BITS Pilani
Hyderabad Campus

Data Structures and Algorithms Design

Febin.A.Vahab
2019-20

CONTACT SESSION 6 -PLAN

Contact Sessions(#)	List of Topic Title	Text/Ref Book/external resource
6	Graphs - Terms and Definitions, Properties, Representations (Edge List, Adjacency list, Adjacency Matrix), Graph Traversals (Depth First and Breadth First Search)	T1: 6.1, 6.2, 6.3

Depth-First Search

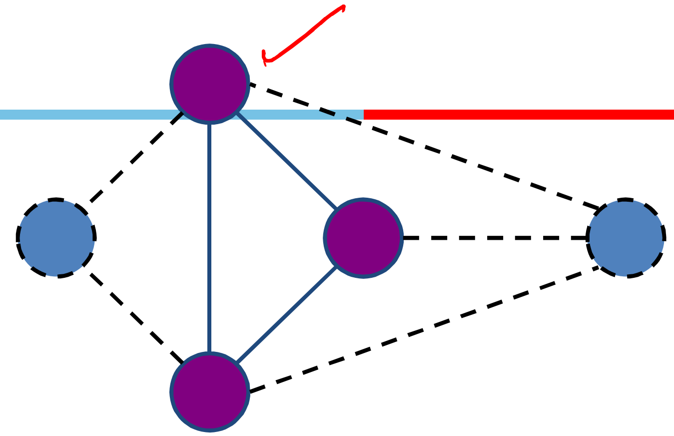
- Definitions
  - Subgraph
  - Connectivity
  - Spanning trees and forests
- Depth-first search
  - Algorithm
  - Example
  - Properties
  - Analysis
- Applications of DFS
  - Cycle finding
  - Path finding

SUBGRAPHS

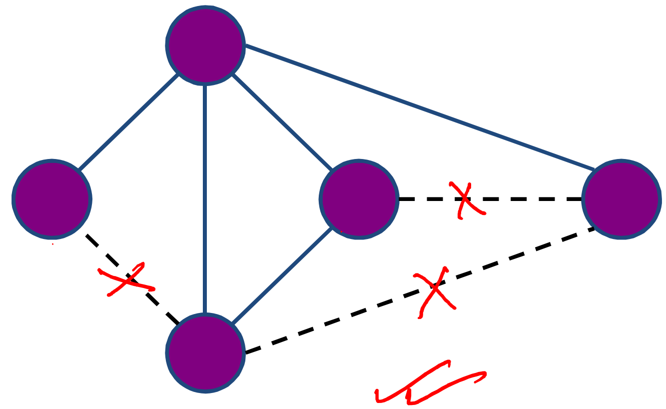
innovate achieve lead

- Subgraphs
- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G

SUBGRAPHS



Subgraph

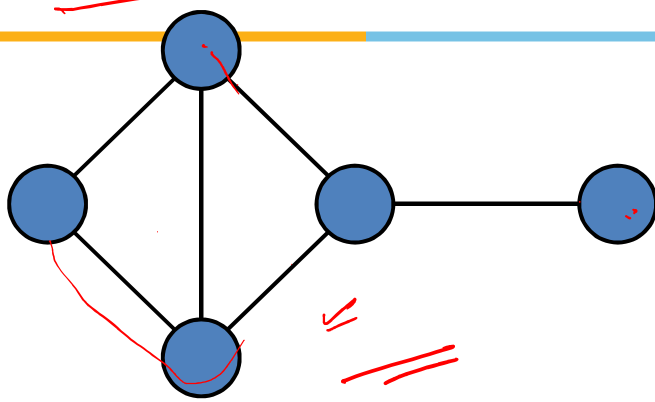


Spanning subgraph

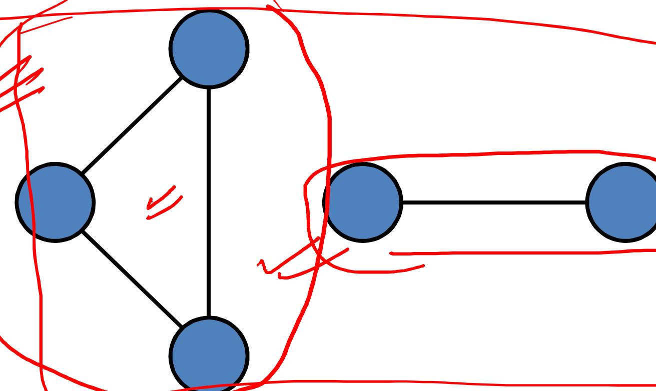
Connectivity

- A graph is **connected** if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G
- A directed graph G is **strongly connected** if:
  - For any two vertices u and v:
  - There is a directed path u → v, and
  - There is a directed path v → u

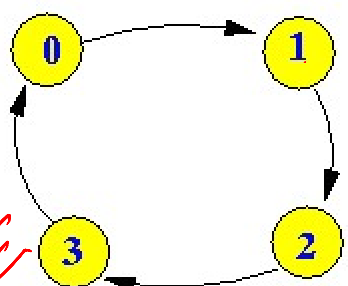
Connected graph



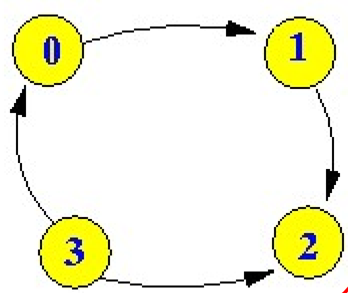
Connected graph



Non connected graph with two connected components



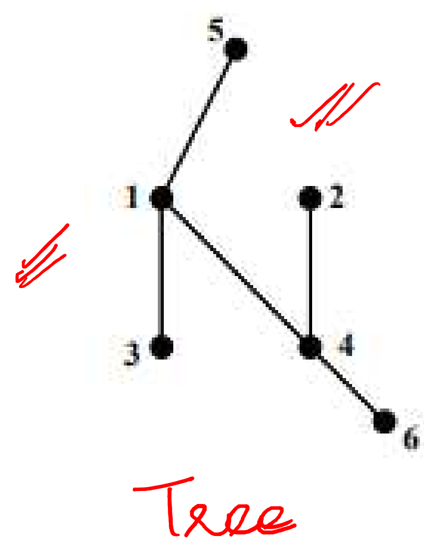
Strongly Connected

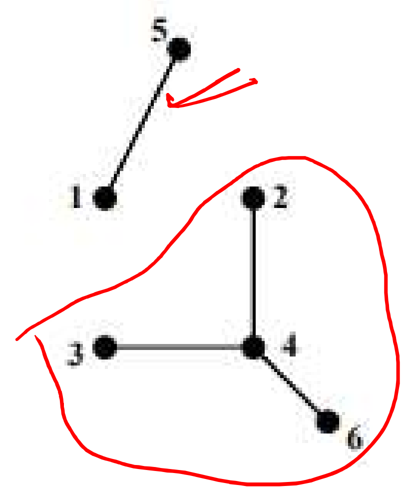


Not Strongly Connected

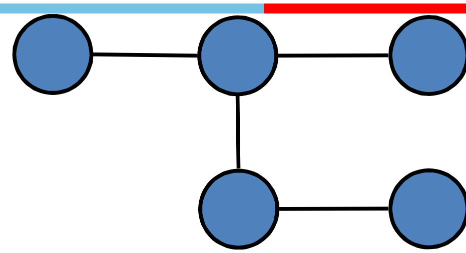
Trees and Forests

- A tree is a connected graph with no cycles.
- A forest is a graph with each connected component a tree

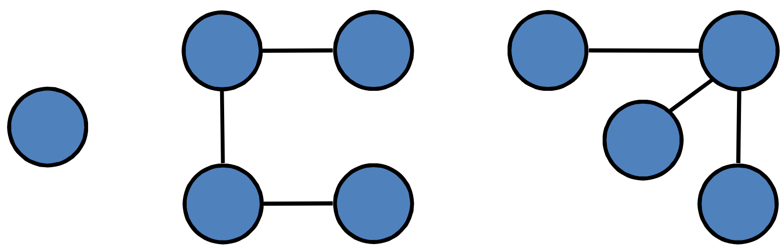




Trees and Forests



Tree

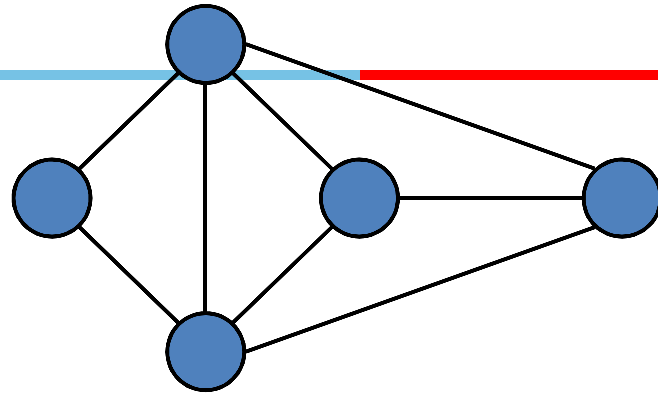


Forest

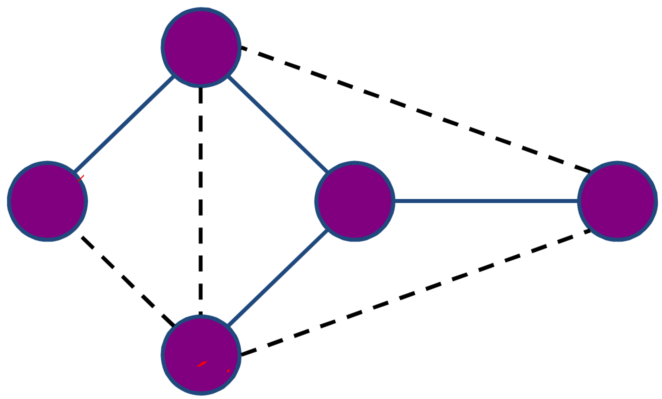
Spanning Trees

- A spanning tree of a connected graph is a spanning subgraph that is a tree:
- which includes all of the vertices of G, with minimum possible number of edges

