Iteration and bisection

There will be no worked-out solutions posted on Canvas. **Post your solutions and questions on the appropriate Slack thread**. Working solutions should emerge from the discussion with the aid of the lecturer and the TA. Engaging in discussions both in the tutorials and on Slack counts toward the 10% participation credit.

In the first lecture we saw the iterative method

$$x^{(k+1)} = \phi(x^{(k)}) = \frac{1}{2}\left(x^{(k)} + \frac{a}{x^{(k)}}\right)$$

We also saw that, starting from $x^{(0)} = 3$ with $a = 5$, after five iterations $x^{(k+1)} = x^{(k)}$ up to at least 15 digits.

Exercise A
In this exercise you will write a function to iterate the function ϕ.

(**a**) Write a function for ϕ. Inputs should be x and a.

(**b**) Write a function that iterates ϕ. Inputs should be an intial point $x^{(0)}$, the parameter a and the maximal number of iterations $k_{\max}$. Your function should print the successive iterates $x^{(k)}$ to the screen.

(**c**) Now modify your function so that it terminates if the maximal number of iterations is reached **or** if the difference between successive iterates is below some threshold, i.e. if

$$|x^{(k+1)} - x^{(k)}| < \epsilon$$

where ϵ is an additional input.

(**d**) Use your function to see from which intial points the iterates converge to $\sqrt{a}$, depending on a. Also, try to formulate *how fast* the iterates converge. How many iterates do you need to compute to achieve a certain error ϵ?

Exercise B
Read the pseudo-code for *bisection* in lecture 2.

(**a**) Show that if $x^{(k+1)} = \phi(x^{(k)}) = x^{(k)}$ then $x^{(k)}$ is the solution to

$$F(x) = x^2 - a = 0$$

(**b**) Write a function for F. Input should be x and a.

(**c**) Write a function that uses bisection to find an approximate solution to $F(x, a) = 0$. Inputs shoud be the parameter a, the intial boundaries a_0 and b_0, the maximal number of iterations $k_{\max}$, the error tolerance ϵ_x and the residual tolerance ϵ_f.

(**d**) Use your function to see for which intial boundaries the iterates converge to $\sqrt{a}$, depending on a. Also, try to formulate *how fast* the iterates converge. How many iterates do you need to compute to achieve a certain error ϵ_x?

Finally, discuss which method is more efficient for approximating $\sqrt{a}$. Suppose every elementary computation (multiplication, division, addition, subtraction) costs you \$10, and you need to compute $\sqrt{5}$ to 25 digits, which method would you choose?