Work Integrated Learning Programmes Division

M.Tech (Data Science and Engineering)

# DSECLZC416 - Mathematical Foundations for Data Science

**INSTRUCTONS**

1. **Assignments have to be handwritten, images to be captured in a doc file. Convert the doc file to a pdf file and name it as your BITSID.pdf and upload the file**
2. **Assignments sent via email would not be accepted**
3. **Submissions beyond 17th of August, 17.00 hrs would not be graded.**

**Assignment 2 - Total Marks – 10**

Problem 1: Write a code in Python for Naïve and Warshall's algorithm for finding the transitive closure for the given relation. Use random matrices of order 10 to 100 and compare the time taken by Naïve method and Warshall's Algorithm. Show the log log plot of the time taken and determine the order

(5 Marks)

Problem 2: Prove that if m and n are positive integers and x is a real number then

$$\left\lfloor \frac{\lfloor x \rfloor + n}{m} \right\rfloor = \left\lfloor \frac{x+n}{m} \right\rfloor$$

(2 Marks)

Problem 3: Derive the volume of greatest rectangular parallelepiped that can be inscribed in an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(2 Marks)

Problem 4: Prove that if $A \subseteq B$ and $C \subseteq D$, then $(A \cup C) \subseteq (B \cup D)$ and $(A \cap C) \subseteq (B \cap D)$

(1 Mark)

**Sol-1: <u>Python code:</u>** (1 mark)

```
# -*- coding: utf-8 -*-
"""naive.ipynb

import matplotlib.pyplot as plt
import numpy as np
import time



def Boolean_Matrix_Multiply(X,Y,n):

  result= [[0 for i in range(n)] for j in range(n)]

  for i in range(n):
     for j in range(n):
         for k in range(n):
            result[i][j] = result[i][j] or  (X[i][k] and Y[k][j])
  return result

def Boolean_Matrix_OR(A,B,n):
# iterate through rows of X
  result= [[0 for i in range(n)] for j in range(n)]
  for i in range(n):
   # iterate through columns of Y
     for j in range(n):
       # iterate through rows of Y
           result[i][j] = A[i][j] or B[i][j]
  return result

def Naive_Algorithm(MR,n):
  A=MR

  B=A

  for i in range(2,n-1):
    A = Boolean_Matrix_Multiply(A,MR,n)

    B = Boolean_Matrix_OR(B,A,n)
  return B

def Warshall(W,n):

  for i in range(n):
     for j in range(n):
         for k in range(n):
            W[i][j] = W[i][j] or (W[i][k] and W[k][j])
  return W


   n_range = np.arange(10, 101, 1).tolist()
   time_naive =np.zeros(91)

   time_warsh =np.zeros(91)
   i=1;

   for n in range(10,100):
```

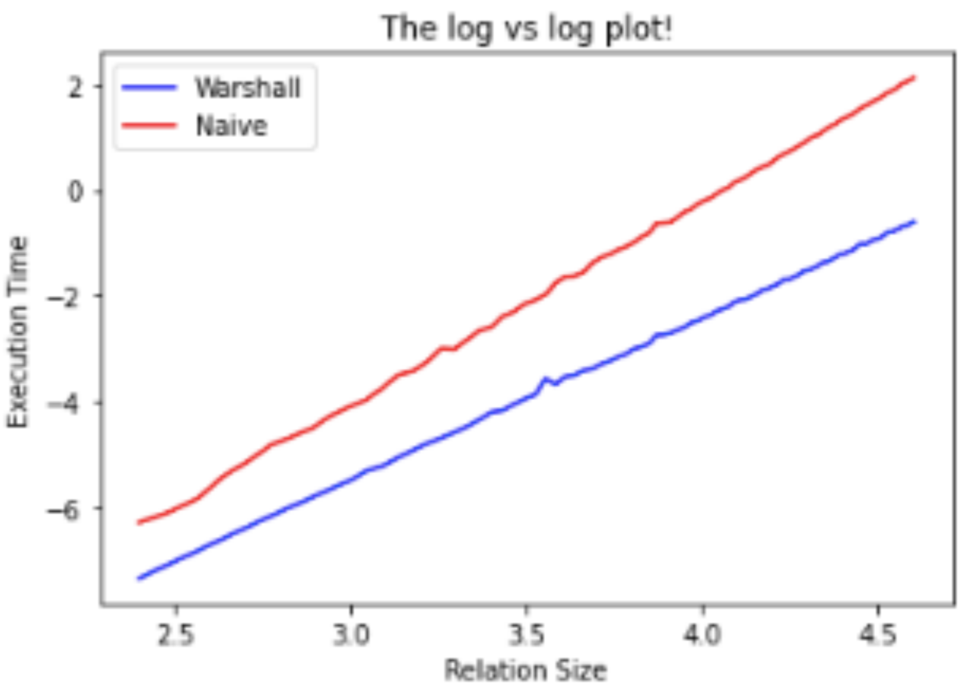
```
    MR = np.random.randint(0,2,size=(n,n))

    start_time_naive= time.time()
    naive=Naive_Algorithm(MR,n)

    execution_time_naive= time.time() - start_time_naive

    time_naive[i] =   execution_time_naive
    start_time_warshall = time.time()
    warsh =Warshall(MR,n)

    execution_time_warshall= time.time() - start_time_warshall
    time_warsh[i] =   execution_time_warshall

    i=i+1


plt.plot(np.log(n_range), np.log(time_warsh), "-b", label="Warshall")
plt.plot(np.log(n_range), np.log(time_naive), "-r", label="Naive")
plt.xlabel('Relation Size')
plt.ylabel('Execution Time')
plt.title('The log vs log plot!')
plt.legend(loc="upper left")
plt.show()

print(time_naive)
```

