Pseudo-code for LU decomposition with partial pivoting:

## LU decomposition with partial pivoting

**Input:** $A \in \mathbb{R}^{n \times n}$

| | |
|---|---|
| $U \leftarrow A$, $L \leftarrow I$, $P \leftarrow I$ | (initialise matrices) |
| **for** $k = 1 : n-1$ | (loop through pivot columns) |
|     Select $i \geq k$ to maximise $\lvert U_{i,k} \rvert$ | (choose pivot element) |
|     $U_{k,k:n} \leftrightarrow U_{i,k:n}$ | (swap rows of $U$) |
|     $L_{k,1:k-1} \leftrightarrow L_{i,1:k-1}$ | (swap rows of $L$ up to pivot) |
|     $P_{k,:} \leftrightarrow P_{i,:}$ | (swap rows of $P$) |
|     **for** $j = k+1 : n$ | (loop through rows under pivot) |
|         $L_{j,k} \leftarrow U_{j,k} / U_{k,k}$ | (store multiplier in $L$ matrix) |
|         $U_{j,k:n} \leftarrow U_{j,k:n} - L_{j.k} U_{k,k:n}$ | (update row $j$ of $U$ matrix) |
|     **end for** | |
| **end for** | |

**Output:** Matrices $L$, $U$ and $P$

Iterate over columns to find pivot elements

$$
\begin{pmatrix}
a_{00} & a_{01} & \cdots & \cdots & \cdots & \cdots & a_{0(n-1)} \\
a_{10} & a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1(n-1)} \\
\vdots & & \ddots & & & & \vdots \\
\vdots & & & a_{kk} & & & \vdots \\
\vdots & & & & \ddots & & \vdots \\
a_{(n-2)0} & a_{(n-2)1} & \cdots & a_{(n-2)k} & \cdots & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\
a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)k} & \cdots & \cdots & a_{(n-1)(n-1)}
\end{pmatrix}
$$

$$
\begin{array}{cc}
\begin{array}{ccccccc} & & & k & & & \\ & & & \downarrow & & & \end{array} & \\
\begin{pmatrix}
a_{00} & a_{01} & \cdots & \cdots & \cdots & \cdots & a_{0(n-1)} \\
a_{10} & a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1(n-1)} \\
\vdots & & \ddots & & & & \vdots \\
\vdots & & & a_{kk} & & & \vdots \\
\vdots & & & & \ddots & & \vdots \\
a_{(n-2)0} & a_{(n-2)1} & \cdots & a_{(n-2)k} & \cdots & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\
a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)k} & \cdots & \cdots & a_{(n-1)(n-1)}
\end{pmatrix}
&
\begin{array}{c} \\ \\ \\ \leftarrow k \\ \\ \\ \\ \end{array}
\end{array}
$$

# Find pivot element

$$\begin{array}{cc}
\begin{array}{ccccccc} \phantom{a_{(n-1)0}} & \phantom{a_{(n-1)1}} & \phantom{\cdots} & k & \phantom{\cdots} & \phantom{a_{(n-2)(n-2)}} & \phantom{a_{(n-1)(n-1)}} \end{array} & \\
\begin{pmatrix}
a_{00} & a_{01} & \cdots & \cdots & \cdots & \cdots & a_{0(n-1)} \\
a_{10} & a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1(n-1)} \\
\vdots & & \ddots & & & & \vdots \\
\vdots & & & a_{kk} & & & \vdots \\
\vdots & & & & \ddots & & \vdots \\
a_{(n-2)0} & a_{(n-2)1} & \cdots & a_{(n-2)k} & \cdots & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\
a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)k} & \cdots & \cdots & a_{(n-1)(n-1)}
\end{pmatrix} & \begin{array}{c} \\ \\ \\ k \\ \\ \\ \\ \end{array}
\end{array}$$

$$\arg\max \begin{pmatrix} a_{kk} \\ a_{(k+1)k} \\ \vdots \\ \vdots \\ \vdots \\ a_{(n-2)k} \\ a_{(n-1)k} \end{pmatrix}$$

| | |
|---|---|
| 0 | k |
| 1 | k + 1 |
| . | . |
| . | . |
| . | . |
| . | . |
| (n - 2) - k | (n - 2) |
| (n - 1) - k | (n - 1) |

## Swap Rows

$$
\begin{pmatrix}
a_{00} & a_{01} & \cdots & \cdots & \cdots & \cdots & a_{0(n-1)} \\
a_{10} & a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1(n-1)} \\
\vdots & & \ddots & & & & \vdots \\
\vdots & & & a_{kk} & & & \vdots \\
\vdots & & & & \ddots & & \vdots \\
a_{(n-2)0} & a_{(n-2)1} & \cdots & a_{(n-2)k} & \cdots & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\
a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)k} & \cdots & \cdots & a_{(n-1)(n-1)}
\end{pmatrix}
\quad \textbf{U, P}
$$

Swap

$$
\begin{pmatrix}
a_{00} & a_{01} & \cdots & \cdots & \cdots & \cdots & a_{0(n-1)} \\
a_{10} & a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1(n-1)} \\
\vdots & & \ddots & & & & \vdots \\
\vdots & & & a_{kk} & & & \vdots \\
\vdots & & & & \ddots & & \vdots \\
a_{(n-2)0} & a_{(n-2)1} & \cdots & a_{(n-2)k} & \cdots & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\
a_{(n-1)0} & a_{(n-1)1} & \cdots & a_{(n-1)k} & \cdots & \cdots & a_{(n-1)(n-1)}
\end{pmatrix}
\quad \textbf{L}
$$

Swap