Spanning Tree



Graph



Spanning tree

Subgraphs,trees-Example

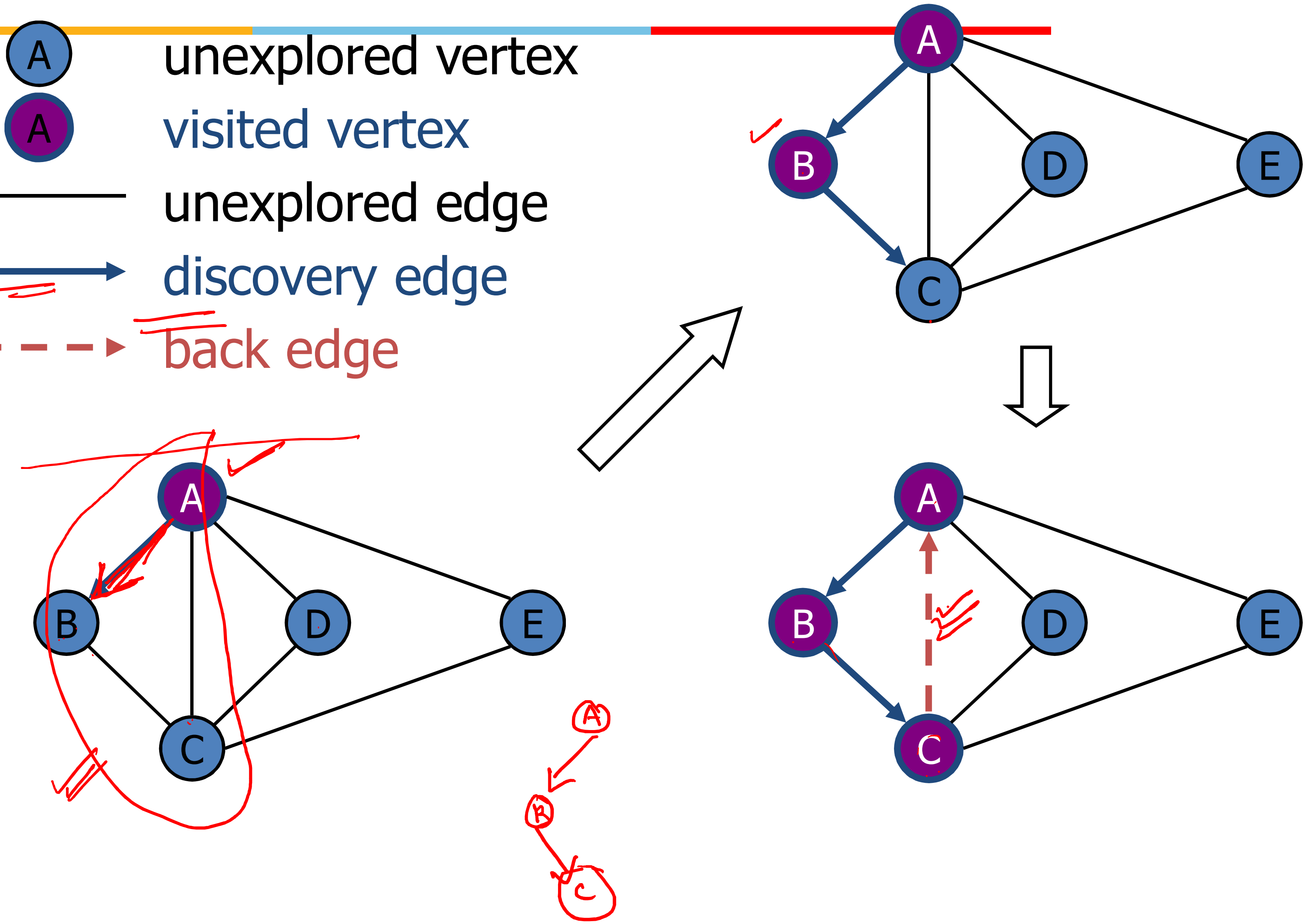
- Perhaps the most talked about graph today is the Internet, which can be viewed as a graph whose vertices are computers and whose (undirected) edges are communication connections between pairs of computers on the Internet.
- The computers and the connections between them in a single domain, like http://www.bits-pilani.ac.in/, form a subgraph of the Internet. If this subgraph is connected, then two users on computers in this domain can send e-mail to one another without having their information packets ever leave their domain.
- Suppose the edges of this subgraph form a spanning tree. This implies that, even if a single connection goes down (for example, because someone pulls a communication cable out of the back of a computer in this domain), then this subgraph will no longer be connected.

Depth-First Search

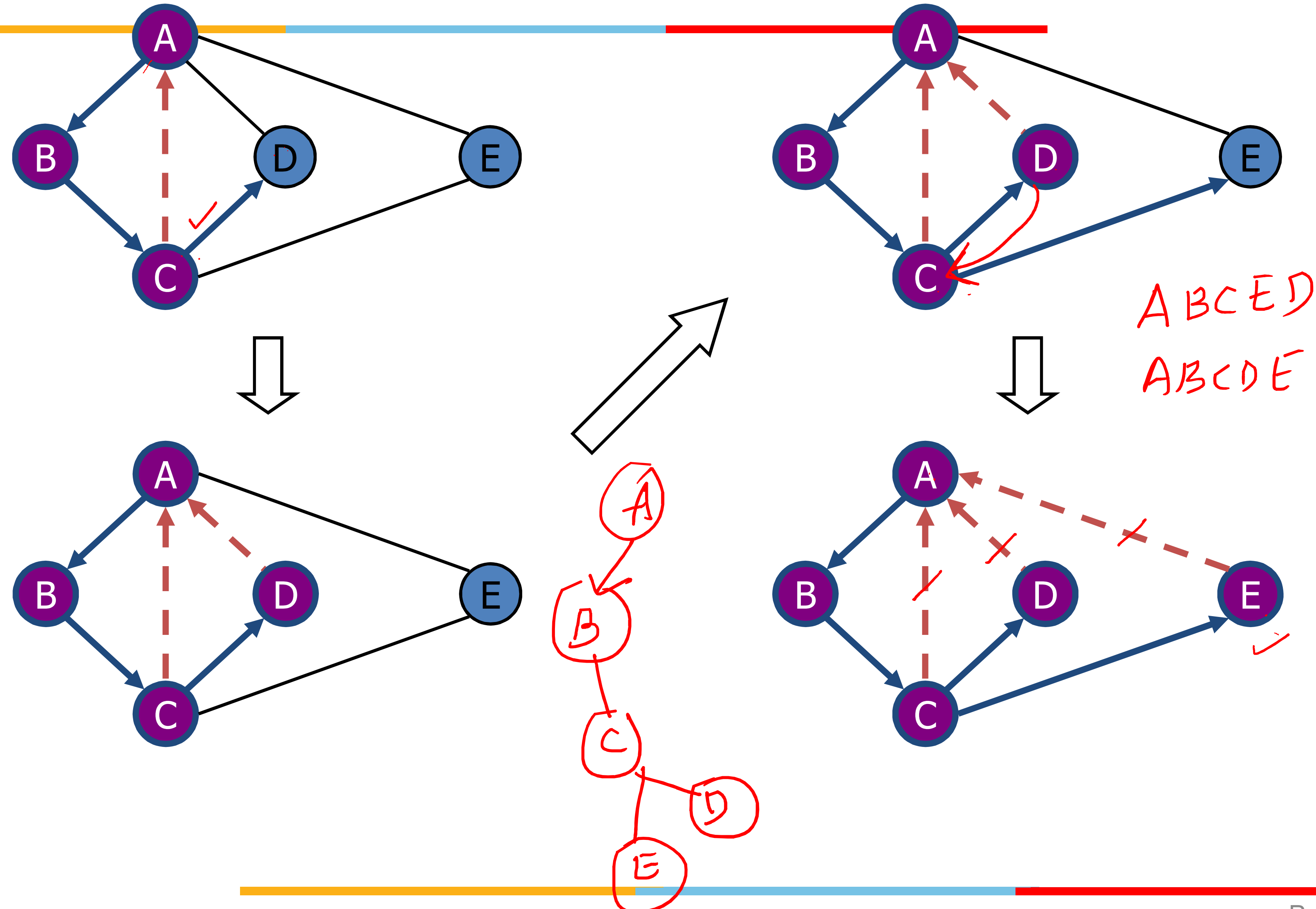
- Depth-first search (DFS) is a general technique for traversing a graph
- Search "deeper" in the graph whenever possible
- Explores edges out of the most recently discovered vertex that still has unexplored edges leaving it.
- Once all of v's edges have been explored, the search "backtracks" to explore edges leaving the vertex from which v was discovered.
- This process continues until we have discovered all the vertices that are reachable from the original source vertex.
- If any undiscovered vertices remain, then depth-first search selects one of them as a new source, and it repeats the search from that source.
- The algorithm repeats this ntire process until it has discovered every vertex

