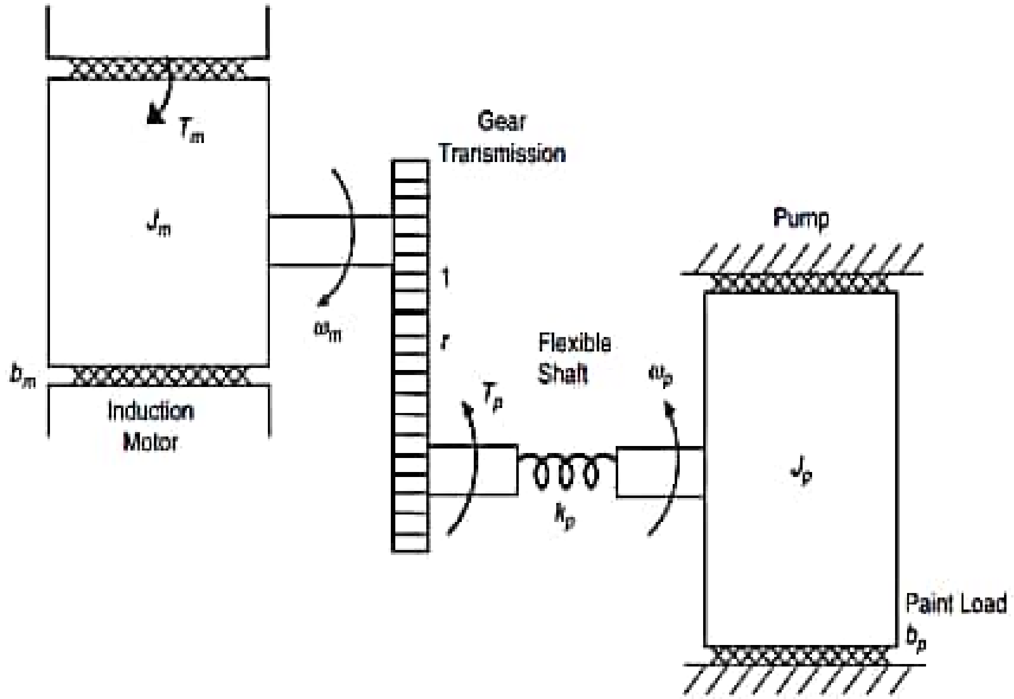
Question 1.



θ_m: motor rotation
θ_p: pump rotation
T_g: Torque transmitted by the motor to the gear

✓ motor speed:

$$\omega_m = \frac{d\theta_m}{dt}$$

✓ Load (pump) speed: $\omega_p = \frac{d\theta_p}{dt}$

✓ Gear efficiency is given by: $\eta = \frac{T_p \, r \, \omega_m}{T_g \, \omega_m} = \frac{\text{output power}}{\text{input power}}$

* $r\omega_m$ is the output speed of the gear & also we know that: power equals to torque × speed *

$T_g = \frac{r}{\eta} T_p$ (I) ,

✓ Newton's second law for the motor: $T_m - T_g - b_m\omega_m = J_m \dot{\omega}_m$ (II)

✓ write the same; for pump: $T_p - b_p\omega_p = J_p \dot{\omega}_p$ (III)

* Hooke's law: Torque = torsional stiffness × angle of twist *

Here is Hooke's Law for the flexible shaft:

$T_p = K_p\left(\frac{\theta_m}{r} - \theta_p\right)$ (IV)

$\longrightarrow J_m \dot{\omega}_m = T_m - b_m\omega_m - \frac{r}{\eta} T_p$

Differentiate eq. (IV): $\dot{T}_p = K_p\left(\frac{\omega_m}{r} - \omega_p\right)$

This is a nonlinear model with state vector $[\omega_m \; T_p \; \omega_p]^T$

input is T_m

&

$$T_m = \frac{T_0 \, q \, \omega_0 (\omega_0 - \omega_m)}{(q\omega_0^2 - \omega_m^2)} \quad VIII$$

