RPOs for bigraphs with sharing

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A B S T R A C T

Bigraphical Reactive System (BRS) is a new formalism designed by Milner. It is intended for modelling systems with locality and connectivity. Bigraphs with sharing is a redefined concept, where the underlying spatial model allows locations to intersect or overlap. This is a significant improvement because overlapping occurs in many domains – mobile communication, wireless systems, social interactions and many others.

The aim of the project was to extend the theory of bigraphs with sharing and implement new algorithms. The main focus was on RPOs (Relative Push Outs), namely identifying classes of bigraphs with sharing which allow RPO construction. RPOs do not exist in general for bigraphs with sharing. The goal was to prove, that RPOs do exists in the sub-precategories of epimorphic and/or monomorphic place graphs with sharing.

1. Introduction

Bigraphs with sharing are intended to model systems whith locality and connectivity that can evolve in time, such as wireless networks, social media, AI, autonomous vehicles. [1] [4] [5] A distinctive property of bigraphs with sharing is that localities can overlap or intersect. Relative Push-Outs (RPOs) are an important concept of this part of the theory: they were introduced by Milner to describe contextual reconfigurations. The problem is that RPOs do not exist in general for bigraphs with sharing. This is caused by the fact that uniqueness of the context derived during a construction of RPOs for concrete place graphs with sharing cannot be ensured.

In this paper we attempt to give a solution to the problem: we give a proposition which states that RPOs exist in the precategory of epimorphic place graphs with sharing and we provide the relative proof.

We will follow the naming convention introduced in [1]. In order to distinguish different types of bigraphs we refer to bigraphs without sharing as standard bigraphs. For the definition not repeated here we also refer the reader to [1].

The paper is organized as follows. In the next Section we provide an informal overview of bigraphs with sharing. We discuss the mechanic of bigraphs composition, how epimorphism is defined for place graphs and the RPOs concept. In Section 3 we explain why RPOs do not exist in general for concrete bigraphs with sharing. In Section 4 we present the proposition, that RPOs exists for epimorphic place graphs of bigraphs with sharing and we give a proof. Finally in Section 5 we comment on areas for future research.

2. Bigraphical Reactive System (BRS)

In this section, we give an informal overview of BRS. For details not included here please refer [1], [2] and [3]. We establish some notational conventions first.

We interpret a natural number as a finite ordinal m = {0, 1, …, m-1}. We use upper-case letters A, B, C, …, for bigraphs and their constituents, *v*, *u* to denote nodes.

This is the simplest formula describing bigraphs. A bigraphs is defined as a pair in the following form:

$$B = \langle B^P, B^L \rangle$$

The two constituents are:

- Place graph B^P, which represents locality (Fig. 1a)
- Link graph B^L, which models connectivity (Fig. 1b)

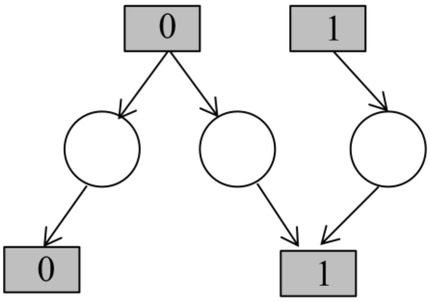

Fig. 1a. Place graph B^P

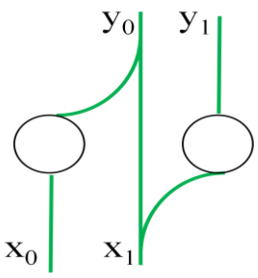

Fig. 1b. Link graph B^L

Place graphs are Directed Acyclic Graphs (DAGs). Shaded squares are roots (on the top) and sites (on the bottom). They are called *outer face* and *inner face*, respectively. They both serve as an interface which allows interaction with an external environment. The nodes are entities and arrows represent their spatial arrangement. A location of child is nested inside the location of its parent; if child has more than one parent then it means that all parents share this location. Observe that children can have more than one parent and the path from children to their ancestors is not always unique.

We will not go over link graphs. They are for the purpose of this paper irrelevant. Link graphs are exactly the same in both kinds of bigraphs. We now take a closer look at place graphs and composition of place graphs. In following, we use the definitions given in [1].

