Linear Algebra **Spring 2023**

Lecture 1: The Geometry of Linear Equations

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Disclaimer: *These notes are based on the freely available online lectures by Prof. Gilbert Strang.*

1.1 Linear Equations

The fundamental problem of linear algebra is to solve n linear equations in n. unknowns.

$$2x - y = 0$$

$$-x + 2y = 3$$

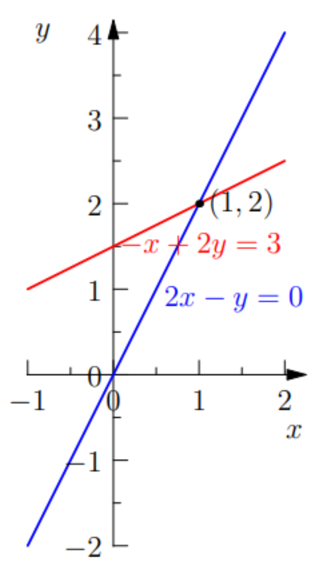
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

The linear equations are Ax=b and the idea now is to solve this particular example,and then step back to see the bigger picture.

1.1.1 Row picture

Plot **all the points** that satisfy that equations.



(1,2) is the all-important point that lies on both lines. because this solves both equations. Seeing the row picture, first of all, for n=2, two equations and two unknowns.

1.1.2 Column picture

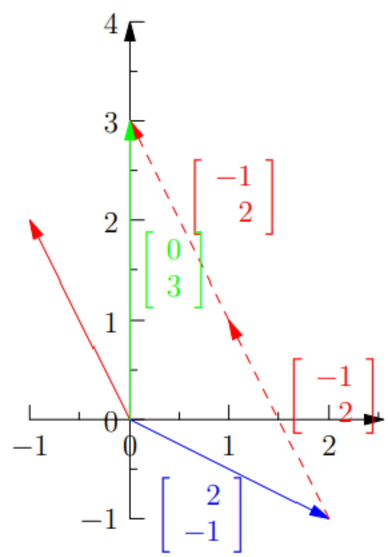
Look at the columns of the matrix.

$$x\begin{bmatrix}2\\-1\end{bmatrix}+\mathrm{y}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

and now what is the equation asking for? It's asking us to find somehow to combine that vector.

It's asking us to find the right **linear combination**.And it's the most fundamental operation in the whole course.

Multiply by some numbers and add. That's a linear combination. This is Algebra. Then what's the geometry(what's the picture that goes with it)?



$$1\begin{bmatrix}2\\-1\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

Combine first column vector and second column vector to get vector **b**.

Then, what are all the combinations? If I took all the **x** and all the **y**, all the combinations, What would be all the results? The results would be that I could get any right-hand side at all.

1.1.3 Extend dimension

Let's do 3x3

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y - z &= -1 \\
-3y + 4z &= 4
\end{aligned}
$$

$$
\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}
$$

$$
A\mathbf{x} = \mathbf{b}
$$

In this case, the points that satisfy all equations make a plane. Each row gives us a plane in three dimensions. those three things meet a point. The main point is that the three planes, because they're not parallel,they're not special.

This row picture getting a little hard to see. When we look at three planes meeting,it's not clear and in four dimensions probably a little less clear.

How about the column picture?

$$
x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \text{y} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}
$$

The left-side is a linear combination of three vectors, so we want to know what combination of those three vectors produces that one. If x=0,y=0,z=1 ,the combinations make **b**.

1.1.4 Linear Independence

It's the next lecture, which is about elimination, which is the systematic way that every body would solve the equations.

Can I solve these equations for every right-hand side? that's the algebra question.

Can I solve Ax=b for every b?

Do the linear combinations of the columns fill three dimensional space?

Every b means all the b in three dimensional space. Elimination will give me a way to find it.

For this matrix A, answer is YES. because I choose good matrix. non-singular matrix, An invertible matrix.

If these three columns all lie in the same plane, then their combinations will lie in that same plane.

It's can solve If b is in the same plane, but most b would be out of the plane and unreachable. So that would be a **singular** case. the matrix is not invertible. there would not be a solution for every b.

If I choose those columns so that they're not independent,

A times x when I multiply a matrix by a vector. It's multiply that Matrix times vector. then how do you multiply a matrix by a vector?

There is two way.

First is columns again!

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = ?$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Second is the idea of a dot product. Do it by rows and each row times x that call a dot product. The point is Ax is a combination of the columns of A.