Office Hours 9/24/25

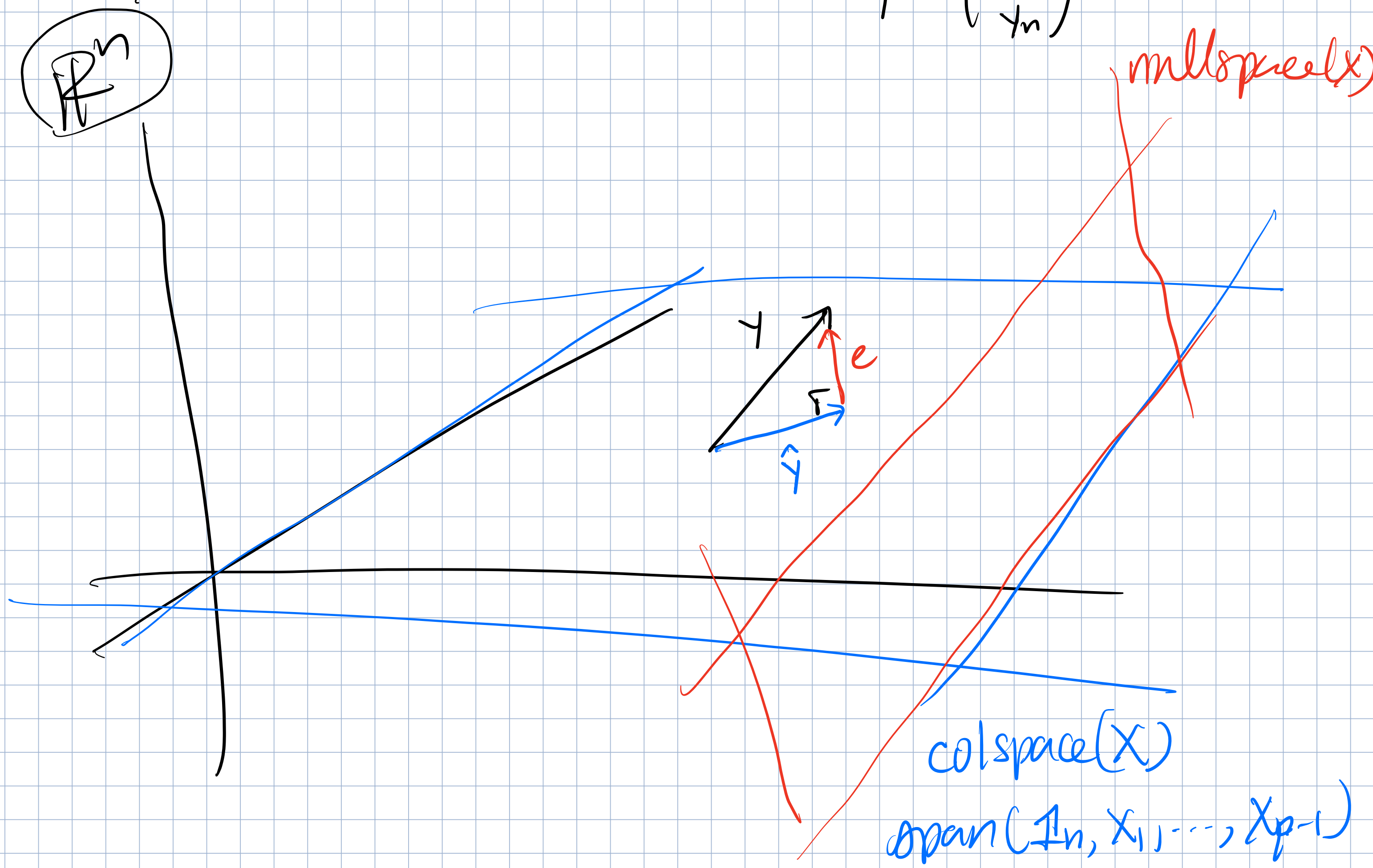
$$\underline{\underline{Y}} \sim N(X\beta, \sigma^2 I_n)$$

$$Var(Y) = E\Big(\underbrace{\underset{n\times 1}{(Y-\mu)}\,\underset{1\times n}{(Y-\mu)^T}}_{n\times n}\Big)$$

$$= \begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) & Cov(Y_1, Y_3) & \cdots & Cov(Y_1, Y_n) \\ \vdots & Var(Y_2) & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & \vdots \\ \cdots & \cdots & \cdots & \cdots & Var(Y_n) \end{bmatrix}$$

