Asian options pricing

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15 Januray 2024

Introduction

An Asian option is any option with payoff of the form:

$$(S_t)_{t\in[0,T]} \mapsto g\left(S_T, A_T\right) \quad \text{with} \quad A_T := \frac{1}{T}\int_0^T S_u du$$

where $(S_t)_t$ denotes the trajectory of the underlying and T is the maturity of the option. For instance, a *fixed-strike* Asian call has $g(x,a) := e^{-rT}\left[a-K\right]_+$ where K denotes the strike and a *floatting-strike* Asian call has $g(s,a) := e^{-rT}\left[s-a\right]_+$. In the first case, the option is exercised by its owner if the underlying has lied above the strike **on average** throughout its lifetime. In the second case, it is worth exercising it when the underlying is above its average value at expiry.

This work studies different pricing techniques for Asian options and will tackle **only fixed-strike Asian calls** for simplicity. Furthermore, we will use Black-Scholes model. In that context, simulating S_T is straightforward and the real challenge consists in simulating A_T. That is why our developments for fixed-strike Asian calls adapt directly to any Asian option of the form given above.

We first introduce and implement different Monte-Carlo approaches as developed by Lambert et al. [3] and B. Bouchard [1]. We then compare them with a PDE approach as presented by Rogers et al. [4]. Our code can be found here.

Assumptions and notations

We consider the case of an arbitrage-free complete market and note $\mathbb{Q}$ the corresponding risk-neutral measure. Therefore, the true price of the option is given by:

$$C := e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[[A_T - K]_+\right]$$

where r is the risk-free interest rate, assumed to be constant. As specified above, we also use Black-Scholes model, which yields:

$$\forall t \in [0,T],\ S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\} \tag{BS}$$

where $W := (W_t)_{t\in[0,T]}$ denotes a Wiener process under $\mathbb{Q}$.

In the context of Monte-Carlo methods, we call **a scheme** a random variable $\bar{A}_T$ made to approximate the quantity A_T for a given set of parameters. Therefore, provided

$\{\bar{A}^i_T\}_{i=1,\dots,n}$ are independent and identically distributed (i.i.d.) copies of $\bar{A}_T$, the Monte-Carlo estimate of the fixed-strike Asian call we get is:

$$\theta_n := e^{-rT} \sum_{i=1}^{n} \left[\bar{A}^i_T - K\right]_+ \xrightarrow[n\to\infty]{\mathbb{P}} e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[[\bar{A}_T - K]_+\right]$$

For $m \in \mathbb{N}^*$ time steps, we note:

- $t_0, \dots, t_m$ the regular subdivision of $[0, T]$, whose mesh is thus $h = \frac{T}{m}$.
- $\bar{W}^m_{t_0}, \dots, \bar{W}^m_{t_m}$ a realization of W at these time steps.
- $\bar{S}^m_{t_0}, \dots, \bar{S}^m_{t_m}$ the corresponding realization of the underlying, obtained with (BS).
- $\mathcal{B}_h$ the σ-field generated by $\{S_{t_0}, \dots, S_{t_m}\}$.

1 Naive approach

The most basic Monte-Carlo approach to the problem consists in approximating the integral of the underlying over its trajectory by a Riemann sum. This is our first scheme:

$$A_T \approx \bar{A}^{r,m}_T := \frac{h}{T} \sum_{k=0}^{m-1} \bar{S}^m_{t_k} = \frac{1}{m} \sum_{k=0}^{m-1} \bar{S}^m_{t_k} \tag{1}$$

Figure 1 shows the result of this method for different values of K, T and σ where we took $n = 10,000$ and $m = 100$. One can already notice that it tallies with the common idea that Asian calls are cheaper than European calls for a given set of parameters. Also, quite intuitively, the gap widens when σ and T grow: the smaller both of them are, the closer to the initial value the underlying will remain, leading to $A_T \approx S_T \approx S_0$ on average.

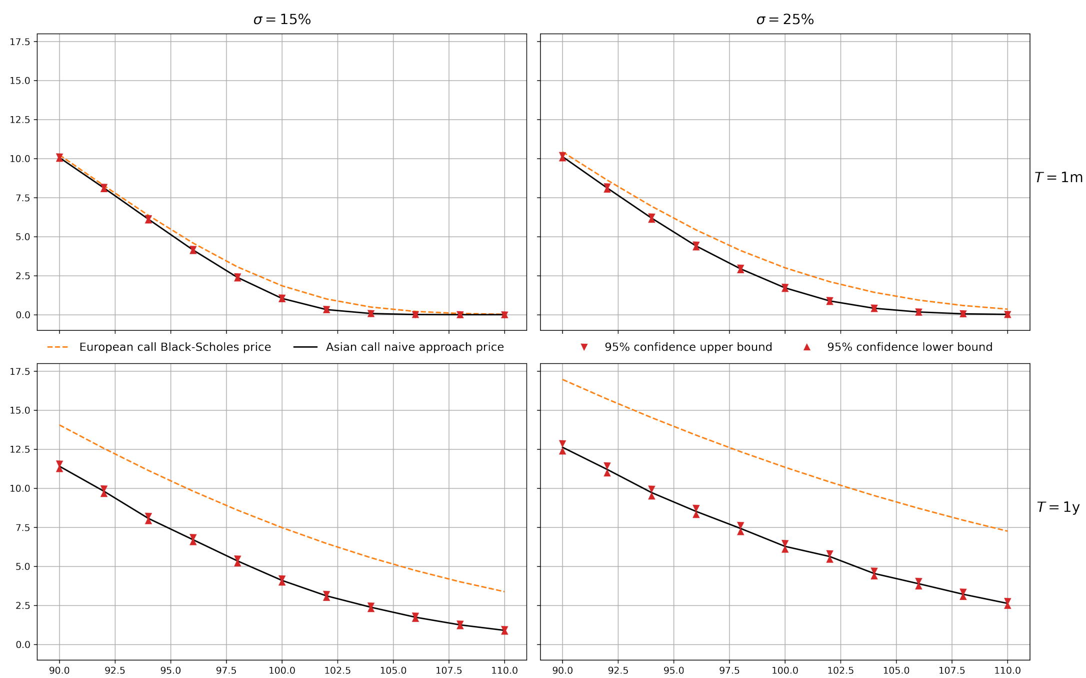


Figure 1: Fixed-strike Asian call naive pricing for different values of K, σ and T

However, these charts, particularly the one for $\sigma = 25\%$ and $T = 1$y, reveal some fluctuations with a greater magnitude than the length of the 95% confidence intervals plotted in red. This stems from the bias introduced by the approximation of A_T. This seems to be the main flaw of the naive approach: even if the variance already looks satisfactory, the bias increases rapidly with the parameters, which yields imprecise results.

