Asian options pricing

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Introduction

An Asian option is any option with payoff of the form:

$$(S_t)_{t\in[0,T]} \mapsto g(S_T, A_T) \quad \text{with} \quad A_T := \frac{1}{T}\int_0^T S_u du$$

where $(S_t)_t$ denotes the trajectory of the underlying and T is the maturity of the option. For instance, a *fixed-strike* Asian call has $g(x,a) := e^{-rT}[a-K]_+$ where K denotes the strike and a *floatting-strike* Asian call has $g(s,a) := e^{-rT}[s-a]_+$. In the first case, the option is exercised by its owner if the underlying has lied above the strike **on average** throughout its lifetime. In the second case, it is worth exercising it when the underlying is above its average value at expiry.

This work studies different pricing techniques for Asian options and will tackle **only fixed-strike Asian calls** for simplicity. Furthermore, we will use Black-Scholes model. In that context, simulating S_T is straightforward and the real challenge consists in simulating A_T. That is why our developments for fixed-strike Asian calls adapt directly to any Asian option of the form given above.

We first introduce and implement different Monte-Carlo approaches as developed by Lambert et al. [3] and B. Bouchard [1]. We then compare them with a PDE approach as presented by Rogers et al. [4].

1 Naive approach

The most basic Monte-Carlo approach to the problem consists in approximating the integral of the underlying over its trajectory by a Riemann sum.

$$A_T \approx \bar{A}_T^{r,m} := \frac{h}{T}\sum_{k=0}^{m-1}\bar{S}_{t_k}^m = \frac{1}{m}\sum_{k=0}^{m-1}\bar{S}_{t_k}^m \quad \text{with} \quad h = \frac{T}{m}$$

where $\bar{S}_{t_k}^m$ denotes the k-th step of an Euler scheme. That gives the following estimate:

$$C_r := e^{-rT}\sum_{i=1}^{n}\left[\frac{1}{m}\sum_{k=0}^{m-1}\bar{S}_{t_k}^{m,i} - K\right]_+ \tag{1}$$

where $\left(\bar{S}^{m,i}\right)_{i=1,\dots,n}$ are independent and identically distributed copies of $\bar{S}^m$.

The variance is in $O(\dots)$ and the biais is in $O(\dots)$. Figure 1 shows the result of this method for different values of K, T and σ where we took $n = 10,000$ and $m = 100$. One can already notice that it tallies with the common idea that Asian call are cheaper than European call for a given set of parameters. Also, quite intuitively, the gap widens when σ and T grow: the smaller both of them are, the closer to the initial value the underlying will remain, leading to $A_T \approx S_T \approx S_0$ on average.