From Eq. VIII we see that when $T_m = 0$, we have $\omega_m = \omega_0$ Hence, ω_0:- no. load speed.

$$\frac{\partial T_m}{\partial T_0} = \frac{q\omega_0 (\omega_0 - \omega_m)}{(q\omega_0^2 - \omega_m^2)} = \beta_1 \; , \; \frac{\partial T_m}{\partial \omega_0} = \frac{T_0 q \left((q\omega_0^2 - \omega_m^2)(2\omega_0 - \omega_m) - \omega_0(\omega_0 - \omega_m) 2q\omega_0 \right)}{(q\omega_0^2 - \omega_m^2)^2}$$

$$= \frac{T_0 q \omega_m \left((\omega_0 - \omega)^2 + (q-1)\omega_0^2 \right)}{(q\omega_0^2 - \omega_m^2)^2} = \beta_2 \qquad \text{we also consider } \beta_1 \& \beta_2 \& b > 0$$

$$\frac{\partial T_m}{\partial \omega_m} = -T_0 q \omega_0 \frac{\left((q-1)(\omega_0^2) + (\omega_0 - \omega_m)^2 \right)}{(q\omega_0^2 - \omega_m^2)^2} = -b$$

Steady state

set $\dot{\omega}_m = 0$, $\dot{T}_p = 0$, $\dot{\omega}_p = 0$

$\rightarrow 0 = \overline{T_m} - b_m \overline{\omega_m} - \frac{r}{\eta} \overline{T_p}$, $0 = K_p \left(\frac{\overline{\omega_m}}{r} - \overline{\omega_p} \right)$, $0 = \overline{T_p} - b_p \overline{\omega_p}$

Hence: $\overline{\omega_p} = \overline{\omega_m}/r$, $\overline{T_p} = b_p \overline{\omega_m}/r$, $\overline{T_m} = b_m \overline{\omega_m} + b_p \overline{\omega_m}/\eta = \frac{T_0 \, q \overline{\omega_0} (\overline{\omega_0} - \overline{\omega_m})}{q\overline{\omega_0}^2 - \overline{\omega_m}^2}$

$\Rightarrow J_m \dot{\hat{\omega}}_m = \hat{T}_m - b_m \hat{\omega}_m - \frac{r}{\eta} \hat{T}_p - b \hat{\omega}_m$, $\dot{\hat{T}}_p = K_p \left(\frac{\hat{\omega}_m}{r} - \hat{\omega}_p \right)$

$J_p \dot{\hat{\omega}}_p = \hat{T}_p - b_p \hat{\omega}_p$ $\rightarrow$ where: $\hat{T}_m = \left[\frac{\partial T_m}{\partial T_0} \right] \hat{T}_0 + \left[\frac{\partial T_m}{\partial \omega_0} \right] \hat{\omega}_0 = \beta_1 \hat{T}_0 + \beta \hat{\omega}_0$

defining the linear

state vector: $x = [\hat{\omega}_m \;\; \hat{T}_p \;\; \hat{\omega}_p]^T$

input vector: $u = [\hat{T}_0 \;\; \hat{\omega}_0]^T$

output vector: $y = [\hat{\omega}_p \;\; \hat{T}_p / K_p]^T$

we have:

$$A = \begin{bmatrix} \frac{-(b+b_m)}{J_m} & \frac{-r}{\eta J_m} & 0 \\ K_p/r & 0 & -K_p \\ 0 & 1/J_p & \frac{-b_p}{J_p} \end{bmatrix} \qquad B = \begin{bmatrix} \beta_1/J_m & \beta_2/J_m \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/K_p & 0 \end{bmatrix} \qquad D = [0]$$

