8.05 PSET 1

1. a) $\psi(x) = N x e^{-\frac{1}{2}\alpha x^2}$

$$\int \psi^*(x)\psi(x) = N^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = 1$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \left[\frac{x}{-2\alpha} e^{-\alpha x^2}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2\alpha} e^{-\alpha x^2} dx = \int_{-\infty}^{\infty} \frac{1}{2\alpha} e^{-\alpha x^2} dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$A = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \qquad A^2 = \iint_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx\,dy$$

$$= \sqrt{\frac{\pi}{\alpha}} \qquad = \iint e^{-\alpha r^2} r\,dr\,d\theta$$

$$= 2\pi \left[\frac{1}{-2\alpha} e^{-\alpha r^2}\right]_0^{\infty} = 2\pi \frac{1}{2\alpha} = \frac{\pi}{\alpha}$$

$$N^2 \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} = N^2 \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} = 1 \Rightarrow N^2 = 2\sqrt{\frac{\alpha^3}{\pi}}$$

$$\Rightarrow \boxed{N = \sqrt{2}\sqrt[4]{\frac{\alpha^3}{\pi}}}$$

$$\langle \hat{x} \rangle = N^2 \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2} dx = N^2\left(\underbrace{\left[\frac{x^2}{-2\alpha} e^{-\alpha x^2}\right]_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} \frac{x}{\alpha} e^{-\alpha x^2} dx\right)$$

$$= \left[C e^{-\alpha x^2}\right]_{-\infty}^{\infty} = \boxed{0}$$

$$\langle \hat{x}^2 \rangle = N^2 \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = N^2\left(\underbrace{\left[\frac{x^3}{-2\alpha} e^{-\alpha x^2}\right]_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} \frac{3x^2}{2\alpha} e^{-\alpha x^2} dx\right)$$

$$= \underbrace{\left(N^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx\right)}_{1} \frac{3}{2\alpha} = \boxed{\frac{3}{2\alpha}}$$

8.05 PSET 1

b) $\langle \hat{p} \rangle = \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\alpha^2}\left(-i\hbar \frac{\partial}{\partial x}\left(x e^{-\frac{1}{2}\alpha x^2}\right)\right) dx$

$= -i\hbar \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\alpha^2}\left(e^{-\frac{1}{2}\alpha x^2} - \alpha x^2 e^{-\frac{1}{2}\alpha x^2}\right) dx$

$= -i\hbar \int_{-\infty}^{\infty} (x - \alpha x^3) e^{-\alpha x^2} dx = -i\hbar \int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = -i\hbar \left[\frac{1}{-2\alpha} e^{-\alpha x^2}\right]_{-\infty}^{\infty}$

$= \boxed{0}$

$\langle \hat{p}^2 \rangle = N \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\alpha^2}\left(-\hbar^2\left(-\alpha \cdot x e^{-\frac{1}{2}\alpha x^2} - 2\alpha x e^{-\frac{1}{2}\alpha x^2} + \alpha^2 x^3 e^{-\frac{1}{2}\alpha x^2}\right)\right) dx$

$= -\hbar^2 N^2 \int_{-\infty}^{\infty} (-3\alpha x^2 + \alpha^2 x^4) e^{-\alpha x^2} dx$

$= -\hbar^2\left(-3\alpha + \frac{3\alpha^2}{2\alpha}\right) = -\hbar^2\left(\frac{-3}{2}\alpha\right) = \boxed{\frac{3}{2}\alpha\hbar^2}$

c) There is uncertainty in both position and momentum, so it is an eigenstate of neither.

d) $\langle \hat{H} \rangle = \langle \hat{V} + \hat{K} \rangle = \langle \hat{K} \rangle = \langle \frac{1}{2m} \hat{p}^2 \rangle = \boxed{\frac{3}{4m}\alpha\hbar^2}$

e) $\frac{\partial^2}{\partial x^2}\psi(x) = (-3\alpha x + \alpha^2 x^3) e^{-\frac{1}{2}\alpha x^2}$