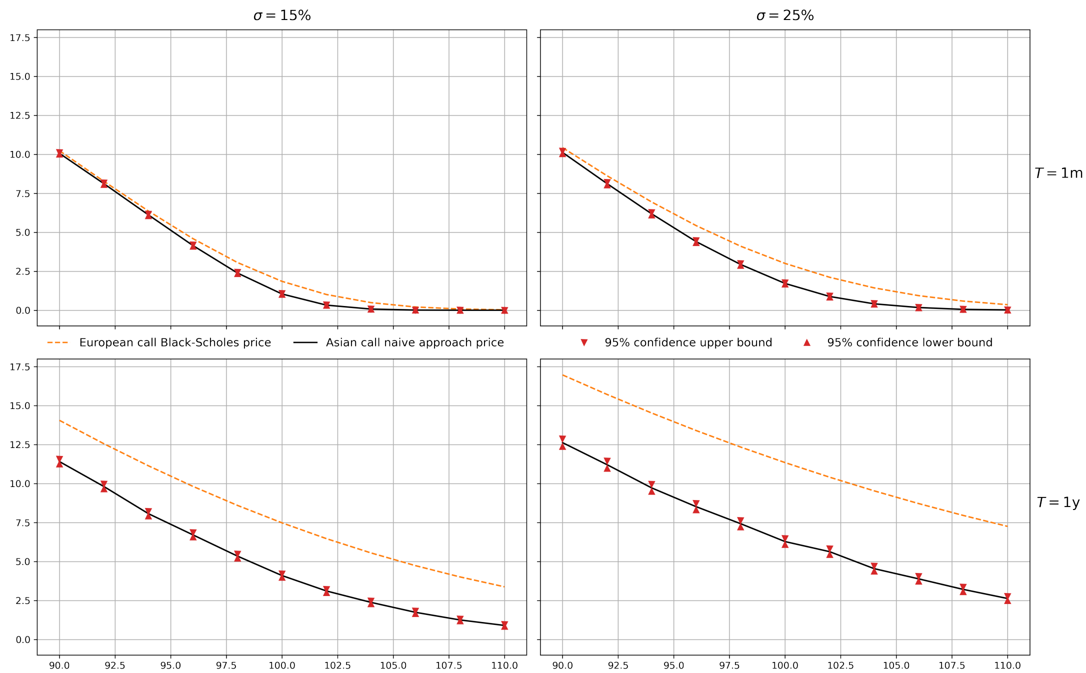

Figure 1: Fixed-strike Asian call naive pricing for different values of K, σ and T

However, these charts, particularly the one for $\sigma = 25\%$ and $T = 1$y, reveal some fluctuations with a greater magnitude than the length of the 95% confidence intervals plotted in red. This stems from the bias introduced by the approximation of A_T, that the figure does not reflect. This seems to be the main flaw of the naive approach: even if the variance already looks satisfactory, the bias increases rapidly with the parameters, which yields imprecise results.

The following models all give similar charts. In the below, we rather focus on the graphical representation of the convergence properties of each scheme. As suggested in the previous paragraph, this is the main challenge that the following approaches mean to address.

2 Improved Monte-Carlo approaches

2.1 Two finer approximations

A first improvement of the above approach proposes to better use the information provided by the simulation $\bar{S}_0^m, \bar{S}_{t_1}^m, \dots, \bar{S}_T^m$ to approximate the integral A_T. It relies on the fact that once the trajectory has been simulated, the best estimation of the price is:

$$\bar{C}^m := \mathbb{E}\left[e^{-rT}\left[A_T - K\right]_+ \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \dots, \bar{S}_T^m\right]$$

By the tower property, the expectation (estimated by a Monte-Carlo method with n

trajectories) of this conditional expectation is the price of the Asian call. At this stage, this quantity is not known either. [3] introduces the following simplification:

$$\bar{C}^m \approx e^{-rT}\left[\mathbb{E}\left[A_T \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m\right] - K\right]_+ \tag{S}$$

A first-order Taylor expansion leads to the additional approximation below:

$$\begin{aligned}
\mathbb{E}\left[\int_{t_k}^{t_{k+1}} S_u du \mid \bar{S}_0^m, \bar{S}_{t_1}^m, \ldots, \bar{S}_T^m\right] &= \int_{t_k}^{t_{k+1}} \mathbb{E}\left[S_u \mid \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right] du \\
&= \bar{S}_{t_k}^m \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\frac{S_u}{\bar{S}_{t_k}^m} \;\middle|\; \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right] du \\
&\approx \bar{S}_{t_k}^m \int_{t_k}^{t_{k+1}} \left\{1 + \ln \mathbb{E}\left[\frac{S_u}{\bar{S}_{t_k}^m} \;\middle|\; \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m\right]\right\} du
\end{aligned}$$

for all $k \in \{0, \ldots, m-1\}$. On the one hand, Black-Scholes model gives:

$$S_u \mid \bar{S}_{t_k}^m, \bar{S}_{t_{k+1}}^m = \bar{S}_{t_k}^m \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(u - t_k) + \sigma\left(\left(W_u \mid \bar{W}_{t_k}^m, \bar{W}_{t_{k+1}}^m\right) - \bar{W}_{t_k}^m\right)\right\}$$

where $\bar{W}_{t_k}^m$ is the k-th step of the Brownian Motion used to simulate $\bar{S}^m$ as part of the Euler scheme, ie: $\bar{S}_{t_k}^m = S_0 \exp\{(r - \frac{\sigma^2}{2})t_k + \sigma \bar{W}_{t_k}^m\}$. On the other hand, $W_u \mid \bar{W}_{t_k}^m, \bar{W}_{t_{k+1}}^m$ follows a Brownian Bridge. Therefore, the integrand has an analytical expression indeed and it yields the following scheme:

$$A_T \approx \bar{A}_T^{e,m} = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \frac{\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m}{2}\right) \tag{2}$$

The above development is actually equivalent to a trapezoidal method in comparison with the more basic Riemann sum used in scheme (1). We note C_e the corresponding estimate of the price.