2.1. Composition for place graphs with sharing

Sites and roots form an interface of place graph.

$$F : m \rightarrow n$$

m is an inner interface and n is an outer interface. They index sites and roots respectively. According to **Definition 2** if $F : k \rightarrow m$ (called *parameter*) and $G : m \rightarrow n$ (called *context*) are two concrete place graphs and their composition is defined, then their composite is

$$G \circ F : k \rightarrow n$$

When composing two concrete place graphs, the sites of the context and the roots of the parameter are merged together. The relations of roots with their children and sites with their parents serve to construct new relations in the composite place graph. Its parent relation is given by a partial operation. The key observation is that all parent relations are invariants of a composition, except those between:

- the roots and their children in a parameter
- the sites and their parents in a context.

This is shown in Fig. 2.

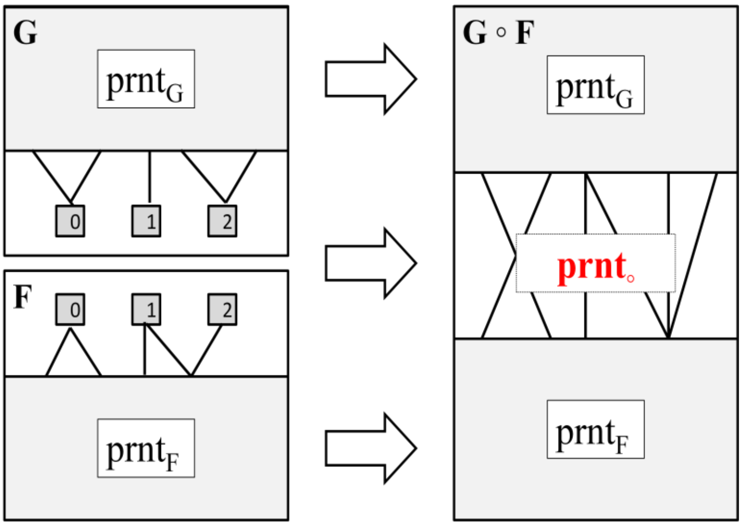

Fig. 2. The mechanism of composition of place graphs. $prnt_f$ and $prnt_g$ are not affected by a composition.

All new (modified) parent relations are defined by $prnt_\circ$ and they are a Cartesian Product of root's children and parents of the sites with the same index as index of roots. We can write this as:

$$prnt_\circ = \sum chld^r_i \times prnt^s_i \quad i \in (0, 1, \ldots, n) \qquad (1)$$

and $chld^r_i$ means children of i-th root and $prnt^s_i$ means parents of i-th site.

Example 1. Let as assume that v_o and v_1 are children of roots with index 0, and w_0, w_1, w_2 are parents of site with index 0. Then $prnt_\circ$ is defined as:

$$prnt_\circ = \{ v_0, v_1 \} \text{ X } \{ w_0, w_1, w_2 \} = \{\{v_0, w_0), (v_0, w_1), (v_0, w_2), \{v_1, w_0), (v_1, w_1), (v_1, w_2)\} .$$

Note that if $chld^r_i$ or $prnt^s_i$ for some i is an empty set, then resulting relation is also an empty set. Roots without children and sites without parents are called *idle*.

Example 2. Let $chld^r_0$ be an empty set (the root is idle), $chld^r_1 = \{ u_0 \}$, $prnt^s_0 = \{ w_0 \}$, $prnt^s_1 = \{ w_0 \}$. Then

$$prnt_\circ = \{ \emptyset \times \{ w_0 \} \} \cup \{ \{ u_0 \} \times \{ w_0 \} \}$$

$$= \{ (u_0, w_0) \}$$

And now, let us make the site with the index 0 orphan, i.e. $prnt^s_0 = \emptyset$. Then

$$prnt_\circ = \{ \emptyset \times \emptyset \} \cup \{ \{ u_0 \} \times \{ w_0 \} \}$$

$$= \{ (u_0, w_0) \}$$

The result is the same - two different contexts give the same composite. The change has no effect for the composition. The intuition is that idle roots "hide" the differences in parent relationships of sites with compatible indexes. The similar thing happens when we make $chld^r_0 = \{ u_0 \}$. The roots with index 0 and 1 become *partners* – they are parents, they share the same node u_0. Again, given the two contexts above we get the same composites after the compositions. This time the "cancelling effect" of idle site with the index 0 is neutralized by "populating" the node into composite parent relation by the second partner, the root with the index 1.

