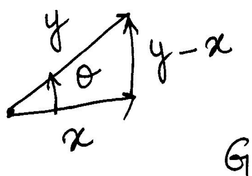
DOT PRODUCT / INNER PRODUCT

Take two vectors $(x_1, \dots x_n)$ $(y_1, \dots y_n) \in \mathbb{R}^n$

inner product $\boxed{x_1 y_1 + x_2 y_2 + \dots + x_n y_n}$



$\overline{x} \cdot \overline{y}$ is $\boxed{\|\vec{x}\| \cdot \|\vec{y}\| \cos\theta}$

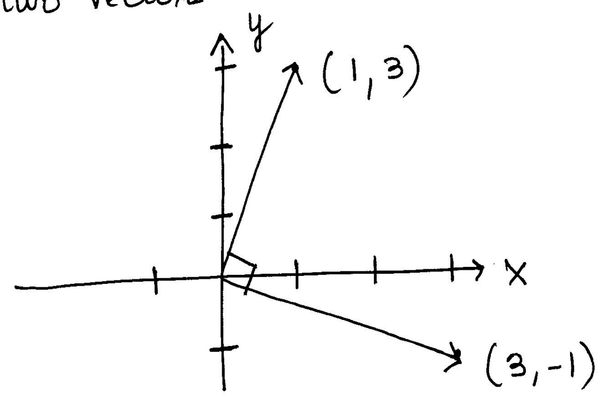
Geometric interpretation

θ is angle between $\overline{x}$ & $\overline{y}$

ORTHOGONAL Two vectors $\vec{v}, \vec{w}$ are orthogonal if their dot product (or inner product) is 0.

When angle θ between them is such that $\cos(\theta) = 0$

These two vectors meet at either a 90° or 270° degree



Two orthogonal vectors

$1 \cdot 3 + 3 \cdot (-1) = 0.$ [algebraic dot product is zero]

Angle between them is $\pi/2$.

ORTHONORMAL BASIS Suppose we have a basis $B = \{\vec{b_1}, \dots \vec{b_N}\}$ for some space $\mathbb{F}^n$

Basis is ORTHONORMAL if

1. ORTHOGONALITY $i \neq j$, $\overline{b_i} \cdot \overline{b_j} = 0$

2. UNIT LENGTH – length $\|\vec{b_i}\|$ of any vector $\overline{b_i}$ is 1

Eg Standard Basis for $\mathbb{R}^n$ is orthonormal but it is not only orthonormal basis for $\mathbb{R}^n$

Non standard Orthonormal Basis

$$S = \left\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{v_1}, \underbrace{\left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}_{v_2}, \underbrace{\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)}_{v_3} \right\}$$

check

Vectors are mutually orthogonal

$$v_1 \cdot v_2 = \frac{-1}{6} + \frac{1}{6} + 0 = 0.$$

$$v_1 \cdot v_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0.$$

|||^{ly} $v_2 \cdot v_3 = 0.$

$$\|v_1\| = \sqrt{v_1 \cdot v_1}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

$$\|v_2\| = \sqrt{v_2 \cdot v_2}$$

$$= \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = 1.$$

$$\|v_3\| = 1.$$

Hence S is orthonormal set

Note:- Orthogonal sets are Linearly Independent

GRAM SCHMIDT Orthonormalization

1. Begin with basis for the inner product space. It need not be orthogonal nor consist of unit vectors
2. Convert the given basis to an orthogonal basis
3. Normalize each vector in the orthogonal basis to form an orthonormal basis.

