

The Formal Semantics of Flix

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1 Model-theoretic Semantics

A Flix program $P = (C, L)$ is a set of constraints C and a set of complete lattices L .

A constraint is a rule $A \Leftarrow A_1, \dots, A_n$ where A is an *atom* (called the *head* of the rule) and A_1, \dots, A_n are atoms (called the *body* of the rule). A fact is a rule with an empty body. An atom is of the form $p_\ell(t_1, \dots, t_n)$ where p is a predicate symbol, $\ell \in L$ is the lattice associated with p , and t_1, \dots, t_n are terms. A term is either a wildcard variable, a named variable or a constant value. The possible values are the unit value $()$, the booleans (**true**, **false**), the integers ($-5, 3, 7$), tagged unions of values (e.g. **Tag** v) and tuples of values (e.g. $(1, \text{true}, 42)$).

A complete lattice $\ell \in L$ is a 6-tuple $\ell = (E, \perp, \top, \sqsubseteq, \sqcup, \sqcap)$ where E is a set of elements $E \subseteq V$, $\perp \in E$ is the least element, $\top \in E$ is the greatest element, \sqsubseteq is the partial order on E , \sqcup is the least upper bound, and \sqcap is the greatest lower bound.

The *Herbrand Universe* \mathcal{U} of a Flix program P is the set of all possible ground terms. A ground term is a non-wildcard, non-variable term. That is, the Herbrand Universe is exactly the set of values.

The *Herbrand Base* \mathcal{B} of P is the set of all possible ground atoms whose predicate symbols occur in P and where the arguments are drawn from the Herbrand Universe.

We define a partition of the Herbrand Base such that two ground atoms $A = p_\ell(v_1, \dots, v_n)$ and $B = p'_\ell(v'_1, \dots, v'_m)$ are in the same subset if they have the same predicate symbol (i.e. $p = p'$) and the same number of terms (i.e. $n = m$). Notice that all predicate symbols in a subset have the same associated lattice ℓ . For each subset S we introduce a complete lattice $L_S = (S, \perp_S, \top_S, \sqsubseteq_S, \sqcup_S, \sqcap_S)$. Given two ground atoms $A = p_\ell(v_1, \dots, v_n)$ and $B = p_\ell(v'_1, \dots, v'_n)$ we define their partial order as: If $n = 1$ then $A \sqsubseteq_S B$ when $v_1 \sqsubseteq v'_1$. If $n > 1$ then $A \sqsubseteq_S B$ when $v_1 = v'_1, \dots, v_{n-1} = v'_{n-1}$ and $v_n \sqsubseteq v'_n$. The other lattice components are defined in a similar way.

Intuitively, any subset in the partition of the Herbrand Base has a complete lattice on the ground atoms in that subset.

An interpretation I of a Flix program P is a subset of the Herbrand Base \mathcal{B} . A ground atom A is true w.r.t. an interpretation if $\exists A' \in I$ such that $A \sqsubseteq A'$. A conjunction of atoms A_1, \dots, A_n is true w.r.t. an interpretation if each atom is true in the interpretation. A ground rule is true if either the body conjunction is false, or the head is true.

A model M of P is an interpretation that makes each ground instance of a each rule in P true.

A model M is *compact* iff (i) every subset S in the partition of M has one unique element, and (ii) this element is non-bottom according the lattice associated with the subset S .

We define a partial order \sqsubseteq_M on compact models. Given two compact models M_1 and M_2 we say that M_1 is less than or equal to M_2 if for every ground atom $A_1 \in M_1$, associated with the subset S , there is a ground atom $A_2 \in M_2$, also associated with S , such that $A_1 \sqsubseteq_S A_2$.

A model M is *minimal* if it is compact and there is no other model M' such that $M' \sqsubseteq_M M$.

Example. The Flix program P with constraints:

$$A_\ell(\text{Even}) \quad A_\ell(\text{Odd}) \quad B_\ell(\text{Odd})$$

and lattices $\{\ell = (\perp, \top, \text{Even}, \text{Odd}), \sqsubseteq, \sqcup, \sqcap\}$ has the Herbrand Universe:

$$\mathcal{U} = \{\perp, \top, \text{Even}, \text{Odd}\}$$

and the Herbrand Base:

$$\mathcal{B} = \{A_\ell(\perp), A_\ell(\text{Even}), A_\ell(\text{Odd}), A_\ell(\top), \\ B_\ell(\perp), B_\ell(\text{Even}), B_\ell(\text{Odd}), B_\ell(\top)\}$$

An interpretation of P is a subset of \mathcal{B} . For example,

$$\begin{aligned} I_1 &= \{A_\ell(\top)\} \\ I_2 &= \{A_\ell(\top), B_\ell(\perp)\} \\ I_3 &= \{A_\ell(\top), B_\ell(\text{Odd}), B_\ell(\top)\} \\ I_4 &= \{A_\ell(\text{Even}), A_\ell(\text{Odd}), B_\ell(\text{Odd})\} \\ I_5 &= \{A_\ell(\top), B_\ell(\top)\} \\ I_6 &= \{A_\ell(\top), B_\ell(\text{Odd})\} \end{aligned}$$

The interpretation I_1 is not a model of P since it does not make $B_\ell(\text{Odd})$ true. I_2 is also not a model of p since it does not make $B_\ell(\text{Odd})$ true. I_3 and I_4 are models of p , but they are not compact. I_5 is a compact model of p , but it is not minimal as evidenced by I_6 .

Example. The Flix program P with constraints:

$$A_\ell(1, \text{Pos}) \quad A_\ell(2, \text{Pos}) \quad A_\ell(2, \text{Neg})$$

and lattices $\{\ell = (\perp, \top, \text{Neg}, \text{Zer}, \text{Pos}), \sqsubseteq, \sqcup, \sqcap\}$ has the Herbrand Universe:

$$\mathcal{U} = \{1, 2, \perp, \top, \text{Neg}, \text{Zer}, \text{Pos}\}$$

and the Herbrand Base:

$$\begin{aligned} \mathcal{B} = \{ & A_\ell(1, 1), A_\ell(1, 2), A_\ell(1, \perp), A_\ell(1, \top), \dots \\ & A_\ell(2, 1), A_\ell(2, 2), A_\ell(2, \perp), A_\ell(2, \top), \dots \\ & A_\ell(\perp, 1), A_\ell(\perp, 2), A_\ell(\perp, \perp), A_\ell(\perp, \top), \dots \\ & \dots \} \end{aligned}$$

An interpretation of P is a subset of \mathcal{B} . For example,

$$\begin{aligned} I_1 &= \{A_\ell(1, \top)\} \\ I_2 &= \{A_\ell(1, \top), A_\ell(2, \top)\} \\ I_3 &= \{A_\ell(1, \text{Pos}), A_\ell(1, \text{Zer}), A_\ell(2, \top)\} \\ I_4 &= \{A_\ell(1, \text{Pos}), A_\ell(2, \top)\} \end{aligned}$$

Here I_2 , I_3 and I_4 are models and I_4 is minimal.