

Binary Index Trees

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Introduction

We often need some sort of data structure to make our algorithms faster. In this article we will discuss the **Binary Indexed Trees** structure. According to [Peter M. Fenwick](#), this structure was first used for data compression. Now it is often used for storing frequencies and manipulating cumulative frequency tables. Let's define the following **problem**: We have n boxes. Possible queries are

1. add marble to box i
2. sum marbles from box k to box l

The naive solution has time complexity of $O(1)$ for query 1 and $O(n)$ for query 2. Suppose we make m queries. The worst case (when all queries are 2) has time complexity $O(n * m)$. Using some data structure (i.e. [RMQ](#)) we can solve this problem with the worst case time complexity of $O(m \log n)$. Another approach is to use Binary Indexed Tree data structure, also with the worst time complexity $O(m \log n)$ -- but Binary Indexed Trees are much easier to code, and require less memory space, than RMQ.

Notation

- BIT - **B**inary **I**ndexed **T**ree
- MaxVal - maximum value which will have non-zero frequency
- $f[i]$ - frequency of value with index i , $i = 1 \dots \text{MaxVal}$
- $c[i]$ - cumulative frequency for index i ($f[1] + f[2] + \dots + f[i]$)
- $\text{tree}[i]$ - sum of frequencies stored in **BIT** with index i (latter will be described what index means); sometimes we will write *tree frequency* instead *sum of frequencies stored in BIT*
- num^- - complement of integer **num** (integer where each binary digit is inverted: $0 \rightarrow 1$; $1 \rightarrow 0$)

NOTE: Often we put $f[0] = 0$, $c[0] = 0$, $\text{tree}[0] = 0$, so sometimes I will just ignore index 0.

Basic idea

Each integer can be represented as sum of powers of two. In the same way, cumulative frequency can be represented as sum of sets of subfrequencies. In our case, each set contains some successive number of non-overlapping frequencies. **idx** is some index of **BIT**. **r** is a position in **idx** of the last digit 1 (from left to right) in binary notation. **tree[idx]** is sum of frequencies from index $(\text{idx} - 2^r + 1)$ to index **idx** (look at the Table 1.1 for clarification). We also write that **idx** is **responsible** for indexes from $(\text{idx} - 2^r + 1)$ to **idx** (note that responsibility is the key in our algorithm and is the way of manipulating the tree).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
f	1	0	2	1	1	3	0	4	2	5	2	2	3	1	0	2
c	1	1	3	4	5	8	8	12	14	19	21	23	26	27	27	29
tree	1	1	2	4	1	4	0	12	2	7	2	11	3	4	0	29

Table 1.1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
tree	1	1..2	3	1..4	5	5..6	7	1..8	9	9..10	11	9..12	13	13..14	15	1..16

Table 1.2 - table of responsibility

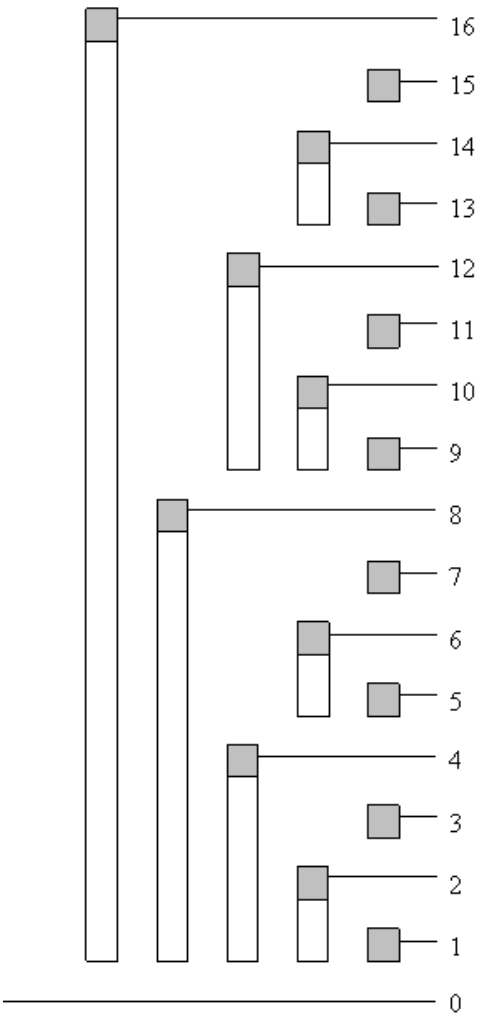


Image 1.3 - tree of responsibility for indexes (bar shows range of frequencies accumulated in top element)

