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4. a) $H = \frac{\hat{p}^2}{2m} + V(\hat{r})$

$\Omega = \hat{r}\cdot\hat{p} = \hat{r}_H \hat{p}_H$

$$\frac{d}{dt}\Omega_H(t) = \frac{1}{i\hbar}[\Omega_H, H_H]$$

$$= \frac{1}{i\hbar 2m}[\hat{r}_H\hat{p}_H, \hat{p}_H^2] + \frac{1}{i\hbar}[\hat{r}_H\hat{p}_H, V(\hat{r}_H)] = \boxed{\frac{\hat{p}_H^2}{m} - \hat{r}_H\frac{d\hat{V}}{d\hat{r}_H}}$$

$$[\hat{r}_H\hat{p}_H, \hat{p}_H^2] = [\hat{r}_H, \hat{p}_H^2]\hat{p}_H = \hat{r}_H\hat{p}_H^3 - \hat{p}_H^2\hat{r}_H\hat{p}_H$$

$\hat{p}_H\hat{r}_H = -i\hbar + \hat{r}_H\hat{p}_H$

$$= \hat{r}_H\hat{p}_H^3 - \hat{p}_H(-i\hbar + \hat{r}_H\hat{p}_H)\hat{p}_H$$

$$= \hat{r}_H\hat{p}_H^3 + i\hbar\hat{p}_H^2 - (-i\hbar + \hat{r}_H\hat{p}_H)\hat{p}_H^2$$

$$= 2i\hbar\hat{p}_H^2$$

$$[\hat{r}_H\hat{p}_H, V(\hat{r}_H)] = -i\hbar\,\hat{r}_H\frac{d\hat{V}}{d\hat{r}_H}$$

$\hat{p}\hat{V} = -i\hbar\frac{dV}{dr} + \hat{V}\hat{p}$

b) $\langle\psi(0)|\frac{d}{dt}O_H|\psi(0)\rangle$

$= \langle\psi(0)|O_H H_H - H_H O_H|\psi(0)\rangle$

$= E_\psi(\langle\psi(0)|O_H|\psi(0)\rangle - \langle\psi(0)|O_H|\psi(0)\rangle) = 0$

c) $V(r) = \frac{c}{r^k}$

~~$\frac{d}{dt}\Omega_H(t) = \ldots$~~

$$\langle\psi(0)|\frac{d}{dt}\Omega_H(t)|\psi(0)\rangle = \langle\psi(0)|\frac{p_H^2}{2m}|\psi(0)\rangle - \langle\psi(0)|\frac{-kc}{r_H^k}|\psi(0)\rangle = 0$$

$\Rightarrow 2\langle T\rangle + k\langle V\rangle = 0$

$\Rightarrow \langle T\rangle = -\frac{k}{2}\langle V\rangle$

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5. $\hat{H} = \frac{\hat{p}^2}{2m} + g\hat{x}$

a) $\frac{d}{dt}\hat{x}_H = \frac{1}{i\hbar 2m}[\hat{x}_H, \hat{p}_H^2] = \frac{\hat{p}_H}{m}$

$[\hat{x}_H, \hat{p}_H^2] = \hat{x}_H \hat{p}_H^2 - \hat{p}_H(-i\hbar + \hat{x}_H \hat{p}_H) = 2i\hbar\hat{p}_H$

$\frac{d}{dt}\hat{p}_H = \frac{g}{i\hbar}[\hat{p}_H, \hat{x}_H] = -g \longrightarrow \frac{d}{dt}(mv) = F = ma$

$\hat{x}_H = \int \frac{\hat{p}_H}{m} dt + \hat{x}$ $\qquad \hat{p}_H = -gt + \hat{p}$

$= \hat{x} + \frac{\hat{p}t}{m} - \frac{gt^2}{2m}$

b) $\langle\psi(0)|\hat{x}_H|\psi(0)\rangle = \langle\psi(0)|\hat{x}|\psi(0)\rangle + \frac{\langle\psi(0)|\hat{p}|\psi(0)\rangle t}{m} - \frac{gt^2}{2m}$

c) $(\Delta x(t))^2 = \langle x_H(t)^2\rangle - \langle x_H(t)\rangle^2$ $\qquad$ 9 terms