The following models all give similar charts. In the below, we rather focus on the graphical representation of the convergence properties of each scheme. As suggested in the previous paragraph, this is the main challenge that the following approaches mean to address.

2 Improved Monte-Carlo approaches

2.1 Two finer approximations

A first improvement of the above approach proposes to better use the information provided by the simulation $\bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m$ to approximate the integral A_T. It relies on the fact that once the trajectory has been simulated, the best estimation of the price is:

$$\bar{C}^m := e^{-rT}\mathbb{E}\left[[A_T - K]_+ \mid \mathcal{B}_h\right]$$

By the tower property, the expectation (estimated by a Monte-Carlo method with n trajectories) of this conditional expectation is the price of the Asian call. At this stage, this quantity is not known either. [3] introduces the following simplification (S):

$$\begin{aligned}
\bar{C}^m &\approx e^{-rT}\left[\mathbb{E}\left[A_T \mid \mathcal{B}_h\right] - K\right]_+ \qquad \text{(S)}\\
&= e^{-rT}\left[\frac{1}{T}\sum_{k=0}^{m-1}\mathbb{E}\left[\int_{t_k}^{t_{k+1}} S_u du \;\Big|\; \mathcal{B}_h\right] - K\right]_+\\
&= e^{-rT}\left[\frac{1}{T}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\mathbb{E}\left[S_u \mid \mathcal{B}_h\right] du - K\right]_+\\
&= e^{-rT}\left[\frac{1}{T}\sum_{k=0}^{m-1}\bar{S}_{t_k}^m\int_{t_k}^{t_{k+1}}\mathbb{E}\left[e^{\left(r-\frac{\sigma^2}{2}\right)(u-t_k)+\sigma\left(W_t - W_{t_k}\right)} \mid \mathcal{B}_h\right] du - K\right]_+
\end{aligned}$$

We need to further simplify this expression. Let us considere the function f defined as:

$$f : (t, w) \mapsto \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma w\right\}$$

A first-order Taylor expansion gives (by Itô's lemma):

$$f(t, W_t) \underset{t\to 0}{\approx} 1 + \partial_t f(0,0)t + \partial_w f(0,0)W_t + \frac{1}{2}\partial_{ww}^2 f(0,0)t = 1 + rt + \sigma W_t$$

It leads to the additional approximation below for all $k \in \{0, \ldots, m-1\}$:

$$\int_{t_k}^{t_{k+1}}\mathbb{E}\left[S_u \mid \mathcal{B}_h\right] du \approx \bar{S}_{t_k}^m\int_{t_k}^{t_{k+1}}\mathbb{E}\left[1 + r(u - t_k) + \sigma\left(W_t - W_{t_k}\right) \;\big|\; \mathcal{B}_h\right] du$$

Finally, the process $\left\{\left(W_u \mid \bar{W}_{t_k}^m, \bar{W}_{t_{k+1}}^m\right); u \in [t_k, t_{k+1}]\right\}$ follows a Brownian bridge, which allows to compute the expectation and then the integral and yields the following scheme:

$$A_T \approx \bar{A}_T^{e,m} = \frac{1}{m} \sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \frac{\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m}{2} \right) \tag{2}$$

The above development is actually equivalent to a trapezoidal method in comparison with the more basic Riemann sum used in scheme (1).

Instead of simplification (S), [1] and [3] suggest a quite similar approach. For each step of the Monte-Carlo estimation, first fix a trajectory with the explicit formula given by Black-Scholes model (same as before). Then, rather than computing the conditional expectation $\bar{C}^m$, simulate a realization of $e^{-rT} \left[A_T - K\right]_+$ conditionally to the trajectory. Similarly:

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} S_u du &\approx S_{t_k} \int_{t_k}^{t_{k+1}} \left\{ 1 + r(u - t_k) + \sigma \left(W_u - W_{t_k} \right) \right\} du \\
&= h S_{t_k} \left\{ 1 + \frac{rh}{2} + \frac{\sigma}{h} \int_{t_k}^{t_{k+1}} \left(W_u - W_{t_k} \right) du \right\}
\end{aligned}$$

Furthermore, the remaining integral of the increment of the Brownian Motion is a Gaussian variable and we can compute its expectation and variance conditionally to the trajectory since the integrand follows a Brownian Bridge. Thus, we can indeed simulate it as stated above. In the following, we note:

$$\bar{I}_k^m := \left(\frac{1}{h} \int_{t_k}^{t_{k+1}} \left(W_u - W_{t_k} \right) du \;\middle|\; \mathcal{B}_h \right)$$

In short, $\bar{I}_k^m \sim \mathcal{N}(\mu_k, \sigma_k^2)$ with the following parameters:

$$\begin{aligned}
\mu_k &= \frac{1}{h} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[W_u - W_{t_k} \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m \right] du \\
&= \frac{1}{h} \left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m \right) \int_{t_k}^{t_{k+1}} \frac{u - t_k}{t_{k+1} - t_k} du \\
&= \frac{1}{2} \left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m \right)
\end{aligned}$$

Interchanging the conditional expectation and the integrals yields:

$$\sigma_k^2 = \frac{1}{h^2} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \operatorname{Cov} \left(W_u, W_v \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m \right) du dv$$

