Q.1 Let B be a skew-symmetric matrix with real entries

1. Prove that (I − B) is non-singular
2. If $A = (I + B)(I - B)^{-1}$ then $A^T = A^{-1}$

<u>Answer: -</u>

Let B be a skew-symmetric matrix of real entries.

This implies the eigen values of B are either 0 or pure imaginary. ………… **(1/2 mark)**

Let the eigen values of B are 0 or *ib*. Then eigen values of I – B are 1 *or* 1– *ib*.

In any case eigen values of I – B are non-zero. Thus det(I – B) is non-zero. Thus I – B is invertible or non-singular. ………….. **(1/2 mark)**

Let $A = (I + B)(I - B)^{-1}$. Consider,

$A^T = [(I + B)(I - B)^{-1}]^T$.

$= [((I - B)^{-1})]^T [(I + B)]^T$.

$= [((I - B)^T)]^{-1} [(I + B)]^T$.

$= [I^T - B^T]^{-1} (I^T + B^T)]$.

$= ((I + B)^{-1}(I - B))$.

Thus, $A^T A = ((I + B)^{-1}(I - B))\big((I + B)(I - B)^{-1}\big) = I \cdot I = I$.

Therefore $A^T = A^{-1}$ ....… **(1 mark).**

Q.2Let M = {m_1, m_2, …., m_r}and N = {m_1, m_2, …., m_r , v}be two sets of vectors from the same vector space V over a field F. Prove that span{M} = span{N } if and only if v ∈ span{M}.

<u>Answer: -</u>

Let M = {m_1, m_2, …., m_r}and N = {m_1, m_2, …., m_r , v} be two set of vectors in a vector space V over a field F.

First we prove that if *span(M) = span(N),* then *v € span(M)*

From given data we can write *span(N)*is contained in*span(M).*

Therefore, any linear combination of vectors in N, say $a_1 m_1 + \cdots + a_r m_r + bv$ can be written as linear combination of vectors in M. Thus, we have-

$a_1 m_1 + \cdots + a_r m_r + bv = b_1 m_1 + \cdots + b_r m_r$ for some scalars $b_1, \dots, b_r$.

That implies

$v = \frac{b_1 - a_1}{b} m_1 + \cdots + \frac{b_r - a_r}{b} m_r$.

Thus $v \in span(M)$. …………………………… **(1 mark)**

Conversely let $v \in span(M)$. Then we will prove that *span(M) = span(N).*

Let, $u \in span(M)$ this implies-

$u = a_1m_1 + \cdots + a_rm_r$. Since $v \in span(M)$, therefore, $v = b_1m_1 + \cdots + b_rm_r$ for some scalars $b_1, \dots, b_r$.

Then $u + v = (a_1 + b_1)m_1 + \cdots + (a_r + b_r)m_r$ that is

$u = (a_1 + b_1)m_1 + \cdots + (a_r + b_r)m_r - v$. Thus, $u \in span(M)$.

Hence, $span(M) \subseteq span(N)$. ……………………………**(1/2 mark)**

Similarly let $u \in span(N)$

Then $u = a_1m_1 + \cdots + a_rm_r + bv$, since $v \in span(M)$, therefore for given scalars $b_1, \dots, b_r$ we have $v = b_1m_1 + \cdots + b_rm_r$

Thus

$u = a_1m_1 + \cdots + a_rm_r + b(b_1m_1 + \cdots + b_rm_r) = (a_1 + bb_1)m_1 + \cdots + (a_r + bb_r)m_r$.

Thus $u \in span(M)$.

Hence, $span(N) \subseteq span(M)$

Thus *span(M) = span(N).*

Hence the proof. …………………………….. **(1/2 mark)**

Q.3 If $T:V \to W$ is a linear transformation from vector space V to a vector space W then prove that $Rank(T)+Nullity(T)=\dim(V)$.

Solution: Let N be the Null space of T. Therefore N is a subspace of V.

Let $\dim(N)=Nullity(T)=k$ and let $\{\alpha_1,\alpha_2,...,\alpha_k\}$ be Basis for N.

Therefore $\{\alpha_1,\alpha_2,...,\alpha_k\}$ is L.I. subset of V. we can extend it to form a basis of V.

(1/2 Mark)

Let $\dim(V)=n$ and $\{\alpha_1,\alpha_2,...,\alpha_k,...,\alpha_n\}$ be basis for V.

