Asian options pricing

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Introduction

An Asian option is any option with payoff of the form:

$$(S_t)_{t\in[0,T]} \mapsto g\left(S_T, A_T\right) \quad \text{with} \quad A_T := \frac{1}{T}\int_0^T S_u du$$

where $(S_t)_t$ denotes the trajectory of the underlying and T is the maturity of the option. For instance, a *fixed-strike* Asian call has $g(x,a) := e^{-rT}\left[a-K\right]_+$ where K denotes the strike and a *floatting-strike* Asian call has $g(s,a) := e^{-rT}\left[s-a\right]_+$. In the first case, the option is exercised by its owner if the underlying has lied above the strike **on average** throughout its lifetime. In the second case, it is worth exercising it when the underlying is above its average value at expiry.

This work studies different pricing techniques for Asian options and will tackle **only fixed-strike Asian calls** for simplicity. Furthermore, we will use Black-Scholes model. In that context, simulating S_T is straightforward and the real challenge consists in simulating A_T. That is why our developments for fixed-strike Asian calls adapt directly to any Asian option of the form given above.

We first introduce and implement different Monte-Carlo approaches as developed by Lambert et al. [3] and B. Bouchard [1]. We then compare them with a PDE approach *(to be chosen)*.

1 Naive approach

The most basic Monte-Carlo approach to the problem consists in approximating the integral of the underlying over its trajectory by a Riemann sum.

$$A_T \approx \bar{A}_T^{r,m} := \frac{h}{T}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \quad \text{with} \quad h = \frac{T}{m}$$

where $\bar{S}_{t_k}^m$ denotes the k-th step of an Euler scheme. That gives the following estimate:

$$C_r := e^{-rT}\sum_{i=1}^{n}\left[\frac{1}{m}\sum_{k=0}^{m-1}\bar{S}_{t_k}^{m,i} - K\right]_+ \tag{1}$$

where $\left(\bar{S}^{m,i}\right)_{i=1,\dots,n}$ are independent and identically distributed copies of $\bar{S}^m$.

2 Improved Monte-Carlo approaches

2.1 Two finer approximations

A first improvement of the above approach proposes to better use the information provided by the simulation $\bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m$ to approximate the integral A_T. It relies on the fact that once the trajectory has been simulated, the best estimation of the price is:

$$\bar{C}^m := \mathbb{E}\left[e^{-rT}\left[A_T - K\right]_+ \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m\right]$$

By the tower property, the expectation (estimated by a Monte-Carlo method with n trajectories) of this conditional expectation is the price of the Asian call. At this stage, this quantity is not known either. [3] introduces the following simplification:

$$\bar{C}^m \approx e^{-rT}\left[\mathbb{E}\left[A_T \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m\right] - K\right]_+ \tag{S}$$

A first-order Taylor expansion leads to the additional approximation below:

$$\begin{aligned}
\mathbb{E}\left[\int_{t_k}^{t_{k+1}} S_u du \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m\right] &= \int_{t_k}^{t_{k+1}} \mathbb{E}\left[S_u \mid \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right] du \\
&= \bar{S}_{t_k}^m \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\frac{S_u}{\bar{S}_{t_k}^m} \;\middle|\; \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right] du \\
&\approx \bar{S}_{t_k}^m \int_{t_k}^{t_{k+1}} \left\{1 + \ln \mathbb{E}\left[\frac{S_u}{\bar{S}_{t_k}^m} \;\middle|\; \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right]\right\} du
\end{aligned}$$

for all $k \in \{0, \ldots, m-1\}$. On the one hand, Black-Scholes model gives:

$$S_u \mid \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m = \bar{S}_{t_k}^m \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(u - t_k) + \sigma\left(\left(W_u \mid \bar{W}_{t_k}^m, \bar{W}_{t_{k+1}}^m\right) - \bar{W}_{t_k}^m\right)\right\}$$

where $\bar{W}_{t_k}^m$ is the k-th step of the Brownian Motion used to simulate $\bar{S}^m$ as part of the Euler scheme, ie: $\bar{S}_{t_k}^m = S_0 \exp\{(r - \frac{\sigma^2}{2})t_k + \sigma \bar{W}_{t_k}^m\}$. On the other hand, $W_u \mid \bar{W}_{t_k}^m, \bar{W}_{t_{k+1}}^m$ follows a Brownian Bridge. Therefore, the integrand has an analytical expression indeed and it yields the following scheme:

$$A_T \approx \bar{A}_T^{e,m} = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \frac{\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m}{2}\right) \tag{2}$$

The above development is actually equivalent to a trapezoidal method in comparison with the more basic Riemann sum used in scheme (1). We note C_e the corresponding estimate of the price.

