Deflection at any point in cantilever beam using Castigliano's theorem

Strain energy techniques are frequently used to analyze the deflection of beam and structures. Castigliano's theorem were developed by the Italian engineer Alberto Castigliano in the year 1873, these theorems are applicable to any structure for which the force deformation relations are linear.

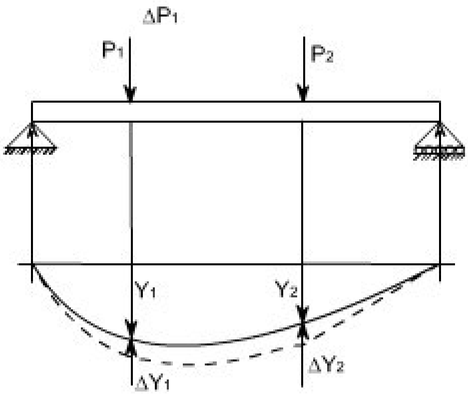


Figure (1)

Consider a loaded beam as shown in figure. Let the two Loads P1 and P2 produce deflections Y1 and Y2 respectively strain energy in the beam is equal to the work done by the forces.

$$\boxed{U = \frac{1}{2}P_1Y_1 + \frac{1}{2}P_2Y_2} \qquad .....(1)$$

Let the Load P1 be increased by an amount Δ P1.

Let Δ P1 and Δ P2 be the corresponding changes in deflection due to change in load to Δ P1. Now the increase in strain energy:

$$\Delta U = \frac{1}{2}\Delta P_1\Delta Y_1 + P_1\Delta Y_1 + P_2\Delta Y_2 \qquad ....(2)$$

Suppose the increment in load is applied first followed by P1 and P2 then the resulting strain energy is

$$\boxed{U+\Delta U = \frac{1}{2}\Delta P_1\Delta Y_1 + \Delta P_1 Y_1 + P_2\Delta Y_2 \frac{1}{2}P_1Y_1 + \frac{1}{2}P_2 Y_2} \qquad ......(3)$$

Since the resultant strain energy is independent of order loading, combining equation 1, 2 and 3. One can obtain:

$$\Delta P_1 Y_1 = P_1 \Delta Y_1 + P_2 \Delta Y_2 \quad .....(4)$$

equations (2) and (4) can be combined to obtain

$$\frac{\Delta U}{\Delta P_1} = y_1 + \frac{1}{2} \Delta Y_1 \quad .....(5)$$

or upon taking the limit as DP1 approaches zero [Partial derivative are used because the strain energy is a function of both P1 and P2]

$$\frac{\partial U}{\partial P} = Y_1 \quad ....(6)$$

For a general case there may be number of loads, therefore, the equation (6) can be written as

$$\frac{\partial U}{\partial P_i} = Y_i \quad ....(7)$$

The above equation is castigation's theorem.

The statement of this theorem can be put forth as follows; if the strain energy of a linearly elastic structure is expressed in terms of the system of external loads. The partial derivative of strain energy with respect to a concentrated external load is the deflection of the structure at the point of application and in the direction of that load.

In a similar fashion, Castigliano's theorem can also be valid for applied moments and resulting rotations of the structure:

$$\frac{\partial U}{\partial M_i} = \theta_i \quad ....(8)$$

Where,

Mi = applied moment

θi = resulting rotation

Strain Energy in Bending:

Consider a beam AB subjected to a given loading as shown in figure.

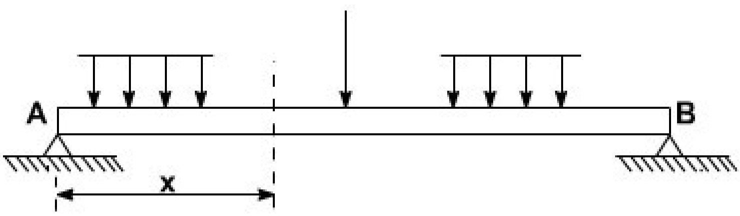


figure (2)

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy, i.e.

$$U = \int \frac{\sigma^2}{2E} dv$$

$$= \int \frac{M^2 . y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dV=dx.dA

Where dA= elemental cross- sectional area

$\frac{M^2 . y^2}{2EI^2}$ is a function of x alone

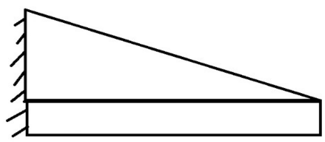
Now substituting for dy in the expression of U.