Example 1 and 2 are just another illustration for **Proposition 3.4.4** and the part of its proof from [1] which says that

A concrete place graph with sharing is epi iff no root is idle and no two roots are partners.

Let us recall some definitions from [1] now:

Definition A.9 (*epimorphism*). An arrow $f : X \to Y$ is an epimorphism (epi) if

$g_0 \circ f = g_1 \circ f$ implies $g_0 = g_1$.

Definition A.10 (*span*, *cospan*). A span is a pair of arrows $(f_0; f_1)$ with the same domain. A cospan is a pair of arrows $(g_0; g_1)$ with the same codomain.

Definition A.11 (*bound*, *consistent*). If $(f_0; f_1)$ is a span and $(g_0; g_1)$ a cospan such that

$g_0 \circ f_0 = g_1 \circ f_1$,

then we call $(g_0; g_1)$ a bound for $(f_0; f_1)$. If $(f_0; f_1)$ has a bound it is said to be consistent.

Definition A.12 (*pushout*). A pushout for a span $(f_0; f_1)$ is a bound (h_0, h_1) for $(f_0; f_1)$ such that, for any bound $(g_0; g_1)$, there is a unique arrow h such that

$h \circ h_0 = g_0$ and $h \circ h_1 = g_1$

Definition A.13 (*relative pushout*). Let $(g_0; g_1)$ be a bound for $(f_0; f_1)$. A bound for $(f_0; f_1)$ relative to $(g_0; g_1)$ is a triple $(h_0; h_1; h)$ of arrows such that $(h_0; h_1)$ is a bound for $(f_0; f_1)$ and

$h \circ h_0 = g_0$ and $h \circ h_1 = g_1$.

A relative pushout (RPO) for $(f_0; f_1)$ relative to $(g_0; g_1)$ is a relative bound $(h_0; h_1; h)$ such that for any relative bound $(k_0; k_1; k)$ (sometimes called candidate triple) there is a unique arrow j for which

$j \circ h_0 = k_0$, $j \circ h_1 = k_1$ and $k \circ j = h$.

We say that a category has RPOs if, whenever a span has a bound, it also has an RPO relative to that bound. (See Fig.3)

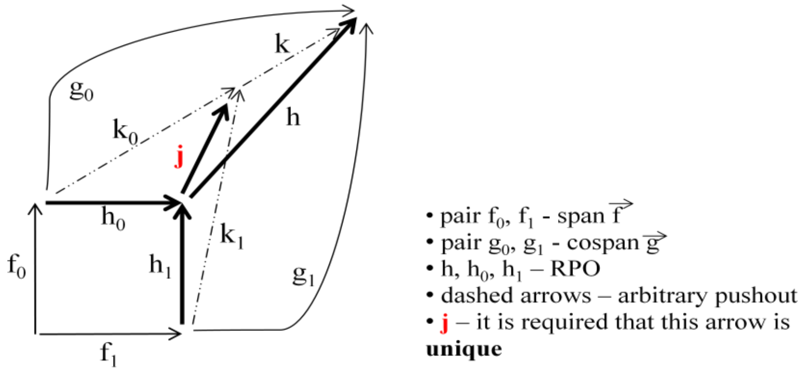

Fig. 3. Relative Push-Outs (RPOs)

We will need these definitions later in this paper.

