EIGEN VALUE & EIGEN VECTORS [defined for square matrices]

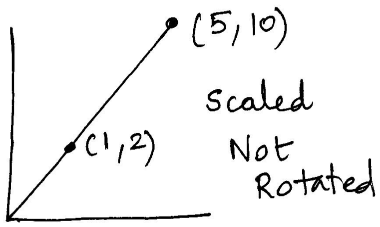
Product of matrix A & Vector $x = Ax$

A

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

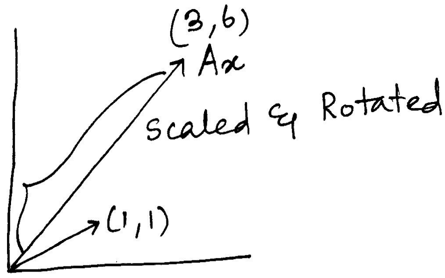
Case 1

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Case 2

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



A vector that undergoes pure scaling without rotation is known as EIGEN VECTOR.

Scaling factor / Stretch Ratio is known as EIGEN VALUE

$$Ax = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$Ax = 5X$ ↳ scalar

Eigen value / characteristic eqn $\boxed{Ax = \lambda x}$ where x is eigen vector & λ is eigen value.

Applications of Eigen Value –

1) Solve System of Linear differential equations
2) describe natural frequencies of vibrations
3) distinguish states of energy.

1) Eigen Vectors must be NON TRIVIAL (they cannot be zero vector)

2) A matrix is SINGULAR if and only if it has ZERO EIGEN VALUE

3) Trace of Matrix $tr(A)$ = Sum of elements on main diagonal
(defined only for square matrix)

$$\boxed{tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n} = \sum_{j=1}^{n} \lambda_j$$ Sum of eigenvalues of a matrix

$$\begin{bmatrix} 2 & -5 & 5 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad tr\, A = 2 + 3 + 3 = 8$$

4) $$\boxed{det(matrix) = \lambda_1 \lambda_2 \cdots \lambda_n} = \text{Product of Eigenvalues of Matrix}$$

$$= \prod_{j=1}^{n} \lambda_j$$

5) The eigenvalues of an upper (lower) triangular matrix are elements on the main diagonal.

6) If λ is Eigenvalue of A and A is invertible then $\frac{1}{\lambda}$ is an Eigenvalue of A^{-1}

7) Eigenvalues of A = A^T Eigenvalues

8) Square matrix $A_{n \times n}$ can have <u>Multiple Eigenvalues</u> but no more than its Number of Rows / Number of columns)

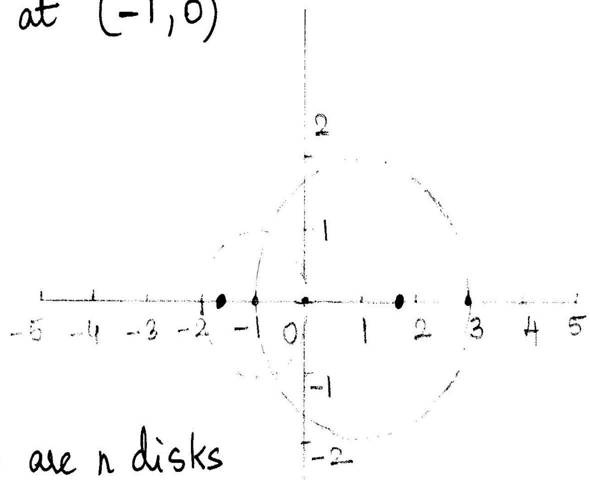
9) Eigenvalues of Real Symmetric matrix are Real.

10) Eigenvalues of Orthogonal matrix have $|\lambda| = 1$

GERSCHGORIN THEOREM – Range of Eigen values defined by union of all discs

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

From the rows of matrix A we get a disc with radius 2 centered at (1, 0) and a disc of radius 1 centered at (−1, 0)



Eigenvalues of matrix A are $\sqrt{3}$, $-\sqrt{3}$

For A_{nn} there are n disks in complex plane, each centered on one of the diagonal entries of the matrix

Every Eigen value must lie within one of these discs

∗ Thm does not state each disc has one eigenvalue

Using Gerschgorin Theorem

- Ranges for Eigenvalues can be found
- Especially Useful for LARGE MATRICES when calculating Eigenvalues can be impractical.

Inner Product / dot Product

$$a \cdot b = \underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{\text{Row Vector}} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\text{column vector}}$$

$$a \cdot b = \sum_{l=1}^{n} a_l b_l = a_1 b_1 + \cdots + a_n b_n$$

Eg) $a = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 2 & 5 & 8 \end{bmatrix}^T$

$$a \cdot b = \begin{bmatrix} 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = 4.2 + (-1).5 + 5.8 = \underline{\underline{43}}.$$

ORTHOGONAL - Two vectors are orthogonal if they are PERPENDICULAR to each other, dot product of two vectors is ZERO

Note :- If the length of vector is ONE, we say vector is NORMALIZED

ORTHONORMAL SET - A set of Orthogonal and Normalized vectors

$\downarrow$

orthogonal unit vectors.

Eg :- Set of vectors $\{v_1, v_2, v_3\}$ where $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ is set of orthogonal vectors

dot product of each two vectors is zero

$$v_1 \cdot v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = 1 - 1 = 0.$$

$v_1 \cdot v_3 = 0$ & $v_2 \cdot v_3 = 0$.

To normalize each vector, calculate Magnitude and divide each vector by magnitude

$$|v_1| = \sqrt{1+1} = \sqrt{2} \Rightarrow \begin{array}{c}\text{Normalized}\\ \text{vector}\end{array} \ \frac{1}{|v_1|} v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$|v_2| = \sqrt{1+2+1} = 2 \Rightarrow \begin{array}{c}\text{Normalized}\\ \text{vector}\end{array} \ \frac{1}{|v_2|} v_2 = \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ -1/2 \end{bmatrix}$$

$$|v_3| = \sqrt{1+2+1} = 2 \Rightarrow \begin{array}{c}\text{Normalized}\\ \text{vector}\end{array} \ \frac{1}{|v_3|} v_3 = \begin{bmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{bmatrix}$$

Orthonormal set is $\left\{ \frac{1}{|v_1|} v_1, \frac{1}{|v_2|} v_2, \frac{1}{|v_3|} v_3 \right\}$

$$= \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{bmatrix} \right\}$$

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vector has length 10 $\Longrightarrow$ Process of Normalization

This vector has a length of 1.

Take vector of any length, keeping it pointing in same direction change its length to 1. (unit vector)

QUADRATIC FORM

A homogeneous polynomial of Second degree in any number of variables

eg i) $ax^2 + 2hxy + by^2$

ii) $ax^2 + by^2 + cz^2 + 2hxy + 2gyz + 2fzx$ and

iii) $ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gyz + 2fzx$
$+ 2lxw + 2myw + 2nzw$.

are quadratic forms in two, three and four variables.

Writing quadratic form in matrix form:-

$$ax^2 + 2hxy + by^2 = [x \ y]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Quadratic Form given by

$$Q = x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$$

Coefficient matrix A is Real Symmetric, therefore has n linearly independent orthonormal set of eigenvectors corresponding to n eigenvalues.

$$Q = X^T \underline{\underline{A}} X = Y^T \underline{\underline{D}} Y.$$ ↳ Canonical form / Sum of squares form / Principal Axes form.

Transformation of Quadratic Form to Principal Axes

Conic Sections

Pbm 20) $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16.$

$$Q = x^T A x.$$