Report 3: Kuramoto Oscillators

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1 The Kuramoto Model

This model is motivated by the phenomenon of collective synchronization in a huge system of weakly coupled phase oscillators. The most general governing equation is given by:

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j)\right) Z(\theta_i) \quad i \in \{1, 2, ..., N\}$$

where θ_i and ω_i is the phase and natural frequency of the i^{th} oscillator, respectively. X and Z are coupling strengths and sensitivity to the coupling. The long term dynamics of such a system can be described by:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma(\theta_j - \theta_i), \quad i \in \{1, 2, ..., N\}$$

where the function Γ can be calculated from the parameters of the original equation. We choose a simple case with equally weighted sinusoidal coupling;

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} \sin(\theta_j - \theta_i)$$

where $K > 0$ is the coupling strength.

1.1 Distribution of natural frequencies

We distribute the frequencies according to a probability density $g(\omega)$ satisfying the following properties:

- $\langle g(w) \rangle = 0$
- $g(\omega) = g(-\omega)$
- Unimodal; $g(\omega)$ is decreasing on $[0, \infty)$

1.2 Order parameter

We define a complex order parameter like so:

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$$

The radius $r(t)$ measures the phase coherence, and $\psi(t)$ measures the average phase. The equation governing long-term dynamics can then be recast into the following form:

$$\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i) \quad i \in \{1, 2, ..., N\}$$

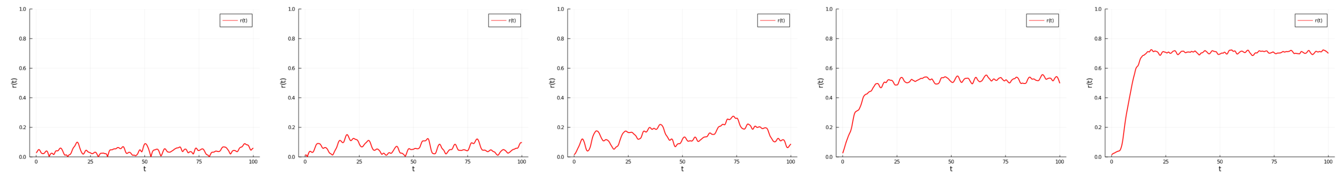
The oscillators interact with each via these mean field quantities, r and ψ. **The calculations performed by Kuramoto indicate the existence of a sudden transition in the phase coherence as we increase the coupling strength.** Observing this phenomena is the main aim of the exploration performed.

2 Analysis

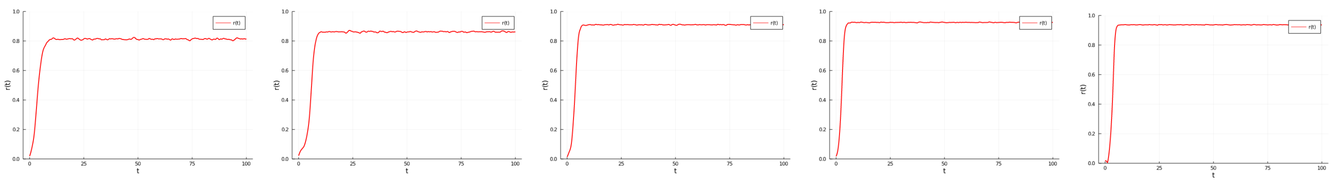
The oscillators are given random initial phases, and random natural frequencies according to a distribution function $g(\omega)$ sampled using rejection sampling. As a result, $r(t = 0)$ is close to 0 since the oscillators are incoherent at the beginning. The system is solved using a simple explicit Euler scheme.

2.1 Variation of order parameter with coupling strength

Below we have used a normal (gaussian) distribution and plotted $r(t)$ vs. t for different K. For low K values, it averages near $r = 0$ but after a specific K_c it increases and saturates near $r = 1$. The fluctuations about the mean value (r_∞) also reduces with higher coupling strength.



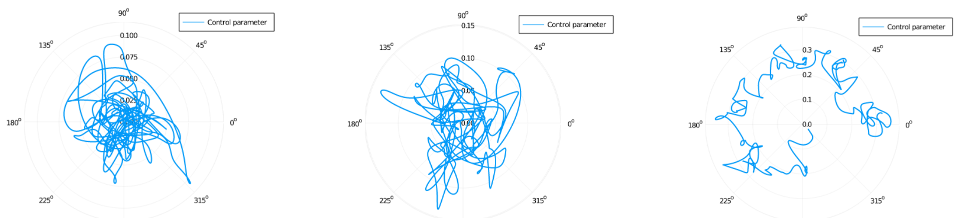
(a) K = 1.00 (b) K = 1.25 (c) K = 1.50 (d) K = 1.75 (e) K = 2.00



