Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

from page 5–17: if strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

10. Unconstrained minimization

- terminology and assumptions
- gradient descent method
- steepest descent method
- Newton's method
- self-concordant functions
- implementation

Unconstrained minimization

$$\text{minimize} \quad f(x)$$

- f convex, twice continuously differentiable (hence $\mathbf{dom}\, f$ open)
- we assume optimal value $p^\star = \inf_x f(x)$ is attained (and finite)

unconstrained minimization methods

- produce sequence of points $x^{(k)} \in \mathbf{dom}\, f$, $k = 0, 1, \ldots$ with

$$f(x^{(k)}) \to p^\star$$

- can be interpreted as iterative methods for solving optimality condition

$$\nabla f(x^\star) = 0$$

Initial point and sublevel set

algorithms in this chapter require a starting point $x^{(0)}$ such that

- $x^{(0)} \in \mathbf{dom}\, f$
- sublevel set $S = \{x \mid f(x) \leq f(x^{(0)})\}$ is closed

2nd condition is hard to verify, except when *all* sublevel sets are closed:

- equivalent to condition that $\mathbf{epi}\, f$ is closed
- true if $\mathbf{dom}\, f = \mathbf{R}^n$
- true if $f(x) \to \infty$ as $x \to \mathbf{bd}\,\mathbf{dom}\, f$

examples of differentiable functions with closed sublevel sets:

$$f(x) = \log(\sum_{i=1}^{m} \exp(a_i^T x + b_i)), \qquad f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

Strong convexity and implications

f is strongly convex on S if there exists an $m > 0$ such that

$$\nabla^2 f(x) \succeq mI \qquad \text{for all } x \in S$$

implications

- for $x, y \in S$,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{m}{2}\|x - y\|_2^2$$

  hence, S is bounded

- $p^\star > -\infty$, and for $x \in S$,

$$f(x) - p^\star \leq \frac{1}{2m}\|\nabla f(x)\|_2^2$$

  useful as stopping criterion (if you know m)

任何梯度足够小的点都是近似最优解

Descent methods

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)} \quad \text{with } f(x^{(k+1)}) < f(x^{(k)})$$

- other notations: $x^+ = x + t\Delta x$, $x := x + t\Delta x$
- Δx is the *step*, or *search direction*; t is the *step size*, or *step length*
- from convexity, $f(x^+) < f(x)$ implies $\nabla f(x)^T \Delta x < 0$
  ($i.e.$, Δx is a *descent direction*)

General descent method.

given a starting point $x \in \mathbf{dom}\, f$.
repeat
1. Determine a descent direction Δx.
2. *Line search.* Choose a step size $t > 0$.
3. *Update.* $x := x + t\Delta x$.

until stopping criterion is satisfied.

Line search types

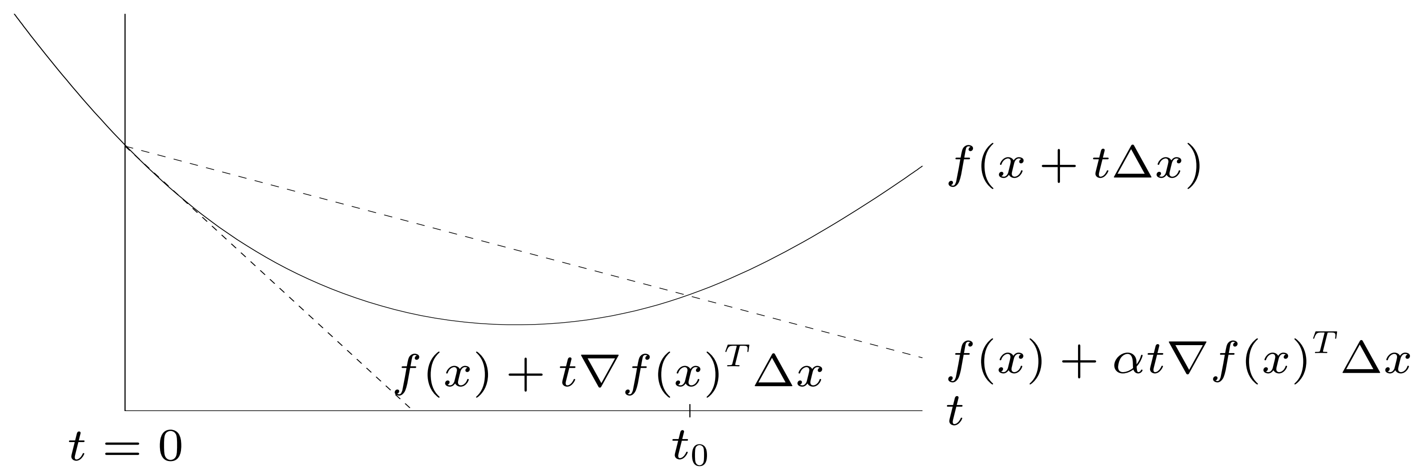
exact line search: $t = \mathrm{argmin}_{t>0} f(x + t\Delta x)$

backtracking line search (with parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$)

- starting at $t = 1$, repeat $t := \beta t$ until

$$f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$$

- graphical interpretation: backtrack until $t \leq t_0$



Gradient descent method

general descent method with $\Delta x = -\nabla f(x)$

---

given a starting point $x \in \mathbf{dom}\, f$.
repeat
1. $\Delta x := -\nabla f(x)$.
2. *Line search.* Choose step size t via exact or backtracking line search.
3. *Update.* $x := x + t\Delta x$.

until stopping criterion is satisfied.