Then, by symmetry of the covariance, we have:

$$\begin{aligned}
\sigma_k^2 &= \frac{2}{h^2} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{v} \operatorname{Cov} \left(W_u, W_v \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m \right) du \right) dv \\
&= \frac{2}{h^2} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{v} \frac{(t_{k+1} - v)(u - t_k)}{t_{k+1} - t_k} du \right) dv \\
&= \frac{1}{h^2} \int_0^h \frac{h - v}{h} \left(\int_0^v 2u du \right) dv \\
&= \frac{1}{h^3} \int_0^h (h - v) v^2 dv \\
&= \frac{h}{12}
\end{aligned}$$

The above finally yields the following scheme:

$$A_T \approx \bar{A}_T^{p,m} = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \bar{I}_k^m\right) \tag{3}$$

Considering the value of μ_k, one can note this scheme is the same as the previous one except the fact that (3) adds a random term with distribution $\mathcal{N}(0, \frac{h}{12})$ in the multiplicative factor. To put it differently, in comparison with (3), (2) approximates $\bar{I}_k^m$ by its mean.

Algorithm 1: Scheme (3) implementation

Data: n (number of independent simulations), m (number of time steps)
Result: Estimation and 95% confidence interval
for $i = 1, \ldots, n$ **do**
    Simulate $\bar{W}^{m,i}$;
    Deduce $\bar{S}^{m,i}$ using Black-Scholes formula;
    for $k = 0, \ldots, m-1$ **do**
        Simulate $\bar{I}_k^{m,i}$ (conditionally to $\bar{W}^{m,i}$);
    end
    Compute $\bar{A}_T^{m,i}$ with $\bar{S}^{m,i}$, $\bar{W}^{m,i}$ and $\bar{I}_0^{m,i}, \ldots, \bar{I}_{m-1}^{m,i}$;
end
Compute the mean and standard error of the prices given by $\bar{A}_T^{m,1}, \ldots, \bar{A}_T^{m,n}$;
return *the MC estimate and* 95% *confidence interval for the price*;

Convergence analysis

Before going further, we suggest comparing the results given by schemes (1)-(3). Noticing that for a given strike K, $a \mapsto [a-K]_+$ is 1-lipschitz, we have for any scheme $\bar{A}_T$ that yields price $\widetilde{C}$ (ie, the price if we were able to compute the expectation exactly):

$$|\widetilde{C} - C|^2 \leqslant e^{-2rT}\mathbb{E}_{\mathbb{Q}}\left[|[A_T - K]_+ - [\bar{A}_T^m - K]_+|^2\right] \leqslant \mathbb{E}_{\mathbb{Q}}\left[|A_T - \bar{A}_T^m|^2\right]$$

Consequently, in the context of fixed-strike Asian calls, the time step error is bounded by the bias due to the approximation of A_T. This provides the following:

(i) The bias of scheme 1 is in $O\left(\frac{1}{m}\right)$.

(ii) The bias of scheme 2 is in $O\left(\frac{1}{m}\right)$ with a lower constant than for scheme 1.

(iii) The bias of scheme 3 is in $O\left(\frac{1}{m\sqrt{m}}\right)$.

On the other hand, the Monte-Carlo error is in $O\left(\frac{1}{\sqrt{n}}\right)$ as usual.

Numerical results

The numerical results, as shown in Figure 2, illustrate the bias provided above for each scheme. They graphically confirm the obtained convergence speeds.

Figure 2 shows the 95% confidence intervals given by the three methods for fixed n and m varying up to 256. As we do not have an analytical solution, it is difficult to compare the impact of the variance and that of the bias by comparing the error to the confidence intervals for different values of m. However, since the proposed methods are asymptotically unbiased, the black dotted lines are good approximations of the solution and we can

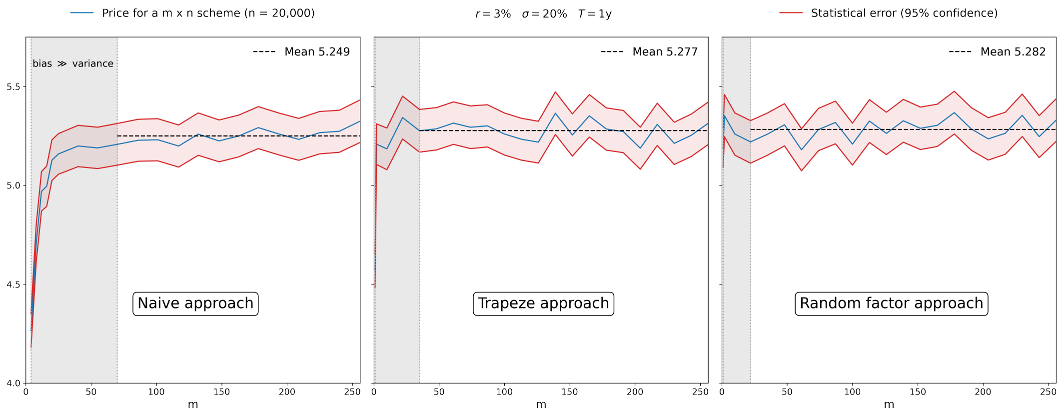


Figure 2: Convergence of schemes (1)-(3) for different values of m

compare the fluctuations of the blue line with the width of the confidence intervals, in red. In the grey area, that is to say for low values of m, the bias is high compared to the variance. For greater values of m, a plateau seems to be reached and the observed deviations are mainly caused by the variance (the bias becomes negligible). Thus, these charts allow to choose m: the one at the edge of the grey zone is the most computationally-effective one that offers a satisfactory level of accuracy. As explained before, there is actually no point in choosing higher values of m since the decrease in the bias would be hidden by the variance. Consequently, one can already notice the advantage of the two improved methods on the right-hand side in comparison with the naive approach: m can be chosen to be significantly lower while achieving as good results. They allow better time complexity. Note however that their space complexity is worse. In particular, the third method requires to simulate an additional $n \times m$ matrix of independent, normally distributed coefficients.

