Autocallable structured products

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Introduction

Autocallables are structured products that were first issued in 2003 by BNP Paribas in the US. Since then, they have become a very popular type of financial instruments. These products aim to furnish investors with potential periodic income, all the while guaranteeing the issuer the ability to redeem the product based on specific conditions.

It integrates elements of structured notes, featuring periodic coupon payments and an autocallable provision. Indeed, in their most general form, autocallables are instruments that pay conditional coupons and that are redeemed when an *autocall event* is triggered. Such events are linked to a given underlying or a basket of underlyings. Considering their structure, autocallables allow yield-enhancement of this underlying (or this basket of underlyings). That has participated in their popularity in the context of low interest rates and therefore low yields that has marked the past decade.

In the classical case, it pays a coupon when an underlying asset S breaches a barrier named the coupon barrier (CB). In such case, the investor gets his initial investment back and and an additional coupon. The autocall barrier (AB) determines the level of the underlying asset S above which the product is automatically redeemed by the issuer. In these circumstances, the autocall barrier defines the autocall events, as defined above. Therefore, if the autocall barrier is equal to the coupon barrier, as soon as the underlying exceeds it, a coupon is paid to the owner of the autocallable and the instrument is redeemed, potentially before maturity. These autocall tests are conducted periodically (for instance every 3 months) so that if the autocall barrier is not reached, the coupons generally accumulate until the next period. This mechanism is summarized in Figure 1. However, in some cases, in particular for the instrument called Phoenix autocallable, the autocall barrier may be strictly greater than the coupon barrier so that the instrument can pay periodic coupons and stop and start again without being redeemed.

In general, if the coupon barrier is never breached throughout the lifetime of the product, the investor might loose part of the principal. To reduce the price of the structured products, a short out-of-the-money put option is often embedded in autocallables. This might also be a down-and-in put option as discussed at the end. That is why the owner of the instrument might suffer a loss in case of drastic decrease in the underlying S and why autocallables do not offer unconditional principal protection.

Autocallable products are especially designed for investors who have an uncertain view on an underlying asset. It should not be necessarily bearish or bullish. Such products could be appealing for investors looking for additional yield in a low interest rate environment and income-generating potential as the coupons are generally high compared to traditional fixed-income securities. It could be also used by investors looking for conditional principal protection while being exposed to potential market gains. The autocall feature provides a level of downside protection, while still allowing participation in positive market movements.

Three types of scenarios are generally observed while using autocallable products:

- If the underlying increases enough, the (AB) barrier is breached. The structured ceases and the investor gets his investment back and an additional coupon.
- If the underlying breaches the (CB) barrier but remains under the (AB) barrier, the investor receives a coupon but the structure pursues.

- The worst-case scenario happens when the underlying falls down. In such case, the investor risks to not get the whole value of his initial investment back at maturity.