SS Decomposition

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

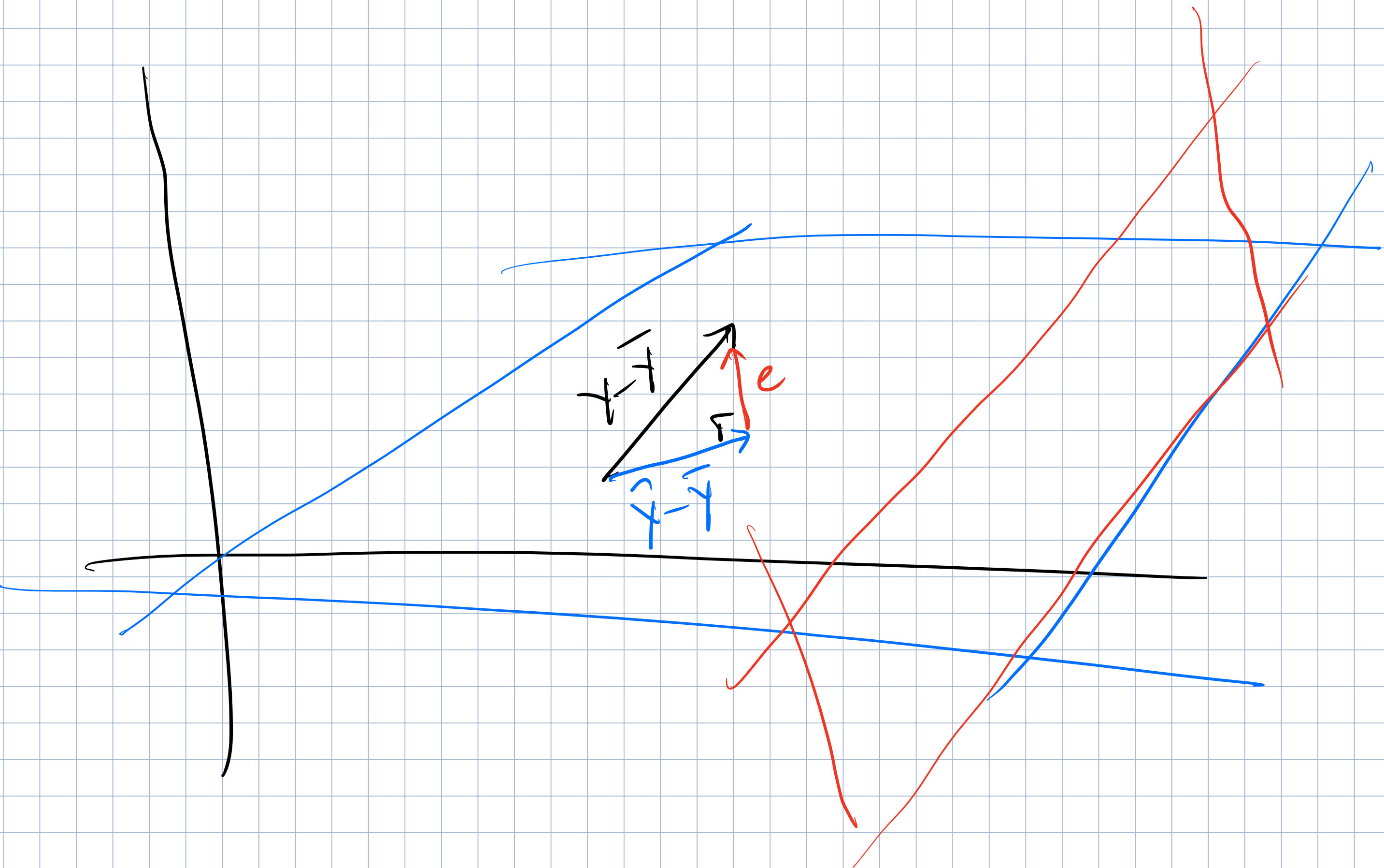


$$\bar{Y} = \begin{pmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix} = \bar{y}\,\mathbb{1}_n$$

$$\hat{Y} = X\hat{\beta} = \underline{X(X^TX)^{-1}X^T}Y$$

$$= HY$$

$$Y = \hat{Y} + e$$



$$Y - \bar{Y} = \hat{Y} - \bar{Y} + e$$

Use Pythagorean theorem:

$$\|Y - \bar{Y}\|^2 = \|\hat{Y} - \bar{Y}\|^2 + \|e\|^2$$

$$SST = SSR + SSE$$

Rigorous Pf:

$$\frac{1}{n}\underbrace{\mathbb{1}_n \mathbb{1}_n^T}_{A} Y = \begin{pmatrix} \frac{1}{n}\sum_i y_i \\ \frac{1}{n}\sum_i y_i \\ \vdots \\ \frac{1}{n}\sum_i y_i \end{pmatrix}$$