Instead of simplification (S), [1] and [3] suggest a quite similar approach. For each step of the Monte-Carlo estimation, first fix a trajectory with the explicit formula given by Black-Scholes model (same as before). Then, rather than computing the conditional expectation $\bar{C}^m$, simulate a realization of $e^{-rT}\left[A_T - K\right]_+$ conditionally to the trajectory. Similarly:

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} S_u du &= S_{t_k} \int_{t_k}^{t_{k+1}} \exp\left\{\left(r - \frac{\sigma^2}{2}\right)(u - t_k) + \sigma\left(W_u - W_{t_k}\right)\right\} du \\
&\approx S_{t_k} \int_{t_k}^{t_{k+1}} \left\{1 + r(u - t_k) + \sigma\left(W_u - W_{t_k}\right)\right\} du \\
&= h S_{t_k}\left\{1 + \frac{rh}{2} + \frac{\sigma}{h}\int_{t_k}^{t_{k+1}} \left(W_u - W_{t_k}\right) du\right\}
\end{aligned}$$

Furthermore, the remaining integral of the increment of the Brownian Motion is a Gaussian variable and we can compute its expectation and variance conditionally to the trajectory since the integrand follows a Brownian Bridge. Thus, we can indeed simulate it as stated above. In the following, we note:

$$\bar{I}_k^m := \frac{1}{h}\int_{t_k}^{t_{k+1}} \left(W_u - W_{t_k}\right) du \;\middle|\; \bar{W}_0^m, \bar{W}_{t_1}^m, \ldots, \bar{W}_T^m$$

In short, $\bar{I}_k^m \sim \mathcal{N}(\mu_k, \sigma_k^2)$ with the following parameters:

$$\begin{aligned}\mu_k &= \frac{1}{h}\int_{t_k}^{t_{k+1}} \mathbb{E}\left[W_u - W_{t_k} \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right] du \\ &= \frac{1}{h}\left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m\right)\int_{t_k}^{t_{k+1}} \frac{u-t_k}{t_{k+1}-t_k} du \\ &= \frac{1}{2}\left(\bar{W}_{t_{k+1}}^m - \bar{W}_{t_k}^m\right)\end{aligned}$$

Interchanging the conditional expectation and the integrals yields:

$$\sigma_k^2 = \frac{1}{h^2}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \mathrm{Cov}\left(W_u, W_v \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right) dudv$$

Then, by symmetry of the covariance, we have:

$$\begin{aligned}\sigma_k^2 &= \frac{2}{h^2}\int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{v} \mathrm{Cov}\left(W_u, W_v \mid \bar{W}_{t_{k+1}}^m, \bar{W}_{t_k}^m\right) du\right) dv \\ &= \frac{2}{h^2}\int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{v} \frac{(t_{k+1}-v)(u-t_k)}{t_{k+1}-t_k} du\right) dv \\ &= \frac{1}{h^2}\int_0^h \frac{h-v}{h}\left(\int_0^v 2udu\right) dv \\ &= \frac{1}{h^3}\int_0^h (h-v)v^2 dv \\ &= \frac{h}{12}\end{aligned}$$

The above finally yields the following scheme:

$$A_T \approx \bar{A}_T^{p,m} = \frac{1}{m}\sum_{k=0}^{m-1} \bar{S}_{t_k}^m \left(1 + \frac{rh}{2} + \sigma \bar{I}_k^m\right) \tag{3}$$

Considering the value of μ_k, one can note this scheme is the same as the previous one except the fact that (3) adds a random term with distribution $\mathcal{N}(0, \frac{h}{12})$ in the multiplicative factor. To put it differently, in comparison with (3), (2) approximates $\bar{I}_k^m$ by its mean.

Algorithm 1: Scheme (3) implementation

Data: n (number of independent simulations), m (number of time steps)
Result: Estimation and 95% confidence interval
for $i = 1, \ldots, n$ **do**
  Simulate $\bar{W}^{m,i}$;
  Deduce $\bar{S}^{m,i}$ using Black-Scholes formula;
  for $k = 0, \ldots, m-1$ **do**
    Simulate $\bar{I}_k^{m,i}$ (conditionally to $\bar{W}^{m,i}$);
  end
  Compute $\bar{A}_T^{m,i}$ with $\bar{S}^{m,i}$, $\bar{W}^{m,i}$ and $\bar{I}_0^{m,i}, \ldots, \bar{I}_{m-1}^{m,i}$;
end
Compute the mean and standard error of the prices given by $\bar{A}_T^{m,1}, \ldots, \bar{A}_T^{m,n}$;
return *the MC estimate and* 95% *confidence interval for the price*;

Results

Before going further, we suggest comparing the results given by schemes (1)-(3).