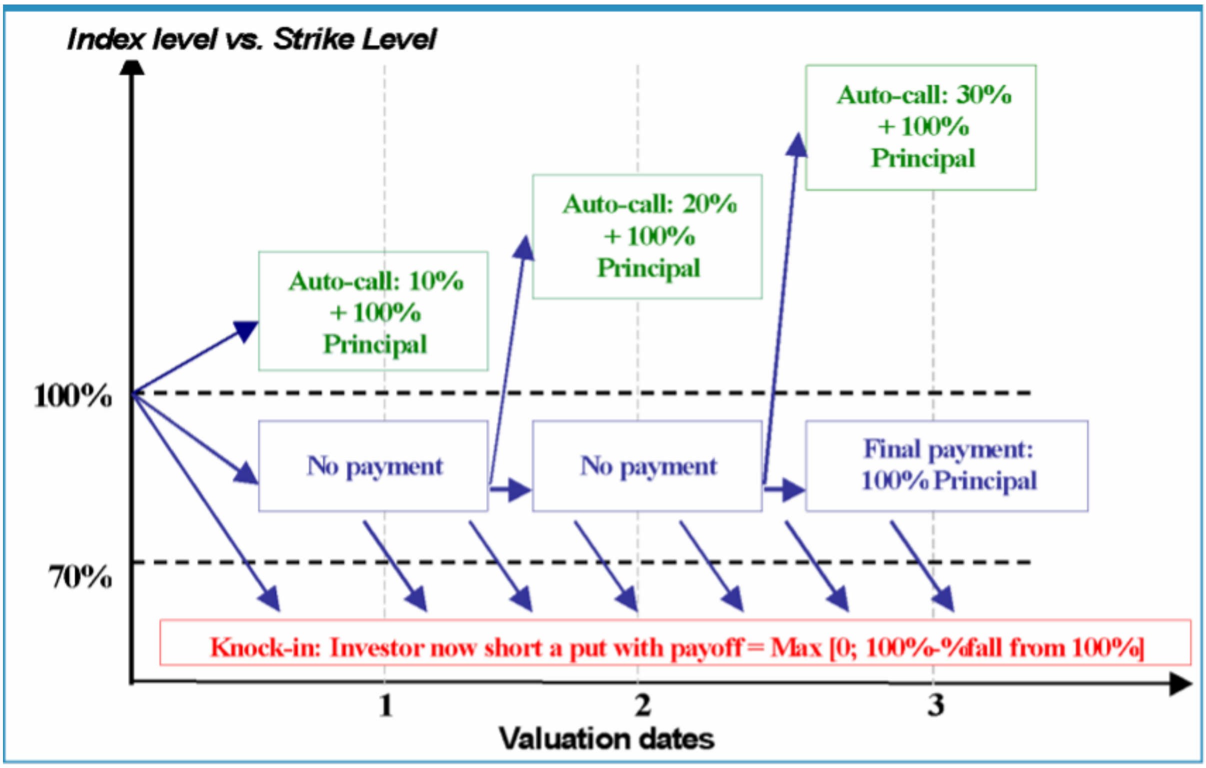

Figure 1: Payoff of an autocallable under different scenarios (source: slides of Aymeric Kalife)

1 Features of autocallable products

In the below, we consider the case we have described where the performance of the autocallable is indexed over a single underlying, namely equity, and the coupon payments and autocalls are characterized by barriers (CB) and (AB). In such case, there exist different factors featuring autocallable products:

- **Barrier levels**: the levels of the barriers (AB) and (CB) should be determined and could vary with the time. The condition (AB) $\leq$ (CB) should be satisfied (otherwise the instrument would be called before paying coupons, which would not make sense for the investor).
- **The choice of the underlying asset**.
- **The value of the coupon**. It can be fixed or not. It might depend on the performance of the asset for instance. It is by definition high in order to allow yield-enhancement.
- **The maturity date**: usually, if the autocall feature is not triggered before maturity (typically some years), the investor gets his investment back with potentially an additional coupon but more generally a partial loss of the principal.
- **Risk of loss**: if the underlying asset's price falls significantly and trigger levels are not reached, investors may not receive the principal protection provided by the autocall feature. The level of the underlying below which loss occurs is often characterized by the embedded put option we introduced earlier.
- **Frequency of autocall tests**: The period between each observation of the underlying asset level (S) is determined initially. Typically, it could be days, months or even years.

In the following, we will give a more precise mathematical definition of such products, and explain how these parameters impact their use.

2 Valuation and greeks

For pricing and risk analysis, we consider an autocallable structured products with fixed barriers (CB) and (AB), linked to a single underlying and with a fixed conditional coupon (that can accumulate). For simplicity, instead of embedding a short put option, when the underlying never breaches the barrier throughout the lifetime of the product, the investors pays the negative performance of the underlying at expiry. That is to say, he gets back his initial principal minus an amount proportional to the decrease in the underlying. This is actually equivalent to an embedded European put with strike 100% and therefore corresponds to a rather bullish investment in comparison with what we have described above. In that case, the instrument is of course path-dependent but only depends on the value of the underlying at the fixed observation dates. More precisely, we take the following parameters:

- (AB) = (CB) = 100% of the spot at time $t = 0$. The reference value to which the underlying is compared at each observation date is therefore constant and equal to its initial value. The barriers (AB) and (CB) are at the same level meaning that the structure automatically ceases when the investor receives a coupon.
- Maturity T is one year and the observation dates are each trimester, which means there are four of them.
- The coupon is $q = 5\%$ so that if the autocallable is not redeemed, it escalates to 5%, 10%, 15% and 20% over the periods. In other words, the value of the coupon increases while getting closer to the expiration date. Indeed, the probability of breaching the (AB) barrier decreases at each observation date, and so with the time.

