HOMEWORK - 2 SOLUTION

Q1. Let $B = (b_1, b_2 \cdots b_{r-1}, b_r, b_{r+1} \cdots, b_n)$ be a non-singular matrix. If column b_r is replace by 'a' and the resulting matrix is called B_a along with $a = \sum_{i=1}^{n} y_i b_i$, then state the necessary and sufficient condition for B_a to be non-singular

Solution:

Given $B = (b_1\ b_2 \cdots b_{r-1}, b_r, b_{r+1} \cdots b_n)$ is a non-singular matrix

The vectors $b_1, b_2 \cdots b_n$ are linearly independent

Take

$$b_r = a = \sum_{i=1}^{n} y_i b_i$$

$$B_a = (b_1, b_2 \cdots b_{r-1}, \sum_{i=1}^{n} y_i b_i, b_{r+1} \cdots b_n)$$

for B_a to be non-singular,

$b_1, b_2, \cdots b_{r-1}, \sum_{i=1}^{n} y_i b_i, b_{r+1}, b_n$ need to be L.I.

$\because (b_1, b_2 \cdots b_{r-1}, b_{r+1} \cdots b_n) \subseteq (b_1\ b_2 \cdots b_r, b_{r+1} \cdots b_n)$

$(b_1\ b_2 \cdots b_{r-1}, b_{r+1} \cdots b_n)$ is also linearly independent

WKT "A set with non-zero vector is linearly independent"

To get B_a is non-singular, the vectors $(b_1, b_2 \cdots \sum_{i=1}^{n} y_i b_i, \cdots b_n)$ should form L.I

$\therefore$ The required conditions are

→ All y_i vectors should not equal

→ Atleast one y_i vector should be non-zero.

Qs2. Let V be a finite dimensional vector space over $\mathbb{R}$. If S is a set of elements in V such that $Span(S) = V$, what is the relationship between S and the basis of V?

Solution:

Given V is finite dimensional vectorspace

Let 'B' be the basis of V

→ B posses the properties

- Vectors in 'B' are linearly independent
- $span(B) = V$

Given $Span(S) = V$

- Vectors in 's' may or may not linearly independent

Case(i): If vectors in 'S' are linearly independent

$span(B) = span(S)$, hence $B = S$

case(ii): If vectors in S are not linearly independent

B is a subset of S, $B \subseteq S$

Qs. 3. Let $T: R^3 \rightarrow R^3$ be defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$

then (1) Show that T is a linear Transformation

(2) What are conditions on a, b, c such that (a, b, c) is in the null space of T. Specifically, find the nullity of T.

Solution:

T is linear transformation if for all vectors $u, v \in R^3$, and Scalar k,

- $T(u+v) = T(u) + T(v)$
- $T(ku) = k\,T(u)$

Let $u = (x_1, x_2, x_3)$ $\quad v = (y_1, y_2, y_3)$

$$u+v = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3)$$

$$T(u+v) = \big(x_1 + y_1 - (x_2 + y_2) + 2(x_3 + y_3),\ 2(x_1 + y_1) + (x_2 + y_2),\ -(x_1 + y_1) - 2(x_2 + y_2) + 2(x_3 + y_3)\big)$$

$$= \big(x_1 - x_2 + 2x_3 + y_1 - y_2 + 2y_3,\ 2x_1 + x_2 + 2y_1 + y_2,\ -x_1 - 2x_2 + 2x_3 - y_1 - 2y_2 + 2y_3\big)$$

$$= (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3) + (y_1 - y_2 + 2y_3,\ 2y_1 + y_2,\ -y_1 - 2y_2 + 2y_3)$$

$$\boxed{T(u+v) = T(u) + T(v)}$$

$$T(ku) = T((kx_1, kx_2, kx_3))$$

$$= (kx_1 - kx_2 + 2kx_3,\ 2kx_1 + kx_2,\ -kx_1 - 2kx_2 + 2kx_3)$$

$$= k(x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x - 2x_2 + 2x_3)$$

$$\boxed{T(ku) = k\,T(u)}$$

$\therefore$ $T: R^3 \rightarrow R^3$ is a linear under the given tranformation.

(2) Kernel of T is known as nullspace of T

Null space of T is the solution space of homogeneous system $AX=0$ where

$$AX = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Consider

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -2 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

Solving, $x_3 = t$

$x_2 = \frac{4}{3} x_3 = \frac{4}{3} t$

$x_1 = -2/3\, x_3 = -2/3\, t$

If (a, b, c) is in nullspace of T, the required Condition, $\boxed{a = -2/3\, c, \quad b = 4/3\, c}$

Since Rank $(A) = 3$, nullity(A) + Rank$(A) = 3$

Hence $\boxed{\text{nullity}(A) = 1}$

Qs. 4. Construct a linear transformation $T: V \to W$ Where V and W are vector spaces over F such that the dimension of the kernel space of T is 666. Is such a transformation unique? Give reasons for your answer.

Solution:

Given dimension of kernel space of T is 666.

Such a transformation is not unique as the kernal space ~~of~~ do not uniquely determined the linear transformation

Coutnter example:

Let $B_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ $B_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$

Where first and last 666 columns of B_1 and B_2 are zero columns respectively.

$T_1(\bar{x}) = B_1 \bar{x}$ and $T_2(\bar{x}) = B_2 \bar{x}$ then dimension of kernel of both T_1 and T_2 is 666.