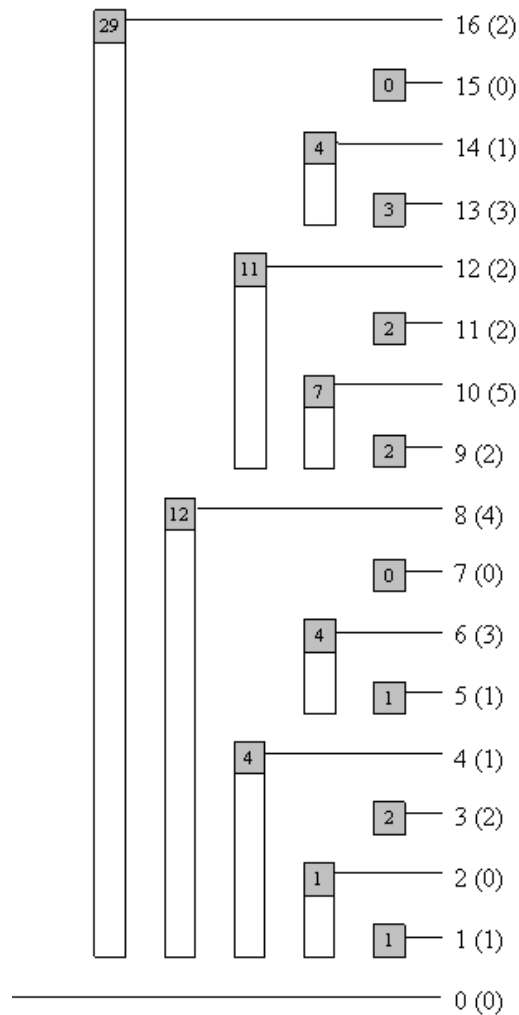


Image 1.4 - tree with tree frequencies

Suppose we are looking for cumulative frequency of index 13 (for the first 13 elements). In binary notation, 13 is equal to 1101. Accordingly, we will calculate $c[1101] = \text{tree}[1101] + \text{tree}[1100] + \text{tree}[1000]$ (more about this later).

Isolating the last digit

NOTE: Instead of "the last non-zero digit," it will write only "the last digit."

There are times when we need to get just the last digit from a binary number, so we need an efficient way to do that. Let **num** be the integer whose last digit we want to isolate. In binary notation **num** can be represented as **a1b**, where **a** represents binary digits before the last digit and **b** represents zeroes after the last digit.

Integer **-num** is equal to $(a1b)^{-} + 1 = a^{-}0b^{-} + 1$. **b** consists of all zeroes, so **b**⁻ consists of all ones. Finally we have

$$-\text{num} = (\text{a1b})^- + 1 = \text{a}^-0\text{b}^- + 1 = \text{a}^-0(0\dots0)^- + 1 = \text{a}^-0(1\dots1) + 1 = \text{a}^-1(0\dots0) = \text{a}^-1\text{b}.$$

Now, we can easily isolate the last digit, using bitwise operator **AND** (in C++, Java it is **&**) with **num** and **-num**:

$$\begin{array}{r} \text{a1b} \\ \& \text{a}^-1\text{b} \\ \hline = (0\dots0)1(0\dots0) \end{array}$$

Read cumulative frequency

If we want to read cumulative frequency for some integer **idx**, we add to **sum tree[idx]**, subtract last bit of **idx** from itself (also we can write - remove the last digit; change the last digit to zero) and repeat this while **idx** is greater than zero. We can use next function (written in C++)

```
int read(int idx){
    int sum = 0;
    while (idx > 0){
        sum += tree[idx];
        idx -= (idx & -idx);
    }
    return sum;
}
```