2.2 The use of a control variate

The two improvements introduced in the above tackle the bias, which they successfully reduce for low values of m. In order to improve the convergence speed with respect to parameter n, Lambert et al. [3] finally propose a variance reduction technique for the three schemes above. It uses a control variate as introduced by Kemna et al. [2].

$$\theta_n = \sum_{i=1}^{n} \left(e^{-rT} \left[A_T^i - K \right]_+ + \beta \left(Z^i - \mathbb{E}[Z] \right) \right) \tag{$*$}$$

Observing that $e^x \approx 1 + x$ and $\ln(1+x) \approx x$ when $|x|$ is small, the idea relies on the approximation:

$$A_T = \frac{1}{T} \int_0^T S_u du \approx \exp \left\{ \frac{1}{T} \int_0^T \ln S_u du \right\} = S_0 \exp \left\{ \frac{1}{T} \int_0^T \ln \frac{S_u}{S_0} du \right\}$$

The equality on the right-hand side justifies the validity of such approximation: if r and σ are small, S_u can be expected to remain near S_0 and $\ln \frac{S_u}{S_0} \ll 1$. Therefore, we would like to use the following as a control variate in the case of a fixed-strike Asian call:

$$Z = e^{-rT} \left[S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) \frac{T}{2} + \frac{\sigma}{T} \int_0^T W_u du \right\} - K \right]_+$$

Note that we can indeed compute the exact expression of $\mathbb{E}[Z]$. First, Itô's lemma gives $d(tW_t) = tdW_t + W_t dt$ and:

$$\frac{1}{T}\int_0^T W_u du = \frac{1}{T}\int_0^T (T-s)dW_s \sim \mathcal{N}\left(0, \frac{T}{3}\right) \quad \text{because} \quad \int_0^T (T-s)^2 ds = \frac{T^3}{3}$$

If we note $a := \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}$, $b := \sigma\sqrt{\frac{T}{3}}$, $\rho := \frac{K}{S_0}$, $x^* := \frac{\ln\rho - a}{b}$ and N the c.d.f of $\mathcal{N}(0,1)$ then:

$$\begin{aligned}\frac{\mathbb{E}[Z]}{e^{-rT}S_0} &= \int_{x^*}^{+\infty}\left(e^{a+bx} - \rho\right)N'(x)dx \\ &= e^{a+\frac{b^2}{2}}\int_{x^*-b}^{+\infty} N'(u)du - \rho\int_{x^*}^{+\infty} N'(x)dx \quad \text{with} \quad u = x - b \\ &= e^{a+\frac{b^2}{2}}N(b-x^*) - \rho N(-x^*)\end{aligned}$$

On can check that the same computations yield Black-Scholes formula for the price of a European call. Thus, we have the analytical expression. We finally need to provide a way to simulate the control variate Z in each scenario. We define:

$$\bar{Z}^{r,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{m}\sum_{i=0}^{m-1}\bar{W}^m_{t_k}\right\} - K\right]_+ \qquad (i)$$

Plugging $\bar{A}^{r,m}_T$ from scheme (1) and $\bar{Z}^{r,m}_T$ in $(*)$ yields a new estimator:

$$e^{-rT}\left[\bar{A}^{r,m}_T - K\right]_+ + \hat{\beta}_r\left(\bar{Z}^{r,m}_T - \mathbb{E}[Z]\right) \qquad (4)$$

where $\hat{\beta}_r$ is estimated with the empirical covariance of the variables.

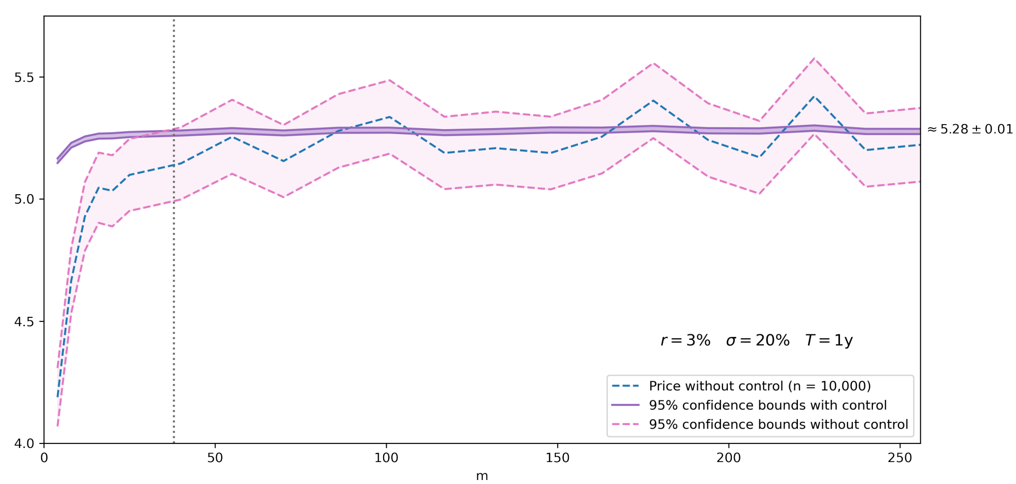


Figure 3: Convergence of scheme (4) for different values of m

As expected, the statistical error (ie: the width of the confidence interval) is much more narrow. With this set of parameters, it is about ten times smaller. The result is quite impressive. Besides, it does not only allow to take a lower n with better results but also to use a lower m. However, considering the assumptions this method relies on, one could wonder whether this variance reduction technique is still as efficient when the parameters (r, σ or T for instance) are high. We testes this scheme for degenerate parameters and obtained really satisfactory results: when $\sigma = r = 50\%$, the variance when using a control variate is still divided by 4 compared to the naive scheme (1).

The enhancement allowed by the control variate looks more interesting than the the one yielded by replacing scheme (1) by scheme (2) or (3). Hopefully, it is possible to combine these improvements in order to obtain really satisfactory results for both low values of m and moderate values of n.