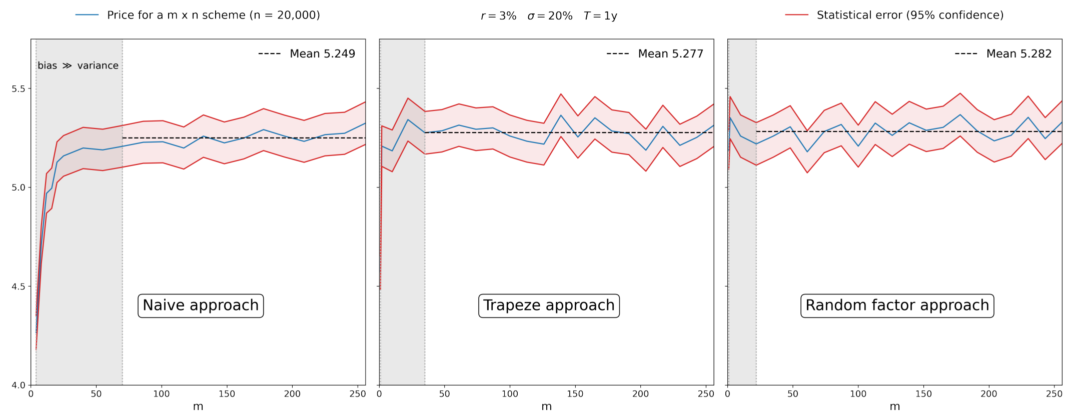

Figure 2: Convergence of schemes (1)-(3) for different values of m

The above charts show the 95% confidence intervals given by the three methods for fixed n and m varying up to 256. As we do not have an analytical solution, it is difficult to compare the impact of the variance and that of the bias by comparing the error to the confidence intervals for different values of m. However, since the proposed methods are asymptotically unbiased, the plotted black dotted lines are good approximations of the solution and we can compare the fluctuations of the blue line with the width of the confidence intervals, in red. In the grey area, that is to say for low values of m, the bias is high compared to the variance. For greater values of m, a plateau seems to be reached and the observed deviations are mainly caused by the variance (the bias becomes negligible). Thus, these charts allow to choose m: the one at the edge of the grey zone is the most computationally-effective one that offers a satisfactory level of accuracy. As explained before, there is actually no point in choosing higher values of m since the decrease in the bias would be hidden by the variance. Consequently, one can already notice the advantage of the two improved methods on the right-hand side in comparison with the naive approach: m can be chosen to be significantly lower while achieving as good results. They allow better time complexity. Note however that their space complexity is worse. In particular, the third method requires to simulate an additional $n \times m$ matrix of independent, normally distributed coefficients.

2.2 The use of a control variate

In order to improve the convergence speed, Lambert et al. [3] finally propose a variance reduction technique for the three schemes above. It uses a control variate as introduced by Kemna et al. [2].

$$\theta = \sum_{i=1}^{n} \left(e^{-rT} \left[A_T^i - K \right]_+ + \beta \left(Z^i - \mathbb{E}[Z] \right) \right) \tag{$*$}$$

Observing that $e^x \approx 1 + x$ and $\ln(1 + x) \approx x$ when $|x|$ is small, the idea relies on the approximation:

$$A_T = \frac{1}{T} \int_0^T S_u du \approx \exp \left\{ \frac{1}{T} \int_0^T \ln S_u du \right\} = S_0 \exp \left\{ \frac{1}{T} \int_0^T \ln \frac{S_u}{S_0} du \right\}$$

The equality on the right-hand side justifies the validity of such approximation: if r and σ are small, S_u can be expected to remain near S_0 and $\ln \frac{S_u}{S_0} \ll 1$. Therefore, we would like to use the following as a control variate in the case of a fixed-strike Asian call:

$$Z = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_u du\right\} - K\right]_+$$

Note that we can indeed compute the exact expression of $\mathbb{E}[Z]$. First, Itô's lemma gives $d(tW_t) = tdW_t + W_t dt$ and:

$$\frac{1}{T}\int_0^T W_u du = \frac{1}{T}\int_0^T (T-s)dW_s \sim \mathcal{N}\left(0, \frac{T}{3}\right) \quad \text{because} \quad \int_0^T (T-s)^2 ds = \frac{T^3}{3}$$