Example for **idx = 13**; **sum = 0**:

iteration	idx	position of the last digit	idx & -idx	sum
1	13 = 1101	0	1 (2 ^0)	3
2	12 = 1100	2	4 (2 ^2)	14
3	8 = 1000	3	8 (2 ^3)	26
4	0 = 0	---	---	---

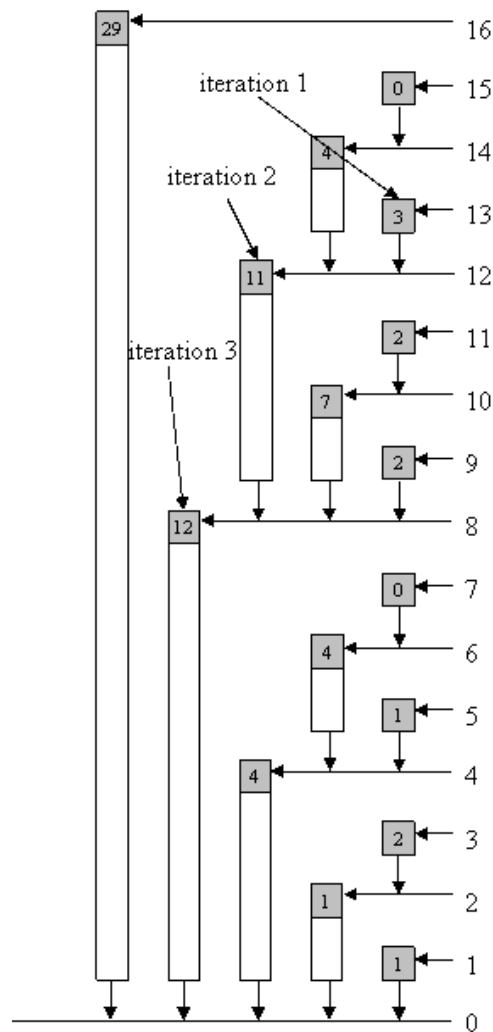


Image 1.5 - arrows show path from index to zero which we use to get sum (image shows example for index 13)

So, our result is 26. The number of iterations in this function is number of bits in **idx**, which is at most **log MaxVal**.

Time complexity: $O(\log \text{MaxVal})$.

Code length: Up to ten lines.

Change frequency at some position and update tree

The concept is to update tree frequency at all indexes which are responsible for frequency whose value we are changing. In reading cumulative frequency at some index, we were removing the last bit and going on. In changing some frequency **val** in tree, we should increment value at the current index (the starting index is always the one whose frequency is changed) for **val**, add the last digit to index and go on while the index is less than or equal to **MaxVal**. Function in C++:

```

void update(int idx ,int val){
    while (idx <= MaxVal){
        tree[idx] += val;
        idx += (idx & -idx);
    }
}

```

Let's show example for **idx** = 5:

iteration	idx	position of the last digit	idx & -idx
1	5 = 101	0	1 (2 ^0)
2	6 = 110	1	2 (2 ^1)
3	8 = 1000	3	8 (2 ^3)
4	16 = 10000	4	16 (2 ^4)
5	32 = 100000	---	---

