Office Hours 9/17/25

charges ~ age + bmi + children

$Y \quad x_1 \quad x_2 \quad x_3$

Partial F-test

$$\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$$

$$\hat{Y} \sim N(X\beta, \sigma^2 X (X^T X)^{-1} X^T)$$

$$= N(X\beta, \sigma^2 H)$$

CI for $\hat{Y}_n$ at a specific point x_n

Ex: $\hat{Y}_h = \hat{\beta}_0 + \hat{\beta}_1 age_i + \hat{\beta}_2 bmi_i + \hat{\beta}_3 children_i$

Make a CI for the mean charges in med. bills for a general pt who is

40 y.o, has a bmi of 20, & has 1 kid.

$$x_n = \begin{bmatrix} 1 \\ 40 \\ 20 \\ 1 \end{bmatrix}$$

$$\hat{y}_n = \underbrace{x_n^T}\hat{\beta} = \underbrace{(1)(\hat{\beta}_0) + (40)(\hat{\beta}_1) + (20)(\hat{\beta}_2) + (1)(\hat{\beta}_3)}_{g}$$

How can I make a CI for this?

$$\hat{SE}(\hat{y}_n) = \sqrt{\hat{Var}(\hat{y}_n)} = \sqrt{\hat{Var}(x_n^T\hat{\beta})}$$

$$= \sqrt{\hat{\sigma}^2 x_n^T (X^TX)^{-1} x_n}$$

$$= \hat{\sigma}\sqrt{x_n^T(X^TX)^{-1}x_n}$$

CI: 95% CI

$$\hat{y}_n \pm t^*_{n-p,\,0.975}\,\hat{\sigma}\sqrt{x_n^T(X^TX)^{-1}x_n}$$

$\boxed{Var(AY) = A\,Var(Y)\,A^T \ \ldots \ \text{here } A = x_n^T \ \& \ Y = \hat{\beta}}$

PI: 95% PI

$$\hat{y}_n \pm t^*_{n-p,\,0.975}\,\hat{\sigma}\sqrt{\underline{\underline{1}} + x_n^T (X^T X)^{-1} x_n}$$

Eigendecomposition of H

$$H = X(X^T X)^{-1} X^T$$

↳ H is a proj. matrix b/c

① $H = H^T$

② $H^2 = H$

Property: All eigenvalues of projection matrices are either 0 or 1.

Why?

$H = U \Lambda U^T$ where U = orthonormal ($\Leftrightarrow U^T U = I$ & $U U^T = I$)

& $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

↑ eigenvals.

$$H^2 = H$$

$$(U\Lambda U^T)(U\Lambda U^T) = U\Lambda U^T$$

$$\Rightarrow (U\Lambda \cancel{U^T U}\Lambda U^T) = U\Lambda U^T$$

$$\Rightarrow (U\Lambda^2 U^T) = U\Lambda U^T$$

$$\Rightarrow \cancel{U^T U}\Lambda^2 \cancel{U^T U} = \cancel{U^T U}\Lambda \cancel{U^T U}$$

$$\Rightarrow \Lambda^2 = \Lambda$$

$$\Rightarrow \lambda_1^2 = \lambda_1 \; \& \; \lambda_2^2 = \lambda_2 \cdots \quad \lambda_n^2 = \lambda_n$$