$= \Delta x^2$ $\qquad$ (1 term)

$+ \; 0$ $\qquad$ (5 terms with $\frac{gt^2}{2m}$)

$+ \left(\langle xp\rangle + \langle px\rangle - 2\langle x\rangle\langle p\rangle\right)\left(\frac{t}{m}\right)$ $\qquad$ (2 terms)

$+ \frac{t^2}{m^2}\Delta p^2$ $\qquad$ (1 term)

$= \cancel{\Delta x^2 + \frac{t^2}{m^2}\Delta p^2 + \langle xp\rangle + \langle px\rangle - 2\langle x\rangle\langle p\rangle}$

$=$ " $\quad 2\langle xp\rangle - i\hbar - 2\cancel{\langle x\rangle}\langle p\rangle$ $\qquad \langle x\rangle = 0$

$\psi(x) = Ne^{-\frac{x^2}{2\Delta^2}}$ $\qquad = i\hbar - i\hbar = 0$

$\langle xp\rangle = -i\hbar\int N^2\left(e^{-\frac{x^2}{2\Delta^2}}\right)x\left(-\frac{x}{\Delta^2}e^{-\frac{x^2}{2\Delta^2}}\right)dx = \int N^2\frac{x^2}{\Delta^2}e^{-\frac{x^2}{\Delta^2}}dx\,(i\hbar)$

$\int N^2 e^{-\frac{x^2}{\Delta^2}}dx = 1$

$\cancel{\int x e^{-\alpha x^2} dx}$

$= N^2\frac{1}{2}\int e^{-\frac{x^2}{\Delta^2}}dx\,(i\hbar) = \frac{1}{2}i\hbar$

$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx = \left[\frac{x}{-2\alpha}e^{-\alpha x^2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{1}{-2\alpha}e^{-\alpha x^2}dx = \int\frac{1}{2\alpha}e^{-\alpha x^2}dx$

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6\.

a) $H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - F\hat{x}$

$= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \left(\hat{x} - \frac{F}{m\omega^2}\right)^2 - \frac{F^2}{2m\omega^2}$

Let $\hat{y} \equiv \hat{x} - \frac{F}{m\omega^2}$

$$H = \underbrace{\left(\frac{\hat{p}}{\sqrt{2m}} + i\sqrt{\frac{m}{2}}\,\omega \hat{y}\right)}_{\sqrt{\hbar\omega}\,a^\dagger} \underbrace{\left(\frac{\hat{p}}{\sqrt{2m}} - i\sqrt{\frac{m}{2}}\,\omega \hat{y}\right)}_{\sqrt{\hbar\omega}\,a_F} - i\frac{\omega}{2}[\hat{y},\hat{p}] - \frac{F^2}{2m\omega^2}$$

$= \hbar\omega a^\dagger_F a_F + \frac{\hbar\omega}{2} - \frac{F^2}{2m\omega^2}$

$[a^\dagger_F, a_F] = \frac{i\omega}{\sqrt{\hbar\omega}\,2}\left([\hat{y},\hat{p}] - [\hat{p},\hat{y}]\right) = \frac{\frac{i\omega}{2}2(i\hbar)}{\hbar\omega} = \frac{-\hbar\omega}{\hbar\omega} = a^\dagger a - a a^\dagger$

$a^\dagger a^2 \psi = (a a^\dagger - \hbar\omega) a\psi = (E - \hbar\omega) a\psi$

Ground state energy: $\boxed{\frac{\hbar\omega}{2} - \frac{F^2}{2m\omega^2}}$

Positive norms tell you that the eigenvalues must be integral

b) $a^\dagger_F a_F = a^\dagger a - \frac{F}{m\omega^2}\frac{a}{\sqrt{\hbar\omega}} + \frac{F}{m\omega^2}\frac{a^\dagger}{\sqrt{\hbar\omega}} - \left(\frac{F}{m\omega^2}\right)^2$