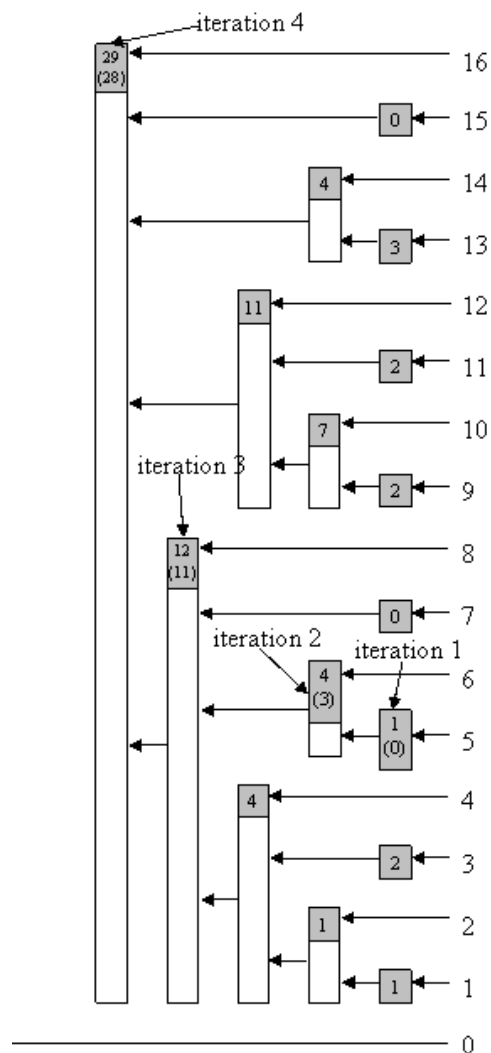


Image 1.6 - Updating tree (in brackets are tree frequencies before updating); arrows show path while we update tree from index to **MaxVal** (image shows example for index 5)

Using algorithm from above or following arrows shown in Image 1.6 we can update **BIT**.

Time complexity: $O(\log \text{MaxVal})$.

Code length: Up to ten lines.

Read the actual frequency at a position

We've described how we can read cumulative frequency for an index. It is obvious that we can not read just **tree[idx]** to get the actual frequency for value at index **idx**. One approach is to have one additional array, in which we will separately store frequencies for values. Both reading and storing take $O(1)$; memory space is linear. Sometimes it is more important to save memory, so we will show how you can get actual frequency for some value without using additional structures.

Probably everyone can see that the actual frequency at a position **idx** can be calculated by calling function **read** twice -- **f[idx] = read(idx) - read(idx - 1)** -- just by taking the difference of two adjacent cumulative frequencies. This procedure always works in $2 * O(\log n)$ time. If we write a new function, we can get a bit faster algorithm, with smaller const.

If two paths from two indexes to root have the same part of path, then we can calculate the sum until the paths meet, subtract stored sums and we get a sum of frequencies between that two indexes. It is pretty simple to calculate sum of frequencies between adjacent indexes, or read the actual frequency at a given index. Mark given index with **x**, its predecessor with **y**. We can represent (binary notation) **y** as **a0b**, where **b** consists of all ones. Then, **x** will be **a1b⁻** (note that **b⁻** consists all zeros). Using our algorithm for getting **sum** of some index, let it be **x**, in first iteration we remove the last digit, so after the first iteration **x** will be **a0b⁻**, mark a new value with **z**.

Repeat the same process with **y**. Using our function for reading **sum** we will remove the last digits from the number (one by one). After several steps, our **y** will become (just to remind, it was **a0b**) **a0b⁻**, which is the same as **z**. Now, we can write our algorithm. Note that the only exception is when **x** is equal to 0. Function in C++:

```
int readSingle(int idx){
    int sum = tree[idx];    // sum will be decreased
    if (idx > 0){           // special case
        int z = idx - (idx & -idx); // make z first
        idx--;              // idx is no important any more,
                            // so instead y, you can use idx
        while (idx != z){ // at some iteration idx (y) will become z
            sum -= tree[idx];
            // subtract tree frequency which is
            // between y and "the same path"
            idx -= (idx & -idx);
        }
    }
    return sum;
}
```

Here's an example for getting the actual frequency for index 12:
First, we will calculate $z = 12 - (12 \& -12) = 8$, $sum = 11$

iteration	y	position of the last digit	y & -y	sum
1	11 = 1011	0	1 (2 ^0)	9
2	10 = 1010	1	2 (2 ^1)	2
3	8 = 1000	---	---	---

