Homework-9

Solution

Q1 Investigate the nature of critical points for the following functions

1. $f(x,y)=x^3-3x^2+y^2$

Solution:

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0$$
$$\Rightarrow 3x^2-6x=0,\ 2y=0$$
$$\therefore 3x(x-2)=0,\ y=0$$
$$x=0,\ 2 \text{ and } y=0$$

Thus critical points are (0,0) and (2,0)

Now Hessian matrix

$$Hf(x,y)=\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}=\begin{bmatrix}6x-6 & 0\\ 0 & 2\end{bmatrix}$$

Case I: For (0,0)

$Hf(0,0)=\begin{bmatrix}-6 & 0\\ 0 & 2\end{bmatrix}$ and the eigen values are -6 and 2 i.e. both positive and negative. Thus (0,0) must be saddle point.

Case II: For (2,0)

$Hf(0,0)=\begin{bmatrix}6 & 0\\ 0 & 2\end{bmatrix}$ and the eigen values are 6 and 2 i.e. all positive. Thus (2,0) must be local minimum point.

Q1 Investigate the nature of critical points for the following functions

2. $f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

Solution:

Hint : when f(x, y) is symmetric function in x and y then critical point occurs at x=y

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x + y - \frac{1}{x^2} = 0, \ 2y + x - \frac{1}{y^2} = 0$$

As f(x, y) is symmetric function in x and y then critical point occurs at x=y

$\therefore$ Put x = y in $\frac{\partial f}{\partial x} = 0$

$$\Rightarrow 2x + x - \frac{1}{x^2} = 0$$

$$\Rightarrow 3x^3 - 1 = 0$$

$$\Rightarrow x = \left(\frac{1}{3}\right)^{1/3} = 0.6934, \ x = -0.35 \pm i \ 0.6$$

Thus critical points is $(0.6934, 0.6934)$

Now Hessian matrix

$$Hf(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 + \frac{2}{x^3} & 1 \\ 1 & 2 + \frac{2}{y^3} \end{bmatrix}$$

at $(0.6934, 0.6934)$

$Hf(0.6934, 0.6934) = \begin{bmatrix} 2 + \frac{2}{1/3} & 1 \\ 1 & 2 + \frac{2}{1/3} \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 1 & 8 \end{bmatrix}$ and the eigen value is 7 , 9 i.e. both positive. Thus $(0.6934, 0.6934)$ must be local minimum point.

Q.2 Using Lagrange multipliers, show that

1. the maximum value of $x^2y^3z^4$ Subject to 2x+3y+4z=a is $\left(\frac{a}{9}\right)^9$.

Solution:

Consider u=f(x, y, z) = $x^2y^3z^4$ and $\emptyset(x, y, z) = 2x + 3y + 4z - a$

Define new function F(x, y, z, λ) =f(x, y, z) + $\lambda\emptyset(x, y, z)$

Let,

u=f(x, y, z) = $x^2y^3z^4$

Taking log on both sides,

log u=2log x + 3log y +4 log z

taking partial differentiation of u w.r.t x, y, z

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} \qquad \frac{\partial u}{\partial x} = \frac{2u}{x}$$

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{3}{y} \qquad \frac{\partial u}{\partial y} = \frac{3u}{y}$$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{4}{z} \qquad \frac{\partial u}{\partial z} = \frac{4u}{z}$$

Now partial differentiation of ∅ w.r.t x, y, z,

$$\frac{\partial\emptyset}{\partial y} = 2, \qquad \frac{\partial\emptyset}{\partial y} = 3, \qquad \frac{\partial\emptyset}{\partial z} = 4$$

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial\emptyset}{\partial x} = \frac{2u}{x} + 2\lambda = 0 \qquad \dots\dots\dots\dots\dots\dots\dots..(1)$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial\emptyset}{\partial y} = \frac{3u}{y} + 3\lambda = 0 \qquad \dots\dots\dots\dots\dots\dots\dots..(2)$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial\emptyset}{\partial z} = \frac{4u}{z} + 4\lambda = 0 \qquad \dots\dots\dots\dots\dots\dots\dots..(3)$$

From equation (1), (2), (3)

$$x = \frac{-u}{\lambda}, y = \frac{-u}{\lambda}, z = \frac{-u}{\lambda}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(*)$$