**Order of the Algorithms:** (2 marks)

Warshall Algorithm: O(n^3)

Naive Algorithm: O(n^4)

**Graph:** (2 marks)



**Sol-2** We know $x \in \mathbb{R}$. Let $x = a + t$, where $a$ is an integer and $0 \leq t < 1$. Then

$$\lfloor x \rfloor = a.$$

Now, we need to show:

$$\left\lfloor \frac{a+n}{m} \right\rfloor = \left\lfloor \frac{a+t+n}{m} \right\rfloor \quad \to \textbf{(1 mark)}.$$

Here, we note that both $t$ and $m$ are positive and $t < m$. Now,

$$\left\lfloor \frac{a+n}{m} \right\rfloor \leq \left\lfloor \frac{a+t+n}{m} \right\rfloor \leq \left\lfloor \frac{a+n}{m} \right\rfloor + 1. \tag{1}$$

Let $s = \left\lfloor \frac{a+t+n}{m} \right\rfloor$. Then

$$s \leq \frac{a+t+n}{m} \;\Rightarrow\; a+t+n \geq sm.$$

All numbers are integers except $t$. So, we get

$$a + n \geq sm.$$

i.e

$$\frac{a+n}{m} \geq s.$$

Therefore, we have

$$\left\lfloor \frac{a+n}{m} \right\rfloor \geq s = \left\lfloor \frac{a+t+n}{m} \right\rfloor. \tag{2}$$

From Equations (1) and (2), we get

$$\left\lfloor \frac{a+n}{m} \right\rfloor = \left\lfloor \frac{a+t+n}{m} \right\rfloor. \quad \to \textbf{(1 mark)}$$

**Sol-3** Volume $= V = 2x \times 2y \times 2z = 8xyz \quad \to$ **(0.5 mark)**.
Maximize $V = 8xyz$ subject to

$$x^2 + y^2 + z^2 = 1. \quad \to \textbf{(0.5 mark)}$$

Let $g = x^2 + y^2 + z^2 - 1 = 0$ then

$$L(x, y, x, \lambda) = 8xyz + \lambda(x^2 + y^2 + z^2 - 1).$$

Now

$$\frac{\partial L}{\partial x} = 8yz + 2\lambda x, \tag{3}$$

$$\frac{\partial L}{\partial y} = 8xz + 2\lambda y, \tag{4}$$

$$\frac{\partial L}{\partial z} = 8xy + 2\lambda z, \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1. \tag{6}$$

Equation (3) $\Rightarrow z = -\frac{\lambda x}{4y}$
Equation (4) $\Rightarrow \lambda = 0$ or $y - \frac{x^2}{y} = 0$
Substituing $z = -\frac{\lambda x}{4y}$ in Equation (5), we get $16y^2 = \lambda^2$. Thus

$$x^2 = y^2 = \frac{\lambda^2}{16}, \quad \rightarrow \textbf{(0.5 mark)}$$

$$z^2 = \frac{\lambda^2 x^2}{16y^2}.$$

Hence $x = y = z = \frac{1}{\sqrt{3}}$ and

$$V = 8xyz = 8(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}) = \frac{8}{3\sqrt{3}}.$$

Now,

$$V = V_{cube} \times a \times b \times c = \frac{8abc}{3\sqrt{3}}. \quad \rightarrow \textbf{(0.5 mark)}$$

**Sol-4** Given $A \subseteq B$ and $C \subseteq D$. If $x \in A \cup C$, then

$$x \in A \text{ or } x \in C.$$

If $x \in A$, then since $A \subseteq B$, we get $x \in B$. So, $x \in B$ or $x \in D$ is true. Therefore,

$$x \in B \cup D.$$

If $x \in C$, Since $C \subseteq D$, we get $x \in D$. So, $x \in B$ or $x \in D$ is true. Therefore,

$$x \in B \cup D.$$

Hence if $A \subseteq B$ and $C \subseteq D$, then

$$A \cup C \subseteq B \cup D. \quad \rightarrow \textbf{(0.5 mark)}$$

Now, given $A \subseteq B$ and $C \subseteq D$. If $x \in A \cap C$, then

$$x \in A \text{ and } x \in C.$$

Since $A \subseteq B$, we get

$$x \in B. \tag{7}$$

Similarly, since $c \subseteq S$, we get

$$x \in D. \tag{8}$$

From Equations (7) and (8), we get

$$x \in B \cap D.$$

Hence, $A \cap C \subseteq B \cap D$. $\rightarrow$ **(0.5 mark)**