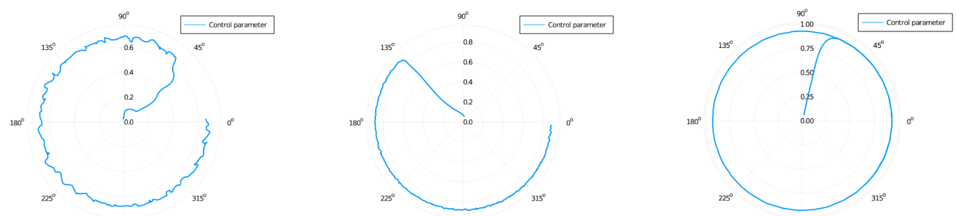
(a) K = 2.25 (b) K = 2.50 (c) K = 2.75 (d) K = 3.00 (e) K = 3.25

Figure 2: 2000 oscillators; Gaussian distribution ($\mu = 0$, $\sigma = 1$); 10^5 timesteps; dt $= 10^{-3}$

We also plot the complete order parameter $r(t)e^{i\psi(t)}$ as polar co-ordinates. It is clear that for $K < K_c$ the coupling is weak and there is no phase coherence. But near K_c, some coupling is observed as seen in (Fig. 3(c)) and eventually a stable orbit is attained in the (r, ψ) polar plane indicating the synchronization of the oscillators.



(a) K = 0.5 (b) K = 1.0 (c) K = 1.5



(a) K = 2.0 (b) K = 2.5 (c) K = 3.0

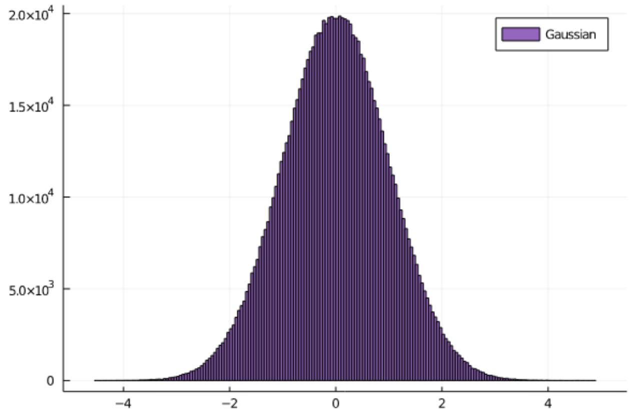
Figure 4: 1000 oscillators; Gaussian distribution ($\mu = 0$, $\sigma = 1$); $1.5 \cdot 10^5$ timesteps; dt $= 10^{-3}$

2.2 Variation of r_∞ with coupling strength K

The theoretical calculations tell us that for a uni-modal distribution $g(\omega)$ of the natural frequencies, we can calculate the transition value K_c below which the oscillators do not synchronize:

$$K_c = \frac{2}{\pi g(0)}$$

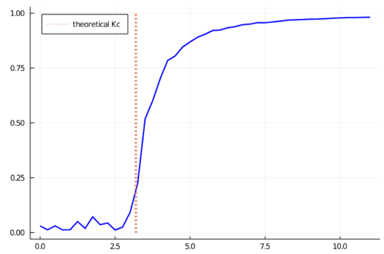
2.3 Gaussian distribution



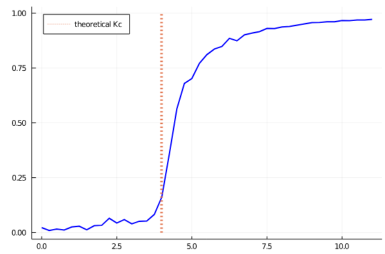
(a) Distribution shape

$$g(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \implies K_c = \sigma\sqrt{\frac{8}{\pi}}$$

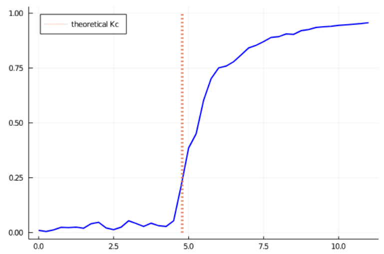
Below we have the plots of r_∞ vs. K for various values of σ of the distribution frequencies. The red line denotes the theoretical prediction of the transition point, K_c. There seems to be fair agreement between the observed and predicted value. (r_∞ is measured by taking the average of $r(t)$ over the last quarter of the run.)