Depth-First Search

- unexplored vertex
- visited vertex
- unexplored edge
- discovery edge
- back edge



Depth-First Search



Depth-First Search

- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)

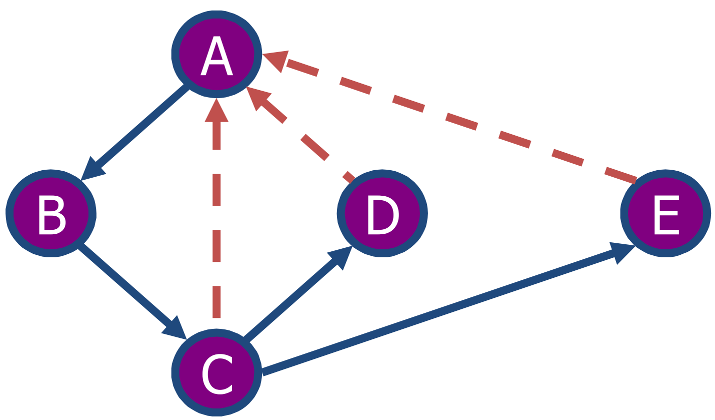
Depth-First Search-Properties

Property 1

DFS(***G***, ***v***) visits all the vertices and edges in the connected component of ***v***

Property 2

The discovery edges labeled by ***DFS***(***G***, ***v***) form a spanning tree of the connected component of ***v***



Depth-First Search

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

DFS Tree

Algorithm ***DFS***(***G***)
  Input graph ***G***
  Output labeling of the edges of ***G*** as discovery edges and back edges
   for all $u \in G.vertices()$
      setLabel(*u, UNEXPLORED*)
   for all $e \in G.edges()$
      setLabel(*e, UNEXPLORED*)
   for all $v \in G.vertices()$
      if *getLabel*(*v*) = *UNEXPLORED*
        DFS(*G, v*)

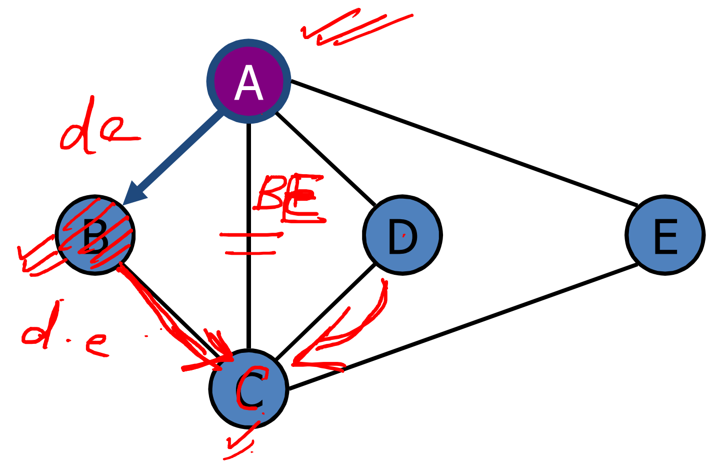
Depth-First Search

Algorithm *DFS*(*G, v*)

Input graph *G* and a start vertex *v* of *G*

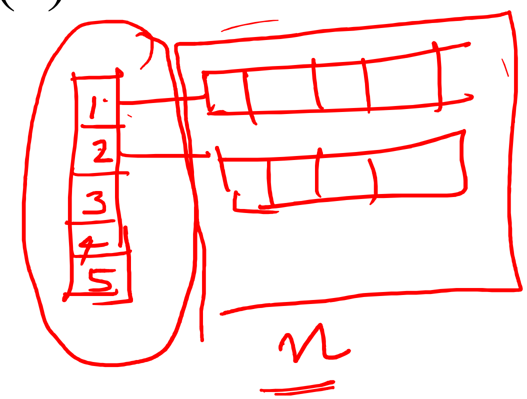
Output labeling of the edges of *G* in the connected component of *v* as discovery edges and back edges

```
setLabel(v, VISITED)
for all e ∈ G.incidentEdges(v)
    if getLabel(e) = UNEXPLORED
        w ← G.opposite(v,e)
        if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            DFS(G, w)
    else
        setLabel(e, BACK)
```



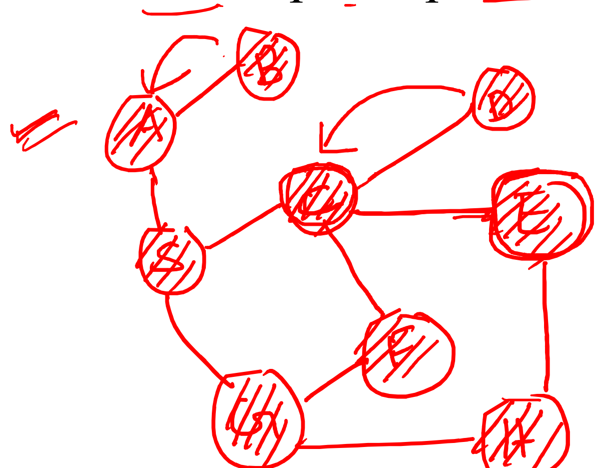
Analysis of DFS

- Setting/getting a vertex/edge label takes $\boldsymbol{O}(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
- DFS runs in $\boldsymbol{O}(\boldsymbol{n}+\boldsymbol{m})$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

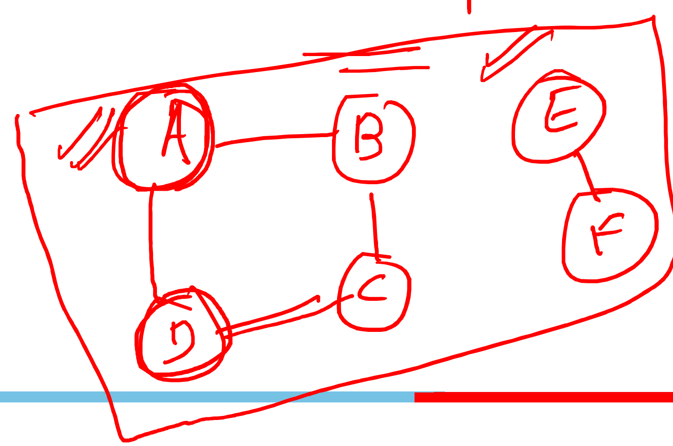


Depth-First Search

- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning tree of G
  - Computing a cycle in G, or reporting that G has no cycles
  - Find and report a path between two given vertices



A B S C D E H G F



Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices u and z using the template method pattern
- We call $DFS(G, u)$ with u as the start vertex
- We use a stack S to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex z is encountered, we return the path as the contents of the stack

Path Finding

```
Algorithm pathDFS(G, v, z)
    setLabel(v, VISITED)
    S.push(v)
     if v = z
         return S.elements()
     for all e ∈ G.incidentEdges(v)
         if getLabel(e) = UNEXPLORED
           w ← opposite(v, e)
           if getLabel(w) = UNEXPLORED
                   setLabel(e, DISCOVERY)
                   S.push(e)
                   pathDFS(G, w, z)
                   S.pop()                { e gets popped }
           else
                   setLabel(e, BACK)
     S.pop()                                  { v gets popped }
```

Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack ***S*** to keep track of the path between the start vertex and the current vertex
- As soon as a back edge (v, w) is encountered, we return the cycle as the portion of the stack from the top to vertex ***w***