2.1 Composition and payoff

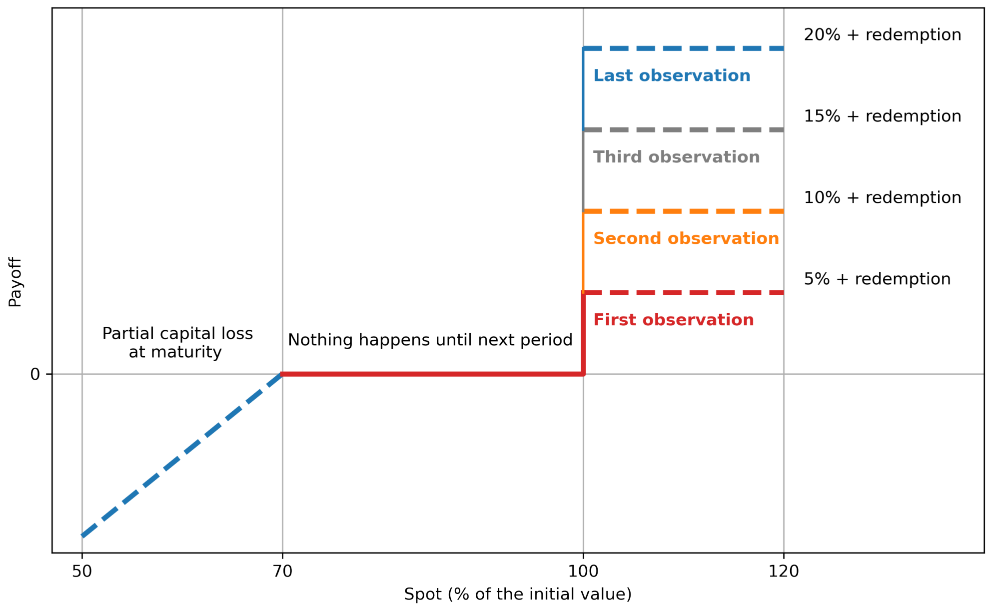

Figure 2: Payoff profile of the instrument under different scenarios (source: authors)

The autocallable described is therefore a composition of:

- Four long European binary options (*cash-or-nothing*) expiring respectively at each observation date.
- A short up-and-out call, which allows the issuer to terminate the instrument if the underlying exceeds the autocall barrier.

- A short European ATM put option, as explained above.

The discounted payoff can be decomposed in two terms:

$$\text{Payoff}_{\text{AC}} = \text{Coupon} + \text{Redemption}$$

Noting t_i the first observation date at which the underlying breaches the barrier (AB) = (CB) (equal to ∞ if it never does before expiry), the terms have the following expressions:

$$\text{Coupon} = \begin{cases} e^{-rt_i} * \text{Notional} * (i * C) \text{ if } t_i \leq T \\ 0 \text{ if } t_i = \infty \end{cases}$$

where r is the risk-free rate (assumed to be constant) and C is the predetermined coupon ($C = 10\%$ here).

$$\text{Redemption} = \begin{cases} e^{-rt_i} * \text{Notional if } t_i \leq T \\ e^{-rT} * \text{Notional} * \frac{S_T}{S_0} \text{ if } t_i = \infty \end{cases}$$

The payoff at maturity (either t_i if it exists or T) is summarized in Figure 2.

2.2 Monte-Carlo approach under Black-Scholes and Heston models

Considering the complexity of the instrument, we do not have an analytical solution to price it. We need to price it with a Monte-Carlo method. The expression of the payoff being straightforward (as given above), the main challenge relies in the model used to simulate the trajectories of the underlying. In practice, this kind of products are often priced with stochastic volatility models, calibrated with market data. We implemented two versions:

- One uses Black-Scholes model. It provides the exact formula

  $$S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$$

  where W is a Brownian motion under the risk-neutral probability. Consequently, it allows to simulate the underlying only at the observation dates and is much less computationally-intensive.