Instead of simplification (S), [1] and [3] suggest a quite similar approach. For each step of the Monte-Carlo estimation, first fix a trajectory with the explicit formula given by Black-Scholes model (same as before). Then, rather than computing the conditional expectation $\bar{C}^m$, simulate a realization of $e^{-rT}\left[A_T - K\right]_+$ conditionally to the trajectory. Similarly:

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} S_u du &= S_{t_k} \int_{t_k}^{t_{k+1}} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(u - t_k) + \sigma\left(W_u - W_{t_k}\right)\right\} du \\
&\approx S_{t_k} \int_{t_k}^{t_{k+1}} \left\{1 + r(u - t_k) + \sigma\left(W_u - W_{t_k}\right)\right\} du \\
&= hS_{t_k}\left\{1 + \frac{rh}{2} + \frac{\sigma}{h}\int_{t_k}^{t_{k+1}} \left(W_u - W_{t_k}\right) du\right\}
\end{aligned}$$

Furthermore, the remaining integral of the increment of the Brownian Motion is a Gaussian variable and we can compute its expectation and variance conditionally to the trajectory since the integrand follows a Brownian Bridge. Thus, we can indeed simulate it as stated above. In the following, we note:

$$\bar{I}_k^m := \frac{1}{h}\int_{t_k}^{t_{k+1}} (W_u - W_{t_k})\, du \;\Big|\; \bar{W}_0^m, \bar{W}_{t_1}^m, \ldots, \bar{W}_T^m$$

$\bar{I}_k^m \sim \mathcal{N}(\mu_k, \sigma_k^2)$ with:

$$\begin{aligned}
\mu_k &= \frac{1}{h}\int_{t_k}^{t_{k+1}} \mathbb{E}\left[W_u - W_{t_k} \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right] du \\
&= \frac{1}{h}\left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m\right)\int_{t_k}^{t_{k+1}} \frac{u - t_k}{t_{k+1} - t_k} du \\
&= \frac{1}{2}\left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m\right)
\end{aligned}$$

Interchanging the conditional expectation and the integrals and Fubini's theorem yield:

$$\begin{aligned}
\sigma_k^2 &= \mathbb{E}\left[\left(\frac{1}{h}\int_{t_k}^{t_{k+1}} W_u du\right)^2 \;\Big|\; \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right] - \mathbb{E}\left[\frac{1}{h}\int_{t_k}^{t_{k+1}} W_u du \;\Big|\; \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right]^2 \\
&= \frac{1}{h^2}\iint_{t_k}^{t_{k+1}} \mathrm{Cov}\left(W_u W_v \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right) dudv \\
&= \frac{1}{h^2}\iint_{t_k}^{t_{k+1}} \frac{(t_{k+1} - u)(v - t_k)}{t_{k+1} - t_k} dudv \\
&= \frac{1}{h^3}\left(\int_0^h v dv\right)^2 \\
&= \frac{h}{4}
\end{aligned}$$

The above finally yields the following scheme:

$$A_T \approx \bar{A}_T^{p,m} = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \bar{I}_k^m\right) \tag{3}$$

Considering the value of μ_k, one can note this scheme is the same as the previous one except the fact that (3) adds a random term with distribution $\mathcal{N}(0, \frac{h}{4})$ in the multiplicative factor. To put it differently, in comparison with (3), (2) approximates $\bar{I}_k^m$ by its mean.

2.2 The use of a control variate

In order to improve the convergence speed, Lambert et al. [3] finally propose a variance reduction technique for the three schemes above. It uses a control variate as introduced by Kemna et al. [2].

$$\theta = \sum_{i=1}^{n}\left(e^{-rT}\left[A_T^i - K\right]_+ + \beta\left(Z^i - \mathbb{E}[Z]\right)\right) \tag{$*$}$$

Observing that $e^x \approx 1 + x$ and $\ln(1 + x) \approx x$ when $|x|$ is small, the idea relies on the approximation:

$$A_T = \frac{1}{T}\int_0^T S_u du \approx \exp\left\{\frac{1}{T}\int_0^T \ln S_u du\right\} = S_0 \exp\left\{\frac{1}{T}\int_0^T \ln \frac{S_u}{S_0} du\right\}$$