$V(x) = -\frac{3\alpha\hbar^2}{2m} + \frac{\alpha^2\hbar^2}{2m} x^2 + C \Rightarrow \boxed{V(x) = \frac{\alpha^2\hbar^2}{2m} x^2 \quad E = \frac{3\alpha\hbar^2}{2m}}$

8.05 PSET 1

3. a) $\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = (V(x) - E)\psi(x)$

→ Integrate from $-\delta$ to δ

$\frac{\hbar^2}{2m}\Delta\frac{\partial}{\partial x}\psi = -V_0 a\psi(0)$ (also $-V_0 a\psi(a)$ and $-V_0 a\psi(-a)$)

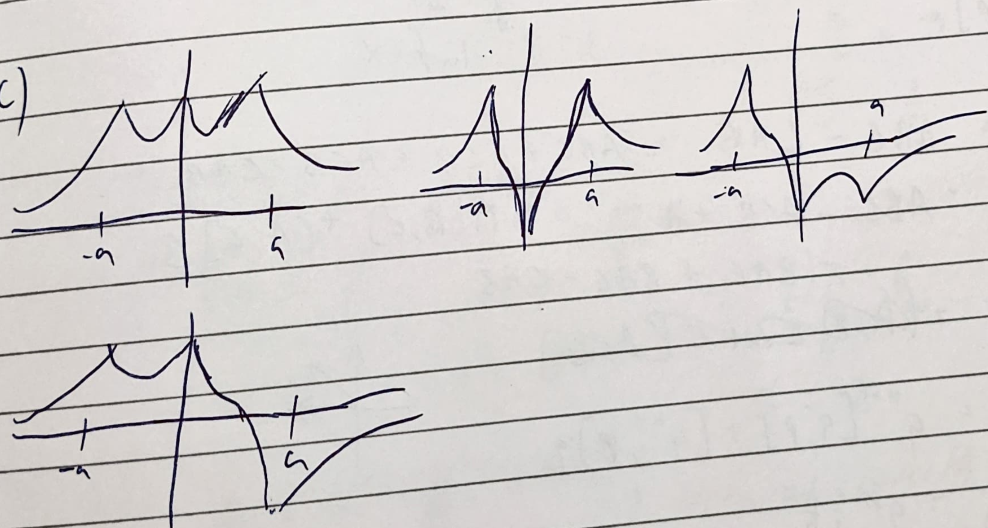
b) i) Because it is a bound state, $E<0$. This means solution on $x>a$ is of the form $Ae^{-\kappa x}$. Thus no nodes are possible

ii) One or zero (see pictures) maybe ??

iii) No, because at $x>a$ the sol is $Ae^{-\kappa x}$ and $A\neq 0$

iv) Yes

c)



8.05 PSET 2

3.a) Since it is orthogonal to the ground state, it must be composed of ~~[illegible]~~ only other eigenstates (when written in eigenbasis.)

$$\psi = \sum_{i=2}^{n} \alpha \psi_i \Rightarrow E_2 \leq \int dx\, \psi^* H \psi$$

b) $$F(\psi) = \frac{\int dx\, (\psi_2^* + \sum \varepsilon_n^* \psi_n^*)\, H\, (\psi_2 + \sum \varepsilon_n \psi_n)}{\int dx\, (\psi_2^* + \sum \varepsilon_n^* \psi_n^*)(\psi_2 + \sum \varepsilon_n \psi_n)} \qquad \psi = \psi_2 + \sum \varepsilon_n \psi_n$$

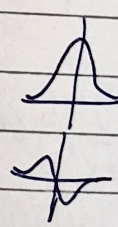
$$\approx \frac{\int dx \left(\psi_2^* H \psi_2 + \varepsilon_2^* \psi_2^* H \psi_2 + \varepsilon_2 \psi_2^* H \psi_2 \right)}{\int dx\, \left(\psi_2^* \psi_2 + (\varepsilon_2^* + \varepsilon_2) \psi^* \psi_2 \right)}$$

$$= \frac{\int dx\, \psi_2^* H \psi_2}{\int dx\, \psi_2^* \psi}$$

Saddle, if you ~~[illegible]~~ add more ψ_1 the energy goes down, if you add more of any of the other eigenstates the energy goes up.

