微分形式を用いたラプラシアンの計算

せき

2024 年 2 月 9 日

目次

1　球座標におけるラプラシアン

1.1　基底の計算

球座標

$$x = r \sin\theta \cos\varphi \tag{1}$$

$$y = r \sin\theta \sin\varphi \tag{2}$$

$$z = r \cos\varphi \tag{3}$$

を考える．このとき，

$$\mathrm{d}x = \frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial x}{\partial \varphi}\,\mathrm{d}\varphi = \sin\theta\cos\varphi\,\mathrm{d}r + r\cos\theta\cos\varphi\,\mathrm{d}\theta - r\sin\theta\sin\varphi\,\mathrm{d}\varphi \tag{4}$$

$$\mathrm{d}y = \frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial y}{\partial \varphi}\,\mathrm{d}\varphi = \sin\theta\sin\varphi\,\mathrm{d}r + r\cos\theta\sin\varphi\,\mathrm{d}\theta + r\sin\theta\cos\varphi\,\mathrm{d}\varphi \tag{5}$$

$$\mathrm{d}z = \frac{\partial z}{\partial r}\,\mathrm{d}r + \frac{\partial z}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial z}{\partial \varphi}\,\mathrm{d}\varphi = \cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi \tag{6}$$

であるから，

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = r^2 \sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi \tag{7}$$

が得られる．ここで，以下のために作為的に，基底を

$$\mathrm{d}r\,,\, r\,\mathrm{d}\theta\,,\, r\sin\theta\,\mathrm{d}\varphi \tag{8}$$

とする．この基底にもホッジ作用素を同様に作用させることができ

$$*(\mathrm{d}r \wedge r\,\mathrm{d}\theta \wedge r\sin\theta\,\mathrm{d}\varphi) = 1 \to *(\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi) = \frac{1}{r^2\sin\theta} \tag{9}$$

$$*(\mathrm{d}r) = r\,\mathrm{d}\theta \wedge r\sin\theta\,\mathrm{d}\varphi = r^2\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \tag{10}$$

$$*(r\,\mathrm{d}\theta) = r\sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}r \to *(\mathrm{d}\theta) = \sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}r \tag{11}$$

$$*(r\sin\theta\,\mathrm{d}\varphi) = r\,\mathrm{d}r \wedge \mathrm{d}\theta \to *(\mathrm{d}\varphi) = \frac{1}{\sin\theta}\,\mathrm{d}r \wedge \mathrm{d}\theta \tag{12}$$

が得られる．

1.2　ラプラシアンの計算

球座標におけるラプラシアンは，

$$\triangle = \mathrm{div}\,\mathrm{grad} \tag{13}$$

で与えられるが，微分形式の表示においては，

$$\triangle = \mathrm{d}\,\delta + \delta\,\mathrm{d} \tag{14}$$

と書くことができる．とくに，0-form においては

$$\triangle = *\,\mathrm{d} * \mathrm{d} \tag{15}$$

と書ける．

微分形式は，外微分をどんな座標で計算しても良い．つまり，

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = \frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \varphi}\,\mathrm{d}\varphi \tag{16}$$

が成立することが知られている．このことを用い，ラプラシアンに 0-form f を作用させると，

$$\begin{aligned}
\triangle f &= *\mathrm{d} * \mathrm{d}f &(17)\\
&= *\mathrm{d} * \left(\frac{\partial f}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial f}{\partial \varphi}\,\mathrm{d}\varphi\right) &(18)\\
&= *\mathrm{d}\left(\frac{\partial f}{\partial r} r^2 \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi + \frac{\partial f}{\partial \theta}\sin\theta\,\mathrm{d}\varphi \wedge \mathrm{d}r + \frac{\partial f}{\partial \varphi}\frac{1}{\sin\theta}\,\mathrm{d}r \wedge \mathrm{d}\theta\right) &(19)\\
&= *\left\{\left(\frac{\partial^2 f}{\partial r^2} r^2 \sin\theta + 2r\sin\theta\frac{\partial f}{\partial r}\right) + \left(\frac{\partial^2 f}{\partial \theta^2}\sin\theta + \cos\theta\frac{\partial f}{\partial \theta}\right) + \left(\frac{\partial^2 f}{\partial \varphi^2}\frac{1}{\sin\theta}\right)\right\}\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi &(20)\\
&= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} &(21)
\end{aligned}$$

つまり，

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2} \tag{22}$$

が得られる．