CSci5512, Spring-2019
ASSIGNMENT 1 :
Assigned: 02/05/19 Due: 02/17/19 at 11:55 PM (submit via Canvas, you may scan or take a picture of your paper answers) Submit only pdf or txt files
On all problems you must show work to receive full credit; all answers found individually

Problem 1. (15 points)
In-class we discussed how instead of using Bayes rule to solve P(a|b) (i.e. P(a|b) = P(b|a)P(a)/P(b)) that instead you could ignore the denominator and instead find P(¬a|b) and normalize. Prove that these methods are theoretically equivalent (i.e. formally prove that this trick will always work).

Solution:
Using the normalize way, we would find:
P(a|b) = α P(b|a)P(a) = α x (let x = P(b|a)*P(a), some number)
P(¬a|b) = α P(b|¬a)P(¬a) = α y (let y = P(b|¬a)P(¬a), some number)

We would then normalize by:
P(a|b) = α x/(α x+α y) = [α x]/[α(x+y)] = x/(x+y), so α = 1/(x+y)
x+y = P(b|a)P(a) + P(b|¬a)P(¬a) = Σ_a P(b|a)P(a) = Σ_a P(b,a) = P(b)

Thus we have shown α is in fact 1/P(b), which is what the basic definition of Bayes rule tells us:
P(a|b) = P(b|a)P(a)/P(b) = α P(b|a)P(a)

Problem 2. (15 points)
In-class we did an example where I said P(a) = 0.2, P(b) = 0.3, P(a or b) = 0.1. I claimed these probabilities were not consistent with each other. What property is violated? Prove this property using only these five facts that I gave in-class:
(1) $0 \leq P(\omega) \leq 1$
(2) $\sum_{\omega \in \Omega} P(\omega) = 1$, where Ω is the set of all possible outcomes
(3) $P(a) + P(\neg a) = 1$
(4) $P(a \text{ or } b) = P(a) + P(b) - P(a, b)$
(5) $P(a) = \sum_b P(a, b)$
Solution:
From (4): P(a or b) = P(a) + P(b) – P(a,b)
0.1 = 0.2 + 0.3 – P(a,b)
P(a,b) = 0.4

From here it should be pretty easy to show this cannot be true, two possible ways are:
(5) P(a) = Σ_b P(a,b)
(1) P(a,b) ≥ 0
(1&5) P(a) ≥ P(a,b), as you have to sum multiple P(a,b)’s to get P(a)... which is not true as P(a) = 0.2 and P(a,b) = 0.4. So the assumption that probabilities are positive is invalidated (or the sum rule)

Otherwise you could directly compute P(a,¬b):
(5) P(a) = P(a,b) + P(a,¬b)

0.2 = 0.4 + P(a,¬b)
P(a,¬b) = -0.2
This invalidates rule (1) that probabilities are non-negative.

Problem 3, 4 & 5 use this table:

Tables for P(a,b,c)

P(a,b,c)	a	¬a
b	0.018837	0.126324
¬b	0.063063	0.160776

when c

P(a,b,¬c)	a	¬a
b	0.011063	0.256476
¬b	0.037037	0.326424

when ¬c

same

$$P(a, b, c) = 0.018837$$
$$P(a, b, \neg c) = 0.011063$$
$$P(a, \neg b, c) = 0.063063$$
$$P(a, \neg b, \neg c) = 0.037037$$
$$P(\neg a, b, c) = 0.126324$$
$$P(\neg a, b, \neg c) = 0.256476$$
$$P(\neg a, \neg b, c) = 0.160776$$
$$P(\neg a, \neg b, \neg c) = 0.326424$$

Problem 3. (20 points)
Find the following probabilities using the table above.
(1) P(a,b)
(2) P(a,b | c)
(3) P(c | ¬a)
(4) P(b)

Solutions:
(1)
P(a,b) = P(a,b,c) + P(a,b,¬c)
P(a,b) = 0.018837 + 0.011063 = **0.0299**

(2) I will use the alpha trick (with multiple variables)
P(a,b|c) = α P(a,b,c) = α * 0.018837 (here α=1/P(c))
P(a,¬b|c) = α P(a,¬b,c) = α * 0.063063
P(¬a,b|c) = α P(¬a,b,c) = α * 0.126324
P(¬a,¬b|c) = α P(¬a,¬b,c) = α * 0.160776
... These four probabilities need to add up to one, so normalize:
P(a,b|c) = 0.018837/(0.018837+0.063063+0.126324+0.160776) = **0.051** ... (note: denominator is in fact P(c) if you computed directly)

(3) Normalize trick...