4.

$$\int -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) = \int -\frac{\hbar^2}{2m} \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \left(-\alpha + \alpha^2 x^2\right) e^{-\alpha x^2}\, dx$$



$$\cancel{\int x^2 e^{-\frac{\alpha x^2}{2}} = \frac{1}{\alpha}\sqrt{\frac{2\pi}{\alpha}} \quad \int e^{-\frac{\alpha x^2}{2}} = \sqrt{\frac{2\pi}{\alpha}}}$$

$$\cancel{-\frac{\hbar^2}{2m} \sqrt{\frac{\alpha}{\pi}} \left(-\sqrt{2\pi\alpha} + \sqrt{2\pi\alpha}\right) = 0}$$

~~Since ψ_α is positive everywhere and $-|V(x)|$ is zero or negative everywhere $\int -|V(x)| \psi_\alpha(x) <$~~

$$\cancel{\int x e^{-\alpha x^2} dx = \int x^3 e^{-\alpha x^2} = 0}$$

$$\int x^2 e^{-\alpha x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \qquad \int e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

$$= -\frac{\hbar^2}{2m} \sqrt{\frac{\alpha}{\pi}} \left(-\sqrt{\pi\alpha} + \frac{1}{2}\sqrt{\pi\alpha} \right) = \frac{\hbar^2}{2m} \sqrt{\frac{\alpha}{\pi}} \left(\frac{1}{2}\sqrt{\pi\alpha}\right)$$

$$= \frac{\hbar^2}{4m}\alpha$$

~~By the conditions,~~

$$\cancel{\int -|V(x)|\,|\psi(x)| \leq -V_0 \sup_{x \in [x_1, x_2]} \psi(x) < 0}$$, so ~~for sufficiently~~

Small alpha the total energy is negative.

8.05 PSET 2

5. a) $x = \beta u$ $\qquad \frac{\partial}{\partial u}\psi = \frac{\partial}{\partial x}\psi \frac{\partial x}{\partial u} = \beta \frac{\partial \psi}{\partial x}$

$\frac{dx}{du} = \beta$

$$-\frac{\hbar^2}{2m\beta^2}\frac{\partial^2\psi}{\partial u^2} + (\alpha\beta^4 u^4 - E)\psi = 0$$

$$-\frac{1}{2}\frac{\partial^2\psi}{\partial u^2} + \left(\frac{m}{\hbar^2\beta^2}\alpha\beta^4 u^4 - \frac{mE}{\hbar^2\beta^2}\right)\psi = 0$$

$$-\frac{1}{2}\frac{\partial^2\psi}{\partial u^2} + \left(\frac{\alpha m}{\hbar^2}\beta^2 u^4 - \frac{mE}{\hbar^2\beta^2}\right)\psi = 0$$

$$\beta^2 = \frac{\hbar^2}{\alpha m} \Rightarrow \beta = \frac{\hbar}{\sqrt{\alpha m}} \qquad e = \underset{\frac{1}{\text{energy}^2}}{\overset{\text{energy}}{\alpha}} \left(\frac{m}{\hbar^2}\right)^2 E^{\text{energy}}$$

β^2 is unit free

8.05 PSET 2

$$iM = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (iM)^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

6. $e^{iM\theta} = \sum_{i=0}^{\infty} \frac{M^i \theta^i i^i}{i!}$

$$M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \dots$$

$$= i \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{(2i+1)!} M + \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{(2i)!} I$$

$$\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^i \quad \cos(\theta) = \sum_{i=0}^{\infty} \frac{\theta^i (-1)^i}{(2i)!}$$

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{(2i+1)!} \quad \sin(\theta) = \sum_{i=0}^{\infty} \frac{\theta^i (-1)^i}{(2i+1)!}$$

~~$i\sin\theta I + \cos\theta M$~~

$$\boxed{\cos\theta I + i\sin\theta M}$$

~~$i\sin(-\theta) I + \cos(-\theta) M$~~

Algebraic property: $M^2 = I$

8.05 PSET 3

3.