- The second one relies on Heston's model and the corresponding diffusion process:

  $$\begin{cases} dS_t = S_t\left(rdt + \sqrt{V_t}dW_t^s\right) \\ dV_t = \kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}dW_t^v \end{cases}$$

  where W^s and W^v are two Wiener processes under the risk-neutral probability with correlation ρ. This time, we need to simulate the trajectories at a greater number of time steps, as part of an Euler-Maruyama scheme. More precisely, considering the initial values (S_0, V_0), the value of the process is simulated at time t_k with the relation:

  $$\begin{cases} V_{t_k} = \kappa\left(\theta - V_{t_{k-1}}\right)(t_k - t_{k-1}) + \sigma_v\sqrt{V_{t_{k-1}}}\left(W_{t_k}^v - W_{t_{k-1}}^v\right) \\ S_{t_k} = S_{t_{k-1}}\left[r(t_k - t_{k-1}) + \sqrt{V_{t_{k-1}}}\left(W_{t_k}^s - W_{t_{k-1}}^s\right)\right] \end{cases}$$

  We first need to simulate the two correlated Brownian motions. This can be done using the Cholesky decomposition of their covariance matrix:

  $$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}\begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix}$$

  This means that if we simulate two independent Wiener process B^1 and B^2, then $W^s := B^1$ and $W^v := \rho B^1 + \sqrt{1-\rho^2}B^2$ are indeed two Wiener processes themselves, with correlation ρ. Once that done, the above recurrence relation therefore allows us to simulate V_{t_k} and S_{t_k} at each time step.

Both methods give very similar results. However, since they are not parameterized the same way, it is difficult to compare the results properly and particularly the sensitivity of the result with respect to the different parameters. These are summarized in the following table. In the absence of market data, the parameters are chosen quite arbitrarily and do not necessarily reflect reality. This is particularly true for Heston's model. To be able to compare both, we decided at least to impose that the Black-Scholes volatility is equal to the long-term volatility in the latter. What is more, we fixed the notional to 100 so that the price of the instrument is expressed as a percentage of the notional.

<table>
<tr><th>Parameter</th><th>Value in Black-Scholes model</th><th>Value in Heston model</th></tr>
<tr><td>Notional</td><td colspan="2">100</td></tr>
<tr><td>Spot (S_0)</td><td colspan="2">100</td></tr>
<tr><td>Barrier ($B = S_0$)</td><td colspan="2">100</td></tr>
<tr><td>Coupon (C)</td><td colspan="2">5% (with snow-balling effect)</td></tr>
<tr><td>Maturity (T)</td><td colspan="2">1 year</td></tr>
<tr><td>Number of observation dates</td><td colspan="2">4 (each trimester)</td></tr>
<tr><td>Risk-free rate (r)</td><td colspan="2">3%</td></tr>
<tr><td>Volatility (σ)</td><td>40%</td><td></td></tr>
<tr><td>Long term volatility (θ)</td><td></td><td>40%</td></tr>
<tr><td>Volatility of volatility (σ_v)</td><td></td><td>3%</td></tr>
<tr><td>Speed of mean reversion (κ)</td><td></td><td>10</td></tr>
<tr><td>Correlation (ρ)</td><td></td><td>-60%</td></tr>
<tr><td>Number of time steps</td><td>4</td><td>40</td></tr>
<tr><td>Number of simulations</td><td colspan="2">10^5</td></tr>
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Our code can be found here. We comment below the main results given by these algorithms.

