1. Proof of Proposition 4.1

To analyze the game by backward induction, we solve the optimization problem in the third stage. Recall that under membership-based pricing, $\pi_i(q, v_i) = P_M \cdot q \cdot v_i$. By taking the derivative of the objective function in (10) with respect to v_i, we have

$$\begin{aligned} & u_i d_i q \cdot (1 + v_i q)^{u_i d_i - 1} - P_M q \\ = \quad & q \left[u_i d_i (1 + v_i q)^{u_i d_i - 1} - P_M \right], \end{aligned}$$

where $q > 0$ (trivial if $q = 0$) and $u_i d_i \in [0, d_{\max}]$.

Next, we leverage the result to analyze the monotonicity of the original function. Since $P_M < 1$, it turns out that when the user's usage pattern $u_i d_i \geq 1$, we always have a non-negative value for the derivative. In this case, the optimal solution to the original problem is always $v_i = 1$ when users have $u_i d_i \geq 1$. Such users are always active in using the service despite the quality of the service. Recall that the parameters u_i and d_i of user usage pattern are distributed uniformly over the intervals $[0, 1]$ and $[1, d_{\max}]$, respectively. Hence, for a sufficiently large number N of users, the probability of a user having $u_i d_i \leq 1$ is $H_{d_{\max}}/d_{\max}$ (by applying the formula of total probability). Note that $H_{d_{\max}}$ is the harmonic number of $d_{\max}$. Accordingly, the number of active users is about $N(1 - H_{d_{\max}}/d_{\max})$.

When $u_i d_i < 1$, we see that $(1 + v_i q)^{u_i d_i - 1}$ always smaller than 1. In other words, provided that $u_i d_i < P_M$, the derivative remains negative and the optimal solution becomes $v_i = 0$. Such users only favor the free trial of the service due to their infrequent usage pattern. We call them the inactive users. The number of such users is about $N \cdot H_{d_{\max}}/d_{\max} \cdot P_M$. For the rest of the users, their decision depends on the quality q of the service. We call them the wavering users. If the quality is high enough such that their usage pattern satisfies $u_i d_i > \log(1 + P_M q)/\log(1 + q)$, they will make the subscription, and stick to free-trial services otherwise. The number of inactive users is thus $N \cdot H_{d_{\max}}/d_{\max} \cdot (1 - P_M)$. The proof is thusly completed.

2. Proof of Corollary 4.2

By taking the result in Proposition 4.1 back to the objective function in (12) where $p = C$ in this case, we have $r^* = \sqrt{\frac{P_M N}{C} \left[1 - \frac{H_{d_{\max}} \log(1+P_M q)}{d_{\max} \log(1+q)} \right]}$. According to the saturation effect of user subscriptions, based on users' rational decisions in the third stage, the FMP invests sufficient resources that lead to service quality $q^* \to 1$. Since $\frac{\log(1+P_M q)}{\log(1+q)}$ is increasing concavely in the value of q, thus we have $r^* = \Theta(\sqrt{N \left[1 - \frac{H_{d_{\max}} \log(1+P_M)}{d_{\max} \log 2} \right]})$. The proof is completed.

3. Proofs of Propositions 4.3 and 4.5

The analysis is similar to the proof of Proposition 4.1, except that the pricing function changes to $\pi_i(q, v_i) = P_T \cdot q \cdot v_i \cdot u$ and $\pi_i(q, v_i) = P_Q \cdot q \cdot v_i \cdot u_i \cdot d_i$. Their corresponding corollaries (4.4 and 4.6) follow similar procedures to Corollary 4.2.

4. Proof of Proposition 4.7

Based on Corollaries 4.2, 4.4, and 4.6, the optimal number r^* of customization resources is proportional to the value of $\sqrt{1/p}$. By setting $r^* = K\sqrt{1/p}$ with K as a fixed constant and taking it back to the objective function in (8), we rewrite it as the form $K(\sqrt{p} - C\sqrt{1/p})$, which is monotonically increasing in the value of p. Hence, given the optimal solution in the second stage, we have $p^* = 1$. The proof is completed.

