BITS Pilani
Pilani Campus

Mathematical Foundations for Data Science

MFDS Section-2 Team

BITS Pilani
Pilani Campus

DSECL ZC416, MFDS

Webinar No.1

Q.1 Let $A_{m \times n}$ be a given matrix with m > n. If the time taken to compute the determinant of a square matrix of size' j ' is j^3. find upper bound on the

a) total time taken to find the rank of A using determinants

Solution:

Definition of Rank of Matrix :

'k' is said to be the rank of Matrix A if

(a) There exist at least one minor of order 'k' which is nonzero and

(b) All minors of order 'k+1', if they exist are zero.

Thus Rank of A is the largest order of any non-zero minor in A

Note:

i) The rank of a matrix would be zero only if the matrix had all zero elements.

ii) If a matrix had even one non-zero element, its minimum rank would be one.

$$\begin{bmatrix} 4 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}_{5 \times 3} \quad [m \times n]$$

(a) 1> 3 order Minor $5C_3 \times 3C_3$

$= 10 \times 1 = 10$

2> 2 order Minor $5C_2 \times 3C_2$

$= 10 \times 3 = 30$

Time for 3 order Minor $= 10\,(3)^3$.

Time for 2 order Minor $= 30\,(2)^3$

Total time $= 10(3)^3 + 30(2)^3$.

Given m > n then maximum rank of A can be n.
In general rank of matrix $\leq$ min(m, n).
Thus minors order starts from n to 2.
As the time taken to compute the determinant of a square matrix of size' j ' is j^3.
Upper bound on the total time taken to find the rank of A using determinants is

$$\sum_{k=2}^{n}\left({}^{m}C_{k} \times {}^{n}C_{k}\right).k^{3}$$

Q.1 Let $A_{m \times n}$ be a given matrix with m > n. If the time taken to compute the determinant of a square matrix of size' j ' is j^3. find upper bound on the

b) number of additions and multiplications required to determine the rank using the elimination procedure.

$$\begin{bmatrix} 4 & 2 & 0 \\ -1 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}_{5\times 2} \quad [m \times n]$$

(b) Below a_{11} 4 multiplications 3 times.

a_{22} 3 multiplications 2 times

a_{33} 2 multiplications 1 time.

Similar for addition.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Solution: Given matrix of order m by n where m > n .

In elimination procedure to get the rank we will make all elements below main diagonal zero.

To make zeros below a_{11} in first column we require (m-1) multiplications n times.

Similar thing is for addition.

Now to make zeros below a_{22} in second column we require (m-2) multiplications (n-1) times.

Similar thing is for addition.