$P(c|\neg a) = \alpha P(c, \neg a) = \alpha[P(b,c,\neg a) + P(\neg b,c,\neg a)] = \alpha [0.126324 + 0.160776] = 0.2871$

$P(\neg c|\neg a) = \alpha P(\neg c, \neg a) = \alpha[P(b,\neg c,\neg a) + P(\neg b,\neg c,\neg a)] = \alpha [0.256476+0.326424] = 0.5829$

Normalize:

$P(c|\neg a) = 0.2871/(0.2871+0.5829) =$ **<u>0.33</u>**

(4)

$P(b) = \Sigma_a \Sigma_c P(a,b,c) = P(a,b,c) + P(a,b,\neg c) + P(\neg a,b,c) + P(\neg a,b,\neg c) = 0.018837 + 0.011063 + 0.126324 + 0.256476 =$ **<u>0.4127</u>**

Problem 4. (20 points)

Using the same table as problem 3, are any of the variables independent? Are any of the variables conditionally independent? (Show a the rationale for your statements.)

Solution:

--Check pure independence:

Pre-work: find P(a), P(b), P(c), P(a,b), P(a,c), P(b,c)

$P(a) = P(a,b,c) + P(a,b,\neg c) + P(a,\neg b,c) + P(a,\neg b,\neg c) = 0.018837 + 0.011063 + 0.063063 +0.037037 = 0.13$

P(b) = Problem 3(4) = 0.4127

$P(c) = P(a,b,c) + P(a,\neg b,c) + P(\neg a,b,c) + P(\neg a,\neg b,c) = 0.018837+0.063063+0.126324+0.160776 = 0.369$

P(a,b) = Problem 3(1) = 0.0299

$P(a,c) = P(a,b,c) + P(a,\neg b,c) = 0.018837+0.063063 = 0.0819$

$P(b,c) = P(a,b,c) + P(\neg a,b,c) = 0.018837+0.126324 = 0.145161$

A independent B?

P(a,b) =?= P(a)P(b)

0.0299 =?= 0.13 * 0.4127

0.0299 =?= 0.053651

No... A& B not independent

A independent C?

P(a,c) =?= P(a)P(c)

0.0819 =?= 0.13 * 0.369

0.0819 =?= 0.048

No... A& C not independent

B independent C?

P(b,c) =?= P(b)P(c)

0.145161 =?= 0.4127 * 0.369

0.145161 =?= 0.1522863

No... B& C not independent

--Check conditional independence

Prework: find P(a,b|c) P(a,c|b) P(b,c|a), P(a|c), P(b|c), P(a|b), P(c|b), P(b|a) and P(c|a):

(Note: this time we will use the base definition of conditional probability as we have computed all the parts for P(a) and such above)

P(a,b|c) = Problem 3(2) = 0.051

$P(a,c|b) = P(a,b,c)/P(b) = 0.018837/0.4127 = 0.045643324$
$P(b,c|a) = P(a,b,c)/P(a) = 0.018837/0.13 = 0.1449$
$P(a|c) = P(a,c)/P(c) = 0.0819/0.369 = 0.22195122$
$P(b|c) = P(b,c)/P(c) = 0.145161/0.369 = 0.393390244$
$P(a|b) = P(a,b)/P(b) = 0.0299/ 0.4127 = 0.072449721$
$P(c|b) = P(b,c)/P(b) = 0.145161/0.4127 = 0.351734916$
$P(b|a) = P(a,b)/P(a) = 0.0299/0.13 = 0.23$
$P(c|a) = P(a,c)/P(a) = 0.0819/0.13 = 0.63$

A conditionally independent B given C?
$P(a,b|c) =?= P(a|c)P(b|c)$
$0.051 =?= 0.22195122*0.393390244$
$0.051 =?= 0.087313445$
No... A & B not conditionally independent (given C)

A conditionally independent C given B?
$P(a,c|b) =?= P(a|b)P(c|b)$
$0.045643324 =?= 0.072449721* 0.351734916$
$0.045643324 =?= 0.025483097$
No... A & C not conditionally independent (given B)

B conditionally independent C given A?
$P(b,c|a) =?= P(b|a)P(c|a)$
$0.1449 =?= 0.23*0.63$
$0.1449 =?= 0.1449$
Yes! B and C are conditionally independent given A

Problem 5. (20 points)
Using the same table as problems 3 and 4, build a Bayesian network (graph and tables) accurately representing the variables in the table.