Therefore vectors $T(\alpha_1),T(\alpha_2),...,T(\alpha_k),...,T(\alpha_n)$ are in Range of T.

We claim that the vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ is a basis for Range of T.

Then we have to show only

a. The vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ span the range of T.
b. The vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ are L.I.

Let $\beta \in$ Range of T. Therefore $\exists \alpha \in V$ such that $\mathrm{T}(\alpha)=\beta$.

Now $\alpha \in V$ $\exists$ $a_1,a_2,...,a_n \in F$ such that $\alpha = a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n$.

$$T(\alpha)=\mathrm{T}(a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n)$$
$$\beta = a_1T(\alpha_1)+a_2T(\alpha_2)+...+a_nT(\alpha_n)$$
$$= a_{k+1}T(\alpha_{k+1})+...+a_nT(\alpha_n) \quad \text{as } a_1T(\alpha_1)+a_2T(\alpha_2)+...+a_kT(\alpha_k)=0$$

Thus vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ span the range of T. **(1/2 Mark)**

Now let $b_{k+1},b_{k+2},...,b_n \in F$ such that $b_{k+1}T(\alpha_{k+1})+b_{k+2}T(\alpha_{k+2})+...+b_nT(\alpha_n)=0$

$$\mathrm{T}(b_{k+1}\alpha_{k+1}+b_{k+2}\alpha_{k+2}+...+b_n\alpha_n)=0$$
$$\Rightarrow \left(b_{k+1}\alpha_{k+1}+b_{k+2}\alpha_{k+2}+...+b_n\alpha_n\right) \in N$$
$$\Rightarrow b_{k+1}\alpha_{k+1}+b_{k+2}\alpha_{k+2}+...+b_n\alpha_n = c_1\alpha_1+c_2\alpha_2+...+c_k\alpha_k$$
$$\therefore c_1\alpha_1+c_2\alpha_2+...+c_k\alpha_k - b_{k+1}\alpha_{k+1} - b_{k+2}\alpha_{k+2} - ... - b_n\alpha_n = 0$$

But $\{\alpha_1,\alpha_2,...,\alpha_k,...,\alpha_n\}$ be basis for V and hence L.I.

$$\Rightarrow c_1=c_2=...=c_k=b_{k+1}=b_{k+2}=...=b_n=0$$
$$i.e.\ b_{k+1}=b_{k+2}=...=b_n=0$$

Thus The vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ are L.I. **(1/2 Mark)**

Hence The vectors $T(\alpha_{k+1}),...,T(\alpha_n)$ is a basis for Range of T.

Therefore Rank of T = dim of Range of T = n-k

Thus Rank of T + Nullity of T = (n-k)+k=n=dim V **(1/2 Mark)**

Question 4.

Find the eigenvalues and eigenvectors for the matrix A nxn whose elements are given by

$$a_{ij} = \begin{cases} \alpha & if\ i = j \\ 1 & if\ |i-j| = 1 \\ 0 & other\,wise \end{cases}$$, where α is a constant.

Solution:

Given $a_{ij} = \begin{cases} \alpha & if\ i = j \\ 1 & if\ |i-j| = 1 \\ 0 & other\,wise \end{cases}$

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & .. & .. & 0 & 0 \\ 1 & \alpha & 1 & 0 & .. & .. & 0 & 0 \\ 0 & 1 & \alpha & 1 & .. & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. & 1 & \alpha \end{pmatrix}$$

A is a tri-diagonal matrix. Let $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, ..., \lambda_n]$ be the eigenvalue of A and $X = [X^1, X^2, X^3, X^4, ..., X^n]$ be the corresponding eigenvectors.

For each pair (λ, X) satisfy AX = λ X implies that (A - λ I)X=0

$$\begin{pmatrix} \alpha-\lambda & 1 & 0 & 0 & .. & .. & 0 & 0 \\ 1 & \alpha-\lambda & 1 & 0 & .. & .. & 0 & 0 \\ 0 & 1 & \alpha-\lambda & 1 & .. & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & .. & .. & 1 & \alpha-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

Equivalently

$$(\alpha-\lambda)x_1 + x_2 = 0$$

$$x_1 + (\alpha-\lambda)x_2 + x_3 = 0$$

.