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$n \times m : 3\times 4$ .2

$$A. \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ -4 & -3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad rank(A) = 2$$

$$AX = x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4$$

$$u_2 = 3u_1 \ , \quad u_4 = (u_3 - 3u_1)\tfrac{1}{3}$$

$$\longrightarrow u_1 (x_1 + 3x_2) + u_3 x_3 + x_4 \cdot \tfrac{1}{3} (u_3 - 3u_1)$$

$$= u_1 (x_1 + \underbrace{3x_2}_{t} - x_4) + u_3 \left(x_3 + \frac{x_4}{3}\right) = AX = 0$$

u_1 & u_3 : مستقل خطی هستند.

$$\left.\begin{aligned} x_1 + 3\cancel{x_2} - x_4 &= 0 \\ 3x_3 + \underbrace{x_4}_{s} &= 0 \end{aligned}\right\} \quad \begin{aligned} x_3 &= -\tfrac{s}{3} \\ x_1 &= -3t + s \end{aligned} \qquad N_R(A) = 3 - 2 = 1$$

$$\vec{X} = \begin{bmatrix} +s-3t \\ t \\ -s/3 \\ s \end{bmatrix} = \begin{bmatrix} -3 \\ +1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} +1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix} s \longrightarrow \text{the nullity of the matrix is "2"}$$

The nullity of a matrix is the demenison of the basis for the null space.

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$N \times M \to 4 \times 3$

B. $\begin{bmatrix} 1 & 2 & 0 \\ -2 & -5 & 5 \\ 0 & -2 & 10 \\ 2 & 2 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 5 \\ 0 & -1 & 5 \\ 2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .2

Rank (B) = 2

$\to \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -5 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$

$x_1 \quad x_2 \quad x_3 \to t$

$x_1 \to -10t$, $x_2 \to 5t$

$\vec{x} = \begin{bmatrix} -10t \\ 5t \\ t \end{bmatrix} = t \begin{bmatrix} -10 \\ 5 \\ 1 \end{bmatrix}$

The nullity of the matrix is "1"

$\to \mathcal{N}_R(B) = 4 - 2 = 2$

سوال 3)

با جایگذاری θ در معادله 2 صورت سوال خواهیم داشت: $\psi = \frac{\pi}{2} - \theta \Longrightarrow \theta = \frac{\pi}{2} - \psi$: داریم

$$r_2 m_2 g \sin\left(\frac{\pi}{2} - \psi\right) - r_1 m_1 g \sin\left(\frac{\pi}{2} - \psi\right) = (m_1 r_1^2 + m_2 r_2^2)(-\ddot{\psi}) + F r_2 \Longrightarrow$$

$$r_2 m_2 g \cos(\psi) - r_1 m_1 g \cos(\psi) = (m_1 r_1^2 + m_2 r_2^2)(-\ddot{\psi}) + F r_2 \overset{*}{\Longrightarrow}$$

F ورودی است و خروجی مطلوب زمانی است که $\psi = 0$ باشد $\Longleftarrow \ast \; \cos(\psi) = 1 - \frac{\psi^2}{2}$

$$r_2 m_2 g \left[1 - \frac{\psi^2}{2}\right] - r_1 m_1 g \left[1 - \frac{\psi^2}{2}\right] = (m_1 r_1^2 + m_2 r_2^2)(-\ddot{\psi}) + F r_2$$

متغیرهای حالت: $\begin{cases} x_1 = \psi \\ x_2 = \dot{x}_1 \end{cases}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{r_1 m_1 g}{m_1 r_1^2 + m_2 r_2^2} - \frac{r_1 m_1 g}{2(m_1 r_1^2 + m_2 r_2^2)} x_1^2 - \frac{r_2 m_2 g}{m_1 r_1^2 + m_2 r_2^2} + \frac{r_2 m_2 g}{2(m_1 r_1^2 + m_2 r_2^2)} x_1^2 + \frac{F r_2}{m_1 r_1^2 + m_2 r_2^2}$$

$$= \frac{r_1 m_1 g - r_2 m_2 g}{m_1 r_1^2 + m_2 r_2^2} + \frac{r_2 m_2 g - r_1 m_1 g}{2(m_1 r_1^2 + m_2 r_2^2)} x_1^2 + \frac{F r}{m_1 r_1^2 + m_2 r_2^2}$$

