The Formal Semantics of Flix

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1 Model-theoretic Semantics

A Flix program $P = (C, L)$ is a set of constraints C and a set of complete lattices L.

A constraint is a rule $A \Leftarrow A_1, \ldots, A_n$ where A is an *atom* (called the *head* of the rule) and $A_1, \ldots, A_n$ are atoms (called the *body* of the rule). A fact is a rule with an empty body. An atom is of the form $p_\ell(t_1, \ldots, t_n)$ where p is a predicate symbol, $\ell \in L$ is the lattice associated with p, and $t_1, \ldots, t_n$ are terms. A term is either a wildcard variable, a named variable or a constant value. The possible values are the unit value (), the booleans (`true`, `false`), the integers (-5, 3, 7), tagged unions of values (e.g. `Tag` v) and tuples of values (e.g. $(1, \texttt{true}, 42)$).

A complete lattice $l \in L$ is a 6-tuple $l = (E, \bot, \top, \sqsubseteq, \sqcup, \sqcap)$ where E is a set of elements $E \subseteq V$, $\bot \in E$ is the least element, $\top \in E$ is the greatest element, $\sqsubseteq$ is the partial order on E, $\sqcup$ is the least upper bound, and $\sqcap$ is the greatest lower bound.

The *Herbrand Universe* $\mathcal{U}$ of a Flix program P is the set of all possible ground terms. A ground term is non-wildcard, non-variable term. That is, the Herbrand Universe is exactly the set of values.

The *Herbrand Base* $\mathcal{B}$ of P is the set of all possible ground atoms whose predicate symbols occur in P and where the arguments are drawn from the Herbrand Universe.

We define a partition of the Herbrand Base such that two ground atoms $A = p_\ell(v_1, \ldots, v_n)$ and $B = p'_{\ell'}(v'_1, \ldots, v'_m)$ are in the same subset if they have the same predicate symbol (i.e. $p = p'$) and the same number of terms (i.e. $n = m$). Notice that all predicate symbols in a subset have the same associated lattice ℓ. For each subset S we introduce a complete lattice L_S on its ground atoms where $L_S = (S, \bot_S, \top_S, \sqsubseteq_S, \sqcup_S, \sqcap_S)$. Given two ground atoms $A = p_\ell(v_1, \ldots, v_n)$ and $B = p_\ell(v'_1, \ldots, v'_n)$ we define their partial order as: If $n = 1$ then $A \sqsubseteq_S B$ when $v_1 \sqsubseteq v'_1$. If $n > 1$ then $A \sqsubseteq_S B$ when $v_1 = v'_1, \ldots, v_{n-1} = v'_{n-1}$ and $v_n \sqsubseteq v'_n$. The other components are defined in a similar way.

What this means is that for any subset in the partion of the Herbrand Base we have a complete lattice on the ground atoms in that subset.

An interpretation I of a Flix program P is a subset of the Herbrand Base $\mathcal{B}$. A ground atom A is true w.r.t. an interpretation if $\exists A' \in I$ such that $A \sqsubseteq A'$. A conjunction of atoms $A_1, \cdots, A_n$ is true w.r.t. an interpretation if each atom is true in the interpretation. A ground rule is true if either the body conjunction is false, or the head is true.

A model M of P is an interpretation that makes each ground instance of a each rule in P true.

A model M is *compact* iff (i) every subset S in the partition of M has one unique element, and (ii) this element is non-bottom according the lattice associated with the subset S.

We define a partial order $\sqsubseteq_M$ on compact models. Given two compact models M_1 and M_2 we say that M_1 is less than or equal to M_2 if for every ground atom $A_1 \in M_1$, associated with the subset S, there is a ground atom $A_2 \in M_2$, also associated with S, such that $A_1 \sqsubseteq_S A_2$.

A model M is minimal if it is compact and there is no other model less than or equal to M according the partial order $\sqsubseteq_M$.

Example. The Flix program P with constraints:

$$A_\ell(\texttt{Even}) \quad A_\ell(\texttt{Odd}) \quad B_\ell(\texttt{Odd})$$

and lattices $\{\ell = (\bot, \top, \texttt{Even}, \texttt{Odd}\}, \sqsubseteq, \sqcup, \sqcap)\}$ has the Herbrand Universe:

$$\mathcal{U} = \{\bot, \top, \texttt{Even}, \texttt{Odd}\}$$

and the Herbrand Base:

$$\mathcal{B} = \{A_\ell(\bot), A_\ell(\texttt{Even}), A_\ell(\texttt{Odd}), A_\ell(\top), B_\ell(\bot), B_\ell(\texttt{Even}), B_\ell(\texttt{Odd}), B_\ell(\top)\}$$

An interpretation of P is a subset of $\mathcal{B}$. For example,

$$\begin{aligned}
I_1 &= \{A_\ell(\top)\} \\
I_2 &= \{A_\ell(\top), B_\ell(\bot)\} \\
I_3 &= \{A_\ell(\top), B_\ell(\texttt{Odd}), B_\ell(\top)\} \\
I_4 &= \{A_\ell(\texttt{Even}), A_\ell(\texttt{Odd}), B_\ell(\texttt{Odd})\} \\
I_5 &= \{A_\ell(\top), B_\ell(\top)\} \\
I_6 &= \{A_\ell(\top), B_\ell(\texttt{Odd})\}
\end{aligned}$$

The interpretation I_1 is not a model of P since it does not make $B_\ell(\texttt{Odd})$ true. I_2 is also not a model of p since it does not make $B_\ell(\texttt{Odd})$ true. I_3 is a model of p, but it is not compact. I_4 is a model of p, but it is not reduced. I_5 is a model of p, it is compact and reduced, but it is not minimal as evidenced by I_6.

Example. The Flix program P with constraints:

$$A_\ell(1, \texttt{Pos}) \quad A_\ell(2, \texttt{Pos}) \quad A_\ell(2, \texttt{Neg})$$

and lattices $\{\ell = (\bot, \top, \texttt{Neg}, \texttt{Zer}, \texttt{Pos}\}, \sqsubseteq, \sqcup, \sqcap)\}$ has the Herbrand Universe:

$$\mathcal{U} = \{1, 2, \bot, \top, \texttt{Neg}, \texttt{Zer}, \texttt{Pos}\}$$

and the Herbrand Base:

$$\begin{aligned}
\mathcal{B} = \{&A_\ell(1,1), A_\ell(1,2), A_\ell(1,\bot), A_\ell(1,\top), \cdots \\
&A_\ell(2,1), A_\ell(2,2), A_\ell(2,\bot), A_\ell(2,\top), \cdots \\
&A_\ell(\bot,1), A_\ell(\bot,2), A_\ell(\bot,\bot), A_\ell(\bot,\top), \cdots \\
&\cdots\}
\end{aligned}$$

An interpretation of P is a subset of $\mathcal{B}$. For example,

$$\begin{aligned}
I_1 &= \{A_\ell(1,\top)\} \\
I_2 &= \{A_\ell(1,\top), A_\ell(2,\top)\} \\
I_3 &= \{A_\ell(1,\texttt{Pos}), A_\ell(1,\texttt{Zer}), A_\ell(2,\top)\} \\
I_4 &= \{A_\ell(1,\texttt{Pos}), A_\ell(2,\top)\}
\end{aligned}$$

Here I_2, I_3 and I_4 are models and I_4 is minimal.