$$Y-\bar{Y} = IY - AY$$
$$= (I-A)Y$$

$$\hat{Y}-\bar{Y} = HY - AY = (H-A)Y$$

$$Y-\hat{Y} = IY - HY = (I-H)Y$$

$$Y-\bar{Y} = Y-\hat{Y}+\hat{Y}-\bar{Y}$$

$$(I-A)Y = (I-H)Y + (H-A)Y$$
$$\shortparallel$$
$$IY - \cancel{HY + HY} - AY$$

Take the norm of "square" both sides

$$SST$$
$$\shortparallel$$
$$\|Y-\bar{Y}\|^2$$

$$\|(I-A)Y\|^2 = \|(I-H)Y + (H-A)Y\|^2$$
$$= \left((I-H)Y + (H-A)Y\right)^T \left((I-H)Y + (H-A)Y\right)$$

$$= \overbrace{\|(I-H)Y\|^2}^{SSE} + \underbrace{\cancel{[(I-H)Y]^T[(H-A)Y]}}_{\to 0} + \underbrace{\cancel{[(H-A)Y]^T[(I-H)Y]}}_{\to 0} + \underbrace{\|(H-A)Y\|^2}_{SSR}$$

WTS these are 0

$$[(I-H)Y]^T (H-A)Y$$

$$= Y^T (I-H)^T (H-A) Y$$

$$= Y^T [(I-H)(H-A)] Y$$

$$= Y^T [H - A - H^2 + HA] Y$$

$$= Y^T [H - A - H + HA] Y$$

$$= Y^T [H - A - H + A] Y = Y^T 0 Y = 0$$

$$\underline{HA} = \underbrace{X(X^TX)^{-1}X^T}_{\text{proj onto colspace of } X} \frac{1}{n} 1_n 1_n^T$$

$= \mathbb{1}_n$

$= \frac{1}{n} (H \mathbb{1}_n) \mathbb{1}_n^T$

b/c $\mathbb{1}_n \in$ colspace (X)

$= \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^T Y = \underline{\underline{A}}$

	$E(\cdot)$	$Var(\cdot)$	CI	Hyp Test.
Y	$\boxed{X\beta}$	$\sigma^2 I_n$	X	X
$\hat\beta$	β	$\underline{\sigma^2 (X^T X)^{-1}}$		
$\boxed{\hat\beta_j}$	...	...	$\hat\beta_j \pm t_{n-p}^{(1-\alpha/2)} \hat\sigma \sqrt{(X^T X)^{-1}_{(j+1, j+1)}}$	
$\hat Y$	$X\beta$	$\sigma^2 H$		
$\boxed{\hat y_n}$	$x_n^T \beta$	$\underline{\sigma^2 x_n^T (X^T X)^{-1} x_n}$		
e	$\underline{\underline{0}}$	$\underline{\underline{\sigma^2 (I - H)}}$	X	X

$\beta_0 + \beta_1 X_{1i} + \cdots + \beta_{p-1} X_{p-1,i}$

$$E(\hat{\beta}) = E(\underline{(X^TX)^{-1}X^TY}) =$$

$$= (X^TX)^{-1}X^T E(Y) = (X^TX)^{-1}X^T(X\beta)$$

$$= \cancel{(X^TX)^{-1}}\, \cancel{X^TX}\,\beta = I\beta = \beta$$

$A = \mathcal{X}_n^T$

$\boxed{Var(AY) = A\,Var(Y)A^T}$

$$Var(\hat{\beta}) = Var(\underbrace{(X^TX)^{-1}X^T}_{A}Y)$$

$$= [(X^TX)^{-1}X^T]\,Var(Y)\,[(X^TX)^{-1}X^T]^T$$

$$= [(X^TX)^{-1}X^T][\sigma^2 I_n][(X^T)^T((X^TX)^{-1})^T]$$

$$= \sigma^2 (X^TX)^{-1}X^T(X[(X^TX)^T]^{-1})$$

$$= \sigma^2 \cancel{(X^TX)^{-1}}\,\cancel{X^TX}\,(X^TX)^{-1}$$

$(j+1, j+1)$

$$= \sigma^2 \underline{(X^TX)^{-1}} = \begin{pmatrix} \boxed{Var(\hat{\beta}_0)} & & & \\ & Var(\hat{\beta}_1) & & \\ & & \ddots & \\ & & & \boxed{Var(\hat{\beta}_j)} \end{pmatrix}$$

$\hat{A}$

CI for $\underline{\hat{\beta}_j}$? $j = 0, 1, 2, \dots, p-1$

$$\hat{\beta}_j \pm \underbrace{\text{crit val}} \times \widehat{SE}(\hat{\beta}_j)$$

$$\underline{\widehat{SE}}(\hat{\beta}_j) = \sqrt{\widehat{Var}(\hat{\beta}_j)} = \sqrt{\hat{\sigma}^2 \underbrace{(X^TX)^{-1}_{(j+1,j+1)}}_{j+1^{st} \text{ diagonal of } (X^TX)^{-1}}}$$

$$\hat{\sigma}^2 = \frac{SSE}{n-p} = \frac{e^Te}{n-p} = \frac{\sum_i e_i^2}{n-p}$$

$$\hat{\beta}_j \pm \underbrace{t^*_{(n-p)}(1-\tfrac{\alpha}{2})} \cdot \hat{\sigma} \sqrt{\underbrace{(X^TX)^{-1}_{(j+1,j+1)}}}$$

