Vasicek Interest Rate Model

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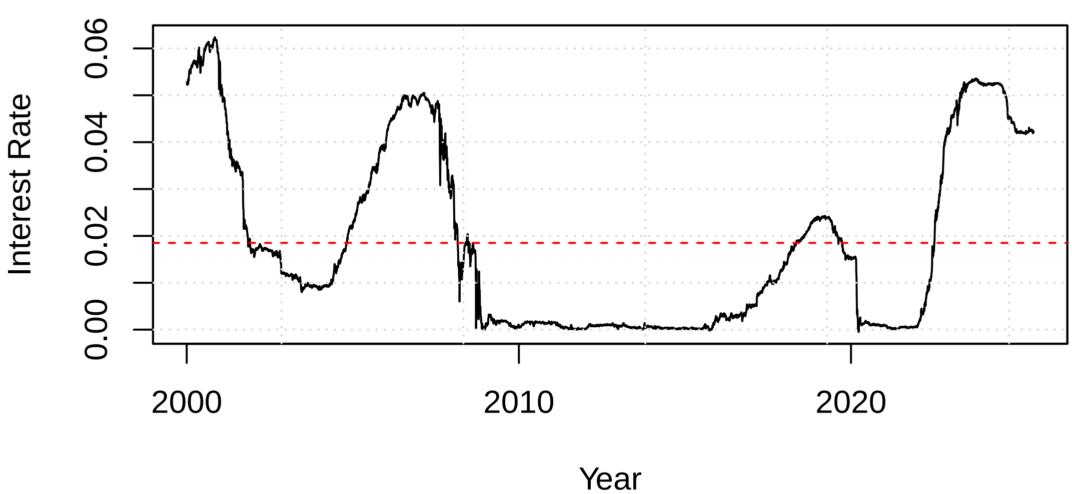
1 Introduction

In the field of quantitative finance, modeling the evolution of interest rates plays a crucial role in the valuation of financial instruments, risk management, and investment strategy development.
Among the most well-known and widely studied models, the Vasicek model holds a prominent place due to its analytical tractability and the mathematical rigor with which it describes the stochastic behavior of interest rates.
Introduced by Oldřich Vašíček in 1977, this model was one of the first to represent interest rate dynamics using a mean-reverting process. The Vasicek model laid the groundwork for more advanced interest rate models and remains a valuable tool in various practical applications, from bond pricing to credit risk assessment.
This model also accounts for negative interest rates: during times of uncertainty having negative rates can help central banks to manage their economies.

One of the most commonly used benchmark interest rates is the U.S.
3-month rate, whose dynamics are illustrated in the graph below.



2 The Model

The model takes the following form:

$$dr(t) = a(b - r(t))dt + \sigma_r \, dW(t)$$

with the following boundary condition:

$$r(t_0) = r_0$$

Where:

- a is the strength of mean reversion (>0)
- b is the long term mean level
- σ is the interest rate volatility (>0)
- W(t) is the standard Brownian Motion

This Stochastic Differential Equation has a closed-form solution that can be easily computed.

$$r(t) = r(t_0)e^{-a(t-t_0)} + b(1 - e^{-a(t-t_0)}) + \int_{t_0}^{t} e^{-a(t-s)}\sigma_r dW(s)$$

3 Parameters Estimate

Vasicek model has a costant volatility: $V_t[dr_t] = \sigma^2 \, dt$, therefore we can easily estimate parameter σ through Method of Moments:

$$\sigma_r = \sqrt{\frac{V_t[dr_t]}{dt}}$$

We can estimate the other 2 parameters by discritizing the Vasicek equation.