$a^\dagger_F a_F N e^{\alpha a^\dagger}|0\rangle$

must be ~~eigenvalue~~ eigenvector of

$a^\dagger a + \frac{F}{m\omega^2}(a^\dagger - a)\frac{1}{\sqrt{\hbar\omega}}$

$[a, e^{\alpha a^\dagger}] = \sum \frac{\alpha^n [a, a^{\dagger n}]}{n!} = \sum \frac{\alpha^n a^{\dagger\, n-1}}{(n-1)!} = \alpha e^{\alpha a^\dagger}$

$a e^{\alpha a^\dagger} = \alpha e^{\alpha a^\dagger} + e^{\alpha a^\dagger} a$

$a^\dagger a e^{\alpha a^\dagger}|0\rangle = \alpha a^\dagger e^{\alpha a^\dagger}|0\rangle$

$(a^\dagger - a) e^{\alpha a^\dagger}|0\rangle = a^\dagger e^{\alpha a^\dagger}|0\rangle - \alpha e^{\alpha a^\dagger}|0\rangle$

8.05 PSET 7

6.b) The ground state $|0'\rangle$ of H can be written as

$$|0'\rangle = N e^{\alpha \hat{a}^\dagger}|0\rangle$$

$\hat{a}^\dagger$ and $|0\rangle$ correspond to the ~~unperturbed~~ unperturbed Hamiltonian $F=0$, Find α.

$$\hat{a}_F^\dagger = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{\hat{p}}{\sqrt{2m}} + i\sqrt{\frac{m}{2}}\,\omega\hat{y}\right) = \hat{a}^\dagger - i\sqrt{\frac{m}{2}}\left(\frac{\omega}{\sqrt{\hbar\omega}}\right)\frac{F}{m\omega^2} = \hat{a}^\dagger - i\sqrt{\frac{\omega}{2\hbar m}}\,\frac{F}{\omega^2}$$

$$\hat{a}_F = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{\hat{p}}{\sqrt{2m}} - i\sqrt{\frac{m}{2}}\,\omega\hat{y}\right) = a + i\sqrt{\frac{\omega}{2\hbar\omega}}\,\frac{F}{\omega^2}$$

$$\hat{a}_F^\dagger\hat{a}_F = a^\dagger a + i\sqrt{\frac{\omega}{2\hbar m}}\,\frac{F}{\omega^2}(a^\dagger - a) \qquad \text{Let } \gamma = i\sqrt{\frac{\omega}{2\hbar m}}\,\frac{F}{\omega^2}$$

$$\hat{a}_F^\dagger\hat{a}_F e^{\alpha a^\dagger}|0\rangle = \alpha a^\dagger e^{\alpha a^\dagger}|0\rangle + \gamma\left(a^\dagger e^{\alpha a^\dagger}|0\rangle - \alpha e^{\alpha a^\dagger}|0\rangle\right) = 0$$

$\left(\text{Using } [a, e^{\alpha a^\dagger}] = \alpha e^{\alpha a^\dagger} \quad a e^{\alpha a^\dagger} = \alpha e^{\alpha a^\dagger} + e^{\alpha a^\dagger} a\right)$

$$\Rightarrow \alpha(\alpha^{n-1}) + \gamma\alpha^{n-1} - \alpha\gamma\alpha^n \neq 0$$

$$\Rightarrow \alpha^n + \gamma\alpha^{n-1} - \gamma\alpha^{n+1} \neq 0$$

$$\Rightarrow \gamma + \alpha - \gamma\alpha^2 \neq 0$$

$$\alpha = \frac{-1 \pm \sqrt{1+4\gamma^2}}{-2\gamma}$$

~~$\hat{a}_F|0'\rangle = 0$~~

$$\hat{a}_F e^{\alpha a^\dagger}|0\rangle = \alpha e^{\alpha a^\dagger}|0\rangle + \gamma e^{\alpha a^\dagger}|0\rangle = 0$$

$$\alpha = -\gamma$$

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2. By the Baker-Hausdorff theorem:

a) $e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{\frac{-|\alpha|^2}{2}}|0\rangle = e^{\alpha a^\dagger} e^{\frac{-|\alpha|^2}{2}}|0\rangle = |\alpha\rangle$

