The error of polynomial interpolation

Consider the function

$$f(x) = \frac{1}{1+x^2},\ x \in [-10, 10]$$

and suppose you want to find a polynomial approximation.

(**a**) Write a script that generates N equally spaced interpolation nodes on the domain and computes the corresponding interpolating polynomial Q_N. Use may in-built functions like `interpolate.lagrange` or `interpolate.barycentric_interpolate` or the functions we wrote in lecture 12 (see the course_codes repository).

(**b**) Plot the interpolant together with f for $N = 4, 8, 16, 32$. Also, plot the error $|f(x) - Q_N(x)|$. Is interpolation on these nodes useful for approximating f on this domain?

(**c**) In lecture 14 we looked at a formula for the error of polynomial interpolation. Taking this formula into consideration, can you explain why the error is so large for this test problem?

Discussion: The error of interpolation depends on three things: the location of the interpolation nodes, the order of the interpolant and the magnitude of the high-order derivatives of the function that generates the interpolation data. In the lecture, we considered the function $f(x) = \sin(x)$. Since all of its derivatives satisfy $|f^{(k)}(x)| \leq 1$ the interpolations works well. What about the derivatives of the current test function (it is sometimes called the *Runge function*)?