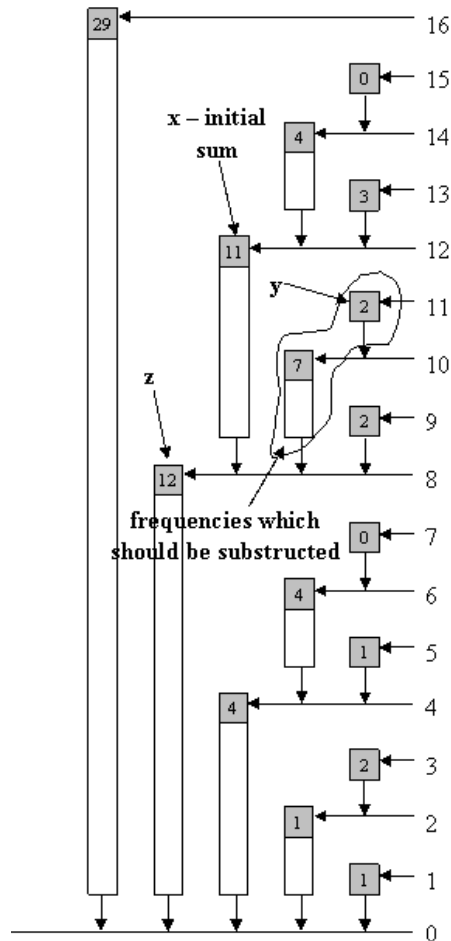


Image 1.7 - read actual frequency at some index in BIT
(image shows example for index 12)

Let's compare algorithm for reading actual frequency at some index when we twice use function **read** and the algorithm written above. Note that for each odd number, the algorithm will work in const time $O(1)$, without any iteration. For almost every even number **idx**, it will work in $c * O(\log \text{idx})$, where c is strictly less than 1, compare to **read(idx) - read(idx - 1)**, which will work in $c1 * O(\log \text{idx})$, where $c1$ is **always** greater than 1.

Time complexity: $c * O(\log \text{MaxVal})$, where c is less than 1.

Code length: Up to fifteen lines.

Scaling the entire tree by a constant factor

Sometimes we want to scale our tree by some factor. With the procedures described above it is very simple. If we want to scale by some factor **c**, then each index **idx** should be updated by **-(c - 1) * readSingle(idx) / c** (because $f[idx] - (c - 1) * f[idx] / c = f[idx] / c$). Simple function in C++:

```
void scale(int c){
    for (int i = 1 ; i <= MaxVal ; i++)
        update(-(c - 1) * readSingle(i) / c , i);
}
```

This can also be done more quickly. Factor is linear operation. Each tree frequency is a linear composition of some frequencies. If we scale each frequency for some factor, we also scaled tree frequency for the same factor. Instead of rewriting the procedure above, which has time complexity $O(\text{MaxVal} * \log \text{MaxVal})$, we can achieve time complexity of $O(\text{MaxVal})$:

```
void scale(int c){
    for (int i = 1 ; i <= MaxVal ; i++)
        tree[i] = tree[i] / c;
}
```

Time complexity: $O(\text{MaxVal})$.

Code length: Just a few lines.

Find index with given cumulative frequency

The naive and most simple solution for finding an index with a given cumulative frequency is just simply iterating through all indexes, calculating cumulative frequency, and checking if it's equal to the given value. In case of negative frequencies it is the only solution. However, if we have only non-negative frequencies in our tree (that means cumulative frequencies for greater indexes are not smaller) we can figure out logarithmic algorithm, which is modification of [binary search](#). We go through all bits (starting with the highest one), make the index, compare the cumulative frequency of the current index and given value and, according to the outcome, take the lower or higher half of the interval (just like in binary search). Function in C++:

```
// if in tree exists more than one index with a same
// cumulative frequency, this procedure will return
// some of them (we do not know which one)

// bitMask - initially, it is the greatest bit of MaxVal
// bitMask store interval which should be searched
int find(int cumFre){
    int idx = 0; // this var is result of function

    while ((bitMask != 0) && (idx < MaxVal)){ // nobody likes overflow :)
        int tIdx = idx + bitMask; // we make midpoint of interval
        if (cumFre == tree[tIdx]) // if it is equal, we just return idx
            return tIdx;
        else if (cumFre > tree[tIdx]){
            // if tree frequency "can fit" into cumFre,
```

```

        // then include it
        idx = tIdx; // update index
        cumFre -= tree[tIdx]; // set frequency for next loop
    }
    bitMask >>= 1; // half current interval
}
if (cumFre != 0) // maybe given cumulative frequency doesn't exist
    return -1;
else return idx;
}

// if in tree exists more than one index with a same
// cumulative frequency, this procedure will return
// the greatest one
int findG(int cumFre){
    int idx = 0;
    while ((bitMask != 0) && (idx < MaxVal)){
        int tIdx = idx + bitMask;
        if (cumFre >= tree[tIdx]){
            // if current cumulative frequency is equal to cumFre,
            // we are still looking for higher index (if exists)
            idx = tIdx;
            cumFre -= tree[tIdx];
        }
        bitMask >>= 1;
    }
    if (cumFre != 0) return -1;
    else return idx;
}