Cycle Finding

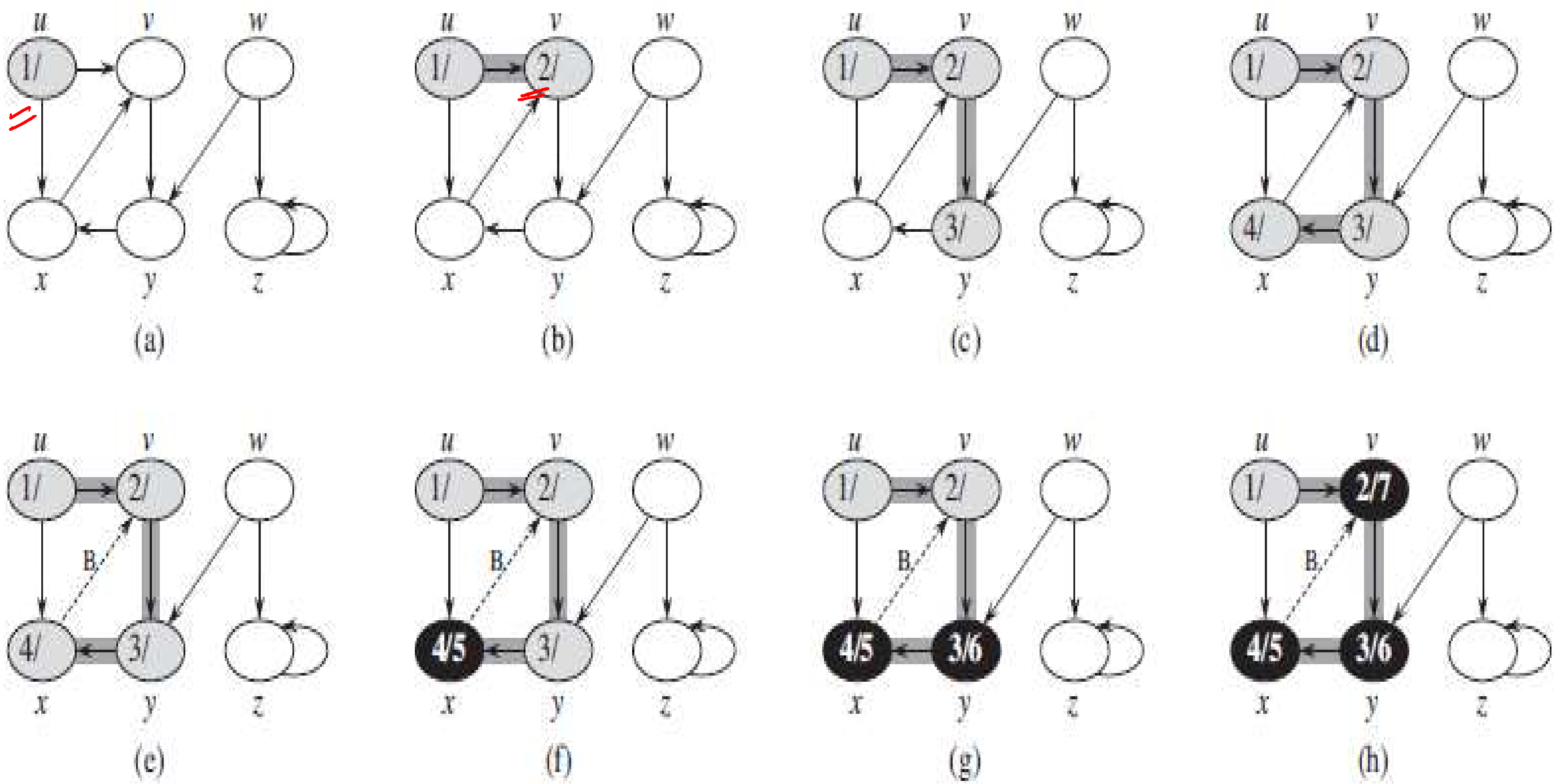
```
Algorithm cycleDFS(G, v, z)
    setLabel(v, VISITED)
    S.push(v)
     for all  e ∈ G.incidentEdges(v)
        if  getLabel(e) = UNEXPLORED
          w ← opposite(v,e)
          S.push(e)
          if getLabel(w) = UNEXPLORED
                   setLabel(e, DISCOVERY)
                  pathDFS(G, w, z)
                  S.pop()
          else
                  C ← new empty stack
                  repeat
                          o ← S.pop()
                          C.push(o)
                  until o = w
                  return C.elements()
     S.pop()
```

DFS:R2-Chapter 22

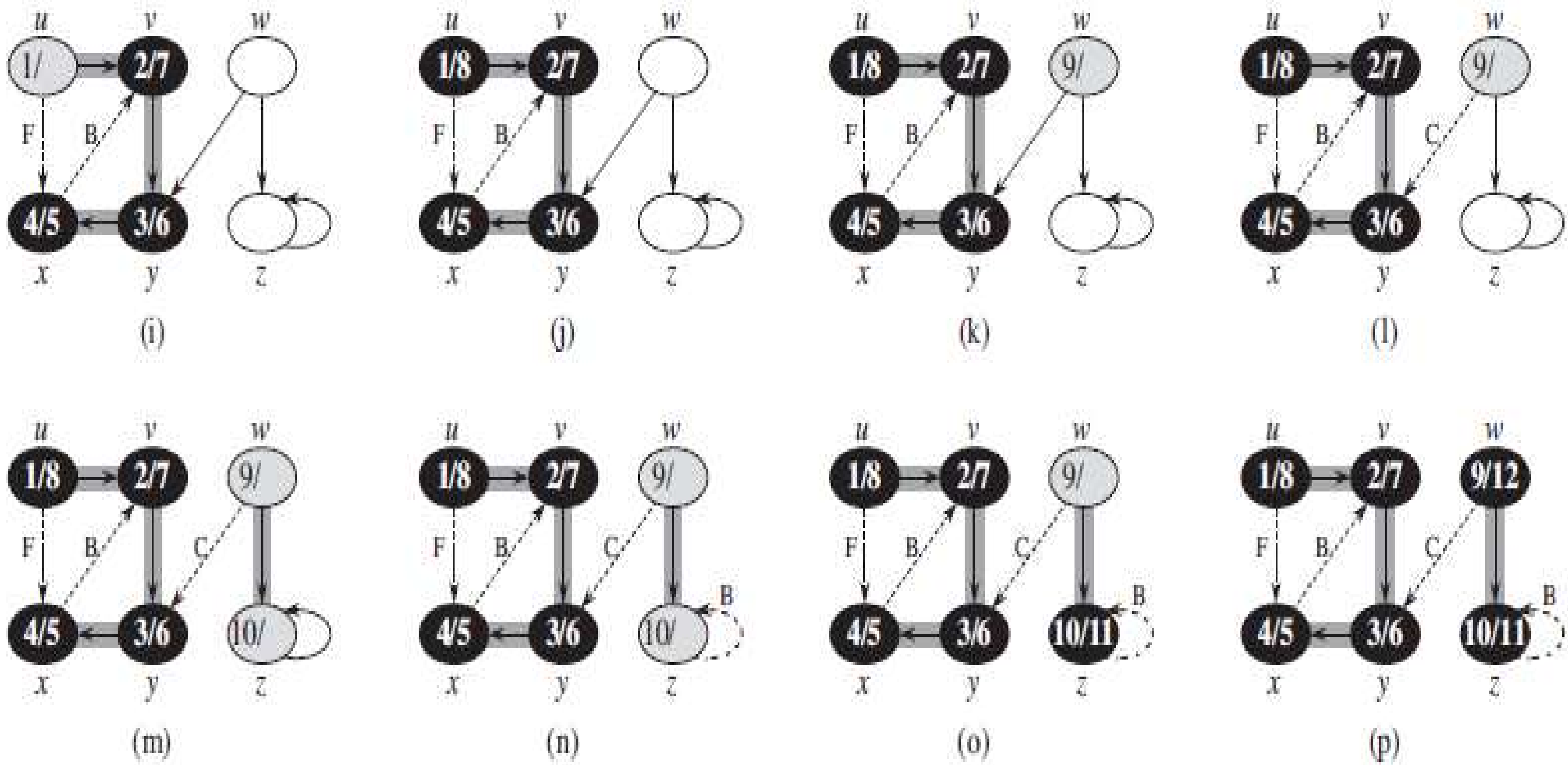
```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1                // white vertex u has just been discovered
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]          // explore edge (u, v)
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK                // blacken u; it is finished
time = time + 1
u.f = time
```