2.3 The combination of both improvements

As we did with scheme (1), we can plug the estimates of schemes (2) and (3) in $(*)$. That requires to adapt the estimates of Z accordingly. Let us define:

$$\bar{Z}^{e,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{m}\sum_{i=0}^{m-1}\frac{\bar{W}^m_{t_{k+1}} + \bar{W}^m_{t_k}}{2}\right\} - K\right]_+ \tag{ii}$$

$$\bar{Z}^{p,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{m}\sum_{i=0}^{m-1}\bar{J}^m_k\right\} - K\right]_+ \tag{iii}$$

With (provided the same computations as for $\bar{I}^m_k$):

$$\bar{J}^m_k := \left(\frac{1}{h}\int_{t_k}^{t_{k+1}} \bar{W}_u du \,\middle|\, \bar{W}^m_{t_{k+1}}, \bar{W}^m_{t_k}\right) \sim \mathcal{N}\left(\frac{\bar{W}^m_{t_{k+1}} + \bar{W}^m_{t_k}}{2}, \frac{h}{12}\right)$$

It yields two new estimators:

$$e^{-rT}\left[\bar{A}^{e,m}_T - K\right]_+ + \hat{\beta}_e\left(\bar{Z}^{e,m}_T - \mathbb{E}[Z]\right) \tag{5}$$

$$e^{-rT}\left[\bar{A}^{p,m}_T - K\right]_+ + \hat{\beta}_p\left(\bar{Z}^{p,m}_T - \mathbb{E}[Z]\right) \tag{6}$$

Where $\hat{\beta}_e$ and $\hat{\beta}_p$ are still estimated with the empirical covariance of $\bar{Z}^{e,m}_T$ (resp. $\bar{Z}^{p,m}_T$) and $\bar{A}^{e,m}_T$ (resp. $\bar{A}^{p,m}_T$). In practice, these two schemes yield expected results, meaning they merge the advantages of scheme (2) (resp. (3)) and those of the control variate observed with scheme (4). Undoubtedly, estimator (4) is the best one we have implemented so far.

2.4 A possible step further

Among the six schemes we have introduced, a great advantage of the third scheme (in particular compared to the second one) is that it can be extended to other diffusion processes, while keeping the same convergence rate.

$$A_T \approx \frac{1}{m}\sum_{k=0}^{m-1}\bar{S}^m_{t_k}\left\{1 + \left(\frac{1}{h}\int_{t_k}^{t_{k+1}}(S_u - S_{t_k})\,du \,\middle|\, \mathcal{B}_h\right)\right\} \tag{3+}$$

This would for instance allow to use stochastic volatility models as long as the conditional integral can be simulated. This is a very important aspect in order to better model the actual stock markets. Unfortunately, it is not straight-forward to adapt the variance reduction technique as it strongly relies on Black-Scholes model and consequently, the closed-form solution of an integral.

3 PDE approach

3.1 Mathematical background

In the general case, it is possible to evaluate the price of an option if it satisfies a PDE. Therefore, finding the price of the option is equivalent to solving a PDE. To start with, we introduce the mathematical background and a well-chosen transformation as developed in [4] that satisfies a simple PDE. As usual when it comes to pricing an option based on a PDE model, we define:

$$\phi(t,x) = \mathbb{E}\left[\left[\int_t^T \frac{S_t}{T}\,dt - x\right]_+ \mid S_t = 1\right] \tag{I}$$

The goal is to find a PDE satisfied by ϕ. Let us define M a well-chosen martingale. Using Itô's formula:

$$\begin{aligned}
M_t &= \mathbb{E}\left[\left[\int_0^T \frac{S_u}{T}\,du - K\right]_+ \Bigm| F_t\right] \\
M_t &= \mathbb{E}\left[\left[\int_t^T \frac{S_u}{T}\,du + \int_0^t \frac{S_u}{T}\,du - K\right]_+ \Bigm| F_t\right] \\
M_t &= S_t\mathbb{E}\left[\left[\int_t^T \frac{S_u}{TS_t}\,du - \frac{K - \int_0^t \frac{S_u}{T}\,du}{S_t}\right]_+ \Bigm| F_t\right] \\
M_t &= S_t\phi(t,\zeta_t)
\end{aligned}$$

Where

$$\zeta_t := \frac{K - \int_0^t \frac{S_u}{T}\,du}{S_t}$$

Then, differentiating with respect to Itô's formula and using the fact that M_t is a martingale, the following term in dt must vanish:

$$dM = cS\left[r\phi + \dot{\phi} - (\rho_t + r\xi)\,\phi' + \frac{1}{2}\sigma^2\xi^2\phi''\right]dt + ...dW_t \tag{II}$$

Then writing f the discounted value of the price ϕ such that $f(x,t) = \exp(-r(T-t))\phi(t,x)$ and replacing f in (II) gives a simple PDE for f:

$$\dot{f} + Gf = 0 \tag{PDE}$$

With G the operator defined by: $G \equiv \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\rho_t + rx)\frac{\partial}{\partial x}$.

It is also important to get the boundary conditions at time T. The initial definition of ϕ from (I) in which the integral vanishes at time $t = T$ provides:

$$f(T,x) = (-x)_+$$

The previous transformation was useful because it helped finding a simple PDE for f. What is more, the price of the Asian option with a fixed strike K is given by:

$$e^{-rT}\mathbb{E}\left[\left[\int_0^T (S_u - K)\,\frac{du}{T}\right]_+\right] = S_0 f\left(0, KS_0^{-1}\right) \equiv e^{-rT}S_0\phi\left(0, KS_0^{-1}\right)$$

Hence, solving the PDE and obtaining the values of f at time $t = 0$ allows to estimate the price of the option. It is possible to solve numerically this linear equation with a classic finite difference method.