5. Proof of Lemma 5.1

For the user-driven mode under membership-based pricing, there always exist users who customize and offer the service to others. The reason is that if there is no service in the market, those users with the heaviest usage patterns (i.e., heavy users) will choose to customize their own in pursuit of the high utility. Among other non-heavy users, given the service price P_M, if the FMP's price $p \geq P_M$, (according to Proposition 4.1) those active users and wavering users that satisfy (11) would stick to making subscriptions instead of customization to save their cost. On the other side, if the FMP price is lower than P_M, those users consider customizing the service if the price p is sufficiently low such that $r > P_M/p - 1$. The result is derived from the optimal solution to the objective function for users. The complete is thusly completed.

6. Proof of Proposition 5.2

When $p \geq P_M$, by Lemma 5.1, most non-inactive users subscribe to the given service except the heaviest user(s). First, let us consider when there is only one heaviest user (HU), i.e., $N_{\max} = 1$. Since the service price is fixed, then the non-inactive users will always subscribe to the one with the highest quality. Suppose, in addition to the HU, there is another user who customizes the service. By Lemma 5.1, given the high FMP price p, the user makes a positive profit only if its service is subscribed by a sufficiently large number of users. This means the user must invest at least as many resources as the HU to achieve a comparable service quality. However, the HU can always add more resources to improve the quality of service than the user until the user's net benefit of customization becomes zero. In the end, the user will acquire no subscribers and pay for a large number of resources for service customization with diminishing returns on the quality of service and its utility. Therefore, only the HU will customize the service with a resource number r of $O(\sqrt{N})$. The FMP's payoff equals $O(\sqrt{N})$.

Next, we consider the case when there is more than one HU, i.e., $N_{\max} > 1$. By following a similar analysis procedure, it turns out that all non-heavy users will stick to subscriptions instead of customization. The situation now becomes the competition among a group of HUs in pursuit of the highest quality of service. Each HU will always add more resources to improve the quality of its service whenever possible, to acquire more market share. The equilibrium will finally be reached at the point in which each HU equally pays for the same number of invested resources with a zero payoff. In this case, the HUs will share the market (payments from all subscribers). By solving the HU's objective function (with $v = 0$), we have $r^* = \Theta(N/N_{\max})$, and the FMP's payoff is $O(N)$. The proof is completed.

7. Proof of Proposition 5.3

When $p < P_M$, by Lemma 5.1, some non-inactive users consider customizing services unless the given service's quality $q(r)$ is high enough, *i.e.*, $r > P_M/p - 1$, or equivalently, the FMP price p is low enough.

When there is only one HU (i.e., $N_{\max} = 1$). Based on the condition $r > P_M/p - 1$ and following a similar analysis for Proposition 5.2, the boundary case is when $r^* = \Theta(\sqrt{N})$. Specifically, If $P_M/p = \Theta(N^\alpha)$ for $\alpha \in [0, \frac{1}{2})$, meaning that the price p is not low enough, then all non-HU users will make subscriptions. This case reduces to the situation in Proposition 5.2 with one HU. The analysis is similar. However, if $P_M/p = \Theta(N^\alpha)$ for $\alpha \geq \frac{1}{2}$, then those non-HU users with high usage (i.e., $u_i d_i \to d_{\max}$) will add resources to customize their services. To maintain the market share, the only HU invests as many resources as possible to improve its quality of service. The equilibrium is reached when the only HU and those non-HU users pay for an equal number of resources with near-zero payoff. In this case, the optimal resource equals $\Theta(N^{1+\alpha})$ and the FMP's payoff is $O(N)$. As for the case with more than one HU (i.e., $N_{\max} > 1$), the analysis is similar and hence omitted here. The proof is completed.