$[\alpha a^\dagger, -\alpha^* a] = -|\alpha|^2 (a^\dagger a - a a^\dagger) = |\alpha|^2$

$[a, e^{\alpha a^\dagger}] = \alpha e^{\alpha a^\dagger}$

$a e^{\alpha a^\dagger} = \alpha e^{\alpha a^\dagger} + e^{\alpha a^\dagger} a$

Thus $a e^{\alpha a^\dagger - \alpha^* a}|0\rangle = \alpha e^{\alpha a^\dagger} e^{\frac{-|\alpha|^2}{2}}|0\rangle = \alpha|\alpha\rangle$

b) $\alpha a^\dagger - \alpha^* a = 2i\,\mathrm{Im}(\alpha a^\dagger)$

$|\alpha\rangle = e^{\alpha a^\dagger} e^{\frac{-|\alpha|^2}{2}}|0\rangle = \sum e^{\frac{-|\alpha|^2}{2}} \frac{(\alpha a^\dagger)^n}{n!}|0\rangle = \boxed{\sum e^{\frac{-|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$

$\Pr(E_n) = \boxed{\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}}$

c) $\langle\beta|\alpha\rangle = \sum e^{\frac{-|\alpha|^2 - |\beta|^2}{2}} \frac{\beta^{*n}\alpha^n}{n!} = \boxed{e^{\frac{-|\alpha|^2 - |\beta|^2}{2}} e^{\alpha\beta^*}}$

d) Calculate $\frac{\Delta H}{\langle H\rangle}$

$\langle H\rangle = \sum \Pr(E_n)\left(\frac{\hbar\omega}{2} + \hbar\omega n\right) = \sum e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}\left(\frac{\hbar\omega}{2}\right) + \sum e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \hbar\omega n$

$= \frac{\hbar\omega}{2} + \hbar\omega|\alpha|^2 \Rightarrow \langle H\rangle^2 = \frac{\hbar^2\omega^2}{4} + \hbar^2\omega^2|\alpha|^2 + \hbar^2\omega^2|\alpha|^4$

$\langle H^2\rangle = \sum \Pr(E_n)\left(\frac{\hbar^2\omega^2}{4} + \hbar^2\omega^2 n + \hbar^2\omega^2 n^2\right) = \frac{\hbar^2\omega^2}{4} + \hbar^2\omega^2|\alpha|^2 + |\alpha|^2 \sum e^{-|\alpha|^2} \frac{|\alpha|^{2(n-1)}}{(n-1)!} \hbar^2\omega^2 n$

$\hbar^2\omega^2|\alpha|^4 = \sum e^{-|\alpha|^2} \frac{|\alpha|^{2(n+2)}}{n!} \hbar^2\omega^2$

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2. d) $\frac{\Delta H}{\langle H \rangle}$

$\langle H \rangle = \frac{\hbar\omega}{2} + \hbar\omega\langle n \rangle = \frac{\hbar\omega}{2} + \hbar\omega \sum P_r(E_n) n = \frac{\hbar\omega}{2} + \hbar\omega|\alpha|^2 \sum e^{-|\alpha|^2} \frac{|\alpha|^{2(n-1)}}{(n-1)!}$

$= \frac{\hbar\omega}{2} + \hbar\omega|\alpha|^2$

$\langle H^2 \rangle = \frac{\hbar^2\omega^2}{4} + \hbar^2\omega^2\langle n \rangle + \hbar^2\omega^2\langle n^2 \rangle$

$\langle n^2 \rangle = \langle \alpha | a^\dagger a a^\dagger a | \alpha \rangle = \langle \alpha | a^\dagger (a^\dagger a + 1) a | \alpha \rangle = |\alpha|^4 + |\alpha|^2$

