IDC621 - Modelling Complex Systems

Report 1: Power law distributions in Sand-pile Model

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This report records an attempt at probing the sand-pile model which exhibits self-organized criticality. We aim to measure certain quantities and check for power law distributions which is a documented characteristic of this particular model.

1 Sand-pile Model

The Sand-pile model is a Cellular Automata in which we keep track of numbers on a grid. We will study the 2D version in this report. Each node carries a non-negative integer $z_{i,j}$, which can be interpreted as average height or slope of a sand-pile. There is a critical value z_c such that when $z_{i,j} \geq z_c$, the pile at that point topples onto its neighbours according to the following rule:

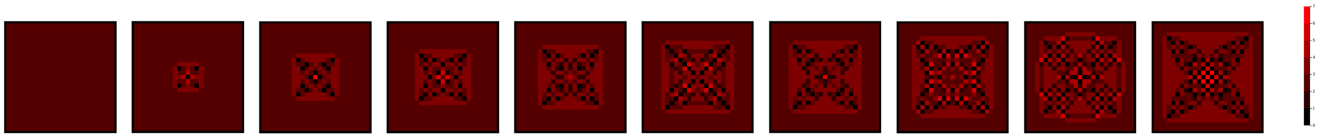
$$z_{i,j}(t+1) = z_{i,j}(t+1) - 4 \qquad \text{if } z_{i,j}(t) >= z_c$$

$$z_{a,b}(t+1) = z_{a,b}(t) + 1 \qquad \forall\ (a,b) \in \{(i+1,j),\ (i-1,j),\ (i,j+1),\ (i,j-1)\}$$

We can externally drive the system by adding a drop of sand at a particular node (i,j) like so:

$$z_{i,j}(t+1) = z_{i,j}(t) + 1$$

Snapshots of a sample run are shown below, where a drop of sand is added in the middle every time-step (external driving).



(a) t = 0 (b) t = 25 (c) t = 50 (d) t = 75 (e) t = 100 (f) t = 125 (g) t = 150 (h) t = 175 (i) t = 200 (j) t = 225

Figure 1: 25x25 grid; $z_c = 4$; Initial Conditions: All nodes = 2; Boundaries = 0 (closed)

For the rest of the analysis, the following parameters are fixed:

- **Critical $\mathbf{z}_c$** : 4
- **Boundary condition:** Closed; all boundaries set to z=0
- **Initial condition:** Random; $0 \leq z < z_c$

2 Analysis

We begin with a random initial condition of the grid and drop a single sand-drop on a random node. The system is then evolved until the perturbation ceases (all topples have occurred) after which a new sand-drop is added on another random node and so on. Each time a sand-drop is added, a few quantities are sampled which are expected to exhibit a power law distribution which will be linear in a log-log plot.

$$D(q) \propto q^{-\alpha}$$

$$\log_{10}(D(q)) \propto -\alpha \cdot \log_{10}(q)$$

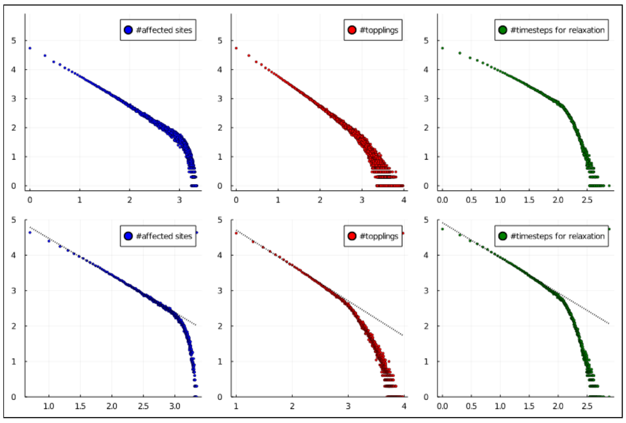
2.1 Quantities measured

The following quantities are measured during each run, whose distributions we wish to analyze.

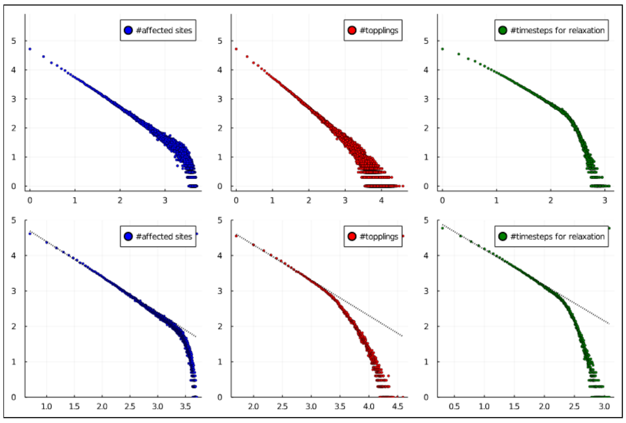
- **Number of sites (area affected)** during the avalanche.
- **Total number of topples** induced during the avalanche.
- **Relaxation time** for the perturbation to die out.