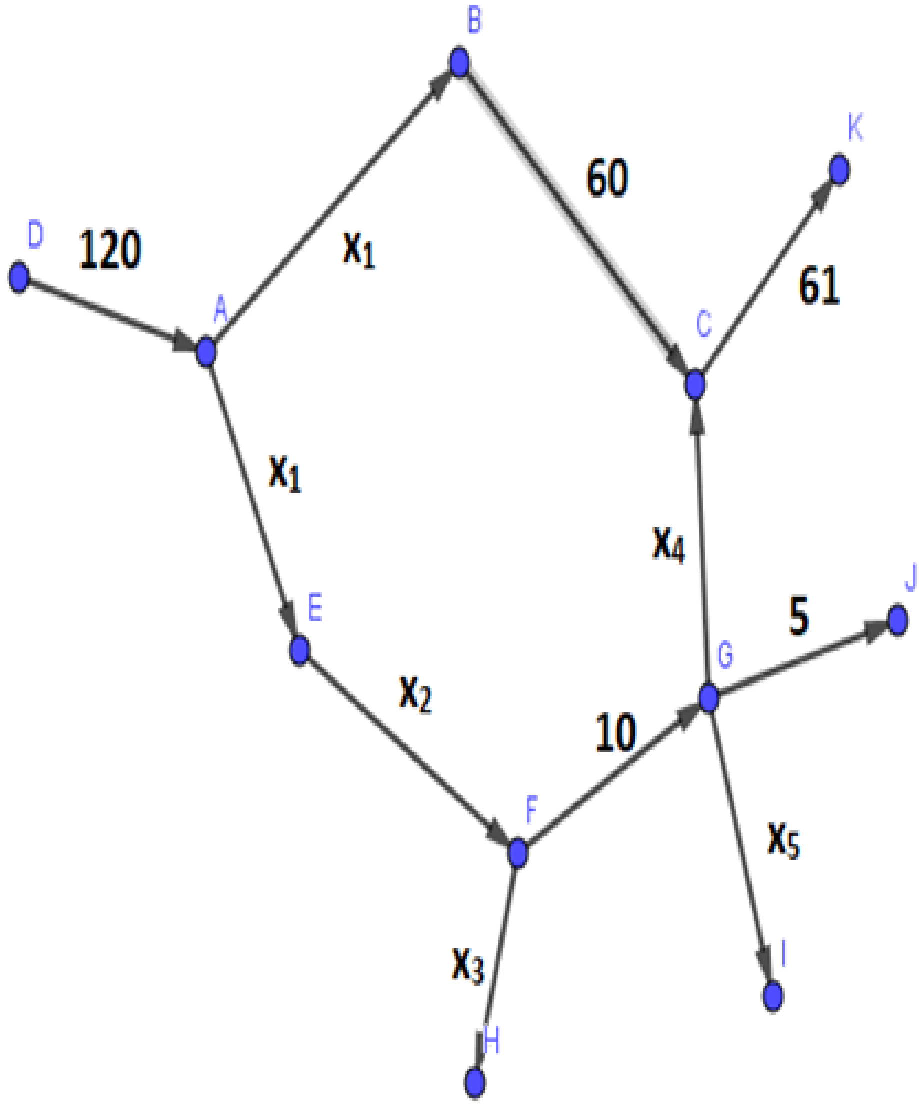
Thus total number of multiplications and additions required to determine the rank by elimination procedure are

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ & \ddots & \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$2.\sum_{k=1}^{n}(m-k)(n-k+1)$$

Q.3 Modelling of electrical / traffic networks would lead to a linear system $Ax = b$. Refer to the text book / other resources and construct a network which has the following properties

a) the number of equations are 6.
b) A has rank 5.
c) the system is consistent.



$$\text{Vertex A} \Rightarrow x_1 + x_1 = 120 \Rightarrow 2x_1 = 120$$

$$\text{Vertex } B \Rightarrow x_1 = 60$$

$$\text{Vertex } C \Rightarrow x_4 + 60 = 61 \Rightarrow x_4 = 1$$

$$\text{Vertex } G \Rightarrow x_5 + 5 = 10 \Rightarrow x_5 = 5$$

$$\text{Vertex } F \Rightarrow x_2 = x_3 + 10$$

$$\text{Vertex } E \Rightarrow x_1 = x_2$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 120 \\ 1 & 0 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 1 & -1 & 0 & 0 & 10 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