Alternatively: $\sum P_r(E_n) n^2 = \sum \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{(n-1)!} \cdot n = \sum \frac{e^{-|\alpha|^2} |\alpha|^{2n} (n-1)}{(n-1)!} + \sum \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{(n-1)!}$

$= |\alpha|^4 + |\alpha|^2$

Thus, $\Delta H^2 = \langle H^2 \rangle - \langle H \rangle^2 = \hbar^2\omega^2 (\langle n^2 \rangle - \langle n \rangle^2) = \hbar^2\omega^2|\alpha|^2$

$\frac{\Delta H}{\langle H \rangle} = \boxed{\frac{\hbar\omega|\alpha|}{\frac{\hbar\omega}{2} + \hbar\omega|\alpha|^2}}$

e) ~~$\Delta x \Delta p$~~

$\langle \hat{x} \rangle = \langle \hat{a} + a^\dagger \rangle \left(\sqrt{\frac{\hbar}{2m\omega}} \right) = (\alpha + \alpha^*) \sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar}{2m\omega}} 2\mathrm{Re}(\alpha)$

$\langle \hat{p} \rangle = \langle a^\dagger - a \rangle \left(i\sqrt{\frac{m\hbar\omega}{2}} \right) = \sqrt{2m\hbar\omega}\, \mathrm{Im}(\alpha)$

$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \langle \hat{a}\hat{a} + \hat{a}^\dagger \hat{a}^\dagger + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} \rangle = \frac{\hbar}{2m\omega} (\alpha^2 + \alpha^{*2} + |\alpha|^2 + \langle a^\dagger a \rangle + 1)$

$= \frac{\hbar}{2m\omega} (2\mathrm{Re}(\alpha^2) + 2|\alpha|^2 + 1)$

$\langle \hat{p}^2 \rangle = -\frac{m\hbar\omega}{2} \langle a^\dagger a^\dagger + aa - a^\dagger a - a a^\dagger \rangle = -\frac{m\hbar\omega}{2} (2\mathrm{Re}(\alpha^2) - 2|\alpha|^2 - 1)$

$\Delta x^2 = \frac{\hbar}{m\omega} (\mathrm{Re}(\alpha^2) + |\alpha|^2 + \frac{1}{2} - 2\mathrm{Re}(\alpha)^2)$

$\Delta p^2 = m\hbar\omega (|\alpha|^2 + \frac{1}{2} - \mathrm{Re}(\alpha^2) - 2\mathrm{Im}(\alpha)^2)$ $\quad \frac{\hbar}{m\omega 2}$

$|\alpha|^2 = \mathrm{Re}(\alpha)^2 + \mathrm{Im}(\alpha)^2$ $\quad = m\hbar\omega (\frac{1}{2} + \mathrm{Re}(\alpha)^2 - \mathrm{Im}(\alpha)^2 - \mathrm{Re}(\alpha^2)) = \frac{m\hbar\omega}{2}$

~~$\Delta x^2 = \frac{\hbar}{m\omega}(\mathrm{Im}(\alpha)^2 + \frac{1}{2})$ $\Delta p^2 = m\hbar\omega(\frac{1}{2} - 2\mathrm{Im}(\alpha)^2)$~~

$\mathrm{Re}(\alpha^2) = \mathrm{Re}(\alpha)^2 - \mathrm{Im}(\alpha)^2$

$\boxed{\Delta x \Delta p = \frac{\hbar}{2}}$

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2.f) $|\alpha\rangle = \sum \frac{e^{-\frac{|\alpha|^2}{2}}(\alpha)^n}{\sqrt{n!}}|n\rangle$