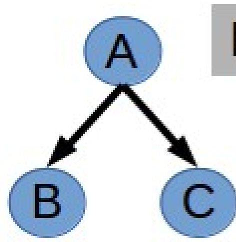
(1) Give the most **efficiently** Bayesian network (least amount of probabilities)
(2) Give the most **inefficient** Bayesian network (maximum amount of probabilities to define network without giving the probabilities for opposite events (e.g. can't give both $P(a|b)$ and $P(\neg a|b)$)

Solution
(1)
Pick A first... nothing else to do
Pick B next. Not purely independent with A, so draw edge from A to B
Pick C last. We want to find the minimal set of parents, so our options are defining C as (ideal is on top, worst is on bottom):
$P(c)$... Not possible as C is not independent with both A and B
$P(c|a)$... Yes, this is possible as C is conditionally independent with B given A
$P(c|b)$... Do not need to check any further down
$P(c|a,b)$... Do not need to check any further down
So we would just draw an edge from A to C. This gives the following Bayesian network (some numbers are form above):
$P(b|\neg a) = P(b,\neg a)/P(\neg a) = [P(\neg a,b,c) + P(\neg a,b,\neg c)] / (1- 0.13) = (0.126324+0.256476)/(1-0.13) =$

P(c|¬a) = P(c,¬a)/P(¬a) = [P(¬a,b,c) + P(¬a,¬b,c)] / (1- 0.13) = (0.126324+0.160776)/(1-0.13) = 0.33

(Note: there are three other valid variations of pick order: [A,C,B], [B,A,C] or [C,A,B]

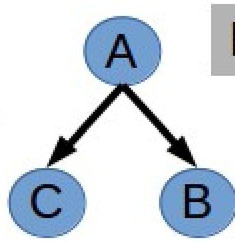
P(a)	0.13



P(b|a)	0.23
P(b|¬a)	0.44

P(c|a)	0.63
P(c|¬a)	0.33

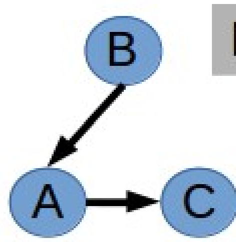
P(a)	0.13



P(c|a)	0.63
P(c|¬a)	0.33

P(b|a)	0.23
P(b|¬a)	0.44

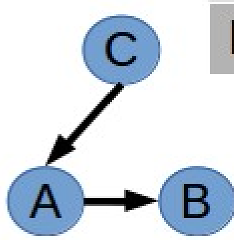
P(b)	0.4127



P(a|b)	0.072449721
P(a|¬b)	0.170441001

P(c|a)	0.63
P(c|¬a)	0.33

P(c)	0.369



P(a|c)	0.22195122
P(a|¬c)	0.076228209

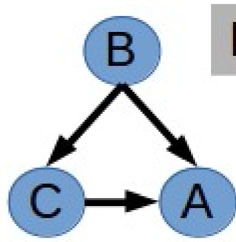
P(b|a)	0.23
P(b|¬a)	0.44

(2)
Pick B first... nothing else to do
Pick C next... C is not independent with B, so we have to draw an edge from B to C
Pick A last... This gives us the following options for parents (we want minimal, so top is better):
$P(a)$... A is not independent with both B and C, so not possible
$P(a|b)$... A is not conditionally independent with C given B
$P(a|c)$... A is not conditionally independent with B given C
$P(a|b,c)$... guess we have to use this one, so add edges [B to A] and [C to A]

(Note: you could also have chosen the pick order [C,B,A])