Algorithm 1: Scheme (3) implementation

Data: n (number of independent simulations), m (number of time steps)
Result: Estimation and 95% confidence interval
for $i = 1, \dots, n$ **do**
  Simulate $\bar{W}^{m,i}$;
  Deduce $\bar{S}^{m,i}$ using Black-Scholes formula;
  for $k = 0, \dots, m-1$ **do**
    Compute the mean and variance of $\bar{I}_k^m$ conditionally to $\bar{W}^{m,i}$;
    Simulate $\bar{I}_k^m$ accordingly;
    $\dots$;
  end
end
return $\dots$;

The equality on the right-hand side justifies the validity of such approximation: if r and σ are small, S_u can be expected to remain near S_0 and $\ln \frac{S_u}{S_0} \ll 1$. Therefore, we would like to use the following as a control variate in the case of a fixed-strike Asian call:

$$Z = e^{-rT} \left[S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) \frac{T}{2} + \frac{\sigma}{T} \int_0^T W_u du \right\} - K \right]_+$$

Note that we can indeed compute the exact expression of $\mathbb{E}[Z]$. First, Itô's lemma gives $d(tW_t) = tdW_t + W_t dt$ and:

$$\frac{1}{T} \int_0^T W_u du = \frac{1}{T} \int_0^T (T-s) dW_s \sim \mathcal{N}\left(0, \frac{T}{3}\right) \quad \text{because} \quad \int_0^T (T-s)^2 ds = \frac{T^3}{3}$$

If we note $a := \left(r - \frac{\sigma^2}{2}\right) \frac{T}{2}$, $b := \sigma \sqrt{\frac{T}{3}}$, $\rho := \frac{K}{S_0}$, $x^* := \frac{\ln \rho - a}{b}$ and N the c.d.f of $\mathcal{N}(0,1)$ then:

$$\begin{aligned}
\frac{\mathbb{E}[Z]}{e^{-rT} S_0} &= \int_{x^*}^{+\infty} \left(e^{a+bx} - \rho \right) N'(x) dx \\
&= e^{a+\frac{b^2}{2}} \int_{x^*-b}^{+\infty} N'(u) du - \rho \int_{x^*}^{+\infty} N'(x) dx \quad \text{with} \quad u = x - b \\
&= e^{a+\frac{b^2}{2}} N(b - x^*) - \rho N(-x^*)
\end{aligned}$$

On can check that the same computations yield Black-Scholes formula for the price of a European call. Thus, we have the analytical expression. We finally need to provide a way to simulate the control variate Z in each scenario. We define:

$$\bar{Z}^{r,m} = e^{-rT} \left[S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} \right) \frac{T}{2} + \frac{\sigma}{m} \sum_{i=0}^{m-1} \bar{W}_{t_k}^m \right\} - K \right]_+ \qquad (i)$$

Plugging $\bar{A}_T^{r,m}$ from scheme (1) and $\bar{Z}_T^{r,m}$ in $(*)$ yields a new scheme:

$$e^{-rT} \left[\bar{A}_T^{r,m} - K \right]_+ + \hat{\beta}_r \left(\bar{Z}_T^{r,m} - \mathbb{E}[Z] \right) \qquad (4)$$

where $\hat{\beta}_r$ is estimated with the empirical covariance of the variables.

2.3 The combination of both improvements

As we did with scheme (1), we can plug the estimates of schemes (2) and (3) in ($*$). That requires to adapt the estimates of Z too.

$$\bar{Z}^{e,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{n}\sum_{i=0}^{n-1}\frac{\bar{W}^m_{t_{k+1}} + \bar{W}^m_{t_k}}{2}\right\} - K\right]_+ \qquad (ii)$$

$$\bar{Z}^{p,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{n}\sum_{i=0}^{n-1}\int_{t_k}^{t_{k+1}}\left(\bar{W}^m_u - \bar{W}^m_{t_k}\right)du\right\} - K\right]_+ \qquad (iii)$$

It yields two new schemes.

3 PDE approaches

Conclusion

References

[1] B. Bouchard. *Méthodes de Monte Carlo en finance*. 2007.

[2] A.G.Z. Kemna and A.C.F. Vorst. A pricing method for options based on average asset values. *Journal of Banking & Finance*, 14:113–129, 1990.

[3] B. Lapeyre and E. Temam. Competitive monte carlo methods for the pricing of asian options. *Journal of Computational Finance*, 2000.