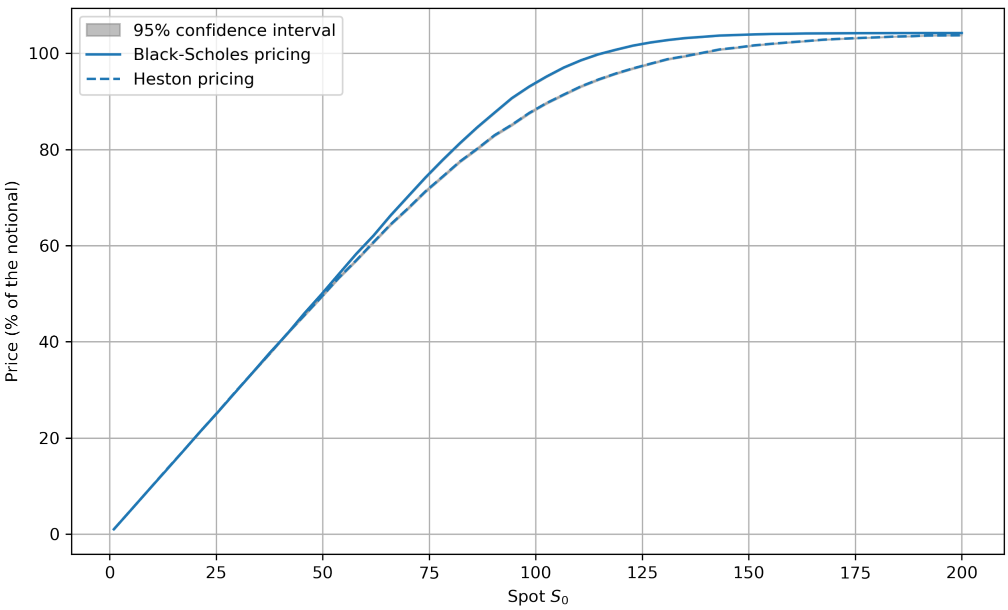

Figure 3: Price against spot S_0 (source: authors)

The two models yield similar results and a high level of accuracy since the 95% confidence intervals are small with regard the price. In particular, the price is linear in the spot when it is much smaller than the barrier (equal to 100) and constant when it is much greater in both cases. This is quite expected. One can notice for instance that when the spot is high compared to the barrier, the probability that the instrument does not early terminate becomes negligible. In this context, the payoff is predetermined which gives the plateau on the right-hand side.

As we are especially interested in the case where $S_0 = 100 = B$, it seems critical to investigate further the behaviour of the price with respect to the volatility under this assumption. This is shown in Figure 4.

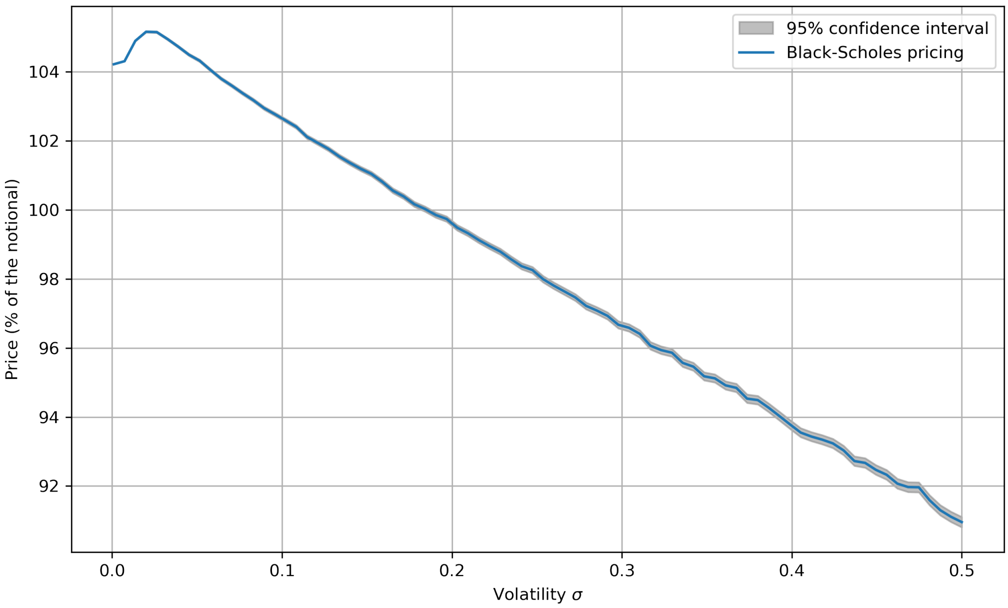

Figure 4: Price sensitivity to the volatility σ under Black-Scholes model (source: authors)