$$a = \frac{1 - \beta_1}{dt}$$

and

$$b = \frac{\beta_0}{1 - \beta_1}$$

```
regr=lm(tail(rate,-1) ~ head(rate,-1))
summary(regr)
beta0=regr$coefficients[[1]]
beta1=regr$coefficients[[2]]
a=(1-beta1)/dt
a
b=beta0/(a*dt)
b
mean(rate)

library(knitr)
library(kableExtra)
library(magrittr)

params <- data.frame(
  Parameter = c("$R^2$", "$a$", "$b$", "$\\sigma$"),
```

```r
  Description = c(
    "Coefficient of determination",
    "Strength of mean reversion $a = \\frac{1 - \\beta_1}{dt}$",
    "Long-term mean level $b =  \\frac{\\beta_0}{1- \\beta_1}$",
    "Volatility"
  ),
  `Estimated Value` = c("0.9901 ", "0.09294512", "0.0138076", "0.007061189")
)

# Tabella LaTeX con booktabs e caption
kable(params, format = "latex", booktabs = TRUE, escape = FALSE,
      caption = "Estimated Vasicek Model Parameters (2000-2025)") %>%
  kable_styling(latex_options = c("hold_position"))
```

```
##
## Call:
## lm(formula = tail(rate, -1) ~ head(rate, -1))
##
## Residuals:
##        Min         1Q     Median         3Q        Max
## -0.0081024 -0.0001046 -0.0000051  0.0000985  0.0073952
##
## Coefficients:
##                 Estimate Std. Error  t value Pr(>|t|)
## (Intercept)    5.671e-06  7.753e-06    0.731    0.465
## head(rate, -1) 9.996e-01  2.900e-04 3447.354   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.000447 on 6373 degrees of freedom
## Multiple R-squared:  0.9995, Adjusted R-squared:  0.9995
## F-statistic: 1.188e+07 on 1 and 6373 DF,  p-value: < 2.2e-16
##
## [1] 0.09847863
## [1] 0.01439642
## [1] 0.01850179
```

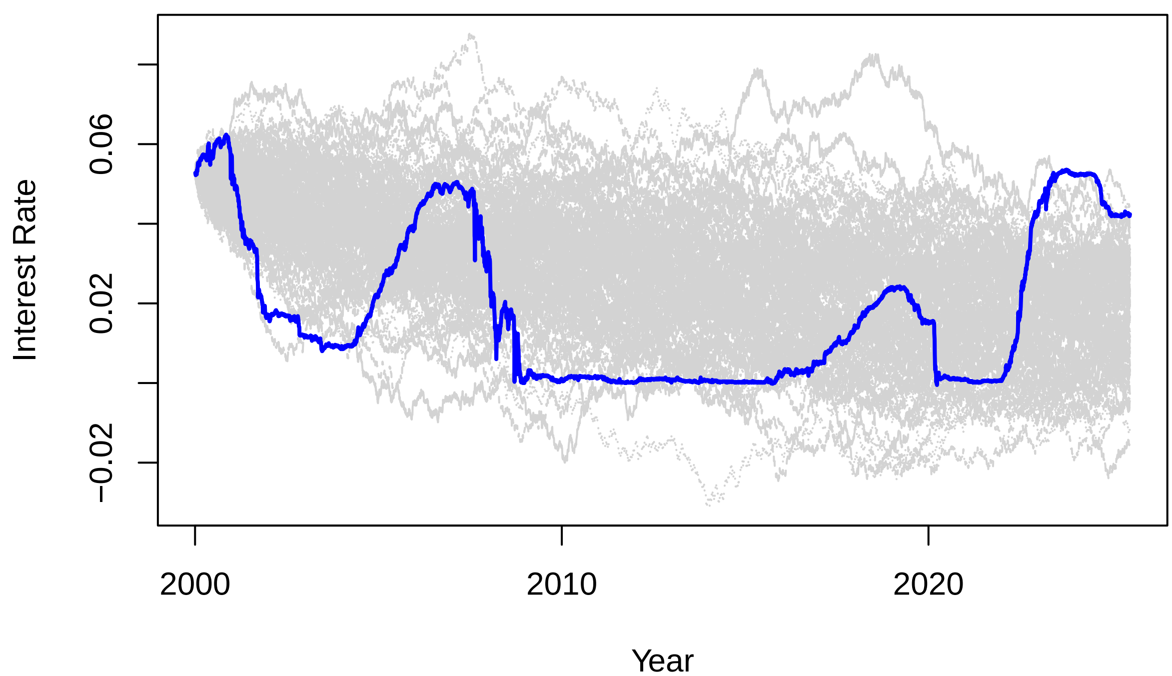
Table 1: Estimated Vasicek Model Parameters (2000-2025)

