BITS Pilani

Pilani Campus

Mathematical Foundations for Data Science

MFDS Team

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Pilani Campus

DSECL ZC416, MFDS

Lecture No.13

MFDS Team

Agenda

- Equivalence relations
- Equivalence classes
- Partial ordering
- Hasse diagram
- Lattices

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

innovate achieve lead

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(c)$ also holds and aRc.

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\,m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

$$a - c = (a - b) + (b - c) = km + lm = (k + l)\,m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a+2m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \varnothing$

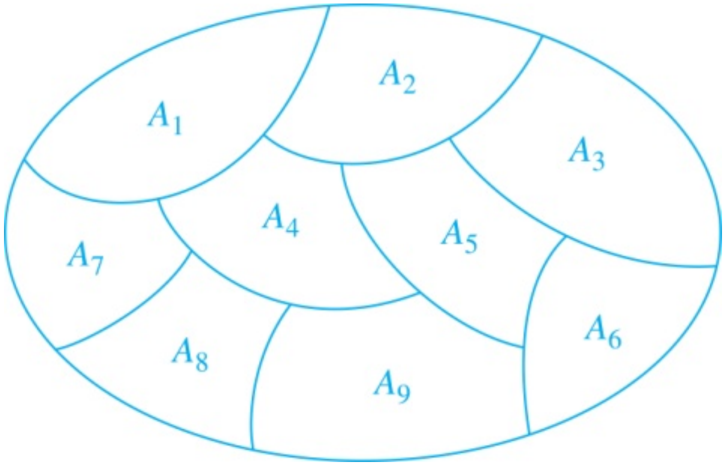
Proof: We show that (i) implies (ii). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and

$$\bigcup_{i \in I} A_i = S.$$



A Partition of a Set

An Equivalence Relation Partitions a Set

Let R be an equivalence relation on a set A. The union of all the equivalence classes of R is all of A, since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

.

From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset \text{ when } [a]_R \neq [b]_R.$$

Therefore, the equivalence classes form a partition of A, because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set (*continued*)

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric and transitive.

- A set together with a partial ordering R is called a *partially ordered set*, or *poset*, I
- It is denoted by (S, ≤)
- Members of S are called *elements* of the poset.

Partial Orderings

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

– *Reflexivity*: $a \geq a$ for every integer a.

– *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.

– *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

Partial Orderings

Example 2: Show that the divisibility relation ($\mid$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a.
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

$(\mathbf{Z}^+, \mid)$ is a poset.

Partial Orderings

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Comparability

Definition 2: The elements a and b of a poset $(S,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