2.3 Greeks with the one-step-survival approach

As explained in the article *A Monte Carlo algorithm for autocallables that allows for stable differentiation*[1], computing the greeks of autocallables is a ill-posed problem, namely: usual algorithms with finite differences return unstable results. They furthermore propose a new algorithm under Black-Scholes model which yields very similar but smoother results than our first algorithm under Black-Scholes model and allows for stable differentiation. The idea consists in estimating analytically the probability that the underlying remains below the barrier at each observation date. As such, only the trajectories of the underlying such that the product does not early terminates are simulated. This is made possible by the fact that the payoff is predetermined when the product early terminates at observation date t_i, unlike the payoff in case the underlying never breaches the barrier. As shown below, this new algorithm enabled us to compute the Delta and the Vega of the instrument (by finite differences).

We obtained the Δ against the spot as shown in Figure 5. This results is fully consistent with the evolution of the price with respect to the spot that we have already provided in Figure 3. Remarkably, the Δ at initiation is close to 1 when the spot is low with regard the barrier and 0 when it is high. However, it is worth noticing that the former value changes a lot with the parameters. It is actually around $\frac{\text{notional}}{\text{barrier}}$ for low values of S_0, which is indeed equal to 1 in our example. In particular, when the spot S_0 is equal to barrier, the Δ at initiation is close to $\frac{\text{notional}}{2S_0}$.

We also show the $\mathcal{V}$ against the volatility σ in Figure 6. It confirms that except for really low values of σ, the owner of such autocallable is short the volatility in that case. What is more, this chart allows to argue that the Vega convexity $\frac{\partial \mathcal{V}}{\partial \sigma}$, or Vomma, is very small at usual levels of volatility $(15\% - 30\%)$.

[1] T.Alm et al., Journal of Computational Finance (2013), link

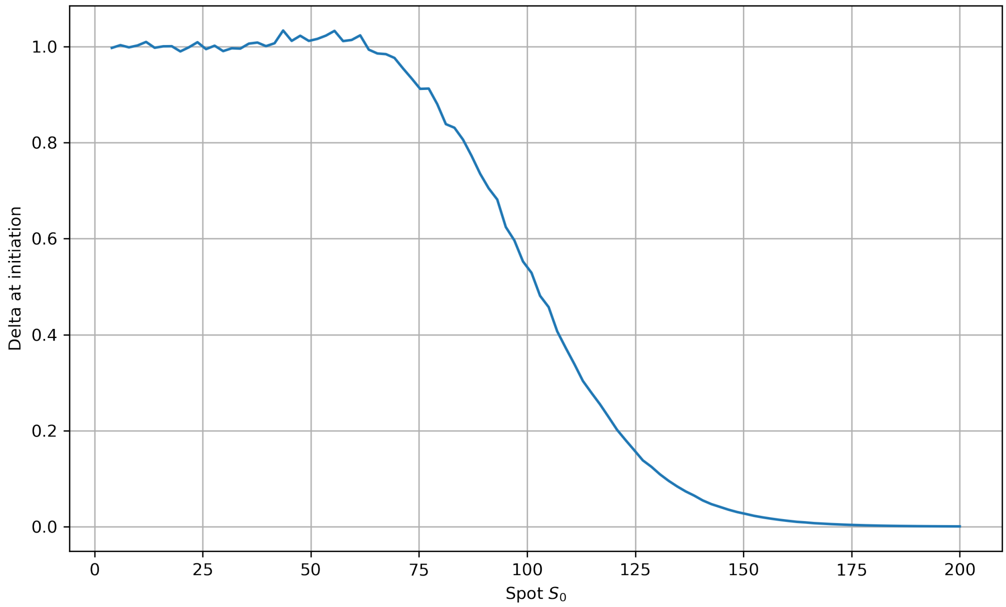

Figure 5: Delta Δ against the spot S_0 under Black-Scholes model (source: authors)