$$y = \psi$$

نقطه تعادل ، $x = [\psi \;\; \dot{\psi}] = [0 \;\; 0]$

خطی سازی: $\Delta \dot{x} = A \Delta x + B \Delta F$
$\Delta y = C \Delta x + D \Delta F$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{r_2 m_2 g - r_1 m_1 g}{r_1 m_1^2 + r_2 m_2^2} x_1 & 0 \end{bmatrix} \xrightarrow[\text{نقطه تعادل}]{\text{جایگذاری}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{r_2}{m_1 r_1^2 + m_2 r_2^2} \end{bmatrix} \;,\; C = [1 \;\; 0] \;\text{و}\; D = 0$$

سوال ۴)

$$\ddot{x} = \frac{1}{m_1}\left(F_u - F_s - \sin\theta\,(F_{pa} - F_{mg})\right) = \frac{1}{m_1}\left(u - K\dot{x}|\dot{x}| - m_2 g\sin\theta + m_1 g\sin\theta\right)$$

$$\dot{\theta}(1+\tan^2\theta) = x^2 - x \Rightarrow \dot{\theta} = \frac{x^2 - x}{1+\tan^2\theta}$$

متغیرهای حالت :
$$\begin{cases} x_1 = x \\ x_2 = \dot{x}_1 \\ x_3 = \theta \end{cases}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m_1}\left(u - Kx_2|x_2| - m_2 g\sin x_3 + m_1 g\sin x_3 + m_2 g\sin x_3\right)$$

$$\dot{x}_3 = \frac{x_1^2 - x_1}{1+\tan^2 x_3}$$

اعمال تقریب گفته شده در سوال و درنظر گرفتن نقطه مورد نظر برای خطی سازی خواهیم داشت :

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m_1}\left(u - kx_2|x_2| + (m_1 - m_2)g\sin x_3\right)$$

$$\dot{x}_3 = \frac{x_1^2 - x_1}{1+x_3^2}$$

$$y = x_1$$

خطی سازی :
$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_3}{\partial x_1} & \cdots & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-2K}{m_1}|x_2| & \frac{-m_2}{m_1}g + g \\ \frac{2x_1 - 1}{1+x_3^2} & 0 & \frac{-2x_3(x_1^2 - x_1)}{(1+x_3^2)^2} \end{bmatrix}$$

جایگذاری نقطه تعادل

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-m_2}{m_1}g + g \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \end{bmatrix}, \quad C = [1\;0\;0], \quad D = 0$$

an operating point

$$0<\gamma_1+\gamma_2<1$$

$$\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix}\begin{bmatrix} X(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y(s) \end{bmatrix}$$

total flow goes to tanks 1 and 3

is equal to the total flow going into tank 2 & 4

in the condition of $\boxed{\gamma_2+\gamma_1=1}$

State space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{a_1}{A_1}\sqrt{2gx_1}+\frac{a_3}{A_1}\sqrt{2gx_3} \\ -\frac{a_2}{A_2}\sqrt{2gx_2}+\frac{a_4}{A_4}\sqrt{2gx_4} \\ -\frac{a_3}{A_3}\sqrt{2gx_3} \\ -\frac{a_4}{A_4}\sqrt{2gx_4} \end{bmatrix}}_{f(x)} + \begin{bmatrix} \frac{\gamma_1 K_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 K_2}{A_2} \\ 0 & G_{32} \\ G_{41} & 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K_C x_1 \\ K_C x_2 \end{bmatrix} \ \& \ G_{41} = \frac{(1-\gamma_1)K_1}{A_4} \ \& \ G_{32} = \frac{(1-\gamma_2)K_2}{A_3}$$