a) $S_n = \frac{\hbar}{2} n \cdot \sigma$

$\cancel{\sigma_z = \begin{pmatrix}1 \\ 0\end{pmatrix}(1\ 0) + \begin{pmatrix}0 \\ 1\end{pmatrix}(0\ 1) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}}$

$$\sigma_z = 1\begin{pmatrix}1 \\ 0\end{pmatrix}(1\ 0) - 1\begin{pmatrix}0 \\ 1\end{pmatrix}(0\ 1) = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$$\sigma_x = 1\begin{pmatrix}1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}(1/\sqrt{2}\ \ 1/\sqrt{2}) - 1\begin{pmatrix}1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}(1/\sqrt{2}\ \ -1/\sqrt{2}) =$$

$$= \begin{pmatrix}1/2 & 1/2 \\ 1/2 & 1/2\end{pmatrix} - \begin{pmatrix}1/2 & -1/2 \\ -1/2 & 1/2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

$$\cancel{\sigma_y = 1\begin{pmatrix}1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix}(1/\sqrt{2}\ \ i/\sqrt{2}) - 1\begin{pmatrix}1/\sqrt{2} \\ -i/\sqrt{2}\end{pmatrix}(1/\sqrt{2}\ \ -i/\sqrt{2})}$$

$$\cancel{= \begin{pmatrix}1/2 & 1/2 \\ i/2 & -1/2\end{pmatrix} - \begin{pmatrix}1/2 & -1/2 \\ -i/\sqrt{2} & -1/2\end{pmatrix} = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}}$$

$\sigma_y = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$

σ_y eigenvectors are not orthogonal so you cannot apply this approach.

$$\{\sigma_z, \sigma_x\} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} + \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} = 0$$

$$\{\sigma_x, \sigma_y\} = \begin{pmatrix}i & 0 \\ 0 & -i\end{pmatrix} + \begin{pmatrix}-i & 0 \\ 0 & i\end{pmatrix} = 0 \qquad \{\sigma_y, \sigma_z\} = 0$$

Thus $S_n^2 = \left(\frac{\hbar}{2}\right)^2 \sum n_i^2 \sigma_i^2 = \left(\frac{\hbar}{2}\right)^2 \sum n_i^2 I = \left(\frac{\hbar}{2}\right)^2 I$

$$\exp\left(-i\frac{\alpha}{2} n\cdot\sigma\right) = \sum \frac{\left(-i\frac{\alpha}{2}\right)^n (n\cdot\sigma)^n}{n!}$$ using $(n\cdot\sigma)^i = I$ when i is even:

$$= \sum \frac{\left(-i\frac{\alpha}{2}\right)^{2n}}{(2n)!} + \sum \frac{\left(-i\frac{\alpha}{2}\right)^{2n+1}}{n!}(n\cdot\sigma) \qquad (-i)^{2n} = (-1)^n$$

$$= \cos\left(\frac{\alpha}{2}\right) - i(n\cdot\sigma)\sin\left(\frac{\alpha}{2}\right)$$

b) $$\hat{R}_y(\alpha)\hat{S}_z\hat{R}_y^\dagger(\alpha) = \left(\cos\frac{\alpha}{2} - i\sin\left(\frac{\alpha}{2}\right)\sigma_y\right)\frac{\hbar}{2}\sigma_z\left(\cos\frac{\alpha}{2} + i\sin\left(\frac{\alpha}{2}\right)\sigma_y^\dagger\right)$$

$$= \cos^2\frac{\alpha}{2}\hat{S}_z + i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\frac{\hbar}{2}\left(\sigma_z\sigma_y - \sigma_y\sigma_z\right)$$
$$+ \frac{\hbar}{2}\sin^2\left(\frac{\alpha}{2}\right)\sigma_y\sigma_z\sigma_y$$