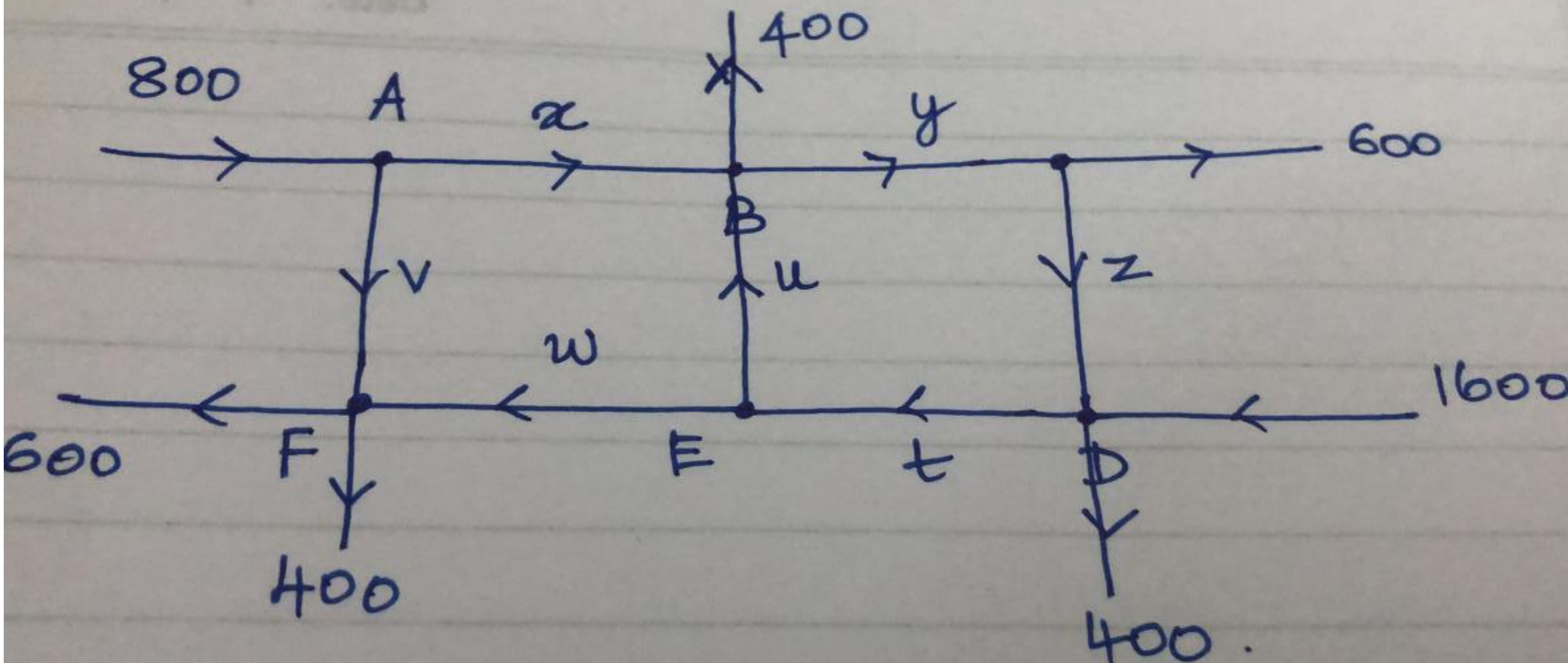
$$x_1 = 60$$

$$x_2 = 60$$

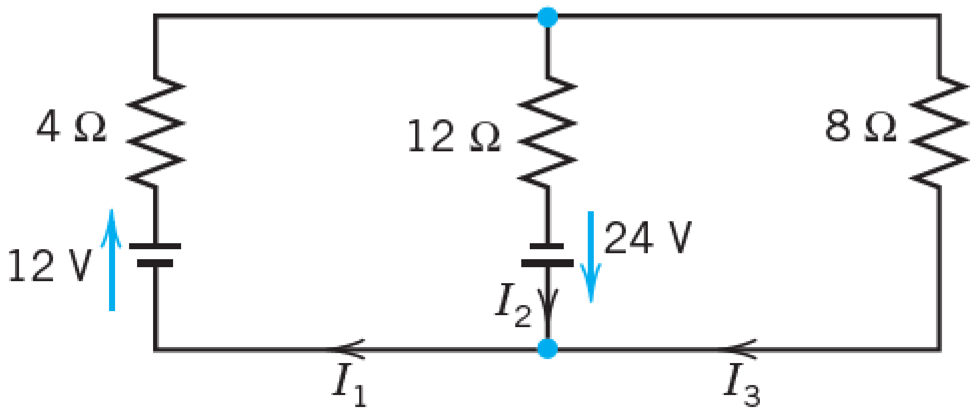
$$x_3 = 50$$

$$x_4 = 1$$

$$x_5 = 5$$


400
800
A
x
y
600
B
v
u
z
w
1600
600
F
E
t
D
400
400.

Exercise Q. 18 :
Using Kirchhoff's laws and showing the details, find the currents:



$$I_1 = I_2 + I_3$$

$$4I_1 + 12I_2 = 12 + 24$$

$$12I_2 - 8I_3 = 24$$

Converting given equations into matrix form

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 4 & 12 & 0 & 36 \\ 0 & 12 & -8 & 24 \end{array}\right]$$