$|n\rangle$ is an energy eigenstate

$H|n\rangle = \hbar\omega\left(\frac{1}{2}+n\right)|n\rangle$

Thus, the time evolution is $e^{i\hat{H}t}|n\rangle = e^{i\hbar\omega(\frac{1}{2}+n)t}|n\rangle$

$$e^{i\hat{H}t}|\alpha\rangle = \sum \frac{e^{-\frac{|\alpha|^2}{2}}\alpha^n}{\sqrt{n!}}e^{i(\hbar\omega)(\frac{1}{2}+n)t}|n\rangle = e^{i\frac{\hbar\omega}{2}t}\sum\frac{e^{-\frac{|\alpha|^2}{2}}\left(e^{i\hbar\omega t}\alpha\right)^n}{\sqrt{n!}}|n\rangle$$

$$= e^{i\frac{\hbar\omega}{2}t}|\alpha e^{i\hbar\omega t}\rangle = |\alpha e^{i\hbar\omega t}\rangle$$ (global phase does not matter)

8.05 PSET8

3. a) $a(\gamma) = S^\dagger(\gamma) a S(\gamma)$

$$\frac{da(\gamma)}{d\gamma} = -\frac{\gamma}{2}(aa - a^\dagger a^\dagger) S^\dagger a S + S^\dagger a S\left(-\frac{\gamma}{2}\right)(a^\dagger a^\dagger - aa)$$

$$= \left[a(\gamma), -\frac{\gamma}{2}(a^\dagger a^\dagger - aa)\right]$$

$$= S^\dagger\left[a, -\frac{\gamma}{2} a^\dagger a^\dagger - aa\right] S = S^\dagger\left[a, -\frac{\gamma}{2} a^\dagger a^\dagger\right] S = -\gamma S^\dagger a^\dagger S$$

$$= -\gamma a^\dagger(\gamma)$$

$(aa^\dagger a^\dagger - a^\dagger a^\dagger a)|n\rangle = (n+2)\sqrt{n+1} - \sqrt{n+1}(n) = 2\sqrt{n+1} = 2a^\dagger$ $\quad (a^\dagger aa - aaa^\dagger)|n\rangle = \sqrt{n}(n-1) - \sqrt{n}(n+1)$

$= -2a$

$$\frac{d^2 a(\gamma)}{d\gamma^2} = -\gamma\, S^\dagger\left[a^\dagger, -\frac{\gamma}{2}(a^\dagger a^\dagger - aa)\right] S$$

$$= -\frac{\gamma^2}{2} S^\dagger [a^\dagger, aa] S = \gamma^2 S^\dagger a S = \gamma^2 a(\gamma)$$

$$\boxed{\begin{aligned} a(\gamma) &= \left(\frac{e^\gamma + e^{-\gamma}}{2}\right) a + \left(\frac{e^\gamma - e^{-\gamma}}{2}\right) a^\dagger \\ a^\dagger(\gamma) &= \left(\frac{e^\gamma + e^{-\gamma}}{2}\right) a^\dagger + \left(\frac{e^\gamma - e^{-\gamma}}{2}\right) a \end{aligned}}$$

$$[a(\gamma), a^\dagger(\gamma)] = \left(\frac{e^\gamma + e^{-\gamma}}{2}\right)[a, a^\dagger] + \left(\frac{e^\gamma - e^{-\gamma}}{2}\right)[a^\dagger, a]$$

$$= \frac{e^\gamma + e^{-\gamma}}{2} - \frac{e^\gamma - e^{-\gamma}}{2} = \boxed{e^{-\gamma}}$$

b) $$\langle N\rangle\big|_{|0_\gamma\rangle} = \langle S^\dagger a^\dagger a S\rangle\big|_{|0\rangle} = \langle a^\dagger(\gamma) a(\gamma)\rangle\big|_{|0\rangle}$$

$$= \left\langle \left(\frac{e^{2\gamma} - e^{-2\gamma}}{4}\right) a^\dagger a^\dagger \right\rangle_{|0\rangle} = \boxed{\frac{e^{2\gamma} - e^{-2\gamma}}{4}}$$

$$\langle N^2\rangle\big|_{|0_\gamma\rangle} = \left\langle a^\dagger(\gamma)\left[a^\dagger(\gamma) a(\gamma) + e^{-\gamma}\right] a(\gamma)\right\rangle\big|_{|0\rangle}$$