Substitute above x, y, z values in $\emptyset(x, y, z) = 0$

$$2x + 3y + 4z - a = 0$$

$$2x + 3y + 4z = a$$

$$2\left(\frac{-u}{\lambda}\right) + 3\left(\frac{-u}{\lambda}\right) + 4\left(\frac{-u}{\lambda}\right) = a$$

$$9\left(\frac{-u}{\lambda}\right) = a$$

$$9\left(\frac{-u}{\mathrm{a}}\right) = \lambda$$

$$\lambda = 9\left(\frac{-u}{\mathrm{a}}\right)\text{....................................}(**)$$

Substitute λ value in equation (*),

$$x = \frac{a}{9}, y = \frac{a}{9}, z = \frac{a}{9}$$

Putting above values in u=f(x, y, z) = $x^2y^3z^4$

Maximum value of u=$x^2y^3z^4$ =$\left(\frac{a}{9}\right)^9$.

Q2 Using Lagrange multipliers, show that

2. the minimum value of $yz+zx+xy$ subject to $xyz = a^2(x+y+z)$ is $9a^2$

Solution: -

Consider the function $f(x,y,z) = xy + yz + zx$ ………………(1)

Let the given condition be $\phi(x,y,z) = xyz - a^2(x+y+z)$……(2)

Construct a new function $F = f + \lambda\phi$.

Thus, $F(x,y,z) = xy + yz + zx + \lambda(xyz - a^2(x+y+z))$………..(3)

Differentiate (3) with respect to x, y, z partially, we get

$F_x = y + z + \lambda yz - a^2\lambda$…………………..(4)

$F_y = x + z + \lambda xz - a^2\lambda$…………………..(5)

$F_z = x + y + \lambda yx - a^2\lambda$…………………..(6)

Using the relations $F_x = 0, F_y = 0, and\ F_z = 0$, we can write

$\lambda = \frac{y+z}{a^2-yz}$, $\lambda = \frac{x+z}{a^2-xz}$, $\lambda = \frac{y+x}{a^2-yx}$ respectively.

Consider first two ratio and cross multiply, we get

$$a^2y + a^2z - xyz - xz^2 = a^2x + a^2z - xyz - yz^2$$

Simplifying we get- $(y - x)(a^2 + z^2) = 0$

This implies $y = x$, as $(a^2 + z^2) \neq 0$.

Similarly considering First and third ratio we get $x = z$

Thus, we get $x = y = z$,

Substituting in equation (2) we get $x^3 = 3xa^2$

Solving this for x we get $x = \pm\sqrt{3}a, x = 0$.

Thus the critical point is obtained as
$(\sqrt{3}a, \sqrt{3}a, \sqrt{3}a), and\ (-\sqrt{3}a, -\sqrt{3}a, -\sqrt{3}a)$

Substituting in the function (1) we get the minimum as $(9a^2)$.

Hence the proof.

Q.3 a Find the minimum of $f(x, y) = \alpha x^2 + \beta y^2$ for various values of α and β, by
a)computing the gradient of f and τ

Solution:

$$f(x, y) = \alpha x^2 + \beta y^2$$

$$\Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2\alpha x, 2\beta y)$$

Let $\left(x_{j+1}, y_{j+1}\right) = \left(x_j, y_j\right) - \tau \nabla f\left(x_j, y_j\right)$ for j=0,1,2...

$$\Rightarrow \left(x_{j+1}, y_{j+1}\right) = \left(x_j - 2\alpha\tau x_j, y_j - 2\beta\tau y_j\right)$$

Let $F(\tau) = f\left(\left(x_{j+1}, y_{j+1}\right)\right)$

Solving $F'(\tau) = 0$ we get

$$\tau = \frac{\alpha x_j^{\,2} + \beta y_j^{\,2}}{2\alpha^3 x_j^{\,2} + 2\beta^3 y_j^{\,2}}$$

Now for $\alpha = 1,\ \beta = 3$

$$\tau = \frac{x_j^2 + 9y_j^2}{2x_j^2 + 54y_j^2}.$$

Starting with $(x_0, y_0) = (1, 2)$, we get the following iterations:

j	x_j	y_j	$f(x_j, y_j)$
0	1.000000	2.000000	13.000000
1	0.660550	−0.036697	0.440367
2	0.033874	0.067749	0.014917
3	0.022376	−0.001243	0.000505
4	0.001147	0.002295	0.000017
5	0.000758	−0.000042	0.000001

For the minimum value of $f(x, y)$, the value of j is 6, x_6 and y_6 are 0.000758 and −0.000042, respectively.