Linear Algebra **Spring 2023**

Lecture 2: An Overview of Key Ideas

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Disclaimer: *These notes are based on the freely available online lectures by Prof. Gilbert Strang.*

2.1 Vectors

What do you do with vectors? Take combinations. We can multiply vectors by scalars, add, and subtract. Given vectors u, v and w we can form the linear combination x1u + x2v + x3w = b.

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The collection of all multiples of u forms a line through the origin. The collection of all multiples of v forms another line. The collection of all combinations of u and v forms a plane. Taking all combinations of some vectors creates a subspace

2.1.1 Matrices

Create a matrix A with vectors **u** , **v** and **w** in its columns:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

equals the sum x1u + x2v + x3w = b.

When we say x1u + x2v + x3w = b we're thinking about multiplying numbers by vectors; when we say Ax = b we're thinking about multiplying a matrix (whose columns are u, v and w) by the numbers. The calculations are the same, but our perspective has changed

For any input vector x, the output of the operation "multiplication by A" is some vector b:

$$A\begin{bmatrix}1\\4\\9\end{bmatrix} = \begin{bmatrix}1\\3\\5\end{bmatrix}$$

A deeper question is to start with a vector b and ask "for what vectors x does Ax = b?" In our example, this means solving three equations in three unknowns. Solving:

$$AX = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix} \times \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2 - x_1\\x_3 - x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

is equivalent to solving:

$$\begin{aligned}x_1 &= b_1\\x_2 - x_1 &= b_2\\x_3 - x_2 &= b_3\end{aligned}$$

We see that x1 = b1 and so x2 must equal b1 + b2. In vector form, the solution is:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_1 + b_2\\b_1 + b_2 + b_3\end{bmatrix}$$

But this just says:

$$X = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

or x = A1b. If the matrix A is invertible, we can multiply on both sides by A1 to find the unique solution x to Ax = b. We might say that A represents a transform x → b that has an inverse transform b x.

$$C = \begin{bmatrix}1 & 0 & -1\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$$

Then:

$$CX = \begin{bmatrix}1 & 0 & -1\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix} \times \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 - x_3\\x_2 - x_1\\x_3 - x_2\end{bmatrix}$$

and our system of three equations in three unknowns becomes circular.

This is **unsolveable**. But

Where before Ax = 0 implied x = 0, there are non-zero vectors x for which Cx = 0. For any vector x with x1 = x2 = x3, Cx = 0. This is a significant difference; we can't multiply both sides of Cx = 0 by an inverse to find a nonzero solution x. The system of equations encoded in Cx = b is:

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \end{aligned}$$

If we add these three equations together, we get:

$$0 = b_1 + b_2 + b_3$$

This tells us that Cx = b has a solution x only when the components of b sum to 0. In a physical system, this might tell us that the system is stable as long as the forces on it are balanced.

2.1.2 Subspaces

Geometrically, the columns of C lie in the same plane (they are dependent; the columns of A are independent). There are many vectors in R3 which do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of C and so correspond to values of b for which Cx = b has no solution x. The linear combinations of the columns of C form a two dimensional subspace of R3.

This plane of combinations of u, v and w can be described as "all vectors Cx". But we know that the vectors b for which Cx = b satisfy the condition b1 + b2 + b3 = 0. So the plane of all combinations of u and v consists of all vectors whose components sum to 0. If we take all combinations of:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the entire space R3; the equation Ax = b has a solution for every b in R3. We say that u, v and w form a basis for R3.
A basis for Rn is a collection of n independent vectors in Rn. Equivalently, a basis is a collection of n vectors whose combinations cover the whole space. Or, a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.
A vector space is a collection of vectors that is closed under linear combinations. A subspace is a vector space inside another vector space; a plane through the origin in R3 is an example of a subspace. A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector. The subspaces of R3 are:

`itemize`

- the Origin(dot)
- a line through the origin,
- a plane through the origin,
- all of R3.

2.1.3 Conclusion

When you look at a matrix, try to see "what is it doing?"
Matrices can be rectangular; we can have seven equations in three unknowns. Rectangular matrices are not invertible, but the symmetric, square matrix ATA that often appears when studying rectangular matrices may be invertible.