3. RPOs for concrete place graphs with sharing

Bigraphs are systems with evolving locality and connectivity. In order to model this we need reaction rules:

Redex → Reactum

The rewritng procedure is as follows (see Fig. 4.):

- find redex (subgraph to be changed)
- decompose bigraph by extracting redex and by splitting the rest into parameter and context
- replace redex with reactum
- compose all the bigraphs –

context ◦ (reactum ◦ parameter)

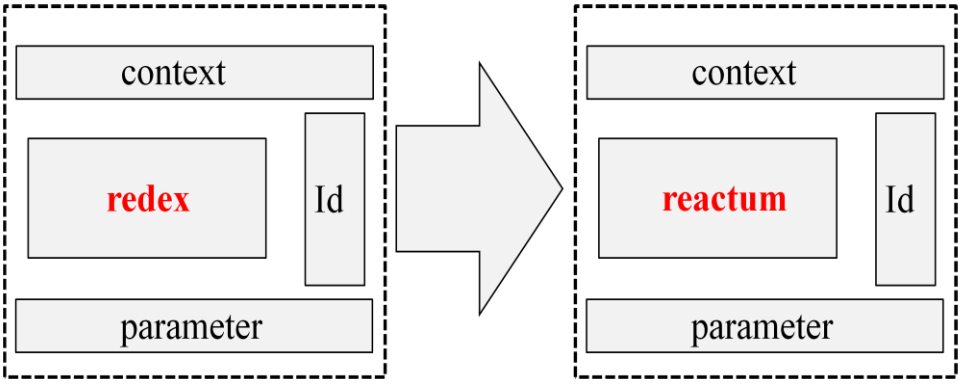

Fig. 4. By applying a reaction rule redex is replaced by reactum

An ability of bigraphs to evolve is an important feature. We can describe this process by the means of labelled transition systems. RPOs are essential to derive such systems. [3]
However, Sevegnani proves that RPOs do not exist in general for concrete bigraphs with sharing [1]. He also conjectures that RPOs always exist for epimorphic bigraphs with sharing. In the following section, we will prove this conjecture.

4. RPOs in epimorphic concrete place graphs with sharing

With all definitions already provided we are ready to present a proposition and its proof.

Proposition. *RPOs exist in the precategory of epimorphic concrete place graphs with sharing.*

Proof. Let (h_0, h_1, h) be a triple of arrows as in a Figure 3 and triple (k_0, k_1, k) be candidate triple for the same bound (g_0, g_1). Let us define j as a context for h_i such that

$$j \circ h_i = k_i \qquad (i = 0, 1) \qquad (1)$$

This is shown in **Fig. 3**.

Now, let us assume an arrow j’ exists such that

$$j' \circ h_0 = k_0 \quad \text{and} \quad j' \neq j \qquad (2)$$

By hypothesis (we are in the precategory of epimorphic concrete place graph with sharing), h_0 is epimorphic. For i = 0 (1)

$$j \circ h_0 = k_0 \qquad (3)$$

RHS for (2) and (3) is the same. Hence

$$j' \circ h_0 = j \circ h_0$$

Therefore, by definition of epimorphism (**Definition A.9**)

$$j' = j$$

This contradicts (2). This proves j is unique and triple (h_0, h_1, h) is an RPO.

5. Conclusion

The future work will be in two areas: designing of algorithm for constructing RPOs and then implement it in the BigraphER tool.
A construction for RPOs for bigraphs without sharing already exists (Construction 5.5, Construction 5.9, [3]). In my opinion, despite the fact that the underlying model of location in bigraphs with sharing is different, some parts of Milner’s construction might be reused. Construction 5.5 is for link graphs. As it is the same in both kinds of bigraphs - this algorithm should deliver proper results also for bigraphs with sharing. However, Construction 5.9 for place graphs must be redefined. According to the proposition we need epimorphic place graphs. Therefore we need to restrict results, where roots of RPOs are idle or partners and in effect the place graph is not epimorphic. Another problem is the fact that decomposition of standard bigraphs proposed by Milner and decomposition of bigraphs with sharing is slightly different. Milner’s RPOs are *ground bigraphs*. They are bigraphs which inner interfaces are two empty sets, <0, Ø>. It means, that ground bigraphs has no inner names and no sites. The bigraphs are decomposed into two parts, without parameter (let compare it with decomposition shown in Fig. 4.). However, in our opinion this approach is less precise and less flexible.

The next step will be extending functionality of BigraphER by implementing the synthesis algorithm. BigraphER is a command-line tool that provides efficient manipulation and visualisation of bigraphs and simulation of BRS and stochastic BRS.

6. References

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