- stopping criterion usually of the form $\|\nabla f(x)\|_2 \leq \epsilon$
- convergence result: for strongly convex f,

$$f(x^{(k)}) - p^\star \leq c^k (f(x^{(0)}) - p^\star)$$

  $c \in (0, 1)$ depends on m, $x^{(0)}$, line search type
- very simple, but often very slow; rarely used in practice

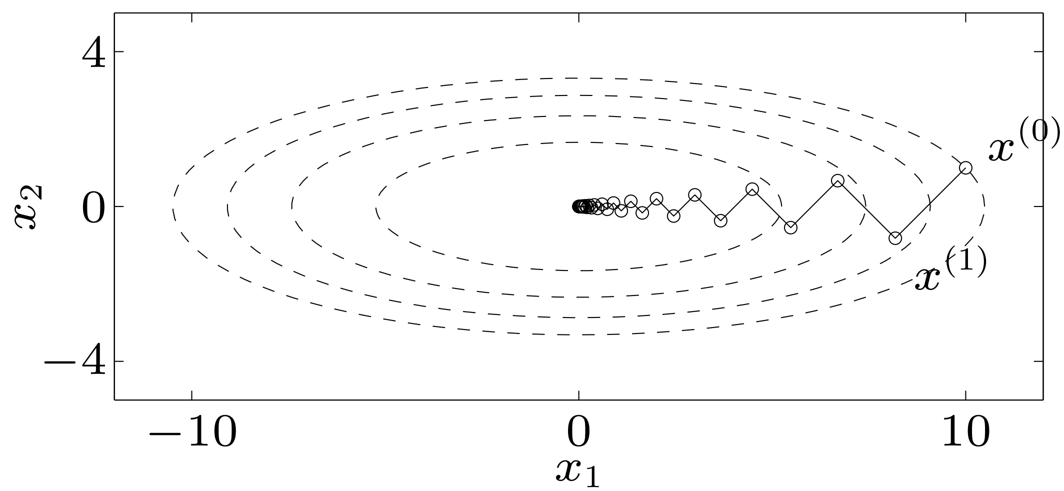
quadratic problem in $\mathbf{R}^2$

$$f(x) = (1/2)(x_1^2 + \gamma x_2^2) \qquad\qquad (\gamma > 0)$$

with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

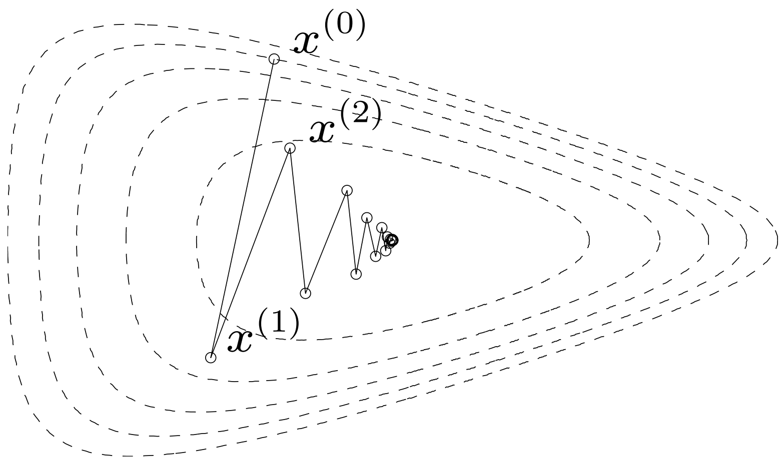
$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

- very slow if $\gamma \gg 1$ or $\gamma \ll 1$
- example for $\gamma = 10$:

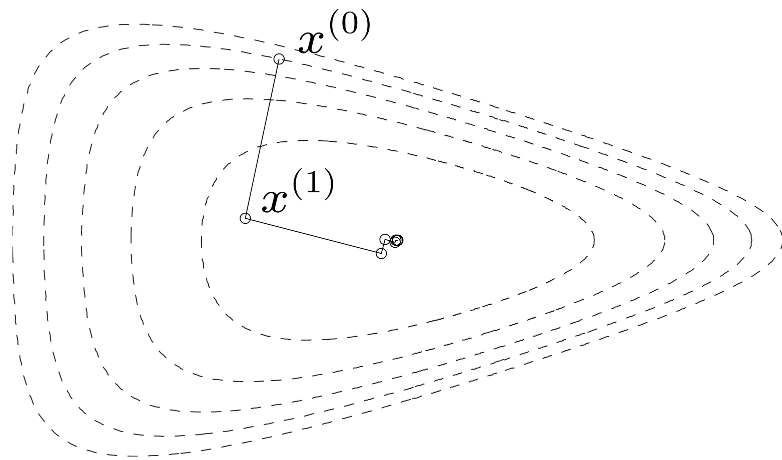


nonquadratic example

$$f(x_1, x_2) = e^{x_1+3x_2-0.1} + e^{x_1-3x_2-0.1} + e^{-x_1-0.1}$$



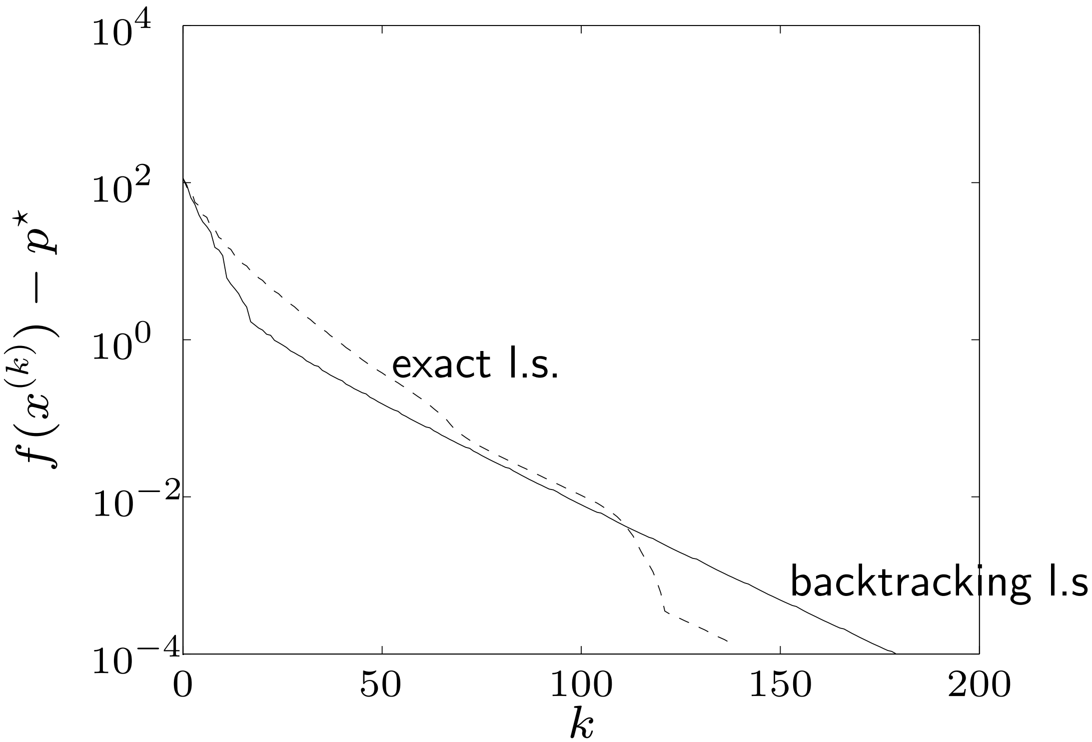
backtracking line search



exact line search

a problem in $\mathbf{R}^{100}$

$$f(x) = c^T x - \sum_{i=1}^{500} \log(b_i - a_i^T x)$$



'linear' convergence, *i.e.*, a straight line on a semilog plot

Newton step

$$\Delta x_{\rm nt} = -\nabla^2 f(x)^{-1} \nabla f(x)$$

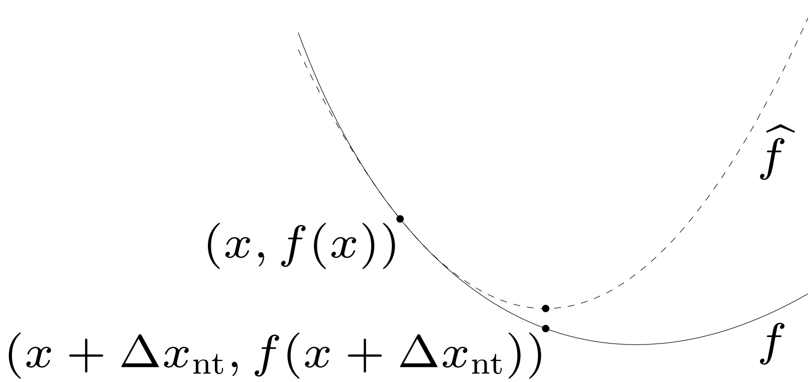
interpretations

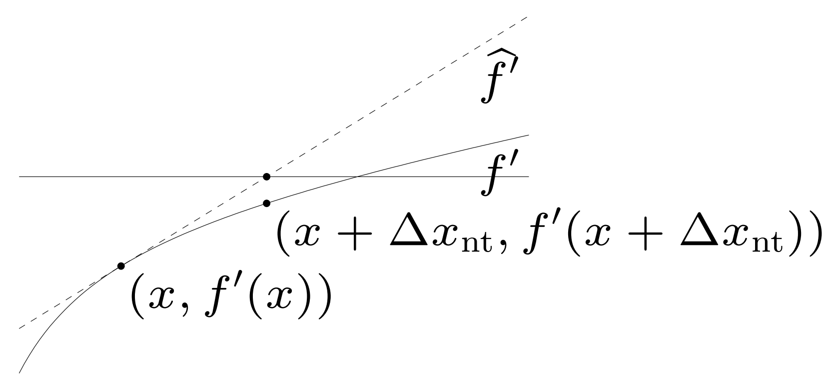
- $x + \Delta x_{\rm nt}$ minimizes second order approximation

$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$

- $x + \Delta x_{\rm nt}$ solves linearized optimality condition

$$\nabla f(x+v) \approx \nabla \widehat{f}(x+v) = \nabla f(x) + \nabla^2 f(x) v = 0$$





Newton decrement

$$\lambda(x) = \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$$

a measure of the proximity of x to $x^\star$

properties

- gives an estimate of $f(x) - p^\star$, using quadratic approximation $\widehat{f}$:

$$f(x) - \inf_y \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

- equal to the norm of the Newton step in the quadratic Hessian norm

$$\lambda(x) = \left(\Delta x_{\mathrm{nt}}^T \nabla^2 f(x) \Delta x_{\mathrm{nt}}\right)^{1/2}$$

- directional derivative in the Newton direction: $\nabla f(x)^T \Delta x_{\mathrm{nt}} = -\lambda(x)^2$
- affine invariant (unlike $\|\nabla f(x)\|_2$)

Newton's method

---

given a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
repeat

1. *Compute the Newton step and decrement.*
   $\Delta x_{\mathrm{nt}} := -\nabla^2 f(x)^{-1}\nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

---

affine invariant, $i.e.$, independent of linear changes of coordinates:

Newton iterates for $\tilde{f}(y) = f(Ty)$ with starting point $y^{(0)} = T^{-1}x^{(0)}$ are

$$y^{(k)} = T^{-1}x^{(k)}$$

11. Equality constrained minimization

- equality constrained minimization
- eliminating equality constraints
- Newton's method with equality constraints
- infeasible start Newton method
- implementation

Equality constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained

optimality conditions: $x^\star$ is optimal iff there exists a $\nu^\star$ such that

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

equality constrained quadratic minimization (with $P \in \mathbf{S}_+^n$)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

optimality condition:

$$\left[\begin{array}{cc} P & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} x^\star \\ \nu^\star \end{array}\right] = \left[\begin{array}{c} -q \\ b \end{array}\right]$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \qquad \Longrightarrow \qquad x^T Px > 0$$

