1.

(a) $\nabla f(x) = \frac{1}{2}\nabla(x^TAx) + b$

where $(x+h)^TA(x+h)$

$= x^TAx + h^TAx + x^TAh + h^TAh$

$= x^TAx + \langle\nabla(x^TAx), h\rangle + o(\|h\|)$

$h^TAx = (Ax)^Th = x^TA^Th = x^TAh$

$= \langle Ax, h\rangle$

so $2Ax = \nabla(x^TAx)$

$\nabla f(x) = Ax + b$

(b) $\nabla f(x) = g'(h(x))\nabla h(x)$

(c) $\nabla^2 f(x) = \nabla(\nabla f(x)) = A$

d) $\nabla f(x) = g'(a^Tx)\,a$

$\frac{\partial \nabla f(x)}{\partial x_i} = g''(a^Tx)\,a_i a$

$\left[\frac{\partial \nabla f(x)}{\partial x_1} \cdots\cdots \frac{\partial \nabla f(x)}{\partial x_n}\right]$

$= g''(a^Tx)\begin{bmatrix} a_1a_1 & \cdots & a_na_1 \\ a_1a_2 & & a_na_2 \\ a_1a_3 & & \vdots \\ a_1a_n & & a_na_n \end{bmatrix} = g''(a^Tx)aa^T$

2. (a)

$x^TAx = (x^Tz)(z^Tx) = \langle x, z\rangle^2 \geq 0$

(b) $Ax = 0$

$z z^Tx = 0$

(z^Tx) is scalar, z is vector

so $N(A) = \{x \mid z^Tx = 0\}$

$\dim N(A) = n-1$

because of $z^Tx = 0$ one straint

So $\mathrm{rank}(A) = n - \dim N(A) = 1$

(c) x^TBAB^Tx

$= (B^Tx)^TA(B^Tx) \geq 0$

so it's PSD

3、(a) $AT = T\Lambda$

$$A[t^{(1)} \cdots t^{(n)}] = [t^{(1)} \cdots t^{(n)}]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$[At^{(1)} \cdots At^{(n)}] = [\lambda t^{(1)} \cdots \lambda t^{(n)}]$$

so $At^{(i)} = \lambda t^{(i)}$

(b) $AU = U\Lambda$

and the same as (a)

(c) $At_i = \lambda_i t_i$

$t_i^T A t_i = t_i^T \lambda_i t_i = \|t_i\| \lambda_i \geq 0$

$\lambda_i \geq 0$