Finite differences

Consider the function

$$f(x) = x \sin(5x)$$

and suppose you need to approximate its derivative on a regular grid of $N + 1$ points $x_k = kh$, $h = \pi/N$, $k = 0, \dots N$.

(**a**) First, compute the centered difference approximation to $f'(x)$ on the grid points x_k, $k = 1, \dots, N - 1$ for $N = 10$.

(**b**) Now construct a more accurate approximation as follows:

1. for each grid point x_k, $k = 2, \dots, N - 2$, compute the fourth order interpolating polynomial on the grid points x_j, $j = k - 2, \dots, k + 2$
2. approximate $f'(x_k)$ by the derivative of this polynomial at x_k.

You may want to write a pseudo-code first to organise the loop over grid points, the interpolation and differentiation.

(**c**) Compare the errors of these approximations, i.e. compute the maximum of the absolute value of the difference between $f'(x)$ and you approximations on all the grid points where the approximations are defined.

(**d**) Now write a script to run this procedure in a loop for $N = 8, 16, \dots, 2^{10}$. Plot the error of both approximations versus N on the right scale (should you make the vertical and/or horizontal scale logarithmic in order to see a straight line?). Can you see the difference in the way the error decreases? Can you explain it using the error of polynomial interpolation and that of round-off?