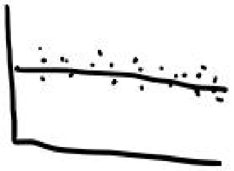
Question_6-HW1

Monday, September 30, 2024 10:20 AM

a) implication: horizontal line $y = \alpha$



because $E(e_i) = 0$

so mean $e_i = 0$

if $\beta = 0$ $y_i = \alpha + e_i$

b) $y_i = \alpha + e_i$ derivative = minimize

least square estimator minimizes SSR

$S(a) = \sum_{i=1}^{n} (y_i - a)^2 \rightarrow SSR$

$\frac{d}{da} = \sum_{i=1}^{n} (y_i - a)^2 = -2/2 \sum_{i=1}^{n} (y_i - a) = 0/2$

$\sum_{i=1}^{n} y_i - na = 0 \quad \sum_{i=1}^{n} y_i = na$

$$\boxed{a = \frac{1}{n} \sum_{i=1}^{n} y_i = \bar{y}}$$

c) $a = 1/n \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (\alpha + e_i)$

take expectation to see if unbiased

$E(a) = \frac{1}{n} \sum_{i=1}^{n} E(\alpha + e_i) = \frac{1}{n} \sum_{i=1}^{n} (\alpha + E(e_i)$

$E(e_i) = 0 \rightarrow \boxed{E(a) = \frac{1}{n} \sum_{i=1}^{n} \alpha = \alpha}$

d) $V(a) = V(\frac{1}{n} \sum_{i=1}^{n} (\alpha + e_i))$

$V(\alpha) = 0 \rightarrow V(\frac{1}{n} \sum_{i=1}^{n} e_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 \rightarrow \boxed{\frac{\sigma^2}{n}}$

e) The estimates are normally distributed by the Central Limit Theorem, which states that with enough n values (a lot of them) $\bar{Y}$ follows a normal distribution, as long as they are independent and identically distributed.

f) i) unbiased $\rightarrow E(\hat{\alpha}) = \alpha$

$\hookrightarrow E\left(\sum_{i=1}^{n} c_i \underline{y_i}\right) = \sum_{i=1}^{n} c_i \underline{E(y_i)} = \sum_{i=1}^{n} c_i \left(\underline{\alpha + E(e_i)}\right) = \alpha \sum_{i=1}^{n} c_i = 1$

implies sum of $C = 1$

ii) $\sum_{i=1}^{n} d_i/n = 0 \quad \hat{\alpha} = \sum_{i=1}^{n} c_i y_i$

$d_i = c_i - 1/n$

$E\left(\sum_{i=1}^{n} c_i y_i\right) = \alpha \quad \sum_{i=1}^{n} c_i \alpha = \alpha \rightarrow \sum_{i=1}^{n} c_i = 1 \rightarrow c_i = 1/n + d_i$

$\sum_{i=1}^{n} \left(\frac{1}{n} + d_i\right) = 1 \rightarrow \sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} d_i = 1$

$\sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1 \quad \sum_{i=1}^{n} d_i = 0$

$\boxed{\sum_{i=1}^{n} d_i/n = 1/n \sum_{i=1}^{n} d_i = 0}$

iii) $y_i = \alpha + e_i$

$V(\hat{\alpha}) = V\left(\sum_{i=1}^{n} c_i y_i\right) \rightarrow V\left(\sum_{i=1}^{n} c_i (\alpha + e_i)\right)$

$\hat{\alpha} = \alpha \underbrace{\sum c_i}_{1} + \sum_{i=1}^{n} c_i e_i$

$V\left(\sum_{i=1}^{n} c_i e_i\right) \rightarrow \sum_{i=1}^{n} c_i^2 \, V(e_i)$ $\rightarrow \sigma^2$

$\sigma^2 \sum_{i=1}^{n} \underline{(d_i + 1/n)^2} = \underline{\cancel{d_i^2}^{=0} + \cancel{2d_i/n}^{=0} + 1/n^2}$

$\sigma^2 \sum_{i=1}^{n} 1/n^2 = \sigma^2 \cdot \cancel{n} \cdot 1/n^2 = \frac{\sigma^2}{n}$

$\boxed{\text{variance is minimized when } V(\hat{\alpha}) = \frac{\sigma^2}{n}}$