Define the output of the system:

$$\dot{y}_1 = L_f h_1 + L_{g_1} u_1 + L_{g_2} h_1 u_2 = K_C\left(\frac{-a_1}{A_1}\sqrt{2gx_1}+\frac{a_3}{A_1}\sqrt{2gx_3}+\frac{\gamma_1 K_1}{A_1}u_1\right)$$

$$\dot{y}_1 = v_1$$

$$\frac{v_1}{K_C} = \frac{-a_1}{A_1}\sqrt{2gx_1}+\frac{a_3}{A_3}\sqrt{2gx_3}+\frac{\gamma_1 K_1}{A_1}u_1 \rightarrow u_1 = \frac{A_1}{\gamma_1 K_1}\left(\frac{v_1}{K_C}+\frac{a_1}{A_1}\sqrt{2gx_1}-\frac{a_3}{A_1}\sqrt{2gx_3}\right)$$

as the first one, for the second one we have:

$$\dot{y}_2 = L_f h_2 + L_{g_1} h_2 u_2 + L_{g_2} h_2 u_2 \rightarrow \dot{y}_2 = K_C\left(\frac{-a_2}{A_2}\sqrt{2gx_2}+\frac{a_4}{A_2}\sqrt{2gx_4}+\frac{\gamma_4 K_2}{A_2}u_2\right)$$

$$\dot{y}_2 = v_2, \quad \frac{v_2}{K_C} = \frac{-a_2}{A_2}\sqrt{2gx_2}+\frac{a_4}{A_4}\sqrt{2gx_4}+\frac{\gamma_2 K_2}{A_2}u_2 \rightarrow u_2 = \frac{A_2}{\gamma_2 K_2}\left(\frac{v_2}{K_C}+\frac{a_2}{A_2}\sqrt{2gx_2}-\frac{a_4}{A_4}\sqrt{2gx_4}\right)$$

Here we have the linear model:

$$\dot{x} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 K_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 K_2}{A_2} \\ 0 & \frac{(1-\gamma_2)K_2}{A_3} \\ \frac{(1-\gamma_1)K_1}{A_4} & 0 \end{bmatrix} u \text{ and } y = \begin{bmatrix} K_C & 0 & 0 & 0 \\ 0 & K_C & 0 & 0 \end{bmatrix} x$$

pump speed $\xrightarrow{(V)}$ tank level (H)

$T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$. $i = 1, \ldots, 4$, $C_1 = \frac{T_1 K_1 K_c}{A_1}$, $C_2 = \frac{T_2 K_2 K_c}{A_2}$

mass balance's & Bernolli's law yield[2]:

$\dot{h}_1 = -\frac{a_1}{A_1}\sqrt{2h_1 g} + \frac{a_3}{A_1}\sqrt{2h_3 g} + \frac{\gamma_1 K_1}{A_1} v_1$

$\dot{h}_2 = -\frac{a_2}{A_2}\sqrt{2h_2 g} + \frac{a_4}{A_2}\sqrt{2h_4 g} + \frac{\gamma_2 K_2 v_2}{A_2}$

$\dot{h}_3 = \frac{-a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2) K_2 v_2}{A_3}$