2.2 Results

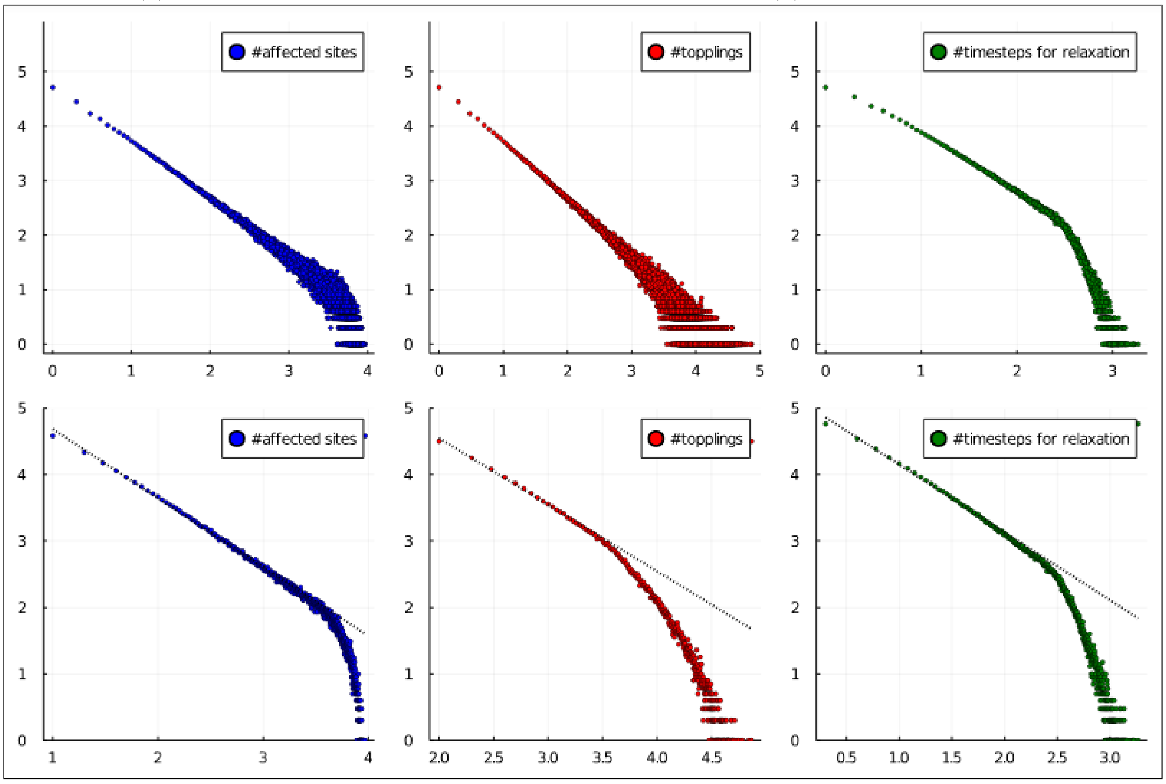
The following plots are the distributions of the 3 quantities. The first row consists of the **raw unbinned data**, and the second row is the data **plotted in $\sqrt{N}$ bins for N data-points**. All the plots are log-log (base 10).



(a) 50x50 grid; 10^6 points



(b) 75x75 grid; 10^6 points



(c) 100x100 grid; 10^6 points

Figure 2: $z_0 = 4$, Initial Conditions: Random; Boundaries = 0 (closed)

2.3 Observations

We note that there is a clear linear (power law) regime for the first 2-3 generations after which the curve drops exponentially due to under-sampling of the larger values. As the grid size is increased, the exponential deviation occurs at a later stage. The linear regimes of the plots were fit (least sq.), and these are the measured slopes of the curves;

50x50 grid:	**75x75 grid:**	**100x100 grid:**
(1) -1.048 ± 0.005	(1) -0.998 ± 0.007	(1) -1.037 ± 0.005
(2) -0.999 ± 0.010	(2) -0.998 ± 0.012	(2) -1.003 ± 0.011
(3) -0.990 ± 0.009	(3) -1.008 ± 0.011	(3) -1.019 ± 0.010

These are the exponents for the power laws of the three quantities we have measured. All the exponents are seen to be fairly close to -1. We also see that roughly the same power law is observed regardless of grid size. The average values of the exponents are as follows;

- **(1) Number of sites (area affected):** -1.027 ± 0.017
- **(2) Total number of topples:** -1.000 ± 0.033
- **(3) Relaxation time:** -1.006 ± 0.030

3 Conclusion

Three power-law distributions were observed for various quantities measured in the evolution of a sand-pile model in response to a small perturbation. The linear regimes of these curves were fitted and the power-law exponent was found to be ≈ -1 for the 2D model.

3.1 Implementation

The model was implemented and analyzed in Julia 1.4.2 and the plots were made using Images.jl and Plots.jl. The relevant Jupyter notebook can be found here:
`https://github.com/20akshay00/ModellingComplexSystems/`