- equivalent condition for nonsingularity: $P + A^T A \succ 0$

Eliminating equality constraints

represent solution of $\{x \mid Ax = b\}$ as

$$\{x \mid Ax = b\} = \{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}$$

- $\hat{x}$ is (any) particular solution
- range of $F \in \mathbf{R}^{n \times (n-p)}$ is nullspace of A ($\mathbf{rank}\, F = n - p$ and $AF = 0$)

reduced or eliminated problem

$$\text{minimize} \quad f(Fz + \hat{x})$$

- an unconstrained problem with variable $z \in \mathbf{R}^{n-p}$
- from solution $z^\star$, obtain $x^\star$ and $\nu^\star$ as

$$x^\star = Fz^\star + \hat{x}, \qquad \nu^\star = -(AA^T)^{-1}A\nabla f(x^\star)$$

example: optimal allocation with resource constraint

$$\begin{array}{ll} \text{minimize} & f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \\ \text{subject to} & x_1 + x_2 + \cdots + x_n = b \end{array}$$

eliminate $x_n = b - x_1 - \cdots - x_{n-1}$, *i.e.*, choose

$$\hat{x} = be_n, \qquad F = \begin{bmatrix} I \\ -\mathbf{1}^T \end{bmatrix} \in \mathbf{R}^{n \times (n-1)}$$

reduced problem:

$$\text{minimize} \quad f_1(x_1) + \cdots + f_{n-1}(x_{n-1}) + f_n(b - x_1 - \cdots - x_{n-1})$$

(variables $x_1, \ldots, x_{n-1}$)

Newton step

Newton step Δx_{nt} of f at feasible x is given by solution v of

$$\left[\begin{array}{cc} \nabla^2 f(x) & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} v \\ w \end{array}\right] = \left[\begin{array}{c} -\nabla f(x) \\ 0 \end{array}\right]$$

interpretations

- Δx_{nt} solves second order approximation (with variable v)

$$\begin{array}{ll} \text{minimize} & \widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v \\ \text{subject to} & A(x+v) = b \end{array}$$

- Δx_{nt} equations follow from linearizing optimality conditions

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x) v + A^T w = 0, \qquad A(x+v) = b$$

Newton decrement

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2} = \left(-\nabla f(x)^T \Delta x_{\rm nt}\right)^{1/2}$$

properties

- gives an estimate of $f(x) - p^\star$ using quadratic approximation $\widehat{f}$:

$$f(x) - \inf_{Ay=b} \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

- directional derivative in Newton direction:

$$\left.\frac{d}{dt} f(x + t\Delta x_{\rm nt})\right|_{t=0} = -\lambda(x)^2$$

- in general, $\lambda(x) \neq \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$

Newton's method with equality constraints

---

given starting point $x \in \textbf{dom}\, f$ with $Ax = b$, tolerance $\epsilon > 0$.

repeat

1. Compute the Newton step and decrement Δx_{nt}, $\lambda(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \leq \epsilon$.
3. *Line search.* Choose step size t by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.

- a feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) < f(x^{(k)})$
- affine invariant

Newton step at infeasible points

2nd interpretation of page 11–6 extends to infeasible x ($i.e.$, $Ax \neq b$)

linearizing optimality conditions at infeasible x (with $x \in \mathbf{dom}\, f$) gives

$$\left[\begin{array}{cc} \nabla^2 f(x) & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} \Delta x_{\rm nt} \\ w \end{array}\right] = -\left[\begin{array}{c} \nabla f(x) \\ Ax - b \end{array}\right] \qquad (1)$$

primal-dual interpretation

- write optimality condition as $r(y) = 0$, where

$$y = (x, \nu), \qquad r(y) = (\nabla f(x) + A^T \nu, Ax - b)$$

- linearizing $r(y) = 0$ gives $r(y + \Delta y) \approx r(y) + Dr(y)\Delta y = 0$:

$$\left[\begin{array}{cc} \nabla^2 f(x) & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} \Delta x_{\rm nt} \\ \Delta \nu_{\rm nt} \end{array}\right] = -\left[\begin{array}{c} \nabla f(x) + A^T \nu \\ Ax - b \end{array}\right]$$

  same as (1) with $w = \nu + \Delta \nu_{\rm nt}$

Infeasible start Newton method

---

given starting point $x \in \mathbf{dom}\, f$, ν, tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$.
repeat

1. Compute primal and dual Newton steps $\Delta x_{\rm nt}$, $\Delta \nu_{\rm nt}$.
2. *Backtracking line search on* $\|r\|_2$.
   $t := 1$.
   while $\|r(x + t\Delta x_{\rm nt}, \nu + t\Delta \nu_{\rm nt})\|_2 > (1 - \alpha t)\|r(x, \nu)\|_2, \quad t := \beta t.$
3. *Update.* $x := x + t\Delta x_{\rm nt}$, $\nu := \nu + t\Delta \nu_{\rm nt}$.

until $Ax = b$ and $\|r(x, \nu)\|_2 \le \epsilon$.

- not a descent method: $f(x^{(k+1)}) > f(x^{(k)})$ is possible
- directional derivative of $\|r(y)\|_2$ in direction $\Delta y = (\Delta x_{\rm nt}, \Delta \nu_{\rm nt})$ is