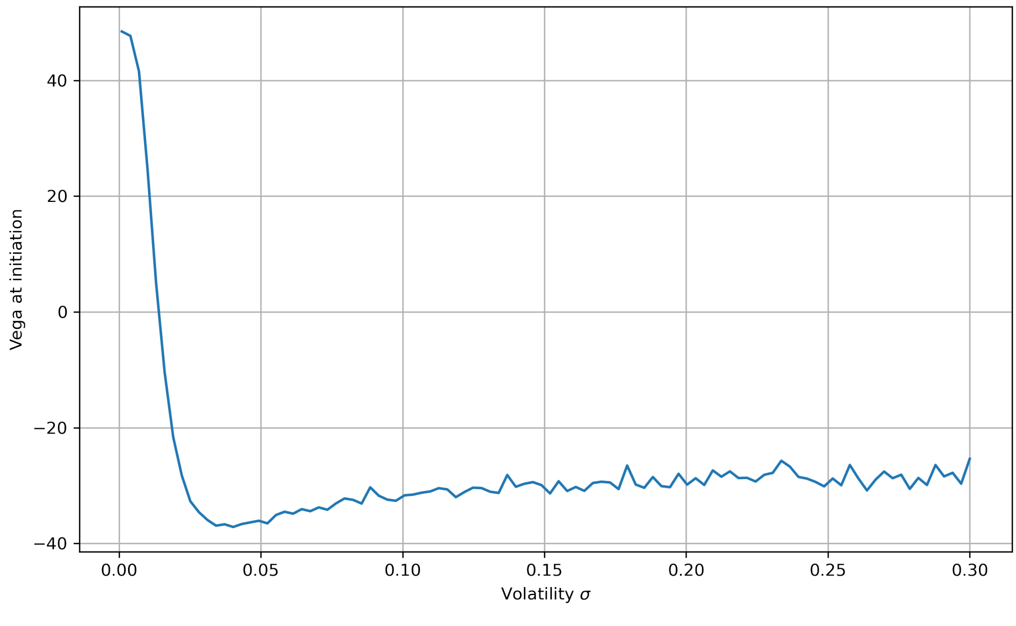

Figure 6: Vega $\mathcal{V}$ against the spot S_0 under Black-Scholes model (source: authors)

2.4 Risk analysis

It is not straightforward at all to analyse qualitatively the greeks of autocallables.

- Concerning the forward, the investor does not want the underlying to increase too fast since the best possible outcome for him is that it breaches the barrier as late as possible. However, considering the possible loss of principal when the underlying never breaches the barrier, the investor is in general still long the forward.
- Concerning the Vega, the binary options are long the volatility at initiation but the redemption at expiry in case of no early termination (which generally takes the form of a short put option) is obviously short the volatility (for the investor). As confirmed by our Vega curve above, the overall product is indeed short the volatility at initiation since it is the payoff at expiry which weights the most on the sensitivity to the volatility. However, this might evolve a lot throughout the lifetime of the product. Close to an observation date and for high values of the spot S_t, the Vega gets closer to 0 because the product is very likely to autocall, meaning the payoff at expiry has less impact.
- On the seller side, he is short the skew with respect to the binary options and long the skew regarding the redemption at expiry (allegedly a put here) so that his overall position with regard the skew is short in general.

3 Other possible features of autocallables

Autocallables may have very diverse compositions that can differ significantly from the product we have studied. We introduce here two common features of autocallables that we have not tackled in our analysis. The first one is about the form of the payoff at maturity when the barrier has not been breached. The second one explains that this kind of instruments can actually be indexed on a basket of underlyings rather than a single one. Knowing the correlation between these assets, the algorithm we have used above can adapt to these features.

3.1 Short down-and-in put

In the general case, it is common for the investor to sell a down-and-in put. In case of no early termination, the investor would then recover his initial principal minus the payoff of this put. The "Put Down-and-In" (PDI) or barrier put option is a contract that grants its holder the possibility – and not the obligation – to sell an underlying asset at a predetermined price on a future date (at maturity), provided that the underlying asset has fallen below a certain level during the option's lifetime. Using such option brings more complexity to the autocallable product.