$\sigma_y^\dagger = \sigma_y$

$$\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} = \cancel{\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix} = \sigma_z} = \begin{pmatrix}-1 & 0 \\ 0 & 1\end{pmatrix} = -\sigma_z$$

$$\sigma_z\sigma_y - \sigma_y\sigma_z = \begin{pmatrix}0 & -i \\ -i & 0\end{pmatrix} - \begin{pmatrix}0 & i \\ i & 0\end{pmatrix} = \cancel{0} = -2i\sigma_x$$

$$\hat{R}_y(\alpha)\hat{S}_z\hat{R}_y^\dagger(\alpha) = \left(\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)\right)\hat{S}_z + 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\hat{S}_x$$

$$= \boxed{\cos\alpha\,\hat{S}_z + \sin\alpha\,\hat{S}_x}$$

8.05 PSET 3

2. ~~$|\langle n;+|\langle n;+|^2 = |$~~

$$|\langle n';+|n;+\rangle|^2 = \left|\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi+\phi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right|^2$$

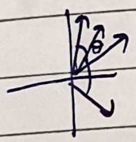
$$= \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \cos\phi+\phi'\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2 + \left(\sin\phi+\phi'\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2$$

$$\left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^2 + \cos(\phi+\phi')\,2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}$$
$$+\left(\cos(\phi+\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2 + \left(\sin(\phi+\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2$$

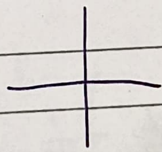
~~$\cos\frac{\theta}{2}\,2\cos$~~

$$= \left(\cos\frac{\theta}{2}\cos\frac{\theta'}{2}\right)^2 + \left(\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\right)^2 + \cos(\phi+\phi')\frac{\sin\theta\sin\theta'}{2}$$

~~$(\cos\theta\cos\theta' +$~~



Just like makes sense actually, ~~forgot~~ since you are doing $\cos\frac{\theta}{2}$ instead of $\cos\theta$.



8.05 PSET 4

1. a) $[A,B]C + B[A,C]$

$= (AB - BA)C + B(AC - CA)$

$= ABC - BCA = [A, BC]$

b) $[[A,B],C] + [[B,C],A] + [[C,A],B]$

$= \cancel{ABC} - \cancel{BAC}\ CAB + BCA - \cancel{ABC} + CAB - BCA = 0$

c)

$\cancel{[q^n, p] = -[p, q^n] = [p, q^{n-1}]q - q[p, q^{n-1}]}$

$\cancel{= [p, q^{n-2}]q^2 \quad = i\hbar(n-1)}$

Induction Base case:

$[q, p] = i\hbar (1) q^0 = i\hbar$

Induction recursion

$[q^n, p] = q^{n-1}[q, p] + [q^{n-1}, p]q$

$= q^{n-1} i\hbar + i\hbar(n-1)q^{n-2}q = i\hbar n q^{n-1}$

d) $f(q) = \sum a_n q^n$

$[f(q), p] = \sum a_n [q^n, p] = \sum a_n i\hbar n q^{n-1} = i\hbar f'(q)$

e) $[x, p]\psi = x \frac{\hbar}{i}\frac{\partial \psi}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x\psi) =$

Detour: why $\hat{p}\hat{x}\psi$ is unambiguously $\frac{\partial}{\partial x}(x\psi)$ and not $\left(\frac{\partial}{\partial x}x\right)\psi = \psi$.

$$\hat{p}: \begin{bmatrix} \frac{-1}{dx} & \frac{1}{dx} & & \\ & \frac{-1}{dx} & \frac{1}{dx} & \\ & & & \ddots \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 1 & & & \\ & 1.1 & & \\ & & 1.2 & \\ & & & \ddots \ 2.2 \end{bmatrix}$$

The point is $\hat{x}$ is a matrix, not a vector. You eval once you apply the operators to a vector

$$\hat{p}\hat{x} = \begin{bmatrix} \frac{1}{dx} & \frac{1.1}{dx} & \\ & -\frac{1.1}{dx} & \frac{1.2}{dx} \\ & & \ddots \end{bmatrix}$$