$$\frac{d}{dt}\|r(y + t\Delta y)\|_2 \bigg|_{t=0} = -\|r(y)\|_2$$

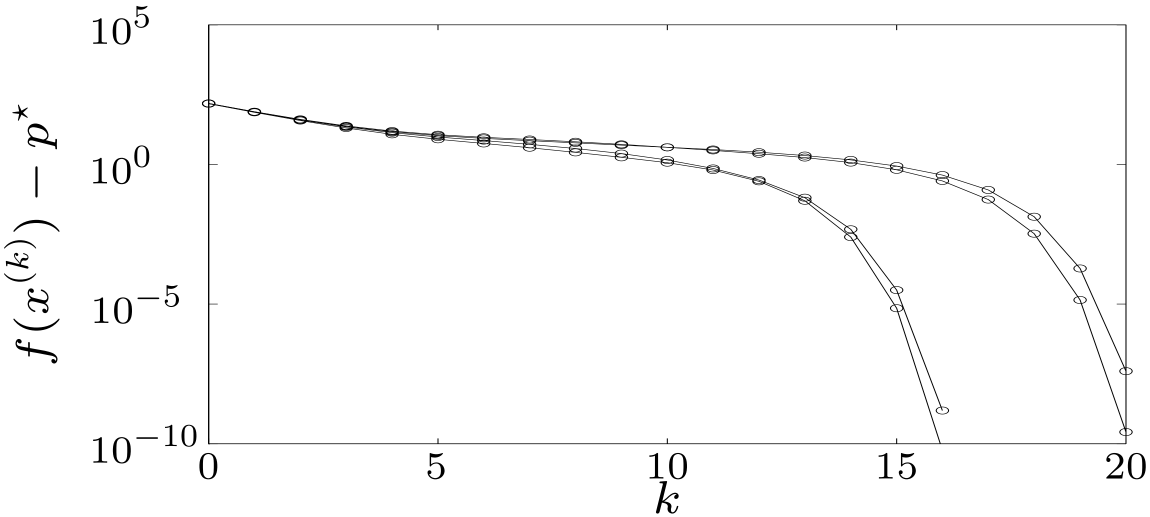
Equality constrained analytic centering

primal problem: minimize $-\sum_{i=1}^{n} \log x_i$ subject to $Ax = b$

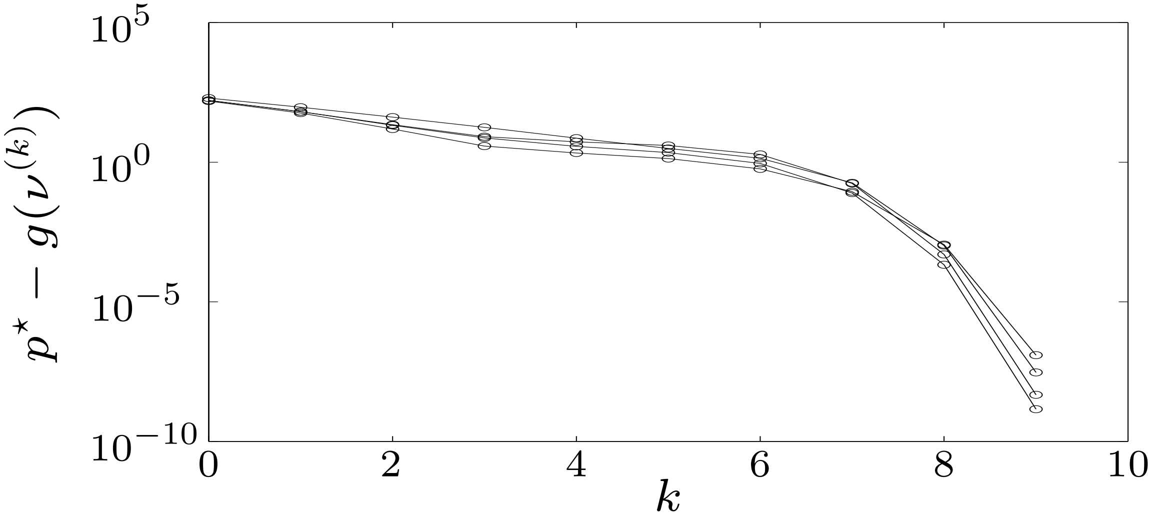
dual problem: maximize $-b^T\nu + \sum_{i=1}^{n} \log(A^T\nu)_i + n$

three methods for an example with $A \in \mathbf{R}^{100\times 500}$, different starting points

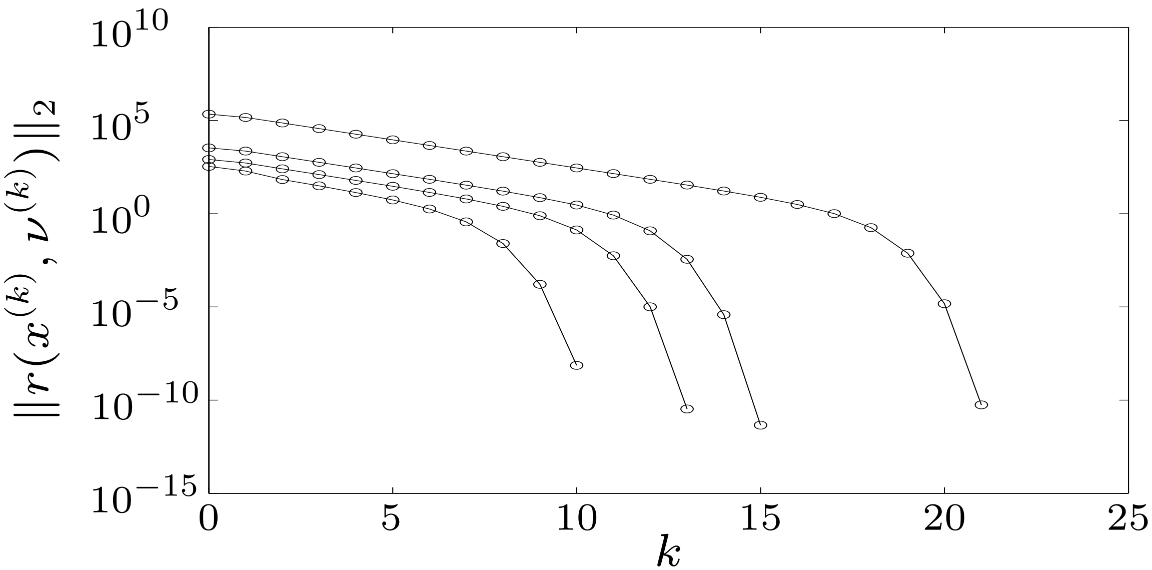
1. Newton method with equality constraints (requires $x^{(0)} \succ 0$, $Ax^{(0)} = b$)



