €) = 1/.769 = 1.30.

	\$	€	£	¥	KD
\$	1	.769	.625	105	.342
€	$\frac{1}{.769}$	1	.813	137	.445
£	$\frac{1}{.625}$	$\frac{1}{.813}$	1	169	.543
¥	$\frac{1}{105}$	$\frac{1}{137}$	$\frac{1}{169}$	1	.0032
KD	$\frac{1}{.342}$	$\frac{1}{.445}$	$\frac{1}{.543}$	$\frac{1}{.0032}$	1

Is it possible to increase the dollar holdings (above the initial \$5 million) by circulating currencies through the currency market?

Problem 3: Currency Arbitrage Model

Currency	\$	€	£	¥	KD
Code	1	2	3	4	5

Define

x_{ij} = Amount in currency i converted to currency j, i and $j = 1, 2, \ldots, 5$

For example, x_{12} is the dollar amount converted to euros and x_{51} is the KD amount converted to dollars. We further define two additional variables representing the input and the output of the arbitrage problem:

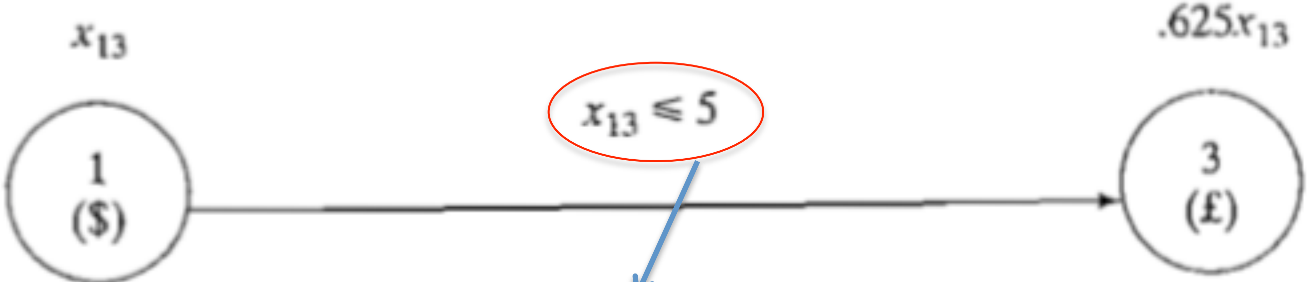
I = Initial dollar amount (= \$5 million)

y = Final dollar holdings (to be determined from the solution)

Problem 3: Currency Arbitrage Model

Our goal is to determine the maximum final dollar holdings, y, subject to the currency flow restrictions and the maximum limits allowed for the different transactions.

$$\boxed{\text{Maximize } z = y}$$



Transacted dollar amount cannot exceed the limit set by dealer $x_{13} \leq 5$

To conserve the flow of money from one currency to another, each currency must satisfy the following input-output equation:

$$\begin{pmatrix}\text{Total sum available}\\ \text{of a currency (input)}\end{pmatrix} = \begin{pmatrix}\text{Total sum converted to}\\ \text{other currencies (output)}\end{pmatrix}$$

1. *Dollar* ($i = 1$):

$$\begin{aligned}\text{Total available dollars} &= \text{Initial dollar amount} + \\ &\quad \text{dollar amount from other currencies} \\ &= I + (€ \rightarrow \$) + (£ \rightarrow \$) + (¥ \rightarrow \$) + (\text{KD} \rightarrow \$) \\ &= I + \tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\end{aligned}$$

$$\begin{aligned}\text{Total distributed dollars} &= \text{Final dollar holdings} + \\ &\quad \text{dollar amount to other currencies} \\ &= y + (\$ \rightarrow €) + (\$ \rightarrow £) + (\$ \rightarrow ¥) + (\$ \rightarrow \text{KD}) \\ &= y + x_{12} + x_{13} + x_{14} + x_{15}\end{aligned}$$

Given $I = 5$, the dollar constraint thus becomes

$$\boxed{y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5}$$

2. *Euro* ($i = 2$):

$$\text{Total available euros} = (\$ \to €) + (£ \to €) + (¥ \to €) + (\text{KD} \to €)$$
$$= .769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}$$

$$\text{Total distributed euros} = (€ \to \$) + (€ \to £) + (€ \to ¥) + (€ \to \text{KD})$$
$$= x_{21} + x_{23} + x_{24} + x_{25}$$

Thus, the constraint is

$$\boxed{x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0}$$

Transaction Limit

British pounds (£), yen (¥), and Kuwaiti dinars (KD). Currency dealers set the following limits on the amount of any single transaction: 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. The table below provides typical spot exchange rates. The

The only remaining constraints are the transaction limits, which are 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. These can be translated as

$$x_{1j} \leq 5, j = 2, 3, 4, 5$$
$$x_{2j} \leq 3, j = 1, 3, 4, 5$$
$$x_{3j} \leq 3.5, j = 1, 2, 4, 5$$
$$x_{4j} \leq 100, j = 1, 2, 3, 5$$
$$x_{5j} \leq 2.8, j = 1, 2, 3, 4$$

Problem 3: Currency Arbitrage Model

The complete model is now given as

$$\text{Maximize } z = y$$

subject to

$$y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5$$

$$x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0$$

$$x_{31} + x_{32} + x_{34} + x_{35} - \left(.625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}\right) = 0$$

$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$

$$x_{51} + x_{52} + x_{53} + x_{54} - \left(.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}\right) = 0$$

$$x_{1j} \leq 5,\ j = 2, 3, 4, 5$$

$$x_{2j} \leq 3,\ j = 1, 3, 4, 5$$

$$x_{3j} \leq 3.5,\ j = 1, 2, 4, 5$$

$$x_{4j} \leq 100,\ j = 1, 2, 3, 5$$

$$x_{5j} \leq 2.8,\ j = 1, 2, 3, 4$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j$$

Problem 3: Currency Arbitrage Model

Solution	Interpretation
$y = 5.09032$	Final holdings = \$5,090,320. Net dollar gain = \$90,320, which represents a 1.8064% rate of return
$x_{12} = 1.46206$	Buy \$1,462,060 worth of euros
$x_{15} = 5$	Buy \$5,000,000 worth of KD
$x_{25} = 3$	Buy €3,000,000 worth of KD
$x_{31} = 3.5$	Buy £3,500,000 worth of dollars
$x_{32} = 0.931495$	Buy £931,495 worth of euros
$x_{41} = 100$	Buy ¥100,000,000 worth of dollars
$x_{42} = 100$	Buy ¥100,000,000 worth of euros
$x_{43} = 100$	Buy ¥100,000,000 worth of pounds
$x_{53} = 2.085$	Buy KD2,085,000 worth of pounds
$x_{54} = .96$	Buy KD960,000 worth of yen

Problem 3: Currency Arbitrage Model

At first it may appear for solution to be nonsensical as it calls for using $x_{12} + x_{15} = 1.46206 + 5 = 6.46206 = \$6,462,060$ to buy Euros or KD but initial dollar amount is only \$5 million

In practice the given solution is submitted to the currency dealer as one order, we do not wait until we accumulate enough currency of certain type before making a buy.

$$I = y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\frac{1}{.769}x_{21} + \frac{1}{.625}x_{31} + \frac{1}{105}x_{41} + \frac{1}{.342}x_{51}\right)$$

$$= 5.09032 + 1.46206 + 5 - \left(\frac{3.5}{.625} + \frac{100}{105}\right) = 5$$