$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx \qquad .....(11)$$

We know ∫y2 dA represents the moment of inertia 'I' of the cross section about its neutral axis.

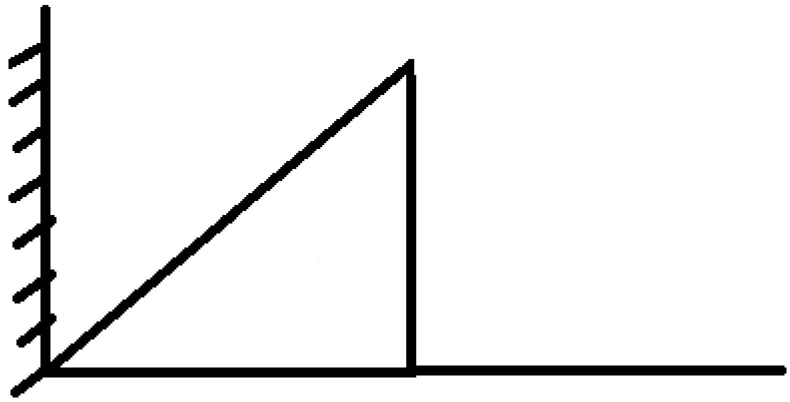
$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

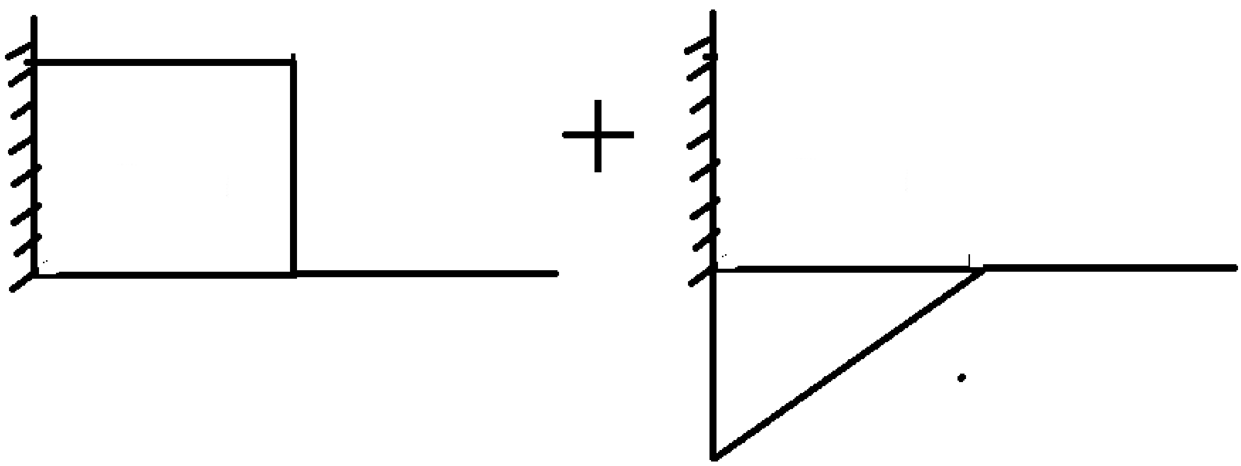
Deflection in the cantilever beam at any point under various loads (point load, uniformly distributed load, and uniformly varying load) is calculated with the help of the code written.
Initially input are taken in order as, point load, UDL, UVL and moment.
Point loads and their respective locations are compiled in form of a matrix of n*2.
Point load=[load, location], where laod and location represent the column containing point load and their respective locations.
UDL loads and their respective locations along with length of UDL are compiled in the form below:
UDL= [UDL load, UDL length, location 1, location 2]
Where coumns, location 1 and location 2 represents the location of edges of UDLs.
For UVL, load value, UVL length and their edge locations are compiled in another matrix in the form given below:
UVL=[load1, load2, UVL length, location1, location2]
If UVL is in this form:



It in converted into UDL+ UVL form:

UVL loads, in which point near fixed end is zero as below in converted as below:



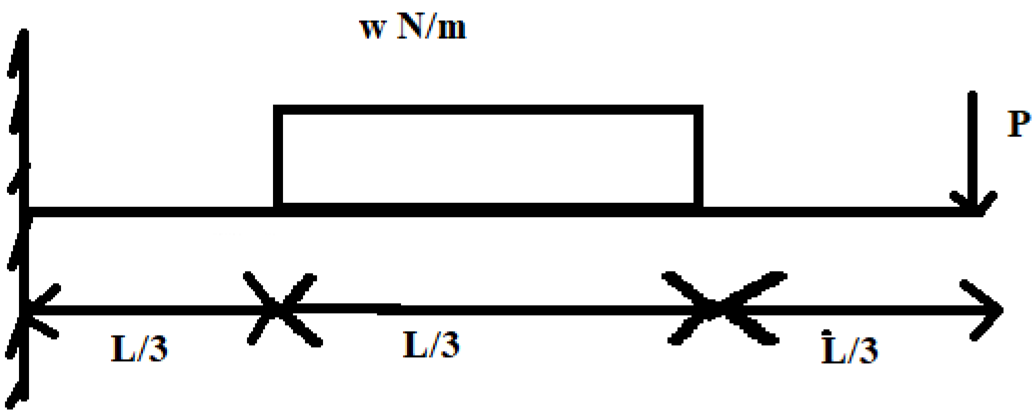


For Moment: Moment whose direction is outward plane (Right hand rule) is taken positive and vice versa.

After compiling all the loads and moments in their respective matrices, moment is calculated taking location of calculating deflection into consideration.
Then by using equation (7) and equation (12) deflection at any point can be calculated.

These are two problems which are calculated with help of the mentioned algorithm.

Problem 1: Determine the deflection at the free and of the cantilever beam with a uniform load w, acting over the middle third of the length and a point load acting at the free end.

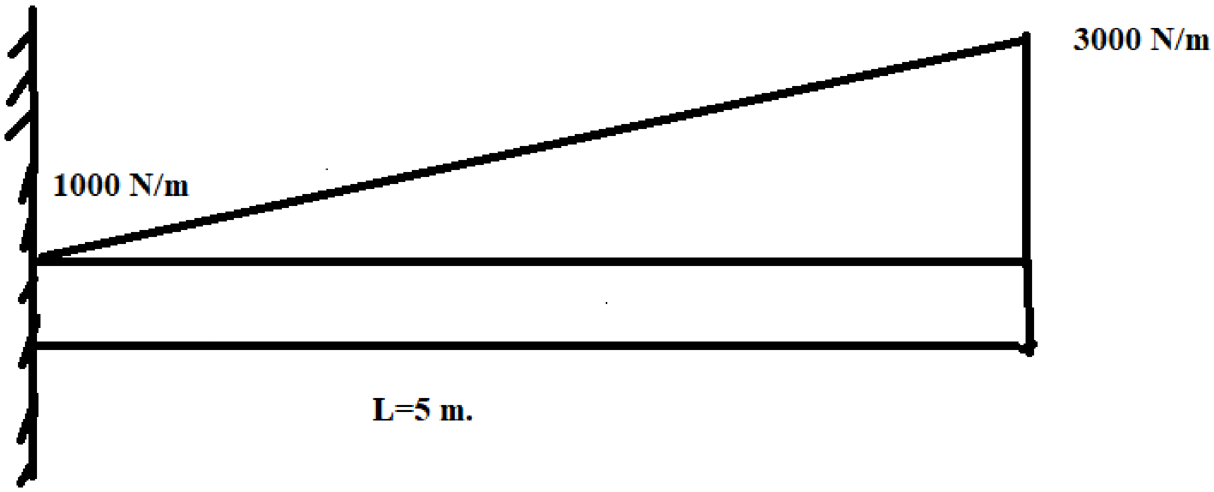


Where w=5000 N/m, P=1000 N, L= 1 m, E=200*10^{9} N/m^{2},I=6.667*10 m^{4}.

Solution: By manually calculating deflection at free end= 3.5415*10^{-5} m.

Deflection with the written code= 3.5373*10^-5 m.

Problem 2: Determine the deflection at the free and of the cantilever beam with a UVL acting over entire length (load values are mentioned in the figure.)



Solution: Deflection calculated manually= 0.1445 m.

deflection calculated with the written code = 0.145 m.