2. Newton method applied to dual problem (requires $A^T\nu^{(0)} \succ 0$)



3. infeasible start Newton method (requires $x^{(0)} \succ 0$)



complexity per iteration of three methods is identical

1. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1}\mathbf{1} \\ 0 \end{bmatrix}$$

   reduces to solving $A\,\mathbf{diag}(x)^2A^Tw = b$

2. solve Newton system $A\,\mathbf{diag}(A^T\nu)^{-2}A^T\Delta\nu = -b + A\,\mathbf{diag}(A^T\nu)^{-1}\mathbf{1}$

3. use block elimination to solve KKT system

$$\begin{bmatrix} \mathbf{diag}(x)^{-2} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\nu \end{bmatrix} = \begin{bmatrix} \mathbf{diag}(x)^{-1}\mathbf{1} - A^T\nu \\ b - Ax \end{bmatrix}$$

   reduces to solving $A\,\mathbf{diag}(x)^2A^Tw = 2Ax - b$

conclusion: in each case, solve $ADA^Tw = h$ with D positive diagonal

12. Interior-point methods

- inequality constrained minimization
- logarithmic barrier function and central path
- barrier method
- feasibility and phase I methods
- complexity analysis via self-concordance
- generalized inequalities

Inequality constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array} \tag{1}$$

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- we assume $p^\star$ is finite and attained
- we assume problem is strictly feasible: there exists $\tilde{x}$ with

$$\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b$$

hence, strong duality holds and dual optimum is attained

Examples

- LP, QP, QCQP, GP

- entropy maximization with linear inequality constraints

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} x_i \log x_i \\ \text{subject to} & Fx \preceq g \\ & Ax = b \end{array}$$

  with $\mathbf{dom}\, f_0 = \mathbf{R}_{++}^n$

- differentiability may require reformulating the problem, $e.g.$, piecewise-linear minimization or ℓ_∞-norm approximation via LP

- SDPs and SOCPs are better handled as problems with generalized inequalities (see later)

Logarithmic barrier

reformulation of (1) via indicator function:

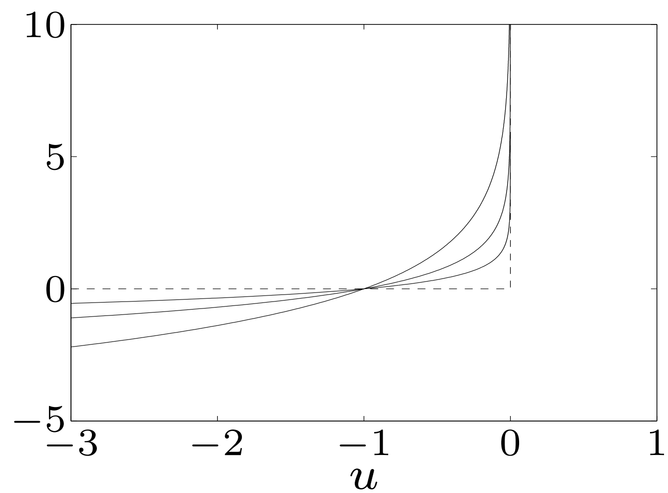
$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise (indicator function of $\mathbf{R}_-$)

approximation via logarithmic barrier

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$



logarithmic barrier function

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \quad \mathbf{dom}\,\phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- convex (follows from composition rules)
- twice continuously differentiable, with derivatives

$$\begin{aligned}
\nabla \phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\
\nabla^2 \phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\end{aligned}$$

Central path

- for $t > 0$, define $x^\star(t)$ as the solution of

$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

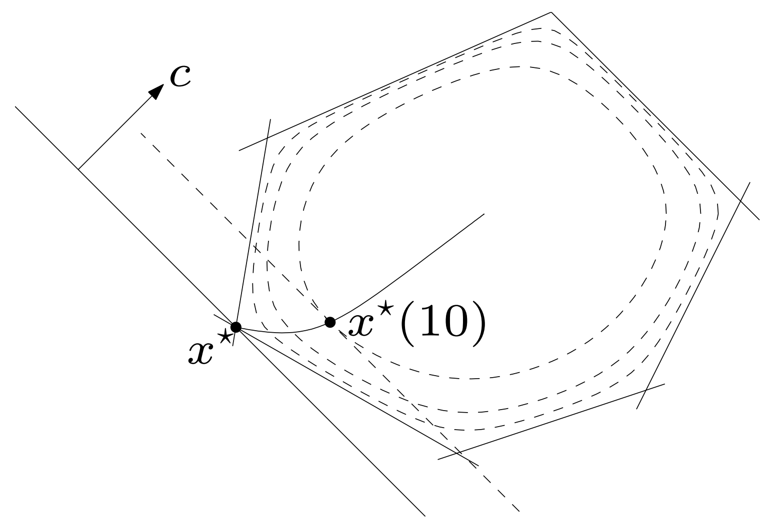
  (for now, assume $x^\star(t)$ exists and is unique for each $t > 0$)

- central path is $\{x^\star(t) \mid t > 0\}$

example: central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, 6 \end{array}$$

hyperplane $c^T x = c^T x^\star(t)$ is tangent to level curve of ϕ through $x^\star(t)$



Dual points on central path

$x = x^\star(t)$ if there exists a w such that

$$t\nabla f_0(x) + \sum_{i=1}^m \frac{1}{-f_i(x)}\nabla f_i(x) + A^T w = 0, \qquad Ax = b$$

- therefore, $x^\star(t)$ minimizes the Lagrangian

$$L(x, \lambda^\star(t), \nu^\star(t)) = f_0(x) + \sum_{i=1}^m \lambda_i^\star(t) f_i(x) + \nu^\star(t)^T(Ax - b)$$

  where we define $\lambda_i^\star(t) = 1/(-t f_i(x^\star(t))$ and $\nu^\star(t) = w/t$