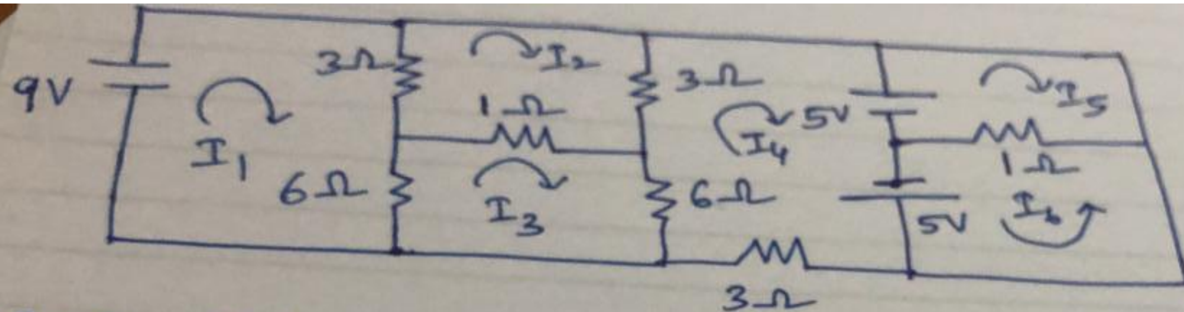
$$R_2 \leftarrow R_2 - 4 \times R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 16 & 4 & 36 \\ 0 & 12 & -8 & 24 \end{array}\right]$$

$$R_3 \leftarrow R_3 - \frac{3}{4} \times R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 16 & 4 & 36 \\ 0 & 0 & -11 & -3 \end{array}\right]$$

$$I_1 = \frac{27}{11}, I_2 = \frac{24}{11}, I_3 = \frac{3}{11}$$



$$3(I_1 - I_2) + 6(I_1 - I_3) = 9.$$

$$3(I_2 - I_1) + 3(I_2 - I_4) + (I_2 - I_3) = 0.$$

$$6(I_3 - I_1) + (I_3 - I_2) + 6(I_3 - I_4) = 0$$

$$3(I_4 - I_2) + 6(I_4 - I_3) + 3I_4 = 0.$$

$$I_5 + I_6 = 5$$

$$I_5 + I_6 = 5.$$

Converting given equations into matrix form

$$\left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ -3 & 7 & -1 & -3 & 0 & 0 & 0 \\ -6 & -1 & 13 & -6 & 0 & 0 & 0 \\ 0 & -3 & -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_2 \leftarrow R_2 + \frac{1}{3} \times R_1$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ -6 & -1 & 13 & -6 & 0 & 0 & 0 \\ 0 & -3 & -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{2}{3} \times R_1$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & -3 & 9 & -6 & 0 & 0 & 6 \\ 0 & -3 & -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_3 \leftarrow R_3 + \frac{1}{2} \times R_2$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} & 0 & 0 & \frac{15}{2} \\ 0 & -3 & -6 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_4 \leftarrow R_4 + \frac{1}{2} \times R_2$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} & 0 & 0 & \frac{15}{2} \\ 0 & 0 & -\frac{15}{2} & \frac{21}{2} & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_4 \leftarrow R_4 + R_3$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} & 0 & 0 & \frac{15}{2} \\ 0 & 0 & 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

$R_6 \leftarrow R_6 - R_5$

$$= \left[\begin{array}{cccccc|c} 9 & -3 & -6 & 0 & 0 & 0 & 9 \\ 0 & 6 & -3 & -3 & 0 & 0 & 3 \\ 0 & 0 & \frac{15}{2} & -\frac{15}{2} & 0 & 0 & \frac{15}{2} \\ 0 & 0 & 0 & 3 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

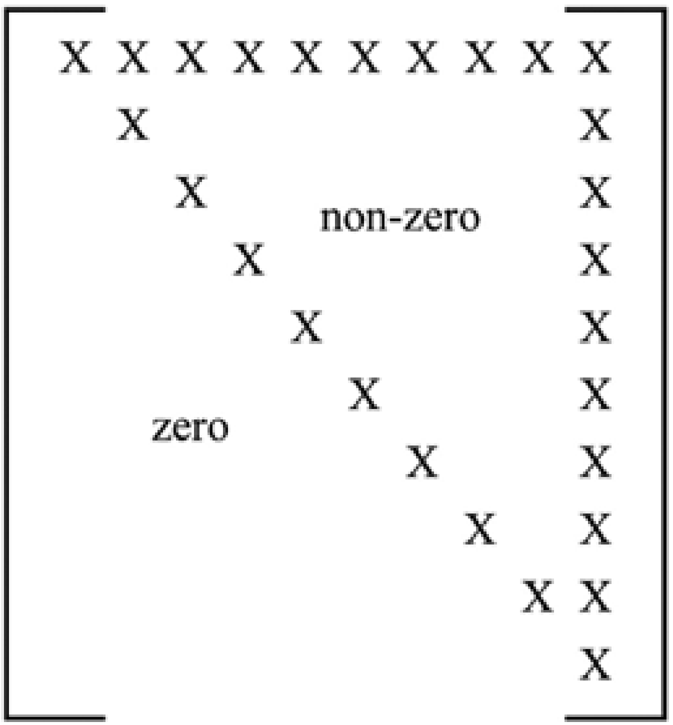
$$I_1 = 5$$
$$I_2 = 4$$
$$I_3 = 4$$
$$I_4 = 3$$
$$I_5 = 5 - t$$
$$I_6 = t$$

