Details of simplification

In the following section, we provide detailed explanations.
If we have a point in the Poincare ball model, represented as $x^{\mathbb{D}}$. In the following, we transfer our idea to Poincar'e ball model.

Step 1: Transform a point from the Poincare ball model to the Lorentz model.
Given a point $\mathbf{x}$ in the Poincare ball model, we can transform it into the Lorentz model using the following formula:

$$\mathbf{y} = \Pi_{\mathbb{D}^{d,\kappa_1} \to \mathbb{L}^{d,\kappa_1}}(x_1, \ldots, x_d) = \frac{\left(1 + \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2, \frac{2}{\sqrt{|\kappa_1|}}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}}x_d\right)}{1 - \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2}$$

Step 2: Apply the HRC operation.
After obtaining the point $\mathbf{y}$ in the Lorentz model, we apply the HRC operation:

$$\mathbf{z} = HRC(\mathbf{y}; f_r, \kappa_1, \kappa_2) = \sqrt{\frac{\kappa_1}{\kappa_2}}\left(\sqrt{\|f_r(\mathbf{y}_{[1:d]})\|_2^2 - \frac{1}{\kappa_1}}, f_r(\mathbf{y}_{[1:d]})\right)^T$$

Step 3: Convert the result back to the Poincare ball model.
Finally, we convert the resulting point back to the Poincare ball model using the following formula:

$$\tilde{\mathbf{x}} = \prod_{\mathbb{H}^{d,\kappa_2} \to \mathbb{D}^{d,\kappa_2}}(z_0, z_1, \ldots, z_d) = \frac{\left(\frac{1}{\sqrt{|\kappa_2|}}z_1, \ldots, \frac{1}{\sqrt{|\kappa_2|}}z_d\right)}{\frac{1}{\sqrt{|\kappa_2|}}z_0 + 1}$$

Let's simplify the expression step by step, considering the curvature.

Starting with the combined expression:

$$\tilde{\mathbf{x}} = \Pi_{\mathbb{H}^{d,\kappa_2} \to \mathbb{D}^{d,\kappa_2}}(HRC(\Pi_{\mathbb{D}^{d,\kappa_1} \to \mathbb{H}^{d,\kappa_1}}(\mathbf{x}); f_r, \kappa_1, \kappa_2))$$

Step 1: Expand $\Pi_{\mathbb{D}^{d,\kappa_1} \to \mathbb{H}^{d,\kappa_1}}(\mathbf{x})$.

$$\mathbf{y} = \frac{\left(1 + \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2, \frac{2}{\sqrt{|\kappa_1|}}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}}x_d\right)}{1 - \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2}$$

Let's denote $\alpha = 1 - \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2$, then:

$$\mathbf{y} = \left(\frac{1 + \frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2}{\alpha}, \frac{2}{\sqrt{|\kappa_1|}\alpha}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\alpha}x_d\right)$$

Step 2: Expand $HRC(\mathbf{y}; f_r, \kappa_1, \kappa_2)$.

$$\mathbf{z} = \sqrt{\frac{\kappa_1}{\kappa_2}}\left(\sqrt{||f_r(\mathbf{y}_{[1:d]})||_2^2 - 1/\kappa_1}, f_r(\mathbf{y}_{[1:d]})\right)^T$$

Substitute $\mathbf{y}$ into the HRC operation:

$$\mathbf{z} = \sqrt{\frac{\kappa_1}{\kappa_2}}\left(\sqrt{||f_r\left(\frac{2}{\sqrt{|\kappa_1|}\alpha}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\alpha}x_d\right)||_2^2 - 1/\kappa_1}, f_r\left(\frac{2}{\sqrt{|\kappa_1|}\alpha}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\alpha}x_d\right)\right)^T$$

Let's denote $\mathbf{u} = f_r\left(\frac{2}{\sqrt{|\kappa_1|}\alpha}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\alpha}x_d\right)$, then:

$$\mathbf{z} = \sqrt{\frac{\kappa_1}{\kappa_2}}\left(\sqrt{||\mathbf{u}||_2^2 - 1/\kappa_1}, \mathbf{u}\right)^T$$

Step 3: Expand $\Pi_{\mathbb{H}^{d,\kappa_2}\to\mathbb{D}^{d,\kappa_2}}(\mathbf{z})$.

$$\tilde{\mathbf{x}} = \Pi_{\mathbb{H}^{d,\kappa_2}\to\mathbb{D}^{d,\kappa_2}}(z_0, z_1, \ldots, z_d) = \frac{\left(\frac{1}{\sqrt{|\kappa_2|}}z_1, \ldots, \frac{1}{\sqrt{|\kappa_2|}}z_d\right)}{\frac{1}{\sqrt{|\kappa_2|}}z_0 + 1}$$

Substitute $\mathbf{z}$ into the above expression:

$$\tilde{\mathbf{x}} = \frac{\frac{1}{\sqrt{|\kappa_2|}}\mathbf{u}}{\frac{1}{\sqrt{|\kappa_2|}}\sqrt{||\mathbf{u}||_2^2 - 1/\kappa_1} + 1}$$

The final simplified expression is:

$$\tilde{\mathbf{x}} = \frac{\frac{1}{\sqrt{|\kappa_2|}} f_r\left(\frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_d\right)}{\frac{1}{\sqrt{|\kappa_2|}}\sqrt{||f_r\left(\frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_d\right)||_2^2 - 1/\kappa_1} + 1}$$

$$\tilde{\mathbf{x}} = \frac{\frac{1}{\sqrt{|\kappa_2|}} f_r\left(\frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_d\right)}{\frac{1}{\sqrt{|\kappa_2|}}\sqrt{||f_r\left(\frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_1, \ldots, \frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}x_d\right)||_2^2 - 1/\kappa_1} + 1}$$

let $\gamma = \frac{2}{\sqrt{|\kappa_1|}\left(1-\frac{1}{|\kappa_1|}\|\mathbf{x}\|_2^2\right)}$, then

$$\tilde{\mathbf{x}}^{\mathbb{D}} = \frac{f_r\left(\gamma\mathbf{x}^{\mathbb{D}}\right)}{\sqrt{||f_r\left(\gamma\mathbf{x}^{\mathbb{D}}\right)||_2^2 - 1/\kappa_1} + \sqrt{|\kappa_2|}}$$

if $\kappa_1 = \kappa_2 = -1$,and $\gamma = \frac{2}{1-\|\mathbf{x}\|_2^2}\mathbf{x}$

$$\tilde{\mathbf{x}}^{\mathbb{D}} = \frac{f_r\left(\gamma\mathbf{x}^{\mathbb{D}}\right)}{\sqrt{||f_r\left(\gamma\mathbf{x}^{\mathbb{D}}\right)||_2^2 + 1} + 1}$$

This expression represents the transformation of a point $\mathbf{x}$ from the Poincare ball model with curvature κ_1, the application of the HRC operation in the Lorentz model, and the conversion of the result back to the Poincare ball model with curvature κ_2, all simplified into a single equation.

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