- this confirms the intuitive idea that $f_0(x^\star(t)) \to p^\star$ if $t \to \infty$:

$$\begin{array}{rcl} p^\star & \geq & g(\lambda^\star(t), \nu^\star(t)) \\ & = & L(x^\star(t), \lambda^\star(t), \nu^\star(t)) \\ & = & f_0(x^\star(t)) - m/t \end{array}$$

Interpretation via KKT conditions

$x = x^\star(t)$, $\lambda = \lambda^\star(t)$, $\nu = \nu^\star(t)$ satisfy

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $Ax = b$
2. dual constraints: $\lambda \succeq 0$
3. approximate complementary slackness: $-\lambda_i f_i(x) = 1/t$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0$$

difference with KKT is that condition 3 replaces $\lambda_i f_i(x) = 0$

Barrier method

---

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.* Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. *Update.* $x := x^\star(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

- terminates with $f_0(x) - p^\star \leq \epsilon$ (stopping criterion follows from $f_0(x^\star(t)) - p^\star \leq m/t$)
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: large μ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10$–20
- several heuristics for choice of $t^{(0)}$

Convergence analysis

number of outer (centering) iterations: exactly

$$\left\lceil \frac{\log(m/(\epsilon t^{(0)}))}{\log \mu} \right\rceil$$

plus the initial centering step (to compute $x^\star(t^{(0)})$)

centering problem

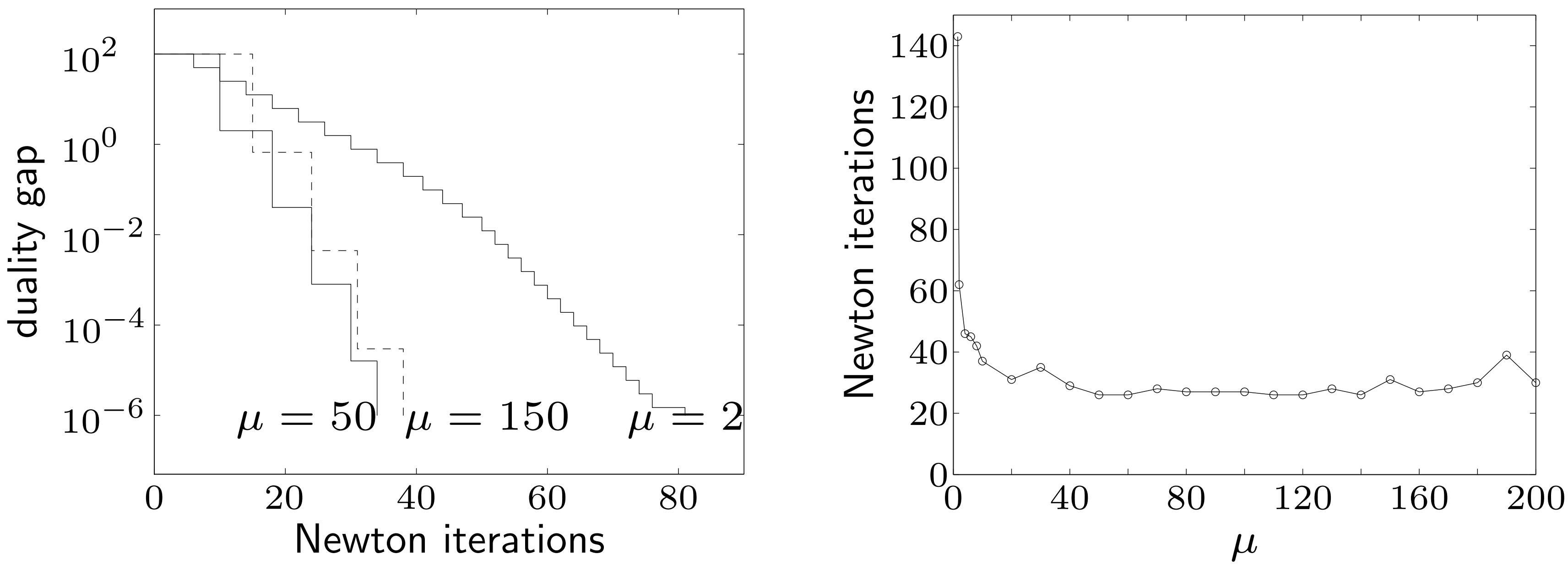
$$\text{minimize} \quad t f_0(x) + \phi(x)$$

see convergence analysis of Newton's method

- $t f_0 + \phi$ must have closed sublevel sets for $t \geq t^{(0)}$
- classical analysis requires strong convexity, Lipschitz condition
- analysis via self-concordance requires self-concordance of $t f_0 + \phi$