Q. Suppose M is an $n \times n$ upper triangular matrix with diagonal entries non- zero, then prove or disprove that the column vectors are linearly independent.

What happens when some of the diagonal entries are zero?

Solution :

Square matrix in which all the entries below the main diagonal are zero, $A = \left[a_{ij}\right]_{n \times n}$, $a_{ij} = 0$ whenever $i > j$.



Consider an upper triangular matrix $A_{n\times n}$ with entries a_{ij} .
If constant α_j is associated with column vector j of A,
then in equation Aα = 0 form, the first column yields,
$\alpha_1 . a_{11} = 0$ which yields, $\alpha_1 = 0$ as $a_{11} \neq 0$.
Similar arguments, in a recursive way, would yield
all $\alpha_j = 0$, ∀j = 1, 2, . . . , n, thus proving Linear Independence of the column vectors.

If $a_{kk} = 0$ then α_k need not be zero and hence we may conclude that the column vectors of A are Linearly Dependent.

Gauss Elimination method Operation Count : Page 849 of Text book

Operation Count

Quite generally, important factors in judging the quality of a numeric method are

Amount of storage

Amount of time ($\equiv$ number of operations)

Effect of roundoff error

For the Gauss elimination, the operation count for a full matrix (a matrix with relatively many nonzero entries) is as follows. In Step k we eliminate x_k from $n - k$ equations. This needs $n - k$ divisions in computing the m_{jk} (line 3) and $(n - k)(n - k + 1)$ multiplications and as many subtractions (both in line 4). Since we do $n - 1$ steps, k goes from 1 to $n - 1$ and thus the total number of operations in this forward elimination is

$$f(n) = \sum_{k=1}^{n-1} (n - k) + 2\sum_{k=1}^{n-1} (n - k)(n - k + 1) \qquad \text{(write } n - k = s\text{)}$$

$$= \sum_{s=1}^{n-1} s + 2\sum_{s=1}^{n-1} s(s + 1) = \tfrac{1}{2}(n - 1)n + \tfrac{2}{3}(n^2 - 1)n \approx \tfrac{2}{3}n^3$$

where $2n^3/3$ is obtained by dropping lower powers of n. We see that $f(n)$ grows about proportional to n^3. We say that $f(n)$ is of *order* n^3 and write

$$f(n) = O(n^3)$$

where O suggests **order**. The general definition of O is as follows. We write

$$f(n) = O(h(n))$$

if the quotients $|f(n)/h(n)|$ and $|h(n)/f(n)|$ remain bounded (do not trail off to infinity) as $n \to \infty$. In our present case, $h(n) = n^3$ and, indeed, $f(n)/n^3 \to \frac{2}{3}$ because the omitted terms divided by n^3 go to zero as $n \to \infty$.

In the back substitution of x_i we make $n - i$ multiplications and as many subtractions, as well as 1 division. Hence the number of operations in the back substitution is

$$b(n) = 2\sum_{i=1}^{n} (n - i) + n = 2\sum_{s=1}^{n} s + n = n(n + 1) + n = n^2 + 2n = O(n^2).$$

We see that it grows more slowly than the number of operations in the forward elimination of the Gauss algorithm, so that it is negligible for large systems because it is smaller by a factor n, approximately. For instance, if an operation takes 10^{-9} sec, then the times needed are:

Algorithm	$n = 1000$	$n = 10000$
Elimination	0.7 sec	11 min
Back substitution	0.001 sec	0.1 sec

Thank You!!