$P(c|\neg b) = P(c,\neg b)/P(\neg b) = [\ P(a,\neg b,c) + P(\neg a,\neg b,c)\ ] / (1-0.4127) = (0.063063+0.160776)/(1-0.4127) = 0.3811323$
$P(b|\neg c) = P(b,\neg c)/P(\neg c) = [\ P(a,b,\neg c) + P(\neg a,b,\neg c)\ ] / (1-0.369) = (0.011063+0.256476)/(1-0.369) = 0.423992076$
$P(a|b,c) = P(a,b,c)/P(b,c) = 0.018837/0.145161 = 0.12976626$
$P(a|b,\neg c) = P(a,b,\neg c)/P(b,\neg c) = P(a,b,\neg c)/(P(a,b,\neg c) + P(\neg a,b,\neg c)) = 0.011063/(0.011063+0.256476) = 0.041350981$
$P(a|\neg b,c) = P(a,\neg b,c)/P(\neg b,c) = P(a,\neg b,c)/(P(a,\neg b,c) + P(\neg a,\neg b,c)) = 0.063063/(0.063063+0.160776) = 0.281733746$
$P(a|\neg b,\neg c) = P(a,\neg b,\neg c)/P(\neg b,\neg c) = P(a,\neg b,\neg c)/(P(a,\neg b,\neg c)+P(\neg a,\neg b,\neg c)) = 0.037037/(0.037037+0.326424) = 0.101900892$

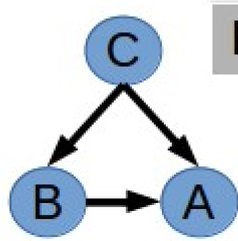
P(c|b)	0.351734916
P(c|¬b)	0.3811323



P(b)	0.4127

P(a|b,c)	0.12976626
P(a|b,¬c)	0.041350981
P(a|¬b,c)	0.281733746
P(a|¬b,¬c)	0.101900892

P(b|c)	0.393390244
P(b|¬c)	0.423992076



P(c)	0.369

P(a|b,c)	0.12976626
P(a|b,¬c)	0.041350981
P(a|¬b,c)	0.281733746
P(a|¬b,¬c)	0.101900892

Problem 6. (20 points)
Pretend there is a slot-machine at the Casino that works as following: 10% of the time it gives a jackpot of $100, 30% of the time it gives a medium reward of $30, 50% of the time it gives a low reward of $5 and 10% of the time you get nothing.

(1) Represent the slot machine as a random variable.
(2) What price should the Casino attach to play this machine?
(3) What is the probability that you get at least one reward from playing the slot-machine 5 times?

Solution:
(1)
The following probability/value pairs (could think of this as a table)
0.1 : 100
0.3 : 30
0.5 : 5
0.1 : 0

(2) The expected value of this random variable is:
0.1*100 + 0.3*30 + 0.5*5 + 0.1*0 = 21.5

... So the casino should charge more than $21.5 to play, otherwise they would lose money.

(3) The probability of <u>not</u> getting a reward is 0.1. We can assume slot pulls are independent (as each pull is represented by the random variable from part (1)). So <u>not</u> getting a reward 5 times in a row would be 0.1^5 = 0.00001. Thus the probability of getting at least one reward = 1 – [probability of not getting any rewards = 1-0.00001 = 0.99999

Problem 7. (5 points)
Suppose a nasty employee modifies the slot-machine from the previous problem. If a jackpot is gotten, the next pull of the slot machine will result in 50% of the time giving the low reward ($5) and 50% of the time giving nothing. What expected amount of money out of two plays of this new slot machine (assuming no jackpot was gotten before the start of these two plays)?

Solution:
Let the random variable from problem 6(1) be called x. We could define a second random variable as y:

RV y is:
0.5 : 5
0.5: 0

If we get a jackpot on the first slot machine, we will then get the expected value of y on the second pull, E[y] = 0.5*5 + 0.5*0 = 2.5. Added with the jackpot this would give $102.5 with a probability 0.1 (jackpot on the first time)

If we don't get a jackpot, we have to re-distribute (i.e. normalize) the probabilities of our original random variable x to a new random variable z:

RV z is:
3/9 : 30
5/9 : 5
1/9 : 0

Expected value of z is: $E[z] = 3/9*30 + 5/9*5 + 1/9*0 = 12+7/9 \approx 12.78$. The second pull will be random variable x again, so we will get 21.5 on the second for a total between the two pulls of $34.28. This happens with a probability of 0.9, as we didn't get a jackpot on z. So we can compute:

value = P(jackpot)*RV(value | jackpot) + P(¬jackpot) * RV(value | ¬jackpot) = 0.1*102.5 + 0.9*34.28 = 41.102 ≈ $41.1 (no fractions of a cent).