Examples

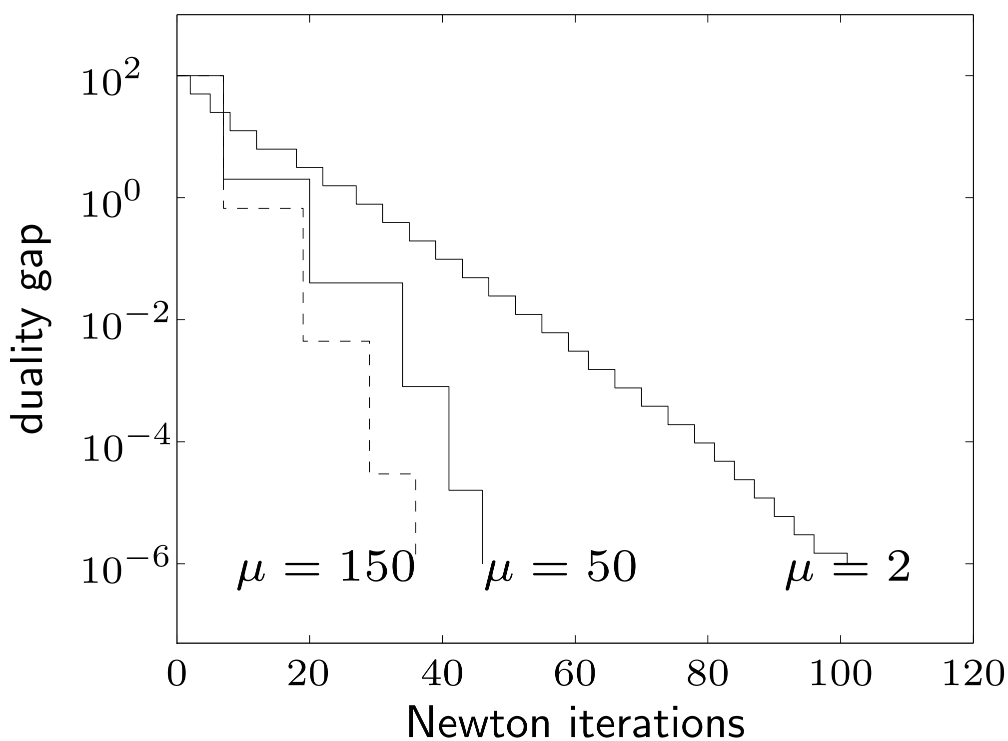
inequality form LP ($m = 100$ inequalities, $n = 50$ variables)



- starts with x on central path ($t^{(0)} = 1$, duality gap 100)
- terminates when $t = 10^8$ (gap 10^{-6})
- centering uses Newton's method with backtracking
- total number of Newton iterations not very sensitive for $\mu \geq 10$

geometric program ($m = 100$ inequalities and $n = 50$ variables)

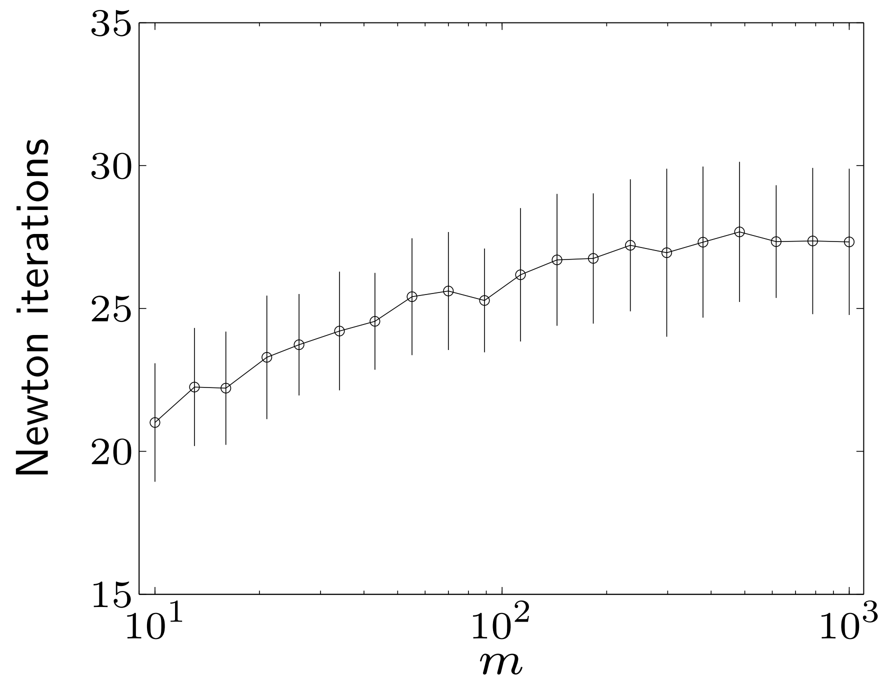
$$\begin{array}{ll} \text{minimize} & \log\left(\sum_{k=1}^{5} \exp(a_{0k}^T x + b_{0k})\right) \\ \text{subject to} & \log\left(\sum_{k=1}^{5} \exp(a_{ik}^T x + b_{ik})\right) \leq 0, \quad i = 1, \ldots, m \end{array}$$



family of standard LPs ($A \in \mathbf{R}^{m \times 2m}$)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0 \end{array}$$

$m = 10, \ldots, 1000$; for each m, solve 100 randomly generated instances



number of iterations grows very slowly as m ranges over a $100 : 1$ ratio

13. Conclusions

- main ideas of the course
- importance of modeling in optimization

Modeling

mathematical optimization

- problems in engineering design, data analysis and statistics, economics, management, . . . , can often be expressed as mathematical optimization problems
- techniques exist to take into account multiple objectives or uncertainty in the data

tractability

- roughly speaking, tractability in optimization requires convexity
- algorithms for nonconvex optimization find local (suboptimal) solutions, or are very expensive
- surprisingly many applications can be formulated as convex problems

Theoretical consequences of convexity

- local optima are global
- extensive duality theory
  - systematic way of deriving lower bounds on optimal value
  - necessary and sufficient optimality conditions
  - certificates of infeasibility
  - sensitivity analysis
- solution methods with polynomial worst-case complexity theory (with self-concordance)

Practical consequences of convexity

(most) **convex problems can be solved globally and efficiently**

- interior-point methods require $20 - 80$ steps in practice
- basic algorithms ($e.g.$, Newton, barrier method, . . .) are easy to implement and work well for small and medium size problems (larger problems if structure is exploited)
- more and more high-quality implementations of advanced algorithms and modeling tools are becoming available
- high level modeling tools like `cvx` ease modeling and problem specification

How to use convex optimization

to use convex optimization in some applied context

- use rapid prototyping, approximate modeling
  - start with simple models, small problem instances, inefficient solution methods
  - if you don't like the results, no need to expend further effort on more accurate models or efficient algorithms

- work out, simplify, and interpret optimality conditions and dual

- even if the problem is quite nonconvex, you can use convex optimization
  - in subproblems, $e.g.$, to find search direction
  - by repeatedly forming and solving a convex approximation at the current point