3.2 Solving the PDE

In this section, we solve the following PDE thanks to numerical finite difference method:

$$\begin{cases} \dot{f} + Gf = 0 \\ f(T,x) = (-x)_+ \end{cases} \qquad \text{with} \qquad G \equiv \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\rho_t + rx)\frac{\partial}{\partial x}$$

To compute the price, we use an explicit scheme:

$$\frac{u_i^{n+1} - u_i^n}{h} \approx \frac{d}{dt}u(t_n, x_i) \approx -Gu^n$$

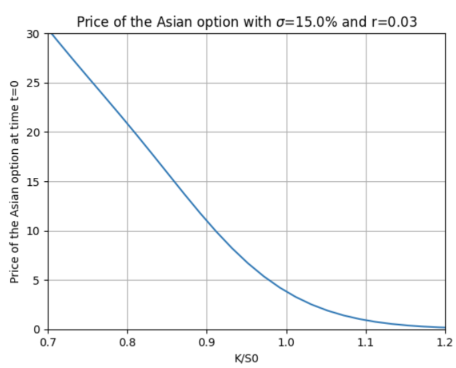
Parameters used

Some parameters are fixed during each simulation:

- $\sigma \in \{0.15; 0.25\}$ the volatility is taken constant
- $r = 0.03$ the interest rate is constant
- $T = 1$ the maturity
- $x \in [-5; 5]$

The space and the time are discretized with space step $\delta \approx 2.10^{-3}$ and time step $h \approx 10^{-3}$. The space step should not be too small with respect to time otherwise it could lead to some instabilities or unsatisfactory results. Intuitively as the initial conditions at $t = T$ vanishes (ie: $\phi(T, x) = 0$ for $x > 0$), the operator G (proportional to the derivatives of f) takes higher number of time iterations t_i to reach and impact $\phi(t_i, .)$ on the whole positive domain $x > 0$.

The results as shown in Figure 4 are satisfactory in the sense that they seem consistent with what we obtained with the Monte Carlo approaches.



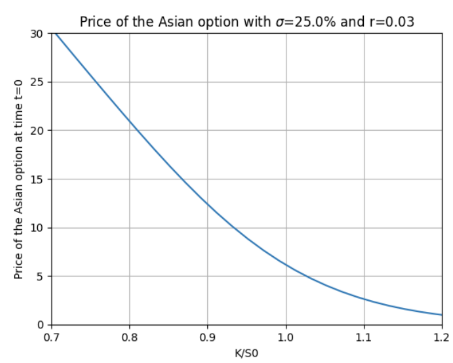


Figure 4: Price curves with respect to the strike with the PDE approach

3.3 Convergence analysis

The Monte-Carlo course (section 2.3 of chapter 7) gives an easy way to check theoretically the convergence of an explicit scheme (case $\theta = 0$). Hence, assuming there exists a smooth solution to the PDE, the scheme is convergent for the L_∞ norm if

$$|b(x_i)|\delta \leqslant \sigma^2(x_i), \quad \sigma^2(x_i)h \leqslant \delta^2$$

Where δ is the space step and h is the time step. In our case, this is equivalent to the following inequalities:

$$0 \leqslant \frac{\sigma^2 x_i^2}{2} - |\rho_t + r x_i|\delta,$$
$$0 \leqslant \delta^2 - \frac{\sigma^2 x_i^2}{2} h$$

Here, $\delta \approx 2.10^{-3}$ and $h \approx 10^{-3}$. We plot these inequalities in Figure 5. We could also play with the step parameters δ and h to ensure that these two functions remains positive.

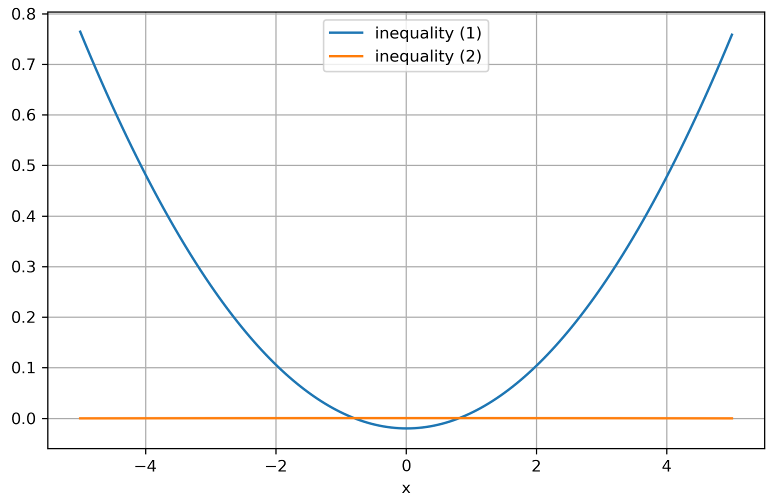


Figure 5: Inequalities ensuring convergence of the scheme for $\sigma = 25\%$ and $r = 3\%$

Graphically, the results are not really convincing, however the inequalities are in both cases almost verified in the whole domain and it is enough for us to observe convergence. Moreover, we observe that issues of convergence could especially occur around 0, while inequalities seem to be verified on the edges. It would be also possible to study more in detail the stability and the consistency of the explicit scheme used to give better arguments to justify the convergence for a wide range of parameters.

4 Lower and upper bounds for the computed price

Computing numerically a price is not enough to have a correct estimation. It is also important to tackle the errors due to the numerical method and thus to find bounds to be able to judge the reliability of the result. In that case, we look for a lower bound for the price that we can establish theoretically and then compute it. Once more, the main ideas of this section were developed in [4]. As in the numerical tests, we choose $T = 1$ and define $Y := A_1 - K = \int_0^1 S_u du - K$ to simplify the notations.