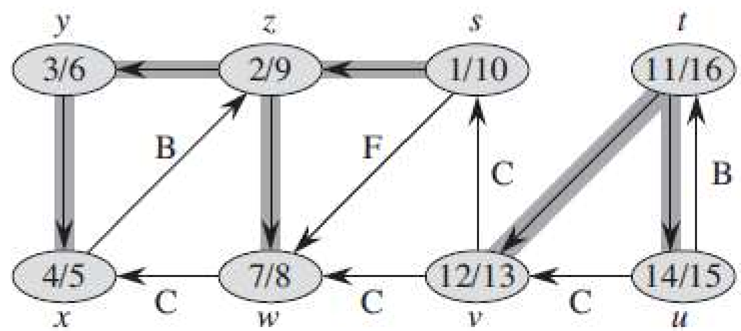
DFS:R2-Chapter 22

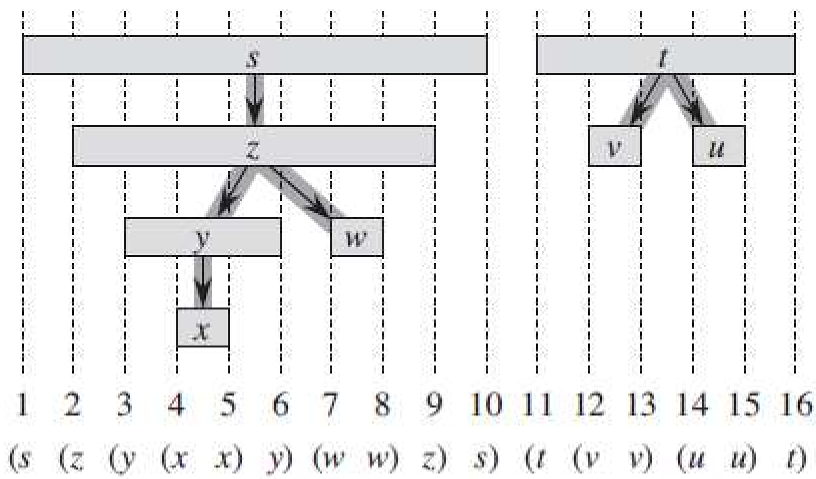


DFS:R2-Chapter 22

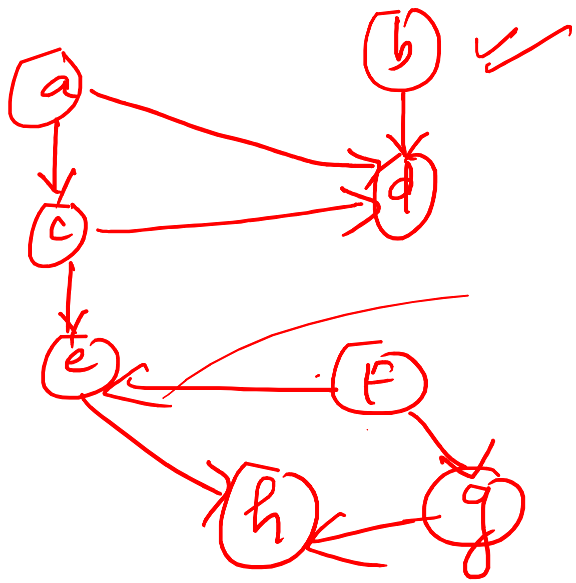


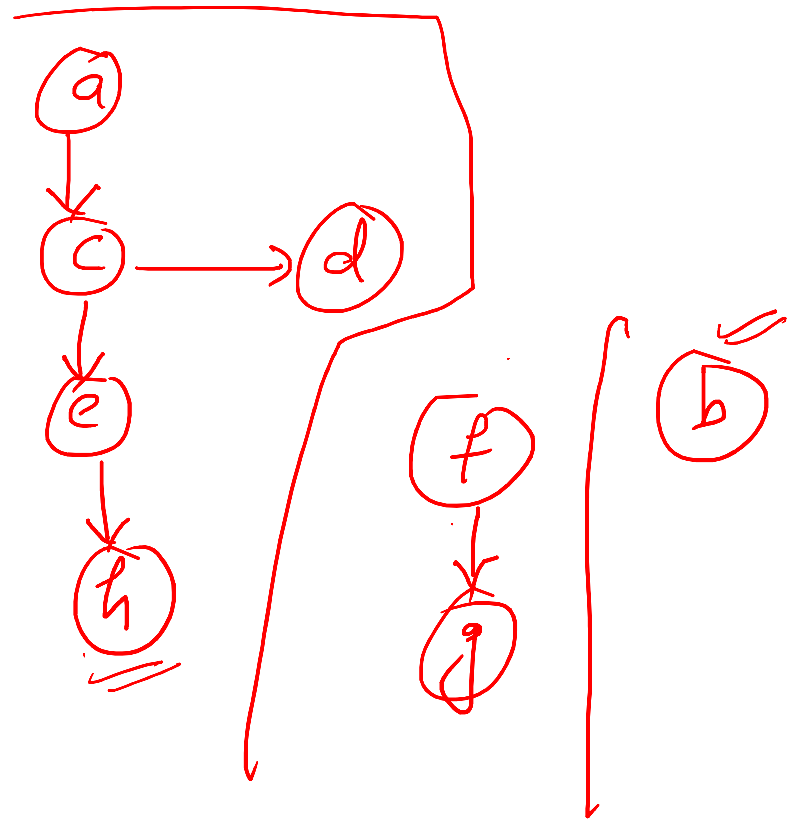
DFS:R2-Chapter 22 Paranthesis Structure





Predecessor subgraph forms a forest of trees





Connected components

How can DFS be used to find the connected components of a graph!

Can you implement it????

What will be the time complexity?

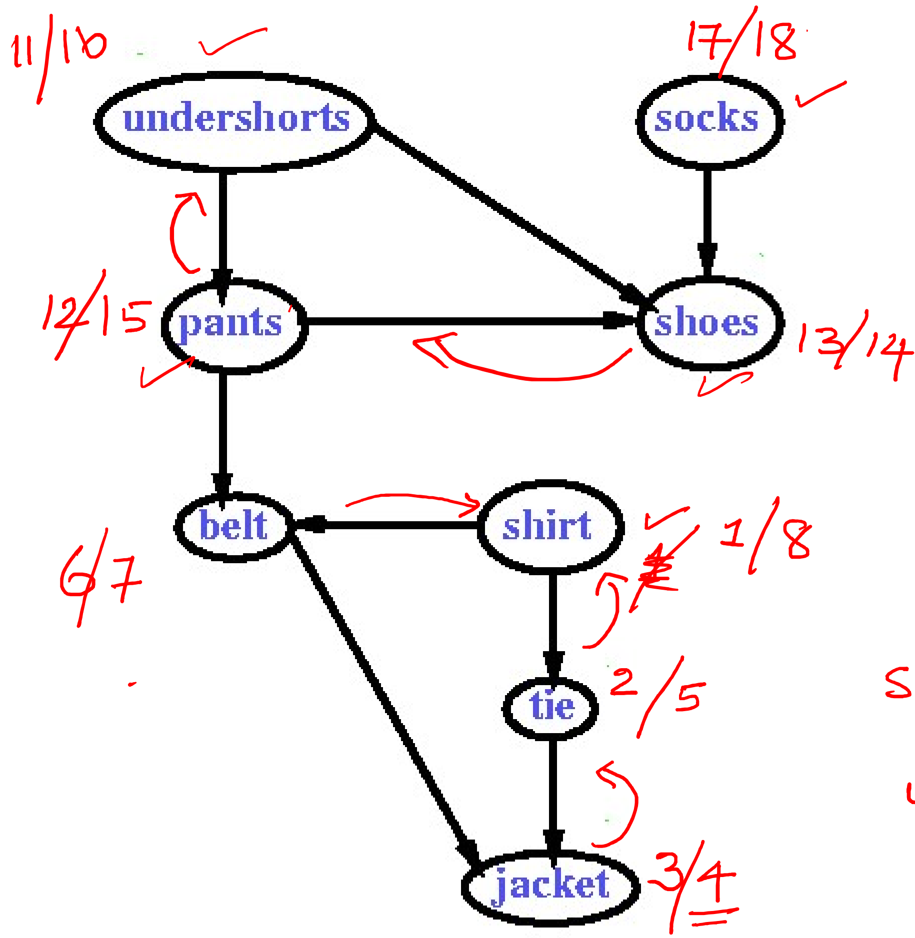
Connected components

How can DFS be used to check whether a graph is connected or not?

Can you implement it????
What will be the time complexity?

DFS for Topological Sort

a/b

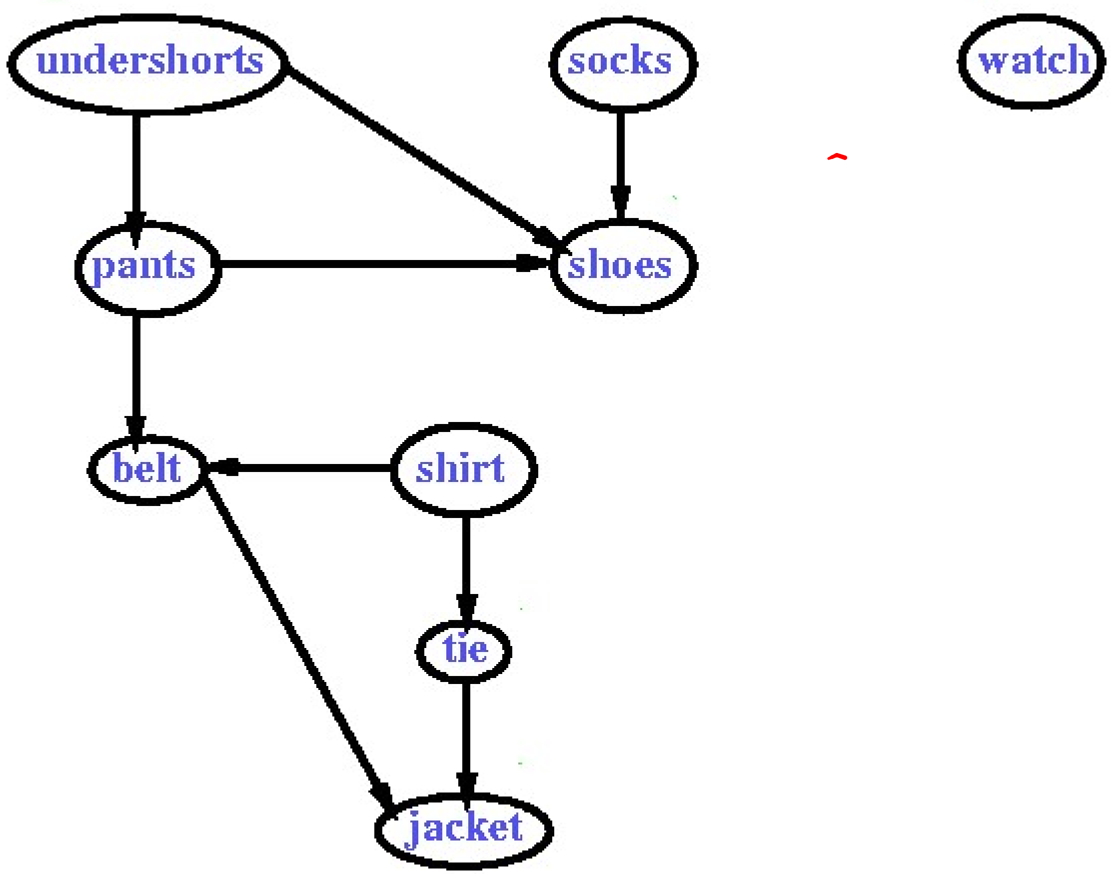


u → v

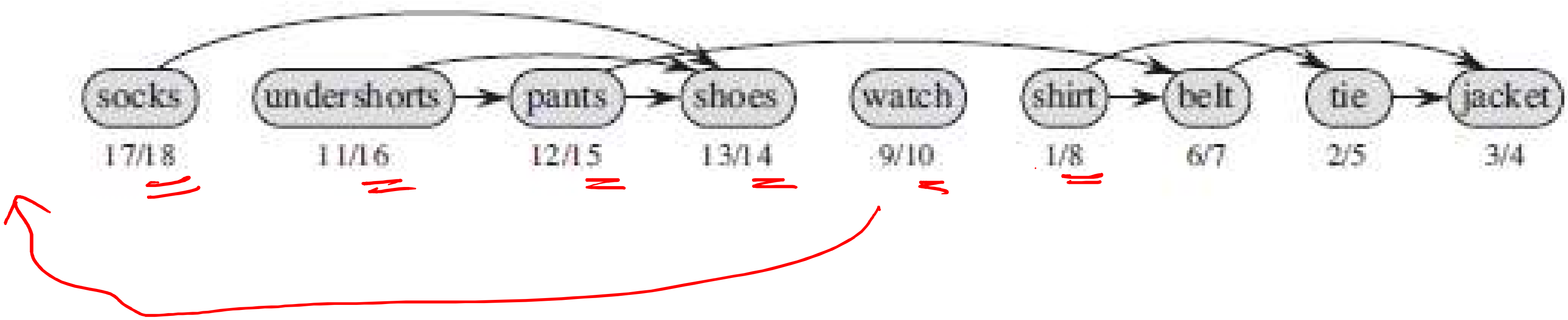
socks → undershorts → pants → shoes

watch → shirt → belt → tie → jacket.