$[x, p]\psi = x \frac{\hbar}{i}\frac{\partial \psi}{\partial x} - \frac{\hbar}{i}\left(x\frac{\partial \psi}{\partial x} + \psi\right) = i\hbar\psi.$

e) $[f(x), p] = i\hbar \frac{df}{dx}.$

8.05 PSET 4

2. a) $\frac{d}{dt} e^{t(A+B)} = \sum \frac{d}{dt} \frac{t^i (A+B)^i}{i!} = \sum \frac{t^{i-1}(A+B)^i}{(i-1)!} = (A+B)e^{t(A+B)}$

b) $F(t) = e^{tA} B e^{-tA}$

$\frac{\partial F}{\partial t} = A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A = AF - FA = [A,F]$

$[e^{tA}, B] = \sum \frac{t^i A^i B}{i!} - \frac{t^i B A^i}{i!} \cancel{= \sum \frac{t^i}{i!}[A^i,B] = \sum \frac{t^i}{i!}[A,B]\, i A^{i-1}}$

$[A^n, B] = A^{n-1}[A,B] + [A^{n-1},B]A = \cancel{A^{n-1}[A,B] + (n-1)[A,B]A^{n-2}}$

$A^{n-1}[A,B] + A^{n-2}[A,B]A + [A^{n-2},B]A^2$

$= cnA^{n-1}$

$[e^{tA}, B] = \sum \frac{t^i}{i!} c\, i A^{i-1} = ct \sum \frac{t^{i-1} A^{i-1}}{(i-1)!} = cte^{tA}$

$e^{tA} B e^{-tA} = [e^{tA}, B] e^{-tA} + B e^{tA} e^{-tA} = ct + B$

3. $F(t) = e^{tA} B e^{-tA}$ $\quad F(0) = B$

$\frac{\partial F}{\partial t} = A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A \quad \frac{\partial F}{\partial t}(0) = [A,B]$

$\frac{\partial^2 F}{\partial t^2} = A^2 e^{tA} B e^{-tA} - 2A e^{tA} B e^{-tA} A + e^{tA} B e^{-tA} A^2$

$\frac{\partial^2 F}{\partial t^2} = A^2 B - 2ABA + BA^2 = [A,[A,B]]$

In general, $\frac{\partial}{\partial t}\left(\frac{\partial^n F}{\partial t^n}\right) = A \frac{\partial^n F}{\partial t^n} - \frac{\partial^n F}{\partial t^n} A = [A, \frac{\partial^n F}{\partial t^n}]$

giving the result.

$\cancel{\mathrm{Ad}A}$ $\mathrm{ad}A(X) = [A,X]$

$e^{\mathrm{ad}A}(B) = \sum \frac{(\mathrm{ad}A)^i (B)}{i!} = \overset{\left(\frac{B}{0!}\right)}{B} + \overset{\left(\frac{\mathrm{ad}A(B)}{1!}\right)}{[A,B]} + \overset{\left(\frac{\mathrm{ad}A(\mathrm{ad}A(B))}{2!}\right)}{\frac{[A,[A,B]]}{2!}} \dots$

8.05 PSET 4

4. $G(t) = e^{t(A+B)}e^{-tA}$

a) $\frac{\partial G(t)}{\partial t} = (A+B)e^{t(A+B)}e^{-tA} - e^{t(A+B)}e^{-tA}A$

$$G^{-1}\frac{d}{dt}G(t) = (e^{-tA})^{-1}\underbrace{(e^{t(A+B)})^{-1}\cancel{e^{t(A+B)}}}_{I}(A+B)e^{-tA} - A$$

$$= \cancel{e^{tA}Ae^{-tA}} + e^{tA}Be^{-tA} \cancel{-A} = B + ct$$

b) $\cancel{\int G^{-1}(t)\, dG(t) = \int (B+ct)\,dt}$

$$\Rightarrow G(t) = e^{\int B+ct\,dt} = e^{Bt+\frac{1}{2}ct^2}$$

c) $\cancel{G(t) = e^{B}e^{\frac{c}{2}}} = e^{A+B}e^{-A} \Rightarrow e^{A+B} = e^{A}e^{B}e^{\frac{c}{2}}$

8.05
PSET 6

1.b) $\langle\psi(0)|\psi(t)\rangle = \langle\psi(0)|e^{\frac{1}{i\hbar}Ht}|\psi(0)\rangle$