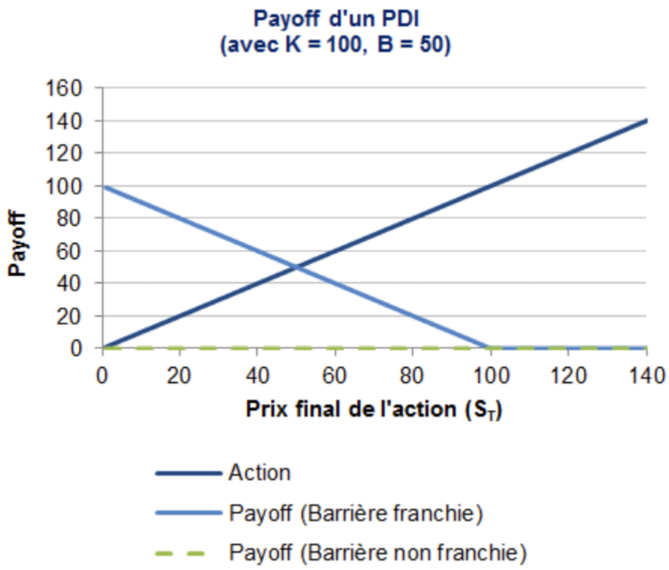

Figure 7: Payoff of a Put Down-and-In option

This type of short put options puts the investor into lower market exposure compared to the payoff-at-expiry we have investigated earlier, since it can be activated only if the asset has fallen below a given barrier (typically 70% of the spot S_0) throughout the lifetime of the product (the put must have the same maturity as the autocallable). In terms of pricing, this option also adds some complexity since one needs to check whether the knock-in barrier is breached at any time and not only at the observation dates of the autocallable.

3.2 Worst-of-basket

Autocallable products also exist in the context of multi-assets $\left(S^i\right)_{i=1,\ldots,I}$. In that case, the payoff remains the same but only the worst asset's performance is taken into account at each observation. At any observation date t_i, the coupon is delivered to the investor when the worst performing underlying asset breaches the (CB) barrier and automatically called when it breaches (AB) too. The product is redeemed at maturity T at a price depending of the worst performing asset.

Let us define:

$$\text{Worst}(t_j) = \min_{i=1,\ldots,I} S^i(t_j)$$

- If $(CB) \leq \text{Worst}(t_j) < (AB)$ the investor gets a coupon.
- If $\text{Worst}(t_j) \geq (AB)$, the product is redeemed and the holder gets the principal back.
- If $\text{Worst}(t_j) < (CB)$, nothing happens until the next observation date.

At maturity date T, if the autocallable has not been redeemed:

- If $(CB) \leq \text{Worst}(t_j)$ the investor gets the principal back and an additional coupon.
- If $\text{Worst}(t_j) < (CB)$ receives no coupon and might loose part of his principal.

Worst of basket autocallables often offer higher coupon payments compared to traditional autocallables. This is because the worst-performing asset scenario allows for a potentially higher barrier level, increasing the likelihood of the product autocalling and delivering coupons to investors. By accepting the additional risk associated with the worst-performing asset, investors may seek higher potential returns.

Conclusion

Autocallable products offer a wide range of features which can meet the requirements of investors and issuers. They are meant to investors with an uncertain view of the market and allow yield-enhancement in return for a possible loss of principal. As a matter of fact, it only offers conditional principal protection.

After introducing the notion and the context in which autocallable products are used, different Monte Carlo pricing methods were implemented. These methods were based on Black-Scholes and Heston models and allowed to have a first overview of the sensitivities of the price with respect to different parameters. In the general case, computing Greeks for such products could be hard to handle as the embedded binary options are strongly non-differentiable and lead to instabilities and numerical errors. However, one of the implemented algorithm was specifically adapted to allow stable differentiation. The results it yielded enabled us to confirm part of the theoretical knowledge we have on autocallables, particularly regarding volatility.