DFS for Topological Sort



DFS for Topological Sort-Result



Breadth-first search

- Algorithm
- Example
- Properties
- Analysis
- Applications

Breadth-first search

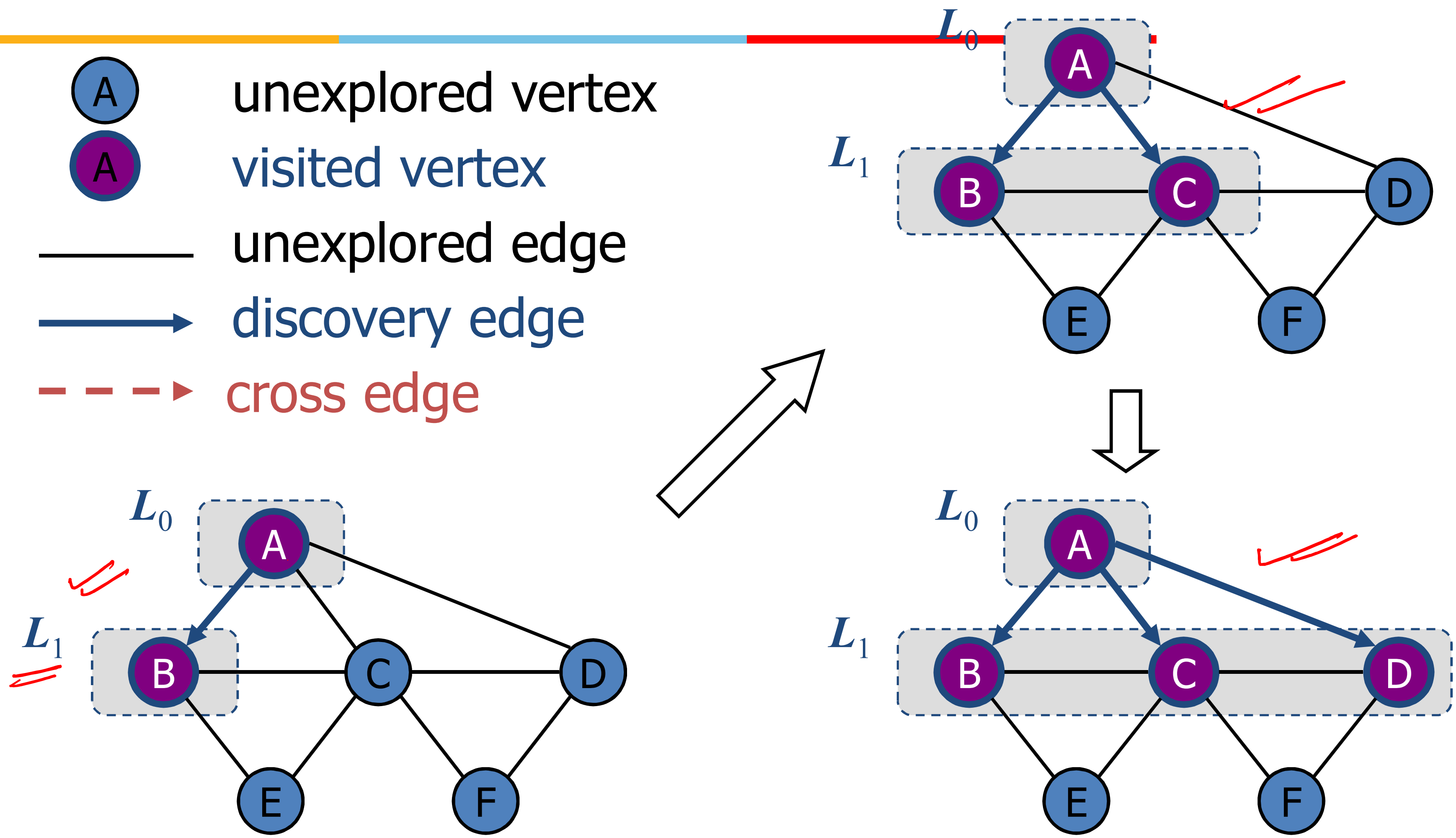
- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - discovers all vertices at distance k from s before discovering any vertices at distance k + 1.

$d=0$

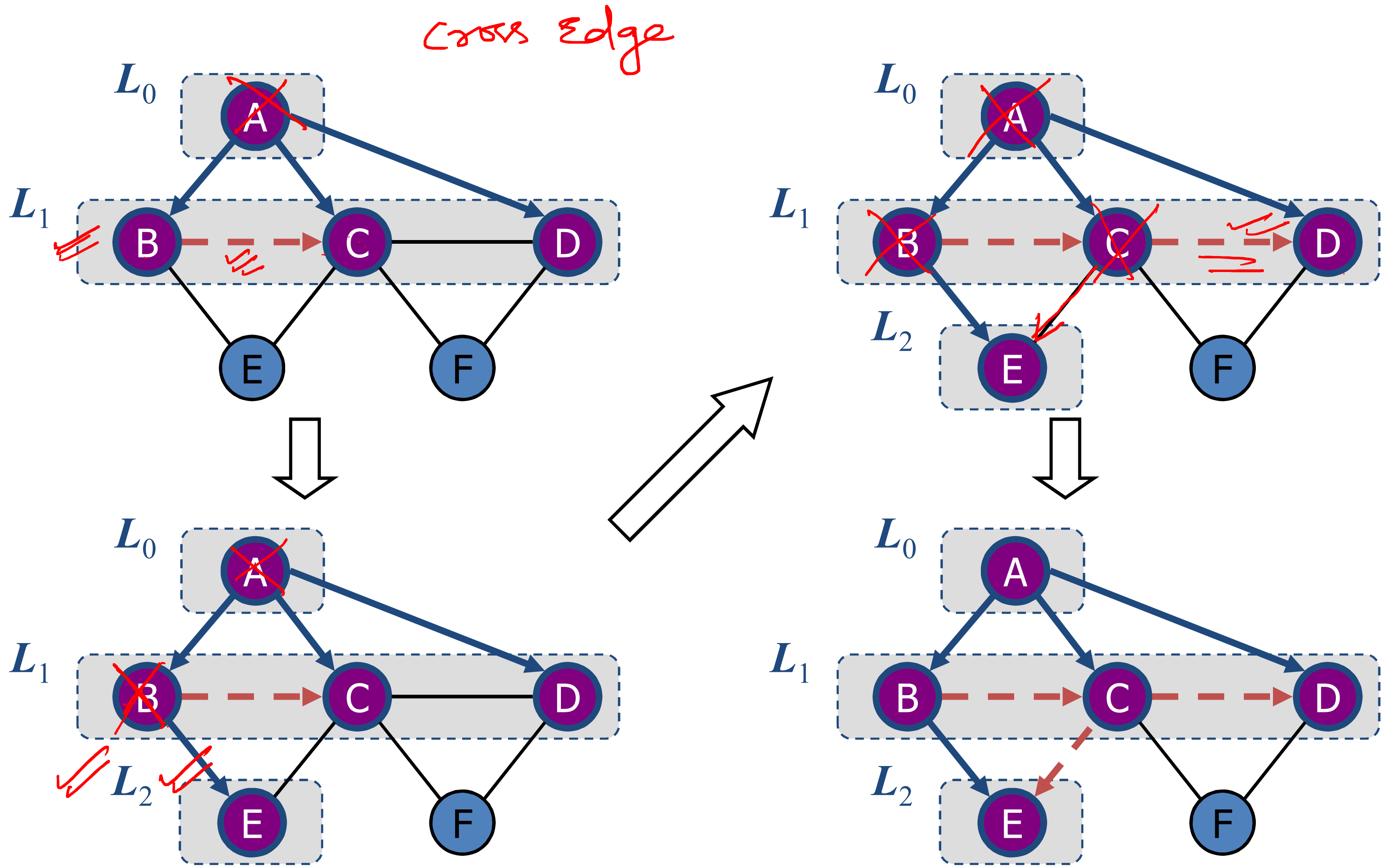
$d=1$

- For any vertex v reachable from vertex s, the simple path in the breadth-first tree from s to v corresponds to a **"shortest path"** from s to v in G, that is, a path containing the smallest number of edges.