Ex: $j=2$

$$(X^TX)^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 3 & 3 & 9 \end{bmatrix}$$

Hyp Test:

$$t = \frac{\hat{\beta}_j - \beta_j^*}{\hat{SE}(\hat{\beta}_j)} = \frac{\hat{\beta}_j - \beta_j^*}{\hat{\sigma}\sqrt{(X^TX)^{-1}_{(j+1),(j+1)}}}$$

$$E(\hat{Y}) = E(\underline{X\hat{\beta}}) = E(\underline{X(X^TX)^{-1}X^T}Y)$$

$$= E(\underline{H}Y)$$

$$= HE(Y) = H[X\beta]$$

$$= HX\beta$$

$$= X\cancel{(X^TX)^{-1}(X^TX)}\beta$$

$$= X\beta$$

$$Var(\hat{Y}) = Var(HY) = H\,Var(Y)H^T$$

$$= H(\sigma^2 I_n)H = \sigma^2 H^2 = \sigma^2 H$$

$$\hat{Y} = \begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_h \\ \vdots \\ \hat{Y}_n \end{pmatrix}$$

$$\hat{y}_h = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \cdots + \hat{\beta}_{p-1} x_{p-1,i}$$

$$\Updownarrow$$

$$\hat{y}_h = x_h^T \hat{\beta}$$

$$E(\hat{y}_h) = E(x_h^T \hat{\beta}) = x_h^T E(\hat{\beta})$$

$$= x_h^T \beta$$

$$\underline{Var(\hat{y}_h)} = Var(\underline{x_h^T} \hat{\beta}) = \underline{x_h^T} Var(\hat{\beta}) (\underline{x_h^T})^T$$

$$= x_h^T \sigma^2 (X^T X)^{-1} x_h$$

$$= \sigma^2 x_h^T (X^T X)^{-1} x_h$$

$$X = \begin{pmatrix} 1 & x_{11} & x_{21} & \cdots & x_{(p-1)1} \\ 1 & x_{12} & x_{22} & & \\ \vdots & & & & \\ 1 & x_{1h} & x_{2h} & & x_{(p-1)h} \\ \vdots & & \vdots & & \\ 1 & x_{1n} & x_{2n} & & x_{(p-1)n} \end{pmatrix} \quad x_h^T$$

$$\underline{\underline{x_h}} = \begin{pmatrix} 1 \\ x_{1h} \\ x_{2h} \\ \vdots \\ x_{p-1h} \end{pmatrix}$$

CI for $E(\hat{Y}_h | x_h) = x_h^T \beta$

$$\hat{y}_h \pm t^*_{n-p}(1-\alpha/2)\, \widehat{SE}(\hat{y}_h)$$

$$\widehat{SE}(\hat{y}_h) = \sqrt{\widehat{Var}(\hat{y}_h)} = \sqrt{\hat{\sigma}^2 x_h^T (X^T X)^{-1} x_h}$$

$$= \hat{\sigma}\sqrt{x_n^T(X^TX)^{-1}x_n}$$

$$\hat{y}_n \pm t^*_{n-p}(1-\alpha/2)\,\hat{\sigma}\sqrt{x_n^T(X^TX)^{-1}x_n}$$

Hyp Test

H_0: $E(y_n|x_n) = x_n^T\beta^*$

H_1: $\ldots \neq x_n^T\beta^*$

$$t = \frac{\hat{y}_n - x_n^T\beta^*}{\widehat{SE}(\hat{y}_n)} = \frac{\hat{y}_n - x_n^T\beta^*}{\hat{\sigma}\sqrt{x_n^T(X^TX)^{-1}x_n}}$$

$$E(e) = E(Y - \hat{Y}) = E((I-H)Y)$$

$$= E(Y) - E(\hat{Y})$$

$$= X\beta - X\beta$$

$$= 0$$

$$= (I-H)E(Y)$$

$$= (I-H)X\beta$$

$$= X\beta - HX\beta$$

$$= X\beta - X\beta = 0$$

$$\mathrm{Var}(e) = \mathrm{Var}((I-H)Y) = (I-H)\,\mathrm{Var}(Y)\,(I-H)^T$$

$$= (I-H)(\sigma^2 I_n)(I-H)^T$$

$$= \sigma^2 (I-H)(I-H)$$

$$= \sigma^2 (I-H)$$