```

Example for cumulative frequency 21 and function **find**:

First iteration	tIdx is 16; tree[16] is greater than 21; half bitMask and continue
Second iteration	tIdx is 8; tree[8] is less than 21, so we should include first 8 indexes in result, remember idx because we surely know it is part of result; subtract tree[8] of cumFre (we do not want to look for the same cumulative frequency again - we are looking for another cumulative frequency in the rest/another part of tree); half bitMask and continue
Third iteration	tIdx is 12; tree[12] is greater than 9 (there is no way to overlap interval 1-8, in this example, with some further intervals, because only interval 1-16 can overlap); half bitMask and continue
Fourth iteration	tIdx is 10; tree[10] is less than 9, so we should update values; half bitMask and continue
Fifth iteration	tIdx is 11; tree[11] is equal to 2; return index (tIdx)

Time complexity: $O(\log \text{MaxVal})$.

Code length: Up to twenty lines.

2D BIT

BIT can be used as a multi-dimensional data structure. Suppose you have a plane with dots (with non-negative coordinates). You make three queries:

1. set dot at (x , y)
2. remove dot from (x , y)
3. count number of dots in rectangle (0 , 0), (x , y) - where (0 , 0) is down-left corner, (x , y) is up-right corner and sides are parallel to x-axis and y-axis.

If **m** is the number of queries, **max_x** is maximum x coordinate, and **max_y** is maximum y coordinate, then the problem should be solved in $O(m * \log(\max_x) * \log(\max_y))$. In this case, each element of the tree will contain array - (**tree[max_x][max_y]**). Updating indexes of x-coordinate is the same as before. For example, suppose we are setting/removing dot (**a , b**). We will call **update(a , b , 1)/update(a , b , -1)**, where **update** is:

```
void update(int x , int y , int val){
    while (x <= max_x){
        updatey(x , y , val);
        // this function should update array tree[x]
        x += (x & -x);
    }
}
```

The function **updatey** is the "same" as function **update**:

```
void updatey(int x , int y , int val){
    while (y <= max_y){
        tree[x][y] += val;
        y += (y & -y);
    }
}
```

It can be written in one function/procedure:

```
void update(int x , int y , int val){
    int y1;
    while (x <= max_x){
        y1 = y;
        while (y1 <= max_y){
            tree[x][y1] += val;
            y1 += (y1 & -y1);
        }
        x += (x & -x);
    }
}
```