$\dot{h}_4 = \frac{-a_4}{A_4}\sqrt{2h_4 g} + \frac{(1-\gamma_1)}{A_4} v_1$

$Q_i = K_i V_i$

$y_1 = K_c h_1$

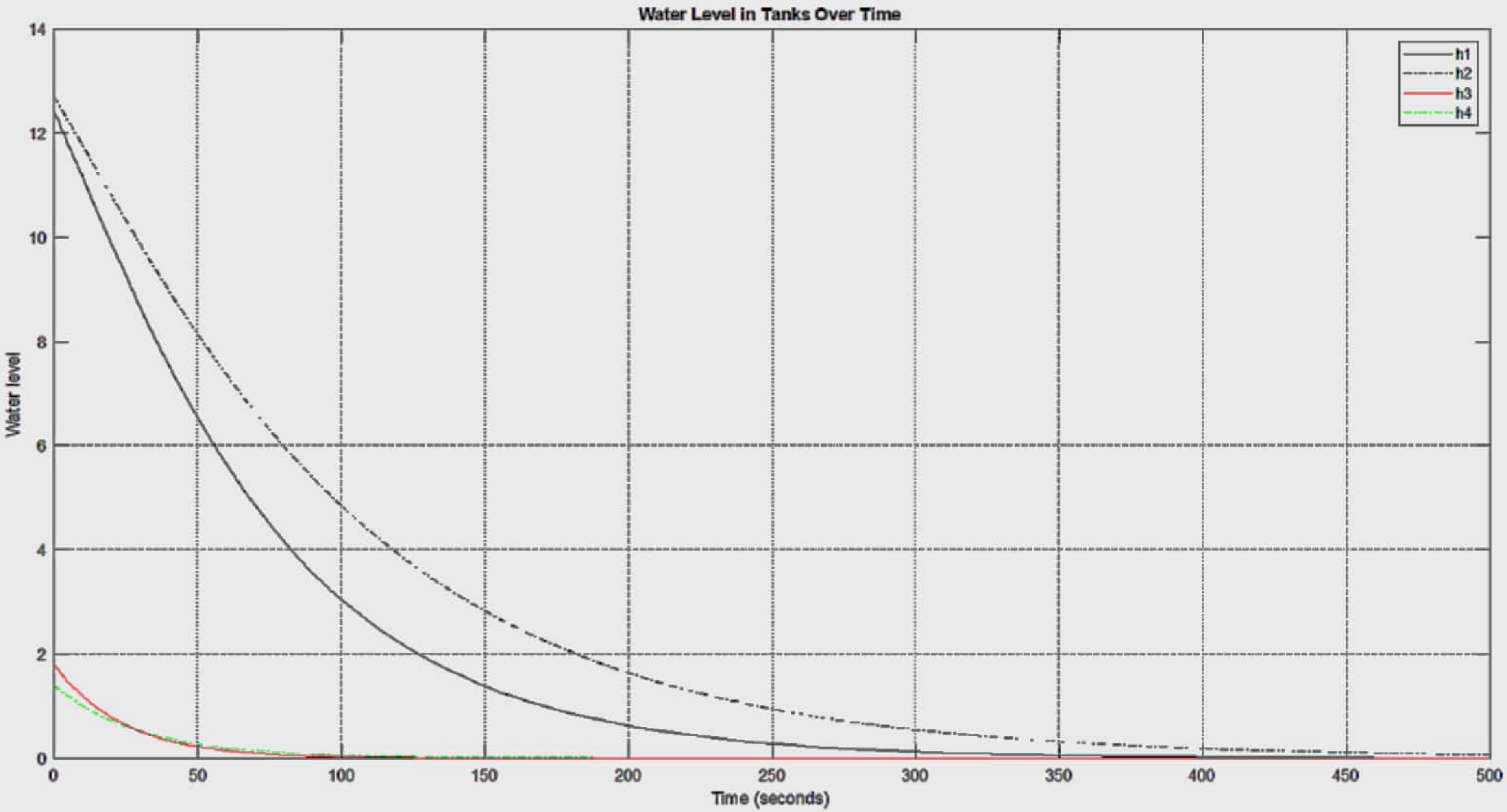
$y_2 = K_c h_2$

gain of measuring device's

$\rightarrow$ zero's of the system can be found after linearization of the plant around

(1)

$$\longrightarrow G(s) = C(SI - A)^{-1}B + D \longrightarrow G(s) = \begin{bmatrix} \dfrac{\gamma_1 T_1}{A_1(1+sT_1)} & \dfrac{(1-\gamma_2)T_1}{(1+sT_3)(1+sT_1)A_1} \\[2em] \dfrac{(1-\gamma_1)T_2}{(1+sT_4)(1+sT_2)A_2} & \dfrac{\gamma_2 T_2}{(1+sT_2)A_2} \end{bmatrix}$$



Water Level in Tanks Over Time
h1
h2
h3
h4
Water level
Time (seconds)
0
2
4
6
8
10
12
14
0
50
100
150
200
250
300
350
400
450
500