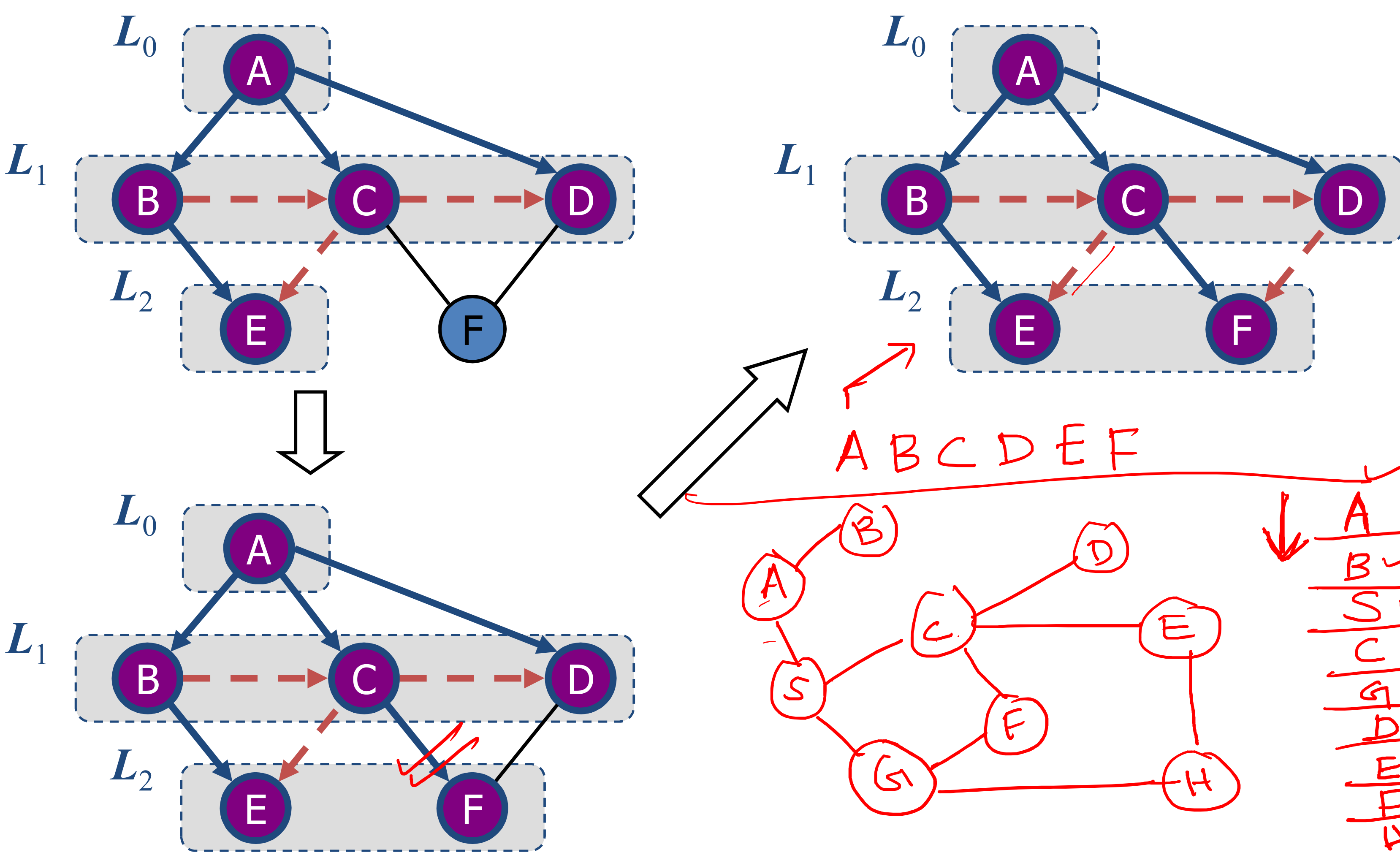
Breadth-first search



Breadth-first search



Breadth-first search



A B C D E F

A
B
S
C
G
D
E
F
H

Breadth-first search

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

Algorithm ***BFS***(***G***)

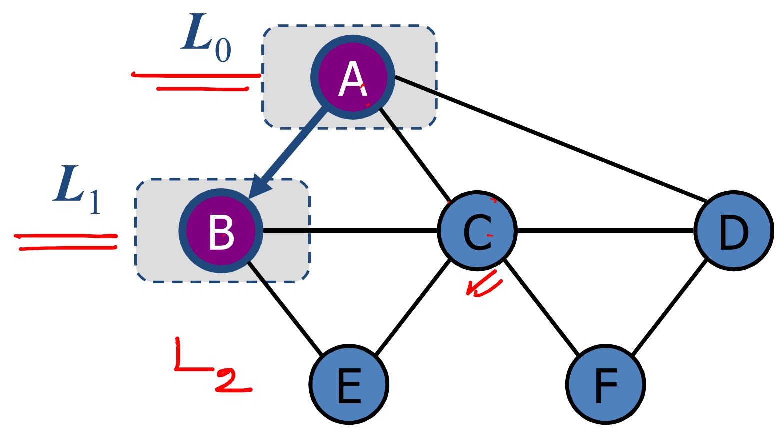
    Input graph ***G***

    Output labeling of the edges and partition of the vertices of ***G***

```
for all u ∈ G.vertices()
    setLabel(u, UNEXPLORED)
for all e ∈ G.edges()
    setLabel(e, UNEXPLORED)
for all v ∈ G.vertices()
    if getLabel(v) = UNEXPLORED
        BFS(G, v)
```

Breadth-first search – Algorithm *BFS*(*G, s*)

$L_0 \leftarrow$ new empty sequence
$L_0.insertLast(s)$
$setLabel(s, VISITED)$
$i \leftarrow 0$
while $\neg L_i.isEmpty()$
  $L_{i+1} \leftarrow$ new empty sequence
  for all $v \in L_i.elements()$
    for all $e \in G.incidentEdges(v)$
    if $getLabel(e) = UNEXPLORED$
      $w \leftarrow opposite(v,e)$
      if $getLabel(w) = UNEXPLORED$
        $setLabel(e, DISCOVERY)$
        $setLabel(w, VISITED)$
        $L_{i+1}.insertLast(w)$
      else
        $setLabel(e, CROSS)$
$i \leftarrow i + 1$



Breadth-first search – **Algorithm *BFS*(*G*, *s*)**

- We use auxiliary space to label edges, mark visited vertices, and store containers associated with levels.
- That is, the containers L0, L 1 , L2, and so on, store the nodes that are in level 0, level 1 , level 2, and so on.

Properties

Notation

G_s: connected component of s

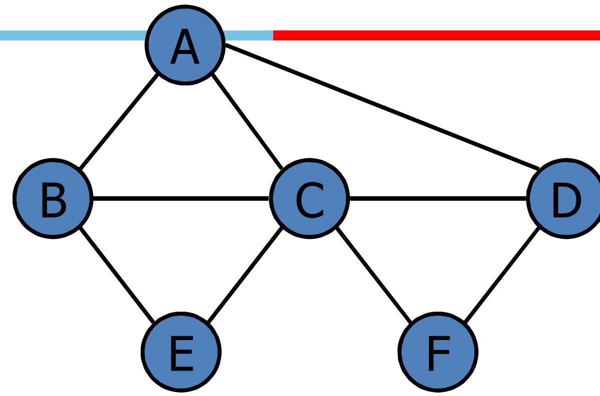
Property 1

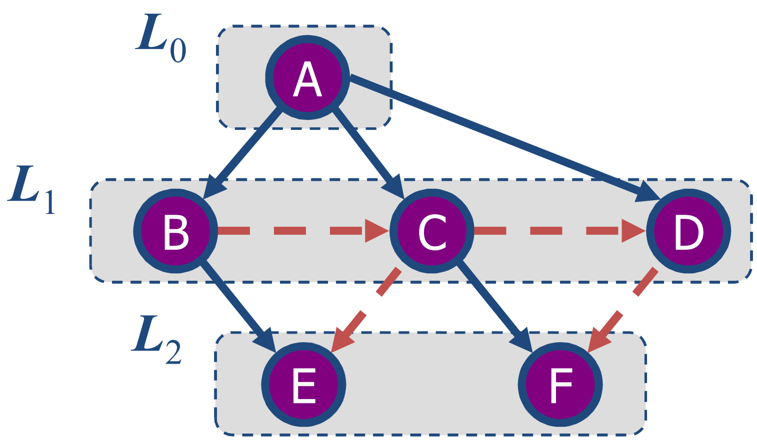
$BFS(G, s)$ visits all the vertices and edges of G_s

Property 2

The discovery edges of a connected component labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Properties





Analysis

- Setting/getting a vertex/edge label takes $\boldsymbol{O}(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED

  $O(n)$

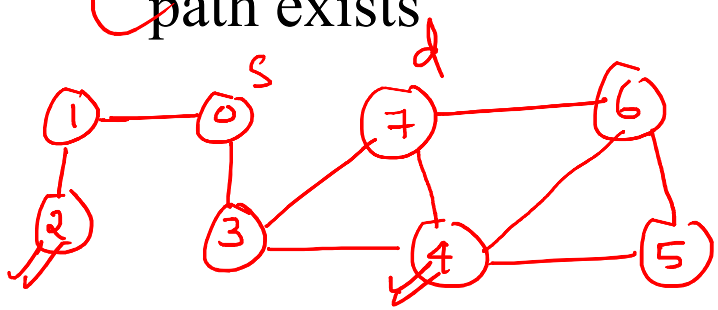
  $O(n+m)$
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS

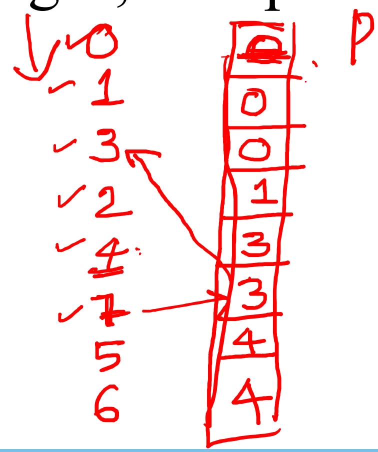
  $O(m)$

  mm x
- Each vertex is inserted once into a sequence $\boldsymbol{L_i}$
- Method incidentEdges is called once for each vertex
- BFS runs in $\boldsymbol{O}(\boldsymbol{n} + \boldsymbol{m})$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2\boldsymbol{m}$

Applications

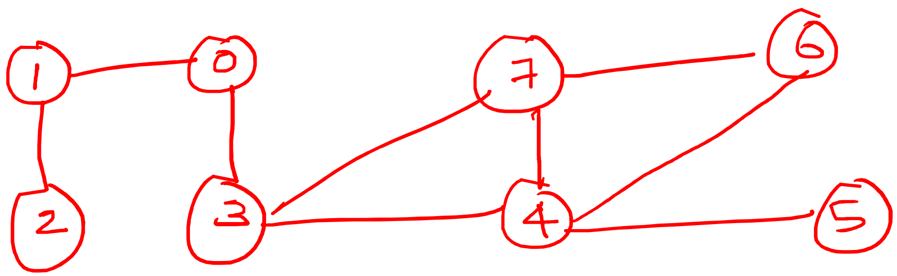
- We can specialize the BFS traversal of a graph $\boldsymbol{G}$ to solve the following problems in $\boldsymbol{O}(\boldsymbol{n} + \boldsymbol{m})$ time
  - Compute the connected components of $\boldsymbol{G}$
  - Compute a spanning forest of $\boldsymbol{G}$
  - Find a simple cycle in $\boldsymbol{G}$, or report that $\boldsymbol{G}$ is a forest
  - Given two vertices of $\boldsymbol{G}$, find a path in $\boldsymbol{G}$ between them with the minimum number of edges, or report that no such path exists





$7 \leftarrow 4 \leftarrow 3 \leftarrow 0$

Find the shortest path from 2 → 4



Start BFS from 2.

Trav d		Predecessor
2	→	-
1	→	2
0	→	1
3	→	0
4	→	3
7	→	3
5	→	4
6	→	4

Trace back from 4 in Predecessor array.

$4 \Leftarrow 3 \Leftarrow 0 \Leftarrow 1 \Leftarrow 2$

Facebook as Graph

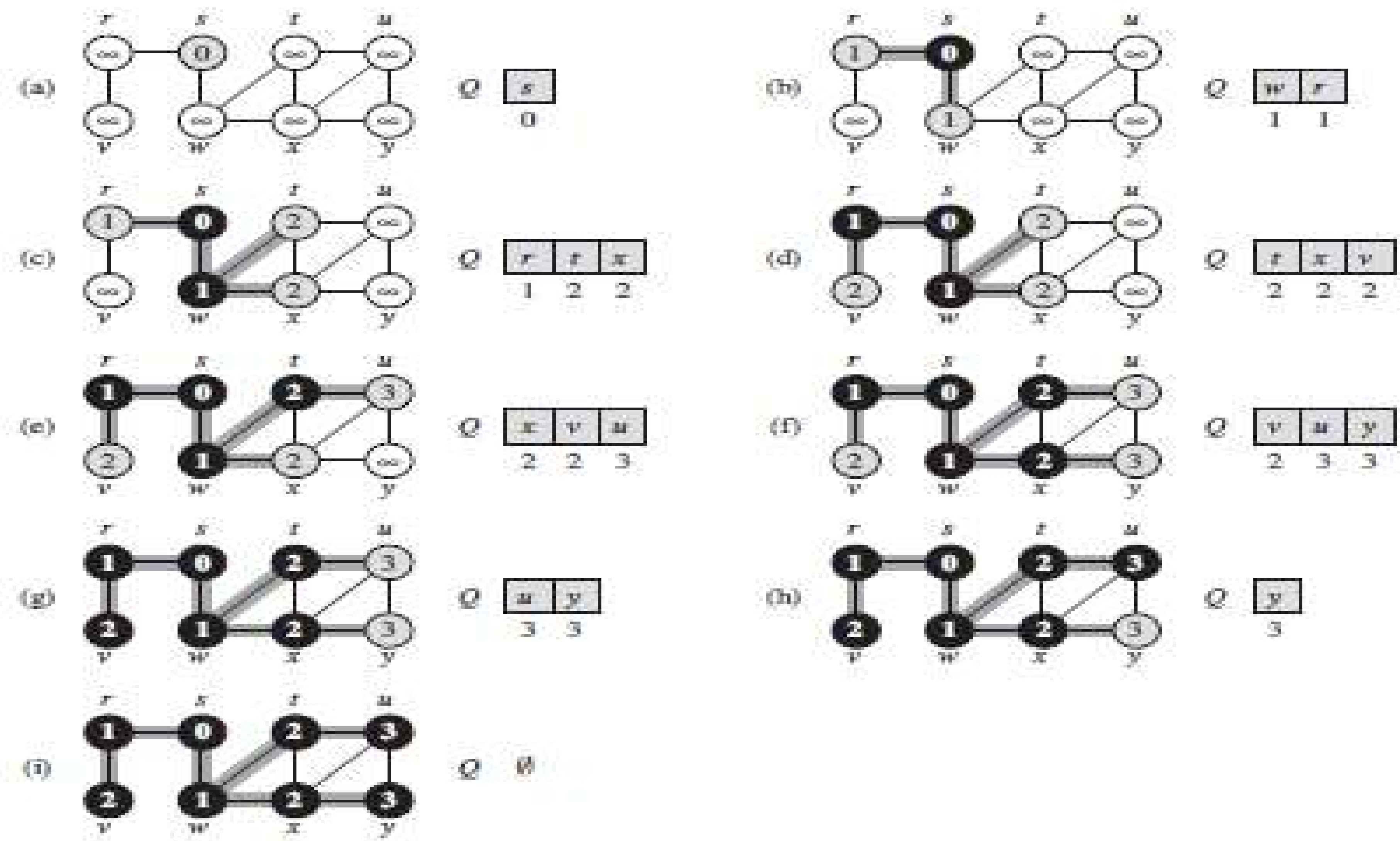
innovate achieve lead

- Traversal: go to ‘Friends’ to display all your friends (like G.Neighbors)
- BFS: the tabs are a queue - open all friends profiles in new tabs, then close current tab and go to the next one
- DFS: the history is a stack - open the first friend profile in the same window; when hitting a dead end, use back button

BFS-CLRS

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

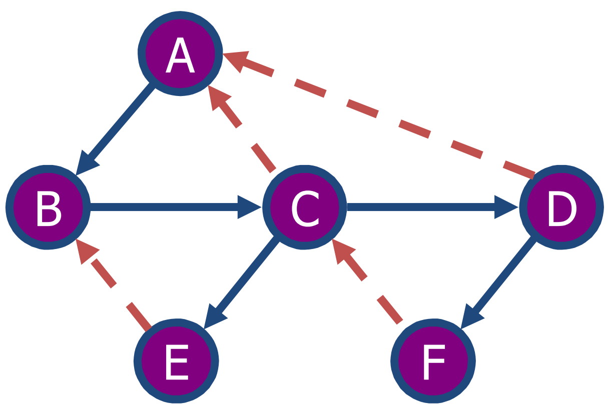
BFS-CLRS



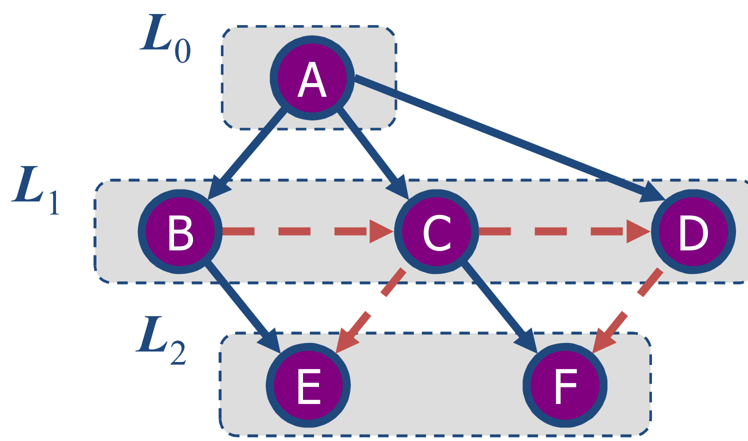
DFS vs. BFS

Application	DFS	BFS
Spanning forest, connected components, paths, cycles	Y	Y
Shortest Paths		Y

DFS vs. BFS



DFS



BFS

DFS vs. BFS

Back edge (***v***,***w***)

- ***w*** is an ancestor of ***v*** in the tree of discovery edges

Cross edge (***v***,***w***)

- ***w*** is in the same level as ***v*** or in the next level in the tree of discovery edges

THANK YOU!

BITS Pilani
Hyderabad Campus