4.1 Theoretical approach

Let $e^{-r}\,\mathbb{E}[Y_+]$ denote the price of the Asian option that we want to bound. For any centered Gaussian variable Z, we get:

$$\mathbb{E}[Y_+] = \mathbb{E}[\mathbb{E}[Y_+ \mid Z]] \geqslant \mathbb{E}\left[\mathbb{E}[Y \mid Z]_+\right] \tag{III}$$

Note the inequality used here is the same as the approximation (S) used in the Monte-Carlo scheme (2). What is more, for any random variable U:

$$\begin{aligned} 0 &\leqslant \mathbb{E}[U_+] - \mathbb{E}[U]_+ \\ &\leqslant \frac{1}{2}\left(\mathbb{E}[|U|] - |\mathbb{E}[U]|\right) \\ &\leqslant \frac{1}{2}\mathrm{Var}[U]^{1/2} \end{aligned} \tag{IV}$$

Hence combining (III) and (IV) provides a framework for the price:

$$\mathbb{E}[\mathbb{E}[Y \mid Z]_+] \leq \mathbb{E}[Y_+] \leqslant \mathbb{E}[\mathbb{E}[Y \mid Z]_+] + \frac{1}{2}\mathbb{E}[\mathrm{Var}[Y \mid Z]^{1/2}]$$

The purpose of this inequality is to find a well-chosen Z which minimizes $\mathbb{E}[\mathrm{Var}[Y \mid Z]^{1/2}]$ to frame the price at best. We are looking for a Gaussian variable with zero-mean so that:

$$\begin{aligned} \mathbb{E}[W_t \mid Z] &= m_t Z \\ \mathrm{Cov}(W_s, W_t \mid Z) &= v_{st} \end{aligned}$$

The goal is now to compute $\mathrm{Var}[Y \mid Z]^{1/2}$. As a first step, we should compute the mean of $Y \mid Z$. The above yields:

$$\mathbb{E}[Y \mid Z] = \mathbb{E}\left[\int_0^1 e^{\sigma W_t - \frac{1}{2}\sigma^2 t + rt}\, dt \;\Big|\; Z\right] = \int_0^1 e^{\sigma m_t Z - \frac{1}{2}\sigma^2 v m_t^2 + rt} dt \tag{V}$$

Where $v = \mathrm{Var}(Z)$. After some calculations:

$$\mathrm{Var}[Y \mid Z] = \int_0^1 ds \int_0^1 \exp(\sigma Z(m_s + m_t) - \tfrac{1}{2}\sigma^2 v(m_s^2 + m_t^2) + r(s+t))(e^{\sigma^2 v_{st}} - 1)dt$$

For common values of σ and provided that for $s, t \in [0,1]$:

$$|v_{st}| = |\mathrm{Cov}(W_s, W_t \mid Z)| \leq \mathrm{Var}(W_s)^{\frac{1}{2}} Var(W_t)^{\frac{1}{2}} \leq 1$$

we can simplify the above expression using $e^{\sigma^2 v_{st}} - 1 \approx \sigma^2 v_{st}$. This approximation is then used to simplify the integral giving $\mathrm{Var}[Y \mid Z]$. As a result, we define the approximate integral V:

$$\begin{aligned} V &\approx \int_0^1 ds \int_0^1 (1 + \sigma Z(m_s + m_t) - \tfrac{1}{2}\sigma^2 v(m_s^2 + m_t^2) + r(s+t))\sigma^2 v_{st} dt \\ &= \int_0^1 ds \int_0^1 (1 + f(s, Z) + f(t, Z)) v_{st} dt \end{aligned}$$

where $f(t, Z) = \frac{1}{2}\sigma^2 v(m_t^2 + rt)\sigma^2$.

What is remarkable about this integral is that it vanishes when we define $Z := \int_0^1 W_u du$. Indeed, by bilinearity of the covariance:

$$\int_0^1 v_{st} ds = \int_0^1 \mathrm{Cov}(W_s, W_t \mid Z) ds = \mathrm{Cov}\left(\int_0^1 W_s ds, W_t \mid Z\right) = \mathrm{Cov}(Z, W_t \mid Z) = 0$$

As a consequence, we just found an ideal candidate for Z : assuming that the initial integral $\text{Var}[Y \mid Z]$ is close to V, vanishing V implies reducing significantly $\text{Var}[Y \mid Z]$. With the above definition of Z:

$$\text{Var}[Y \mid Z] = \text{Var}[Y \mid Z] - V \leq A + B$$

Where:

$$A = \int_0^1 ds \int_0^1 (e^{f(s,Z)+f(t,Z)} - 1 - f(s,Z) - f(t,Z)).|e^{\sigma^2 v_{st}} - 1|dt$$
$$B = \int_0^1 ds \int_0^1 (e^{\sigma^2 v_{st}} - 1 - \sigma^2 v_{st})|1 + f(s,Z) + f(t,Z)|dt$$

After some computations detailed in [4], we finally get:

$$\mathbb{E}[\text{Var}[Y \mid Z]^{1/2}] \leq \frac{1}{2}\left[c\sigma^2 e^{c\sigma^2+\gamma_2}\left[\frac{1}{2}\sigma^2\gamma_1 v e^{\frac{1}{2}\sigma^2\gamma_1 v} + \tfrac{1}{2}\gamma_2^2\right] + \tfrac{1}{2}\sigma^4 c^2 e^{\sigma^4 c^2}(1+\gamma_2)\right]^{1/2} \tag{VI}$$

Where c, γ_1 and γ_2 are constants such that for all $s, t \in [0,1]$:

$$|v_{st}| \leq c, \quad (m_s + m_t)^2 \leq \gamma_1, \quad |g_s + g_t| \leq \gamma_2, \quad g_s \equiv rs - \tfrac{1}{2}\sigma^2 v m_s \tag{VII}$$

It is then necessary to find optimal parameters minimizing the right-term of (VI). As $Z = \int_0^1 W_u du$, we deduce that $v = \frac{1}{3}$.

4.2 Computation of the theoretical bounds

Thanks to the previous part, we are now able to compute a lower and upper bounds for the price of the fixed-strike Asian call. Indeed, the price $\mathbb{E}[Y^+]$ is framed by:

$$\underbrace{\mathbb{E}[\mathbb{E}[Y \mid Z]_+]}_{\text{lower bound}} \leqslant \mathbb{E}[Y_+] \leqslant \underbrace{\mathbb{E}[\mathbb{E}[Y \mid Z]_+] + \frac{1}{2}\mathbb{E}[A+B]^{1/2}}_{\text{upper bound}} \tag{VIII}$$

Where the upper bound of $\mathbb{E}[A+B]$ itself is given in (VI).