$= \langle\psi(0)|I + \frac{1}{i\hbar}Ht - \frac{1}{2\hbar^2}H^2t^2 + O(t^3)|\psi(0)\rangle$

$= 1 + \frac{t}{i\hbar}\langle\hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle H^2\rangle + O(t^3)$

$|\langle\psi(0)|\psi(t)\rangle|^2 = \left(1 + \frac{t}{i\hbar}\langle\hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle\hat{H}^2\rangle\right)\left(1 - \frac{t}{i\hbar}\langle\hat{H}\rangle - \frac{t^2}{2\hbar^2}\langle H^2\rangle\right) + O$

$= 1 + \frac{t^2}{\hbar^2}\langle\hat{H}\rangle^2 - \frac{t^2}{\hbar^2}\langle\hat{H}^2\rangle + O(t^3)$

$= \boxed{1 - \frac{t^2}{\hbar^2}(\Delta H)^2 + O(t^3)}$

2. a) $\Delta H\,\Delta Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right| \qquad \cos^2\varphi(t) = |\langle\psi_0|\psi_t\rangle|^2$

$d\,|\langle\psi_0|\psi_t\rangle|^2 = \frac{d\langle\psi_t|\psi_0\rangle}{dt}\langle\psi_0|\psi_t\rangle + \langle\psi_t|\psi_0\rangle\frac{d\langle\psi_0|\psi_t\rangle}{dt}$

$= \frac{d}{dt}\langle\psi_t|Q|\psi_t\rangle = \frac{d\langle Q\rangle}{dt} = -2\cos\varphi(t)\sin\varphi(t)\frac{d\varphi(t)}{dt}$

Also, ~~$\Delta Q = 2\ldots|Q|\psi\rangle - \langle Q\rangle|\psi\rangle|$~~

$\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2} = \sqrt{\cos^2\varphi(t) - \cos^4\varphi(t)} = \sqrt{\sin^2\varphi(t)\cos^2\varphi(t)}$

$= \cos\varphi(t)\sin\varphi(t)$

Thus,

$\Delta H \cancel{\cos\varphi(t)\sin\varphi(t)} \geq \frac{\hbar}{\cancel{2}}\cancel{2\cos\varphi(t)\sin\varphi(t)}\left|\frac{d\varphi(t)}{dt}\right|$

$\Rightarrow \frac{\Delta H}{\hbar} \geq \left|\frac{d\varphi(t)}{dt}\right|$

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5. c)

$$\hat N_{mn} = \delta_{mn} n$$

$$\hat x = i\sqrt{\frac{\hbar}{2m\omega}}(\hat a - \hat a^\dagger)$$

$$\hat p = \sqrt{\frac{m\omega\hbar}{2}}(\hat a + \hat a^\dagger)$$

$$\hat a_{mn} = \delta_{m+1,n}\sqrt{n} \qquad \hat a^\dagger_{mn} = \delta_{m-1,n}\sqrt{n+1}$$

$$\hat N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \hat a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \hat a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt2 & 0 & 0 \\ 0 & 0 & \sqrt3 & 0 \end{pmatrix}$$

$$\hat x = i\sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \sqrt2 & 0 \\ 0 & -\sqrt2 & 0 & \sqrt3 \\ 0 & 0 & -\sqrt3 & 0 \end{pmatrix} \qquad \hat p = \sqrt{\frac{m\omega\hbar}{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt2 & 0 \\ 0 & \sqrt2 & 0 & \sqrt3 \\ 0 & 0 & \sqrt3 & 0 \end{pmatrix}$$

$$\hat p^2 = \frac{m\omega\hbar}{2}\left(\hat a^2 + \hat a^{\dagger 2} + \{\hat a, \hat a^\dagger\}\right) = \frac{m\omega\hbar}{2}\left(\hat a^2 + \hat a^{\dagger 2}\right)$$

$$= \frac{m\omega\hbar}{2}\begin{pmatrix} 0 & 0 & \sqrt2 & 0 \\ 0 & 0 & 0 & \sqrt5 \\ \sqrt2 & 0 & 0 & 0 \\ 0 & \sqrt5 & 0 & 0 \end{pmatrix} \qquad \hat x^2 = \frac{-\hbar}{2m\omega}\left(\hat a^2 + \hat a^{\dagger 2}\right)$$