.

$$x_k + (\alpha-\lambda)x_{k+1} + x_{k+2} = 0$$

.

$$x_{n-1} + (\alpha-\lambda)x_n = 0$$

From above set of equations, it can be identified a relation

$x_k + (\alpha-\lambda)x_{k+1} + x_{k+2} = 0$ where k = 0,1, 2,….n-1 with $x_0 = x_{n+1} = 0$ ------------(1) **(0.5)**

Which is a second order difference equation

To solve (1), set $x_k = r^k$

(1) Reduced to the quadratic equation $r^2 + (\alpha - \lambda)r + 1 = 0$ -----------------------(2)

Let the root of the equation (2) be r_1 and r_2

The general solution of (1) is

$$x_k = \begin{cases} c_1\rho^k + c_2 k\rho^k, \ r_1 = r_2 & \text{(roots are same)} \\ c_1 r_1^k + c_2 r_2^k, \ r_1 \neq r_2 & \text{(roots are distinct)} \end{cases}, \text{where } c_1 \text{ and } c_2 \text{ are arbitrary constants.}$$

Case (i):

Suppose r_1 and r_2 are same and take $r_1 = r_2 = \rho$

$x_k = c_1\rho^k + c_2 k\rho^k$

By initial conditions, $x_0 = 0$ implies $c_1 = 0$ and hence $c_2 = 0$

Therefore $x_k = 0$, for all k

$x_k = 0$ gives zero eigenvector which is impossible.

Therefore the roots must be distinct in our case. That we will discuss as case (ii)

Case (ii) : Let r_1 and r_2 be two distinct roots of equation (2)

Therefore, $x_k = c_1 r_1^k + c_2 r_2^k$

Using initial condition $x_0 = 0 \Rightarrow c_1 + c_2 = 0$ and we get $c_1 = -c_2$

Taking $x_{n+1} = 0 \Rightarrow c_1 r_1^{n+1} + c\, r_2^{n+1} = 0$

Using $c_1 = -c_2$ we get $\dfrac{r_1^{n+1}}{r_2^{n+1}} = 1$

Therefore, $\dfrac{r_1}{r_2} = (1)^{\frac{1}{n+1}} = (e^{i2\pi j})^{\frac{1}{n+1}} = e^{\frac{i2\pi j}{n+1}}$ -------------(3)

$r_2 = r_1 e^{-\frac{i2\pi j}{n+1}}, \qquad i \leq j < n$

(0.5)

We have $r^2 + (\alpha - \lambda)r + 1 = (r - r_1)(r - r_2)$

$r_1 r_2 = 1, \ and \ r_1 + r_2 = -(\alpha - \lambda)$

Since $r_1 r_2 = 1$ *and subsituting the result* $r_2 = r_1 \, e^{\frac{-2i\pi j}{n+1}}$ *we get*

$r_1 = e^{\frac{i\pi j}{n+1}}$

Therefore $r_2 = e^{\frac{-i\pi j}{n+1}}$

We have $r_1 + r_2 = -(\alpha - \lambda)$

Substituting r_1 and r_2 values we get the eigenvalues of A as

$$\lambda_j - \alpha = e^{\frac{i\pi j}{n+1}} + e^{-\frac{i\pi j}{n+1}}$$

$$\lambda_j = \alpha + 2\cos\frac{\pi j}{n+1}; \quad j = 1,2,\ldots.n,$$

Hence, the eigenvalues of A are given by

$$\lambda_j = \alpha + 2\cos\frac{\pi j}{n+1}; \quad j = 1,2,\ldots.n, \qquad \textbf{(0.5)}$$

Determination of Eigenvector:

Let $X^j = (x_1, x_2, \ldots, x_k, \ldots, x_n)$ is the eigenvector associated with eigenvalue λj.

Here x_k is k^{th} component of eigenvector X^j associated with eigenvalue λj and is given by

$$x_k = c_1 r_1^k + c_2 r_2^k, \qquad \text{where } c_1 + c_1 = 0, \ c_1 = -c_2$$

$$= c_1(r_1^k - r_2^k)$$

$$= c_1\left(e^{\frac{i\pi jk}{n+1}} - e^{-\frac{i\pi jk}{n+1}}\right)$$

$$x_k = 2ic_1 \sin\left(\frac{j\pi k}{n+1}\right) \text{ where } c_1 \text{ is an arbitray constan ts and } k = 1, 2.....n, \ j = 1, 2.....n$$

This is k^{th} component of eigenvector X^j associated with λj.