Definition 4: $(S,\preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

This definition can be easily extended to a lexicographic ordering on strings

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

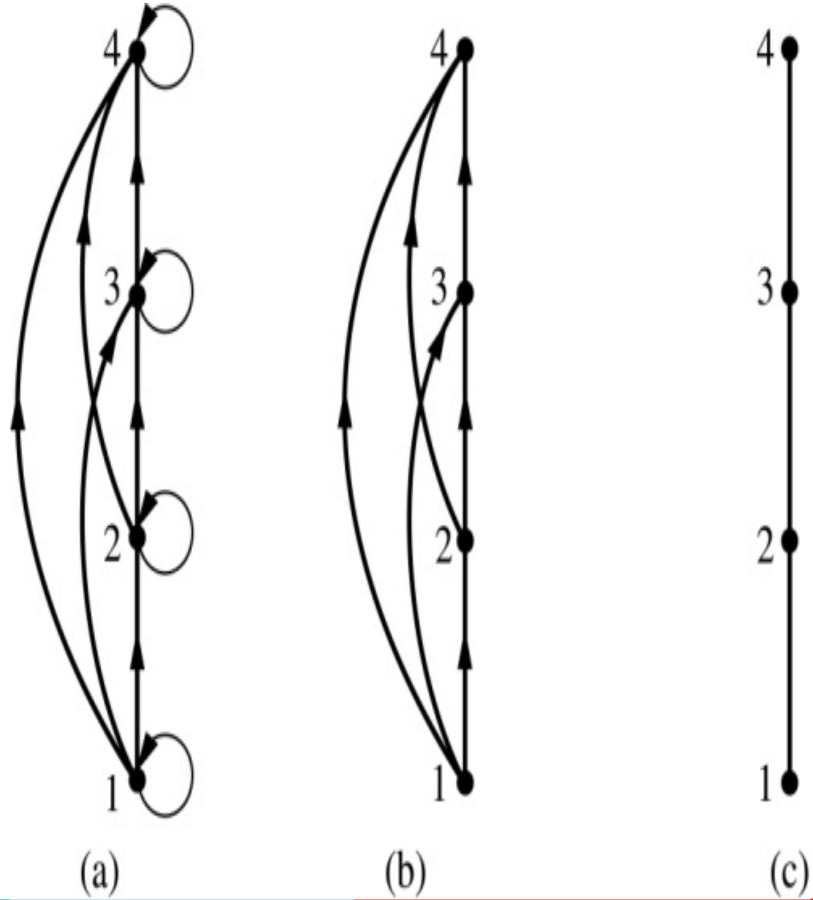
Hasse Diagram

A *Hasse diagram* is a **visual representation** of a partial ordering

It leaves out edges that must be present because of

- the reflexive
- and transitive properties.

$\{(1,2,3,4), \leq\}$



A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

Procedure for Constructing a Hasse Diagram

To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Totally Ordered Sets

$(S, \leq)$ is called a Totally ordered ser if

1. $(S, \leq)$ is a POSET
2. Every two elements in S are comparable using $\leq$

Example $(Z, \leq)$

Maximal and Minimal Element

Maximal element of a POSET $(P, \leq)$ is an element M in P that satisfy the following property:
a is a maximal element in the POSET $(P, \leq)$ if there is no $b \in P$ such that $a < b$

Minimal element of a POSET $(P, \leq)$ is an element M in P that satisfy the following property:
a is a minimal element in the POSET $(P, \leq)$ if there is no $b \in P$ such that $b < a$

Greatest and Least Element

Greatest element of a POSET (P, ≤) is an element a in P that satisfy the following property:
a is a greatest element in the POSET (P, ≤) if b ≤ a for all b ∈ P

Least element of a POSET (P, ≤) is an element a in P that satisfy the following property:
a is a least element in the POSET (P, ≤) if a ≤ b for all b ∈ P

Upper Bound of a Subset A of P

- If u is an element of P such that for all elements $a \in A$, a $\preccurlyeq$ u, then u is called an **upper bound** of A.

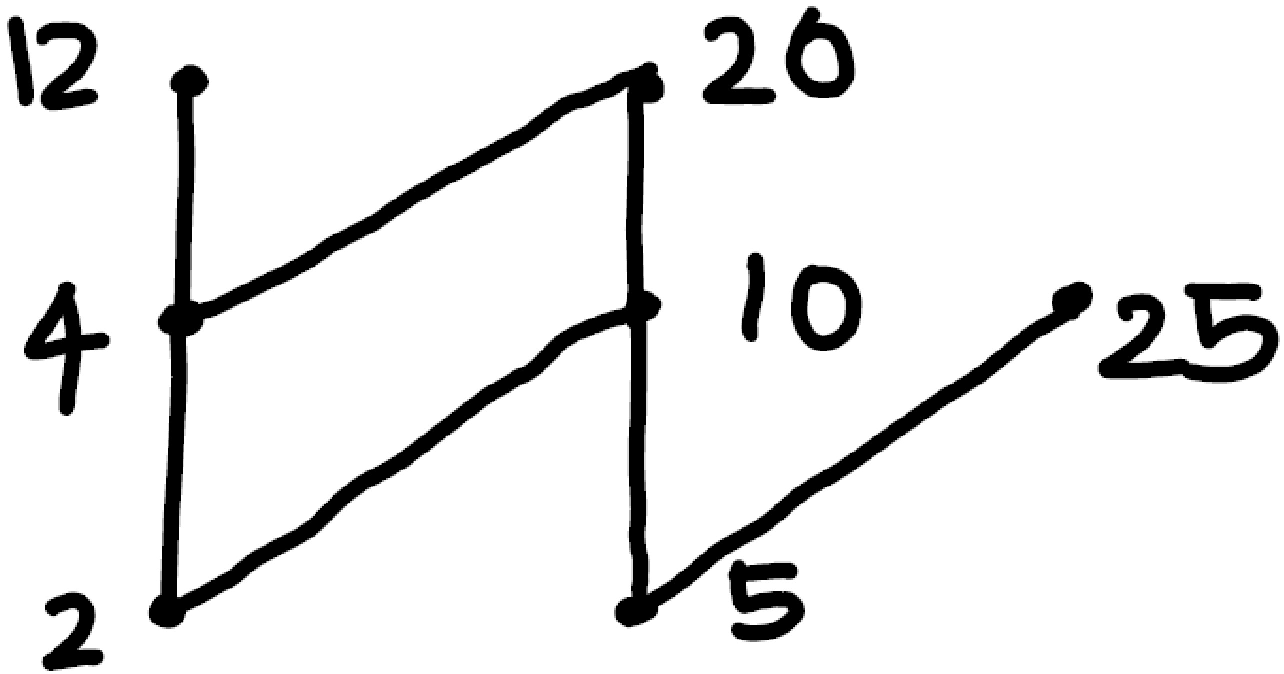
- x is the least upper bound of A if whenever a $\in$ A, x $\preccurlyeq$ z whenever z is an upper bound of A