$$\Rightarrow \lambda_1^2 = \lambda_1 \Rightarrow \lambda_1^2 - \lambda_1 = 0$$

$$\underset{p\times n}{X^T}\,\underset{n\times p}{X}$$

$$\Rightarrow \lambda_1(\lambda_1 - 1) = 0$$

$$\Rightarrow \lambda_1 = 0 \text{ or } \lambda_1 - 1 = 0$$

$$\Leftrightarrow \lambda_1 = 1$$

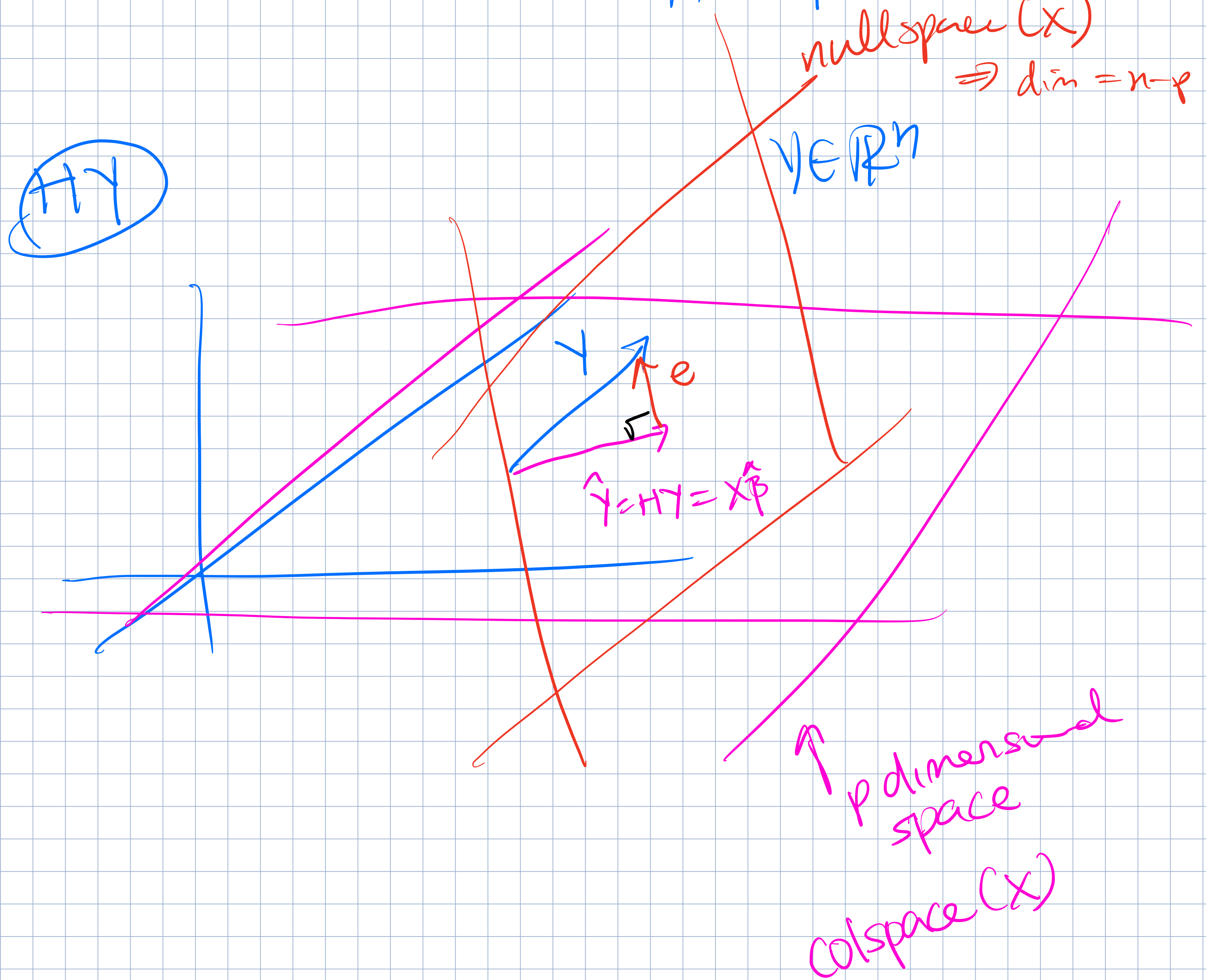
rank(H) = # of non zero eigenvals of H

$$\text{rank}(H) = tr(H) = tr(X(X^TX)^{-1}X^T)$$

idempotenty

$$= tr((X^TX)^{-1}X^TX)$$

$$= tr(I_p) = p$$



HY
nullspace(X)
⇒ dim = n−p
Y∈ℝⁿ
Y
e
Ŷ=HY=Xβ̂
p dimensional space
colspace(X)