4.3 Computing the lower bound

Let us first compute the lower bound with $Z = \int_0^1 W_u du \sim \mathcal{N}(0, \frac{1}{3})$.

Let us note $\mathbb{E}[Y \mid Z] =: \varphi(Z)$ so that $\mathbb{E}[Y \mid Z = z] = \varphi(z)$. Using (V) we find:

$$\varphi(z) = \int_0^1 \exp(\sigma m_t z - \tfrac{1}{2}\sigma^2 v m_t^2 + rt)dt - K$$

Taking into account the density of Z $f_Z(x) = \sqrt{\frac{3}{2\pi}} e^{-\frac{3x^2}{2}}$, we finally find:

$$\mathbb{E}[\mathbb{E}[Y \mid Z]_+] = \int_{-\infty}^{+\infty} \varphi(x)_+ f_Z(x)dx \tag{IX}$$

The last integral can easily be estimated with the python scipy module given the expression of m_t and v_{st} we obtain with this well-chosen Z:

$$m_t = \frac{3}{2}t(2-t) \quad \text{and} \quad v_{st} = s \wedge t - \frac{3}{4}st(2-s)(2-t)$$

4.4 Computing the upper bound

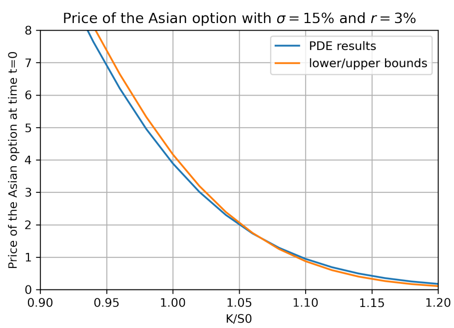
In fact, computing the upper bound is quite simple as we only need to add the following quantity to the lower bound:

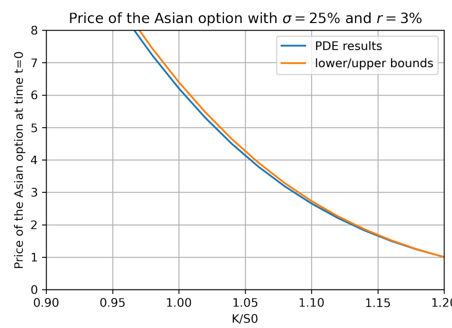
$$\frac{1}{2}\left[c\sigma^2 e^{c\sigma^2+\gamma_2}\left[\frac{1}{2}\sigma^2\gamma_1 v e^{\frac{1}{2}\sigma^2\gamma_1 v}+\tfrac{1}{2}\gamma_2^2\right]+\tfrac{1}{2}\sigma^4 c^2 e^{\sigma^4 c^2}(1+\gamma_2)\right]^{1/2}$$

With well-chosen parameters satisfying (VIII):

$$c=\frac{1}{3},\quad \gamma_1=9,\quad \gamma_2=r+\frac{\sigma^2}{4}$$

We finally get really convincing results close to the PDE solution computed previously. Notice that the bounds are not supposed to directly frame the PDE solution computed but only the price in the general case. However, we could estimate an upper bound of the relative error conveyed by the PDE solution thanks to these bounds.





Conclusion

The following table provides a summary of the different methods we have implemented. The Monte-Carlo approaches were run with $n = 10^5$ and with different values of m (between 20 and 100 depending on the results presented in Figures 2 and 3). Moreover, we took $\sigma = 20\%$ and $r = 3\%$. It turns out that the most accurate methods are the Monte-Carlo approaches with control variates. The three corresponding algorithms yield similar results. What is more, the PDE approach has several flaws: it tends to underestimate the value of the option and does not allow to construct a confidence interval. However, it can still be useful to study in a computationally-effective way the behaviour of the solution for different values of the parameters and the corresponding sensitivities. Regarding the last technique, consisting in computing bounds for the solution, it appears to be the fastest method. Unfortunately, unlike the Monte-Carlo approaches with control, its accuracy is fundamentally limited since it does not allow to achieve a better accuracy than the one written below. Consequently, it is very useful if one does not seek a really high level of precision. Otherwise, the Monte-Carlo with control variates and high values of n and m should be used, requiring more processing time.

Method	Results		Processing time (s)
	Value	**Interval** ($\pm$)	
MC_1 (naive)	5.24	$4.7 \cdot 10^{-2}$ (0.91%)	0.52
MC_2 (trapeze)	5.25	$4.8 \cdot 10^{-2}$ (0.91%)	0.32
MC_3 (conditional)	5.27	$4.8 \cdot 10^{-2}$ (0.91%)	0.31
MC_1 (+ control)	5.28	$3.4 \cdot 10^{-3}$ (0.06%)	0.39
MC_2 (+ control)	5.28	$3.4 \cdot 10^{-3}$ (0.06%)	0.27
MC_3 (+ control)	5.28	$3.4 \cdot 10^{-3}$ (0.06%)	0.22
PDE approach	5.05	*NA*	0.51
Lower-upper bounds	5.29	$7.7 \cdot 10^{-3}$ (0.14%)	0.19

Table 1: Comparison of the implemented methods

References

[1] B. Bouchard. *Méthodes de Monte Carlo en finance*. 2007.

[2] A.G.Z. Kemna and A.C.F. Vorst. A pricing method for options based on average asset values. *Journal of Banking & Finance*, 14:113–129, 1990.

[3] B. Lapeyre and E. Temam. Competitive monte carlo methods for the pricing of asian options. *Journal of Computational Finance*, 2000.

[4] L. C. G. Rogers and Z. Shi. The value of an asian option. *Journal of Applied Probability*, 32(4):1077–1088, 1995.