Apply Gram Schmidt orthonormalization to basis for $\mathbb{R}^3$

$$B = \{\overset{v_1}{(1,1,0)}, \overset{v_2}{(1,2,0)}, \overset{v_3}{(0,1,2)}\}$$

$$w_1 = v_1 = (1,1,0)$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1,2,0) - \frac{3}{2}(1,1,0)$$

$$= \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}\left(\frac{-1}{2}, \frac{1}{2}, 0\right)$$

$$= (0,0,2)$$

set $B' = \{w_1, w_2, w_3\}$ is an orthogonal basis for $\mathbb{R}^3$

Normalizing each vector

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{1/\sqrt{2}}\left(\frac{-1}{2}, \frac{1}{2}, 0\right) = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{2}(0,0,2) = (0,0,1)$$

$B'' = \{u_1, u_2, u_3\}$ is orthonormal basis for $\mathbb{R}^3$.

QR Decomposition

Factorization of a matrix A into Q and R

Q → Orthogonal matrix; R → Upper triangular matrix

If A is a square matrix, A will always have a decomposition.

In case of Rectangular Matrices, QR decomposition will exist only if $A_{m \times n}$, $\underline{\underline{m > n}}$ (over determined system)

Solving a system of linear equations

$A = QR$ We want to solve $Ax = b$

$$Ax = b$$

$$QRx = b$$

$$Q^T QRx = Q^T b \quad \text{(multiplying both sides by } Q^T\text{)}$$

$$Rx = Q^T b \quad (Q \text{ is orthogonal so } QQ^T = I)$$

R is a triangular matrix, final equation is very easy to solve using backward substitution

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LU Decomposition

$A = LU$

Solving system $Ax = b$

$$LUx = b$$

2 step solution

1) Solve $Ly = b$ for y by forward substitution
2) Solve $Ux = y$ for x by back substitution

LU Decomposition and Gaussian Elimination.

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ \boxed{2} & 6 & 13 \end{bmatrix} \qquad \begin{array}{l} R_2 \to R_2 - \alpha R_1 \\ R_3 \to R_3 - \beta R_1 \end{array}$$

α is l_{21} and β is l_{31}

Matrix $A = LU$ substitution step can be carried out efficiently for DIFFERENT VALUES of b.

Note $U_{1j} = a_{ij}$ [for $j = 1$ to n]

1st Row entries U_{1j} is same as matrix A.

Take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ fails to have LU decomposition

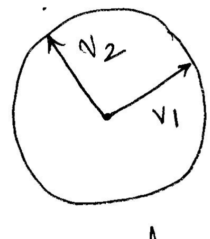
$$A = LU = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$l_{11} u_{11} = 0 \Rightarrow$ either L or U is singular

but $A = LU$ is Non singular

∴ No zero at diagonal place at any stage is required for LU decomposition to work.

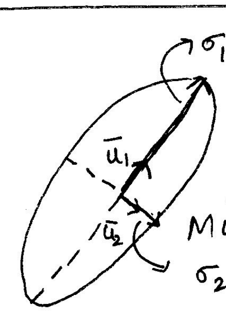
Transformation of Circle Under Matrix Operation



orthogonal $\bar{v}_1$ & $\bar{v}_2$

Multiply with matrix A

Rotate sphere & stretch it



ellipse.

$\bar{u}_1$ & $\bar{u}_2 \rightarrow$ unit vectors (principal axes)

Stretching factors $\longrightarrow$ singular values.

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$ (Rotation & stretching)

New vector space. different vector is produced in SVD.

$$A v_j = \sigma_j u_j \quad ; \quad 1 \leq j \leq n$$

$$AV = \hat{U} \hat{\Sigma}$$ V is orthogonal.

$$AVV^{-1} = \hat{U} \hat{\Sigma} V^{-1}$$

$$\boxed{A = \hat{U} \hat{\Sigma} V^T} \quad [V^{-1} = V^T]$$

Reduced SVD $\quad A_{m \times n} = U_{m \times n} \Sigma_{n \times n} V^T_{n \times n}$

$\downarrow$ Add $m-n$ columns to U, it is of size $m \times m$ now.

SVD. $\quad A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V^T_{n \times n}$

Convert Σ to have m rows, others are padded with 0.