Lower Bound of a Subset A of P

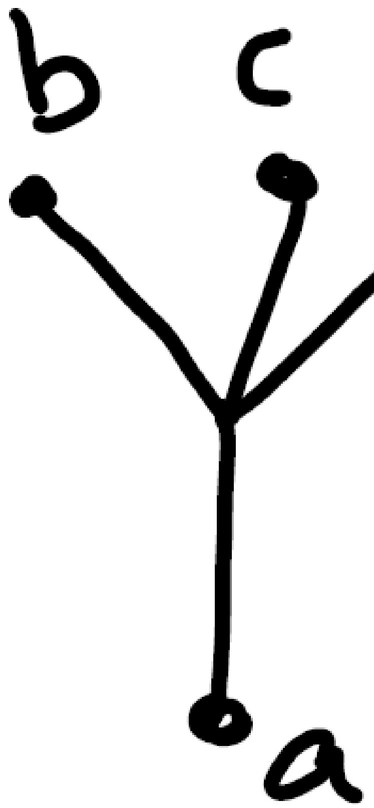
- If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

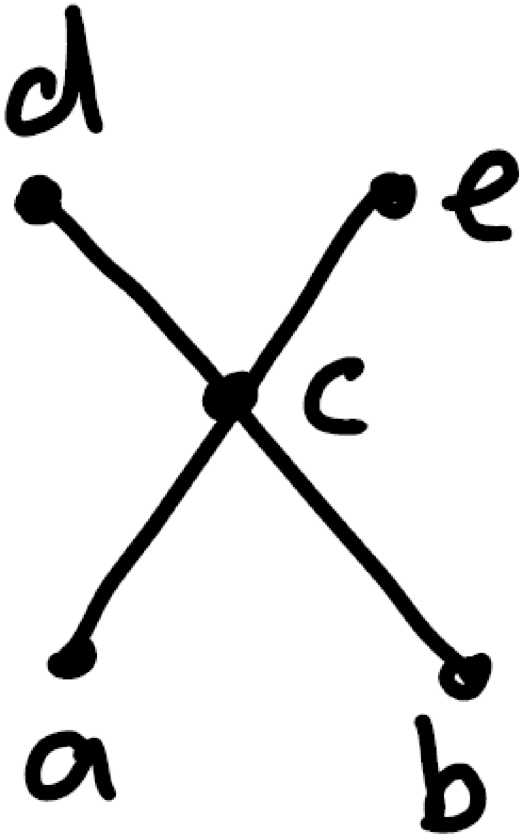
- The element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq l$ whenever z is a lower bound of A

Examples



Examples





Lattice

A POSET is said to be a lattice if it satisfies the following property:

Every pair of elements in P will have a least upper bound and greatest lower bound.

Example

The POSET (Z, |) forms a lattice :
Let a and b be two integers.:

- Least Upper Bound of pair a,b is their LCM
- Greatest Lower Bound of pair a,b is their GCD

Example

Poset with the Hasse diagram shown in the figure below is NOT A LATTICE as elements b and c have no least upper bound.