Therefore the eigenvector X^j corresponding to the eigenvalue λj is given by

$$X^j = \begin{pmatrix} 2ic_1 \sin(\frac{j\pi}{n+1}) \\ 2ic_1 \sin(\frac{2j\pi}{n+1}) \\ 2ic_1 \sin(\frac{3j\pi}{n+1}) \\ . \\ . \\ . \\ 2ic_1 \sin(\frac{(n-1)j\pi}{n+1}) \\ 2ic_1 \sin(\frac{nj\pi}{n+1}) \end{pmatrix}$$

(0.5)

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Q.5 Find the singular value decomposition of A= $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}$ and determine the angle of rotation induced by U and V. Also write the rank 1 decomposition of A in terms of the columns of U and rows of V. Can we do dimensionality reduction in this case?

<u>Solution:</u> We compute

$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 16 \\ 16 & 17 \end{bmatrix}$

The eigenvalues of AA^T

$$|A.A^T - \lambda I| = 0$$

$$\begin{vmatrix} 17-\lambda & 16 \\ 16 & 17-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 33) = 0$$

$$\lambda = 1, \lambda = 33$$

The eigenvector for $\lambda = 33$, $\quad X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvector for $\lambda = 1$, $\quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Consider $A^TA = \begin{bmatrix} 13 & 12 & 10 \\ 12 & 13 & 10 \\ 10 & 10 & 8 \end{bmatrix}$

Eigenvalues of A^TA are computed as

$$|A^TA - \lambda I| = 0$$

$$\lambda = 0,1,33$$

The eigenvectors for $\lambda = 33$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \\ \frac{4}{5} \end{bmatrix}$

The eigenvectors for $\lambda = 1$ $\quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

The eigenvectors for $\lambda = 0$ $\quad v_3 = \begin{bmatrix} 1 \\ 1 \\ \frac{-5}{2} \end{bmatrix}$

Normalizing, we get

$$v_1 = \begin{bmatrix} \frac{5}{66}\sqrt{66} \\ \frac{5}{66}\sqrt{66} \\ \frac{2}{33}\sqrt{66} \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} \frac{2}{33}\sqrt{33} \\ \frac{2}{33}\sqrt{33} \\ -\frac{5}{33}\sqrt{33} \end{bmatrix}$$

Therefore, we get

$$V = \begin{bmatrix} \frac{5}{66}\sqrt{66} & \frac{1}{2}\sqrt{2} & \frac{2}{33}\sqrt{33} \\ \frac{5}{66}\sqrt{66} & -\frac{1}{2}\sqrt{2} & \frac{2}{33}\sqrt{33} \\ \frac{2}{33}\sqrt{66} & 0 & -\frac{5}{33}\sqrt{33} \end{bmatrix}$$

Finally

$$\sum = \begin{bmatrix} \sqrt{33} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix}$$

$$A = U \sum V^T$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{33} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{66}\sqrt{66} & \frac{1}{2}\sqrt{2} & \frac{2}{33}\sqrt{33} \\ \frac{5}{66}\sqrt{66} & -\frac{1}{2}\sqrt{2} & \frac{2}{33}\sqrt{33} \\ \frac{2}{33}\sqrt{66} & 0 & -\frac{5}{33}\sqrt{33} \end{bmatrix}^T$$ ________________(0.5M)

- $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ Comparing with rotation matrix $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

Angle of rotation induced by U=45^0

- V is orthogonal matrix similar to $\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, their traces are same

  Hence 1+2 $cos\theta$=trace of V

Angle of rotation induced by V = 2.964 radians
__________________________(0.5M)

The rank one decomposition of A is

$$A = \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T$$

$$= \sqrt{33} U_1 V_1^T + 1 U_2 V_2^T$$

___________(0.5M)

The dimensionality reduction depends on the application for which SVD is used. The two singular values of A are

$$\sigma_1 = \sqrt{33}\ and\ \sigma_2 = 1$$

$$\sigma_1 = \sqrt{33}\ and\ \sigma_2 = 1$$

We note that $\sigma_1 \approx 6 >> \sigma_2$. Due to this reason, we can write

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \approx \sigma_1 u_1 v_1^T$$

Hence, the dimension is reduced to 1 and so dimensionality reduction is possible. However, for some applications, the difference between σ_1 and σ_2 is not significant enough to replace σ_2 by 0 in the rank one decomposition of A. Hence, in those cases, dimensionality reduction is not possible.

_____________(0.5M)