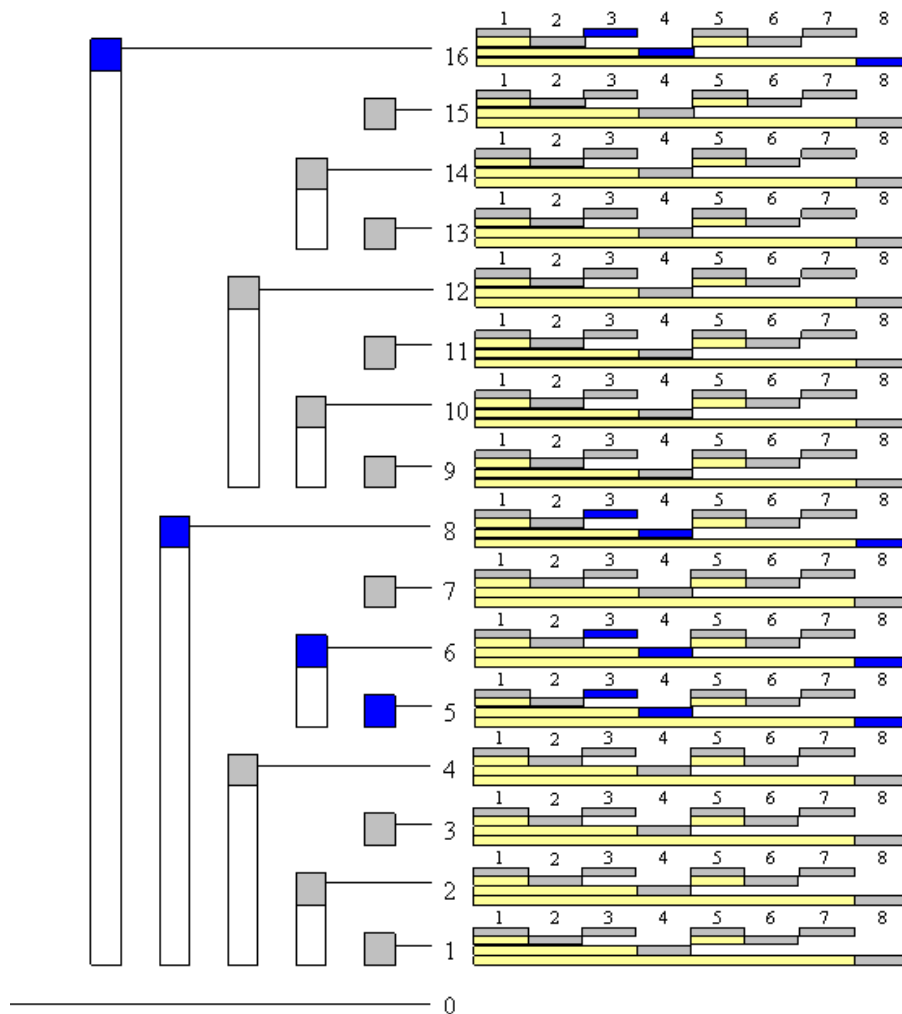


Image 1.8 - BIT is array of arrays, so this is two-dimensional BIT (size 16 x 8). Blue fields are fields which we should update when we are updating index (5 , 3).

The modification for other functions is very similar. Also, note that BIT can be used as an n-dimensional data structure.

Sample problem

- [SRM 310 - FloatingMedian](#)

- Problem 2:

Statement:

There is an array of n cards. Each card is putted face down on table. You have two queries:

1. $T\ i\ j$ (turn cards from index i to index j , include i -th and j -th card - card which was face down will be face up; card which was face up will be face down)
2. $Q\ i$ (answer 0 if i -th card is face down else answer 1)

Solution:

This has solution for each query (and 1 and 2) has time complexity $O(\log n)$. In array f (of length $n + 1$) we will store each query $T(i, j)$ - we set $f[i]++$ and $f[j + 1]--$. For each card k between i and j (include i and j) sum $f[1] + f[2] + \dots + f[k]$ will be increased for 1, for all others will be same as before (look at the image 2.0 for clarification), so our solution will be described sum (which is same as cumulative frequency) module 2.

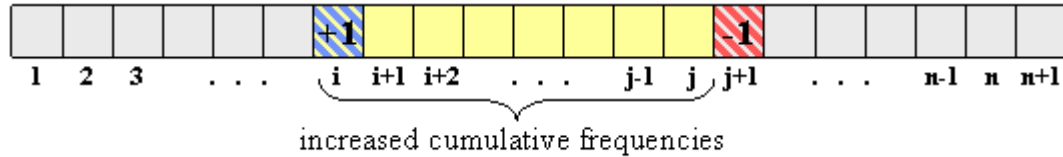


Image 2.0

Use **BIT** to store (increase/decrease) frequency and read cumulative frequency.

Conclusion

- Binary Indexed Trees are very easy to code.
- Each query on Binary Indexed Tree takes constant or logarithmic time.
- Binary Indexed Tree require linear memory space.
- You can use it as an n-dimensional data structure.

References

- [1] [RMQ](#)
- [2] [Binary Search](#)
- [3] [Peter M. Fenwick](#)

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