Boundary value problems

Consider the Poisson problem:

$$\begin{aligned} -u'' &= x\sin(x) \\ u(0) &= 0 \\ u'(\pi) &= 1 \end{aligned}$$

(**a**) Convince yourself that, for any a and b, the following function is a solution to the differential equation

$$2\cos(x) + x\sin(x) + a + bx$$

and that the solution to the boundary value problem is

$$u(x) = -2 + 2\cos(x) + x\sin(x) + (\pi + 1)x$$

(**b**) Form the matrix-vector problem

$$Mv = R$$

where $v_i = u(x_i)$ for $x_i = hi$, $h = \pi/n$, $i = 0, \ldots, n$, such that the solution v is a finite-difference approximation to the solution u from part (**a**). Make sure to include the right boundary condition properly. Turn to lecture 19 for inspiration.

(**c**) Compute the finite-difference approximation for $n = 10$, 100, 1000, 10000 and for each result, compute the error as

$$E_n = \max_{0 \le i \le n} |u(x_i) - v_i|$$

Plot the errors versus the grid size on a logarithmic scale.

(**d**) Consider the error of your finite difference approximation and compare it to the slope in the graph of part (**c**). Does is coincide? If not, why not? Carefully consider the order of your finite difference approximations.

(**e**) For given n, what is the order of the flop count for computing v? Can you bring the flop count down by using the fact that M is *sparse*? For inspiration, turn to tutorial 5.