If we note $a := \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}$, $b := \sigma\sqrt{\frac{T}{3}}$, $\rho := \frac{K}{S_0}$, $x^* := \frac{\ln\rho - a}{b}$ and N the c.d.f of $\mathcal{N}(0,1)$ then:

$$\begin{aligned}
\frac{\mathbb{E}[Z]}{e^{-rT}S_0} &= \int_{x^*}^{+\infty}\left(e^{a+bx} - \rho\right)N'(x)dx \\
&= e^{a+\frac{b^2}{2}}\int_{x^*-b}^{+\infty} N'(u)du - \rho\int_{x^*}^{+\infty} N'(x)dx \quad \text{with} \quad u = x - b \\
&= e^{a+\frac{b^2}{2}}N(b - x^*) - \rho N(-x^*)
\end{aligned}$$

On can check that the same computations yield Black-Scholes formula for the price of a European call. Thus, we have the analytical expression. We finally need to provide a way to simulate the control variate Z in each scenario. We define:

$$\bar{Z}^{r,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{m}\sum_{i=0}^{m-1}\bar{W}^m_{t_k}\right\} - K\right]_+ \tag{i}$$

Plugging $\bar{A}^{r,m}_T$ from scheme (1) and $\bar{Z}^{r,m}_T$ in $(*)$ yields a new scheme:

$$e^{-rT}\left[\bar{A}^{r,m}_T - K\right]_+ + \hat{\beta}_r\left(\bar{Z}^{r,m}_T - \mathbb{E}[Z]\right) \tag{4}$$

where $\hat{\beta}_r$ is estimated with the empirical covariance of the variables.

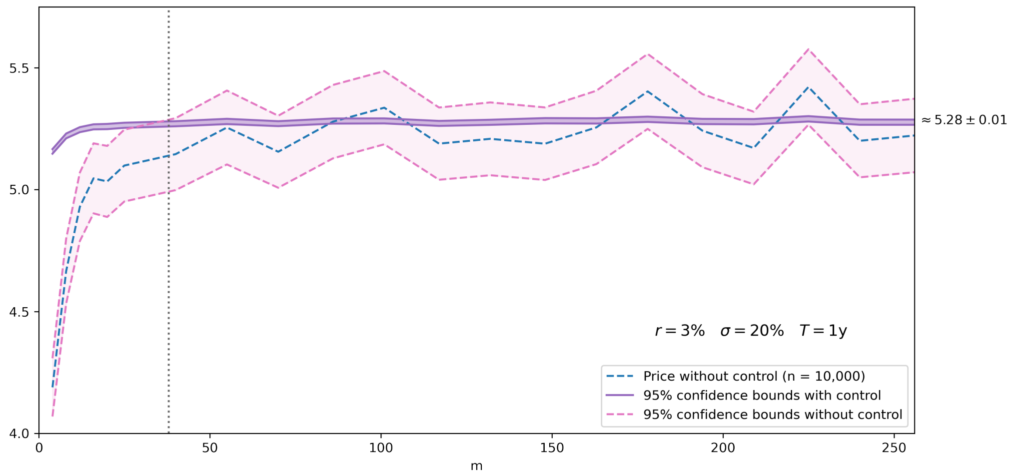

Figure 3: Convergence of scheme (4) for different values of m

As expected, the statistical error (ie: the width of the confidence interval) is much more narrow. With this set of parameters, it is about ten times smaller. The result is quite impressive. Besides, it does not only allow to take a lower n with better results but also to use a lower m.

Description to be completed.

Considering the assumptions this method relies on, one could wonder whether this variance reduction technique is still as efficient when the parameters (r, σ or T for instance) are high. We investigate this question in the next subsection.

2.3 The combination of both improvements

As we did with scheme (1), we can plug the estimates of schemes (2) and (3) in $(*)$. That requires to adapt the estimates of Z too.

$$\bar{Z}^{e,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{n}\sum_{i=0}^{n-1}\frac{\bar{W}^m_{t_{k+1}} + \bar{W}^m_{t_k}}{2}\right\} - K\right]_+ \tag{ii}$$

$$\bar{Z}^{p,m} = e^{-rT}\left[S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{n}\sum_{i=0}^{n-1}\int_{t_k}^{t_{k+1}}\left(\bar{W}^m_u - \bar{W}^m_{t_k}\right)du\right\} - K\right]_+ \tag{iii}$$

It yields two new schemes.

2.4 A possible step further

Adapt to other diffusion processes. **To be completed**. However, it is not straightforward to adapt the variance reduction technique.

3 PDE approaches

Conclusion

References

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