$\beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 Z_i$

Q.2

a. $Y_i = \underline{\beta_1} X_{1i} + \underline{\beta_2} X_{2i} + \underline{\beta_3} X_{1i}^2 + \varepsilon_i$ ← Z_i

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \underbrace{\begin{bmatrix} X_{11} & X_{21} & X_{11}^2 \\ & \vdots & \\ X_{1i} & X_{2i} & X_{1i}^2 \\ & \vdots & \\ X_{1n} & X_{2n} & X_{1n}^2 \end{bmatrix}}_{= X} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}}_{\beta} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Q.23

$Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$

$E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = \sigma^2$

$$Q(\beta) = Q(\beta_1, \beta_2) = \sum_{i=1}^{n} (Y_i - \beta_1 X_{1i} - \beta_2 X_{2i})^2$$

$$\boxed{\frac{\partial Q}{\partial \beta_1} \overset{!}{=} 0 \quad \& \quad \frac{\partial Q}{\partial \beta_2} \overset{!}{=} 0 \quad \leftarrow \cdots \hat{\beta}_1 \quad \hat{\beta}_2 \cdots}$$

Define $X = \begin{bmatrix} x_{11} & x_{21} \\ \vdots & \\ x_{1n} & x_{2n} \end{bmatrix}$ & $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$

$\langle Y, X\beta \rangle$
$\shortparallel$
$\langle X\beta, Y \rangle$
$\shortparallel$
$\beta^T X^T Y$

then $Q(\beta) = (Y - X\beta)^T (Y - X\beta)$

$= \|Y - X\beta\|^2$

$$\frac{\partial Q}{\partial \beta} = \frac{\partial}{\partial \beta}\Big(Y^T Y - \underbrace{Y^T X\beta}_{\overset{\beta^T X^T Y}{\shortparallel}} - \underbrace{(X\beta)^T Y}_{\overset{\beta^T X^T Y}{\shortparallel}} + (X\beta)^T X\beta \Big)$$

$$= \frac{\partial}{\partial \beta}(Y^T Y) - \frac{\partial}{\partial \beta}(2\beta^T X^T Y) + \frac{\partial}{\partial \beta}(\beta^T X^T X \beta)$$

$$= 0 - 2X^T Y + 2X^T X \beta \overset{!}{=} 0$$

$$\Rightarrow (X^T X)\beta = X^T Y$$

$$\Rightarrow \hat{\beta} = (X^T X)^{-1} X^T Y$$

$\frac{\partial(\beta^T A)}{\partial \beta} = A$

" $A\beta^2 = 2A\beta$ "

$\frac{\partial(\beta^T A \beta)}{\partial \beta} = 2A\beta$

A sym

... plugs in $X = \begin{pmatrix} x_{11} & x_{21} \\ \vdots & \\ x_{1n} & x_{2n} \end{pmatrix}$ ...

$$Y_i = \beta_0 + \beta_1 X_{1i} + \boxed{\beta_2 X_{2i}} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i$$

" $Y \sim X_1 + X_2 + X_3 + X_4$ "

Given $\beta_2 = 4$

New model:

$$Y_i = \beta_0 + \beta_1 X_{1i} + 4 X_{2i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i$$

$$Y_i - 4X_{2i} = \beta_0 + \beta_1 X_{1i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i$$

//

$\tilde{Y}_i$

$$\tilde{Y} = \beta_0 + \beta_1 X_{1i} + \beta_3 X_{3i} + \beta_4 X_{4i} + \varepsilon_i$$

Define $\tilde{Y} = \begin{bmatrix} Y_1 - 2X_{21} \\ \vdots \\ Y_n - 2X_{2n} \end{bmatrix}$ &

$X = \begin{bmatrix} 1 & X_{11} & X_{31} & X_{41} \\ \vdots & \vdots & & \\ 1 & X_{1n} & \cdots & X_{4n} \end{bmatrix}$ &

$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_3 \\ \beta_4 \end{bmatrix}$ ... solve like usual

$$\hat{\beta} = (X^T X)^{-1} X^T \tilde{Y}$$

or

" $\tilde{Y} \sim X_1 + X_3 + X_4$ "