Parameter	Description	Estimated.Value
R^2	Coefficient of determination	0.9901
a	Strength of mean reversion $a = \frac{1-\beta_1}{dt}$	0.09294512
b	Long-term mean level $b = \frac{\beta_0}{1-\beta_1}$	0.0138076
σ	Volatility	0.007061189

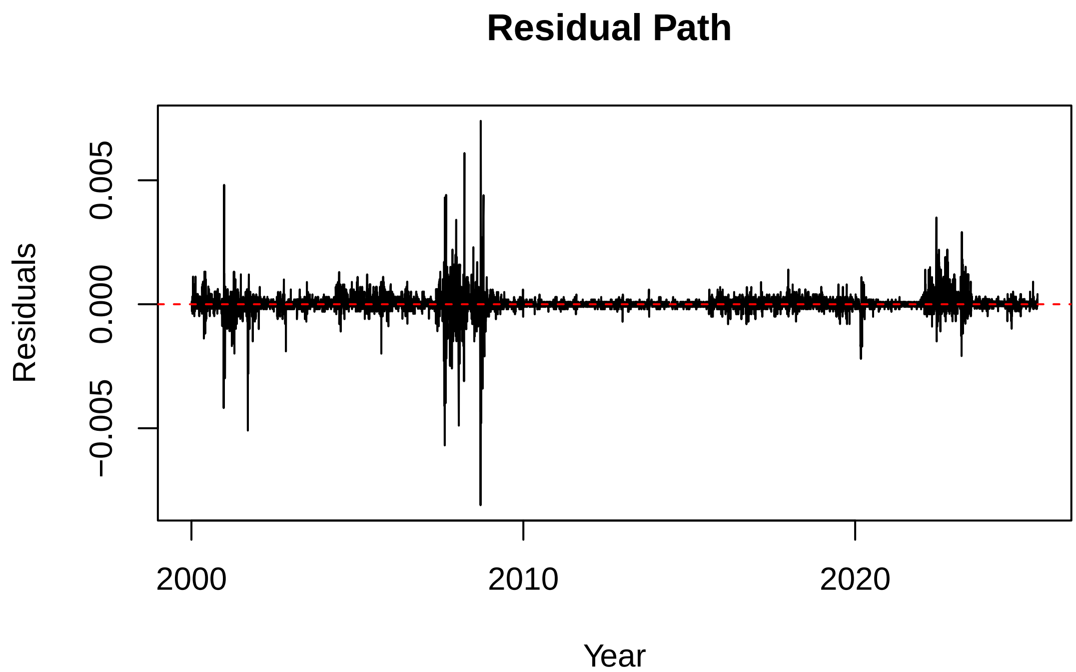
3 Empirical Evidence

Now we can simulate the Vasicek process using the estimated parameters to assess whether the model fits the empirical data well. The actual path followed by the interest rate will be highlighted.

```r
library(quantmod)
library(zoo)
library(xts)
dt=1/250
T=length(rate)*dt
N=100

dW=array(rnorm(T/dt*N,0,sqrt(dt)),dim=c(T/dt,N))
r=array(NA,dim=c(T/dt,N))

r[1, ] <- rate[1]

for (i in 2:(T/dt)){
  r[i,]=r[i-1,]+a*(b-r[i-1,])*dt+sigma_r*dW[i-1,]
}


matplot(time(rate),r,type='l',col='lightgray', xlab = 'Year', ylab = 'Interest Rate')
lines(rate, lwd=2, col='blue')
```



The last plot clearly highlights the phenomenon of heteroskedasticity, that is, when the variance of the error terms is not constant. However, this violates a fundamental assumption of the Ordinary Least Squares (OLS) methodology, and therefore more suitable statistical techniques should be taken into consideration.



Residual Path
Residuals
0.005
0.000
−0.005
2000
2010
2020
Year