d)

$$\hat x\hat p = \frac{i\hbar}{2}\begin{pmatrix} 1 & 0 & \sqrt2 & 0 \\ 0 & 1 & 0 & \sqrt6 \\ -\sqrt2 & 0 & 1 & 0 \\ 0 & -\sqrt6 & 0 & -3 \end{pmatrix} \qquad \hat p\hat x = \frac{i\hbar}{2}\begin{pmatrix} -1 & 0 & \sqrt2 & 0 \\ 0 & -1 & 0 & -\sqrt6 \\ -\sqrt2 & 0 & -1 & 0 \\ 0 & -\sqrt6 & 0 & 3 \end{pmatrix}$$

$\hat x\hat p \approx i\hbar$, except ~~at~~ for the bottom right term, solely because we truncated the matrices.

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6. a) $\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2}m\omega_x^2\hat{x}^2 + \frac{1}{2}m\omega_y^2 y^2$

$= \hbar\omega(\hat{a}_x^\dagger\hat{a}_x) + \hbar\omega(\hat{a}_y^\dagger\hat{a}_y) + \frac{\hbar\omega_x}{2} + \frac{\hbar\omega_y}{2}$

$\hat{a}_x^\dagger\hat{a}_x\hat{a}_x^\dagger\hat{a}_y^\dagger|n\rangle = (n+1)\hat{a}_x^\dagger a_y^\dagger|n\rangle$

$\cancel{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_x^\dagger)^n(\hat{a}_y^\dagger)^n|0\rangle \quad E_n = \hbar\omega(2n+1)}$

$|n_x\rangle = \frac{1}{\sqrt{n!}}[\hat{a}_x^\dagger]^n|0\rangle$

$|n_y\rangle = \frac{1}{\sqrt{n!}}[\hat{a}_y^\dagger]^n|0\rangle$

$E_n = \omega_x(n_x + \frac{1}{2}) + \omega_y(n_y + \frac{1}{2})$

b) $\hat{a}_x = \frac{1}{\sqrt{2m\hbar\omega_x}}\hat{p}_x - i\sqrt{\frac{m\omega_x}{2\hbar}}\hat{x} \qquad \hat{a}_y = \frac{1}{\sqrt{2m\hbar\omega_y}}\hat{p}_y - i\sqrt{\frac{m\omega_y}{2\hbar}}\hat{y}$

$\hat{a}_x^\dagger = \frac{1}{\sqrt{2m\hbar\omega_x}}\hat{p}_x + i\sqrt{\frac{m\omega_x}{2\hbar}}\hat{x} \qquad \hat{a}_y^\dagger = \frac{1}{\sqrt{2m\hbar\omega_y}}\hat{p}_y + i\sqrt{\frac{m\omega_y}{2\hbar}}\hat{y}$

$\hat{a}_x^\dagger\hat{a}_y^\dagger - \hat{a}_x\hat{a}_y = i\sqrt{\frac{\omega_x}{\omega_y}}\frac{1}{\hbar}\hat{x}p_y + i\sqrt{\frac{\omega_y}{\omega_x}}\frac{1}{\hbar}\hat{p}_x\hat{y}$

$\hat{a}_x^\dagger\hat{a}_y - \hat{a}_y^\dagger\hat{a}_x = i\sqrt{\frac{\omega_x}{\omega_y}}\frac{1}{\hbar}\hat{x}\hat{p}_y - i\sqrt{\frac{\omega_y}{\omega_x}}\frac{1}{\hbar}\hat{p}_x\hat{y}$

$i\hbar(\hat{a}_y^\dagger\hat{a} - \hat{a}_x^\dagger\hat{a}_y) = \sqrt{\frac{\omega_x}{\omega_y}}\hat{x}\hat{p}_y - \sqrt{\frac{\omega_y}{\omega_x}}\hat{p}_x\hat{y}$

since $\omega_x = \omega_y$ this is $\hat{x}\hat{p}_y - \hat{p}_x\hat{y}$