The Formal Semantics of Flix

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1 Model-theoretic Semantics

A Flix program $P = (C, L)$ is a set of constraints C and a set of complete lattices L.

A constraint is a rule $A \Leftarrow A_1, \ldots, A_n$ where A is an *atom* (called the *head* of the rule) and $A_1, \ldots, A_n$ are atoms (called the *body* of the rule). A fact is a rule with an empty body. An atom is of the form $p_\ell(t_1, \ldots, t_n)$ where p is a predicate symbol, $\ell \in L$ is the lattice associated with p, and $t_1, \ldots, t_n$ are terms. A term is either a wildcard variable, a named variable or a constant value. The possible values are the unit value (), the booleans (`true`, `false`), the integers (-5, 3, 7), tagged unions of values (e.g. `Tag` v) and tuples of values (e.g. $(1, \texttt{true}, 42)$).

A complete lattice $l \in L$ is a 6-tuple $l = (E, \perp, \top, \sqsubseteq, \sqcup, \sqcap)$ where E is a set of elements $E \subseteq V$, $\perp \in E$ is the least element, $\top \in E$ is the greatest element, $\sqsubseteq$ is the partial order on E, $\sqcup$ is the least upper bound, and $\sqcap$ is the greatest lower bound.

The *Herbrand Universe* $\mathcal{U}$ of a Flix program P is the set of all possible ground terms. A ground term is non-wildcard, non-variable term. That is, the Herbrand Universe is exactly the set of values.

The *Herbrand Base* $\mathcal{B}$ of P is the set of all possible ground atoms whose predicate symbols occur in P and where the arguments are drawn from the Herbrand Universe.

We introduce a partial order on ground atoms. Given two ground atoms $A = p_\ell(v_1, \ldots, v_n)$ and $B = p_\ell(v'_1, \ldots, v'_n)$ with the same (i) predicate symbol, (ii) associated lattice and (iii) number of ground terms, we define their partial order as: If $n = 1$ then $A \sqsubseteq B$ when $v_1 \sqsubseteq v'_1$. If $n > 1$ then $A \sqsubseteq B$ when $v_1 = v'_1, \ldots, v_{n-1} = v'_{n-1}$ and $v_n \sqsubseteq v'_n$. This definition is implicitely stating that the last term of a ground atom is a lattice element.

An interpretation I of a Flix program P is a subset of the Herbrand Base $\mathcal{B}$. A ground atom A is true w.r.t. an interpretation if $\exists A' \in I$ such that $A \sqsubseteq A'$. A conjunction of atoms $A_1, \cdots, A_n$ is true w.r.t. an interpretation if each atom is true in the interpretation. A ground rule is true if either the body conjunction is false, or the head is true.

A model M of P is an interpretation that makes each ground instance of a each rule in P true.

A model is *compact* if it has no two ground atoms $A, A' \in M$ such that $A \sqsubseteq A'$. A model is *reduced* if it has no two ground atoms $A, A' \in M$ such that $A \sqsubseteq (A \sqcup A')$. Finally, a model M_1 is *minimal* if it is compact and reduced, and there is no other model M_2 such that there are two ground atoms $A \in M_1$ and $A' \in M_2$ where $A' \sqsubset A$.

Example. The Flix program P with constraints:

$$A_\ell(\texttt{Even}) \quad A_\ell(\texttt{Odd}) \quad B_\ell(\texttt{Odd})$$

and lattices $\{\ell = (\perp, \top, \texttt{Even}, \texttt{Odd}\}, \sqsubseteq, \sqcup, \sqcap)\}$ has the Herbrand Universe:

$$\mathcal{U} = \{\perp, \top, \texttt{Even}, \texttt{Odd}\}$$

and the Herbrand Base:

$$\begin{aligned}\mathcal{B} = \{&A_\ell(\perp), A_\ell(\texttt{Even}), A_\ell(\texttt{Odd}), A_\ell(\top),\\ &B_\ell(\perp), B_\ell(\texttt{Even}), B_\ell(\texttt{Odd}), B_\ell(\top)\}\end{aligned}$$

An interpretation of P is a subset of $\mathcal{B}$. For example,

$$\begin{aligned}
I_1 &= \{A_\ell(\top)\}\\
I_2 &= \{A_\ell(\top), B_\ell(\perp)\}\\
I_3 &= \{A_\ell(\top), B_\ell(\texttt{Odd}), B_\ell(\top)\}\\
I_4 &= \{A_\ell(\texttt{Even}), A_\ell(\texttt{Odd}), B_\ell(\texttt{Odd})\}\\
I_5 &= \{A_\ell(\top), B_\ell(\top)\}\\
I_6 &= \{A_\ell(\top), B_\ell(\texttt{Odd})\}
\end{aligned}$$

The interpretation I_1 is not a model of P since it does not make $B_\ell(\texttt{Odd})$ true. I_2 is also not a model of p since it does not make $B_\ell(\texttt{Odd})$ true. I_3 is a model of p, but it is not compact. I_4 is a model of p, but it is not reduced. I_5 is a model of p, it is compact and reduced, but it is not minimal as evidenced by I_6.

Example. The Flix program P with constraints:

$$A_\ell(1, \texttt{Pos}) \quad A_\ell(2, \texttt{Pos}) \quad A_\ell(2, \texttt{Neg})$$

and lattices $\{\ell = (\bot, \top, \texttt{Neg}, \texttt{Zer}, \texttt{Pos}\}, \sqsubseteq, \sqcup, \sqcap)\}$ has the Herbrand Universe:

$$\mathcal{U} = \{1, 2, \bot, \top, \texttt{Neg}, \texttt{Zer}, \texttt{Pos}\}$$

and the Herbrand Base:

$$\begin{aligned}\mathcal{B} = \{&A_\ell(1,1), A_\ell(1,2), A_\ell(1,\bot), A_\ell(1,\top), \cdots \\ &A_\ell(2,1), A_\ell(2,2), A_\ell(2,\bot), A_\ell(2,\top), \cdots \\ &A_\ell(\bot,1), A_\ell(\bot,2), A_\ell(\bot,\bot), A_\ell(\bot,\top), \cdots \\ &\cdots\}\end{aligned}$$

An interpretation of P is a subset of $\mathcal{B}$. For example,

$$\begin{aligned}I_1 &= \{A_\ell(1, \top)\} \\ I_2 &= \{A_\ell(1, \top), A_\ell(2, \top)\} \\ I_3 &= \{A_\ell(1, \texttt{Pos}), A_\ell(1, \texttt{Zer}), A_\ell(2, \top)\} \\ I_4 &= \{A_\ell(1, \texttt{Pos}), A_\ell(2, \top)\}\end{aligned}$$

Here I_2, I_3 and I_4 are models and I_4 is minimal.