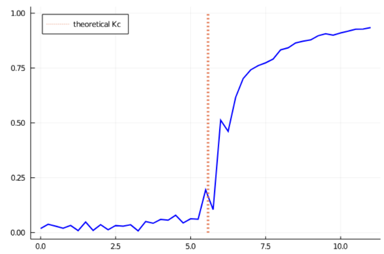
(a) $\sigma = 2.0$



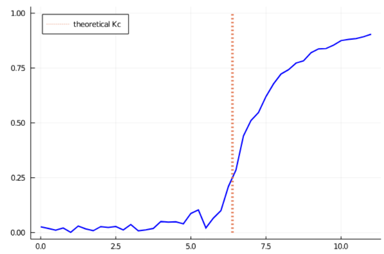
(b) $\sigma = 2.5$



(a) $\sigma = 3.0$



(b) $\sigma = 3.5$



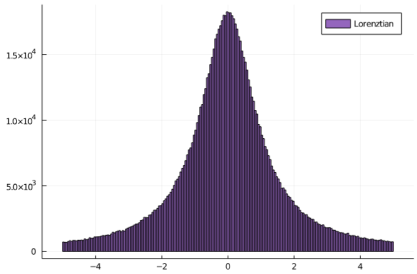
(c) $\sigma = 4.0$

Figure 7: 2500 oscillators; Gaussian distribution ($\mu = 0$); $5 \cdot 10^4$ timesteps; dt $= 10^{-3}$

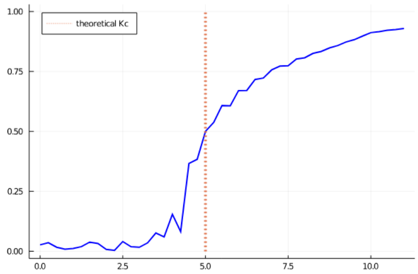
2.4 Lorentzian distribution

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)} \implies K_c = 2\gamma$$

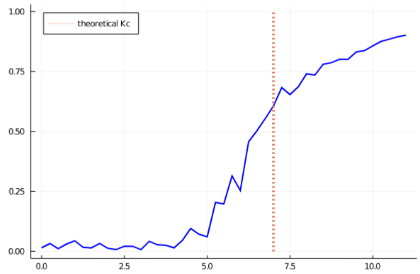
In this case, the distribution is characterized by its half-width at half-maximum parameter, γ. We see that the typical transition curve is still observed, but the predicted K_c are significantly deviating from the observed value.



(a) Distribution shape



(b) $\gamma = 2.5$



(c) $\gamma = 3.5$

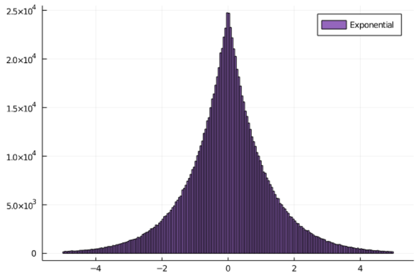
Figure 8: 2500 oscillators; Lorentzian distribution ($\mu = 0$); $0.5 \cdot 10^5$ timesteps; dt $= 10^{-3}$

2.5 Exponential distribution

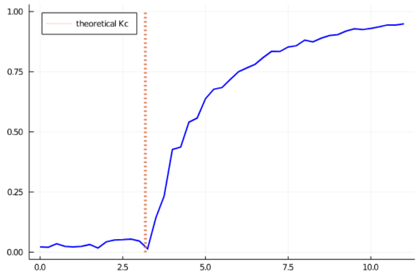
Here we have constructed a symmetric exponential distribution which is also uni-modal and satisfies the other conditions on $g(\omega)$.

$$g(\omega) = \frac{1}{2a} \exp\left(-\frac{|x|}{a}\right) \implies K_c = \frac{4\pi}{a}$$

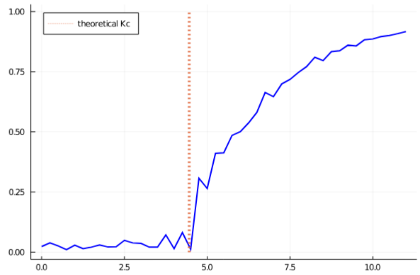
In this case we see that the prediction of K_c is almost exactly the same value as we observe from the simulation runs.



(a) Distribution shape



(b) $a = 2.5$



(c) $a = 3.5$

Figure 9: 2500 oscillators; $e^{-k|x|}$ distribution ($\mu = 0$); $0.5 \cdot 10^5$ timesteps; dt $= 10^{-3}$

3 Conclusion

We have numerically simulated a system of coupled oscillators and studied the transition from random drifting to complete synchronization by changing the coupling strength. We have verified that this characteristic transition is observed for gaussian, lorentzian and exponential distributions of natural frequencies. Furthermore, we also noted that our predicted values of K_c were fairly good even though many approximations were made to obtain it.

4 Implementation

The model was implemented and analyzed in Julia 1.4.2 and the plots were made using Images.jl and Plots.jl. The relevant Jupyter notebook can be found here:
`https://github.com/20akshay00/ModellingComplexSystems/`