Iteration and recursion

Question 1 **50 marks**

You wish you use bisection for finding a root of the continuous functions below. For each function, use tools from calculus to find a domain (a, b) that contains precisely one root (so that you satisfy the sufficient condition for bisection to converge). if the function has more than one root, select the one closest the to origin.

Note that the solutions below are not unique - other intervals are valid but your proof needs to match your interval.

(a) $f(x) = (x-1)\cos(x) + 1/2$

Looking at the graph of this function, we guess that the root we are after lies between $x = 0$ and $x = 1$. Indeed, $f(0) = -1/2 < 0$ and $f(1) = 1/2 > 0$ so there is at least one root in that interval. In order to show there is only one root, we consider the derivative $f'(x) = \cos(x) - (x-1)\sin(x)$. For $x \in (0, 1)$ we have $\cos(x) > 0$, $x - 1 < 0$ and $\sin(x) > 0$ and therefore $f'(x) > 0$. Since the function is monotonically increasing, it can only have one root on this domain.

(b) $f(x) = \sqrt{x^2+1} - x^3$

Looking at the graph of this function, we guess that the root we are after lies between $x = 1$ and $x = 2$. Indeed, $f(1) = \sqrt{2} - 1 > 0$ and $f(1) = \sqrt{5} - 8 < 0$ so there is at least one root in that interval. In order to show there is only one root, we consider the derivative $f'(x) = x/\sqrt{x^2+1} - 3x^2$. Step by step, we can see that, for $x \in (1, 2)$,

$$\begin{aligned} x^2 + 1 > x^2 \quad &\Leftrightarrow \\ \sqrt{x^2+1} > \sqrt{x^2} = x \quad &\Leftrightarrow \\ 1/\sqrt{x^2+1} < 1/x \quad &\Leftrightarrow \\ x/\sqrt{x^2+1} < 1 \quad &\Rightarrow \\ f'(x) < 1 - 3x^2 < -2 & \end{aligned}$$

So that f is monotonically decreasing and can only have one root in this domain.

(c) $f(x) = x\ln(x) - 2x + x^2 + 1$

Looking at the graph of this function, we guess that the root we are after lies between $x = 0$ and $x = 2/3$. Indeed, $f(0) = 1 > 0$ and $f(2/3) = 2\ln(2/3)/3 + 1/9 < 0$ so there is at least one root in that interval. Of course, $\ln(0)$ does not exist, but in the limit

$$\lim_{x \downarrow 0} x\ln(x) = 0$$

In order to show there is only one root, we consider the derivative $f'(x) = \ln(x) - 1 + 2x$. Using the standard inequality $\ln(x) \leq x - 1$, we have

$$f'(x) = \ln(x) - 1 + 2x \leq x - 1 - 1 + 2x = 3x - 2 \leq 0$$

for $x \in (0, 2/3)$. Thus, the function f is monotonically decreasing on this domain and can only have one root.

(d) $f(x) = 1/(x+1)^2 + 4/(x+2)^2 - 1$

Looking at the graph of this function, we guess that the root we are after lies between $x = 0$ and $x = 1$. Indeed, $f(0) = 1 > 0$ and $f(1) = 25/36-1 < 0$ so there is at least one root in that interval. In order to show there is only one root, we consider the derivative $f'(x) = -2/(x+1)^3-8/(x+2)^3 < 0$ for $x \in (0, 1)$. Thus, the function f is monotonically decreasing on this domain and can only have one root.

(**e**) $f(x) = \exp(-x^3 + x - 2) - 1/10$

Looking at the graph of this function, we guess that the root we are after lies between $x = -1/2$ and $x = 0$. Indeed, $f(-1/2) = \exp(-19/8) - 1/10 < 0$ and $f(0) = \exp(-2) - 1/10 > 0$ so there is at least one root in that interval. In order to show there is only one root, we consider the derivative $f'(x) = (-3x^2 + 1) \exp(-x^3 + x - 2)$. Since the exponential function is always positive, f' is zero only at $x = \pm 1/\sqrt{3}$, so outside $(-1/2, 0)$. Therefore, f is a monotonic function on this interval and can only have one root.

Question 2 **50 marks**

(**a**) Write a function that implements the following pseudo-code:
Input: f, f', x, ϵ, N.
Output: x^*.

1. Repeat N times:
   (a) Set $y_1 = x$.
   (b) Take one Newton step, starting from y_1. Call the result y_2.
   (c) Take one Newton step, starting from y_2. Call the result y_3.
   (d) Set
   $$x = y_1 - \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1}$$
   (e) Display $|f(x)|$.
   (f) If $|f(x)| < \epsilon$ print "converged!", break.
2. Output $x^* = x$.

This algorithm is called Steffensen's iteration.

An example can be found in the repository on GitHub.

(**b**) Test your routine on the problem

$$\exp(-x^2 + x) - \frac{1}{2}x = 1.0836 \quad \text{(with initial guess } x = 1\text{)}$$

Show that Newton iteration does not converge quadratically, but your new iterative algorithm does.

Here is the output for both Newton and Steffenson iteration:

```
Calling Newton function...
it=1 x=6.109333e-01 err=3.890667e-01 res=1.207459e-01
it=2 x=4.564079e-01 err=1.545255e-01 res=3.021620e-02
it=3 x=3.785844e-01 err=7.782351e-02 res=7.656717e-03
it=4 x=3.388631e-01 err=3.972124e-02 res=1.916870e-03
it=5 x=3.190605e-01 err=1.980267e-02 res=4.619788e-04
it=6 x=3.098768e-01 err=9.183678e-03 res=9.759948e-05
it=7 x=3.065214e-01 err=3.355400e-03 res=1.291781e-05
it=8 x=3.059175e-01 err=6.038397e-04 res=4.171155e-07
```

```
it=9 x=3.058967e-01 err=2.084388e-05 res=4.967664e-10
No convergence!
Calling Steffensen function...
Iteration 0 x=0.3546001088245211 err=0.6453998911754789 res=0.003735449493816878
Iteration 1 x=0.30063322455862135 err=0.05396688426589974 res=7.347055530781965e-05
Iteration 2 x=0.30740358520394606 err=0.006770360645324713 res=3.268169558090506e-05
Iteration 3 x=0.3058878909726443 err=0.0015156942313017674 res=1.750423115343125e-07
Iteration 4 x=0.3058966634180304 err=8.772445386107108e-06 res=4.418687638008123e-14
Iteration 5 x=0.30589666341581706 err=2.213340621892712e-12 res=0.0
Converged, exiting...
```

Looking at the residuals of Newton iteration, it is clear that they do not converge quadratically. The first six or seven iterations yield a residual decreasing approximately by a factor of 3 or 4, the hallmark of linear convergence. Only in the last two steps does the convergence speed up. In contrast, Steffensen's method converges quadratically after 3 steps and takes only 5 steps in total.