Properties of Matrices

① $AB \neq BA$

② $(AB)^T = B^T A^T$

③ $(AB)^{-1} = B^{-1} A^{-1}$ (why?)

$$\left[\begin{array}{l} (AB)(AB)^{-1} = I \\ ABB^{-1}A^{-1} \\ AA^{-1} = I \end{array}\right.$$

④ $(A^T)^{-1} = (A^{-1})^T$

⑤ $\left((AB)^T\right)^{-1} = \left((AB)^{-1}\right)^T$

$$\begin{array}{ccc} \left((AB)^T\right)^{-1} & = & \left((AB)^{-1}\right)^T \\ \parallel & & \parallel \\ (B^T A^T)^{-1} & = & (B^{-1}A^{-1})^T \end{array}$$

⑥ rank(A) = # of pivots in $\underline{\underline{\text{RREF}}}$

= dim of the image of the linear map of A

⑦ For idempotent A:

$$\text{rank}(A) = tr(A)$$

⑧ trace: $tr(A) = \sum_i a_{ii}$ = sum of diagonal elements

⑨ symmetry: $A^T = A$

⑩ idempotency: $A^2 = A$

⑪ orthonormal matrix: U has $U^T U = I$

& $UU^T = I$

$$\begin{array}{cc} U^T & U \\ \begin{bmatrix} - u_1^T - \\ - u_2^T - \\ \vdots \\ - u_p^T - \end{bmatrix} & \begin{bmatrix} u_1 & u_2 & \cdots & u_p \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

⑫ Every square matrix has an eigendecomposition:

$$A = U \Lambda U^T$$

Λ contains the eigenvalues on its diagonal:

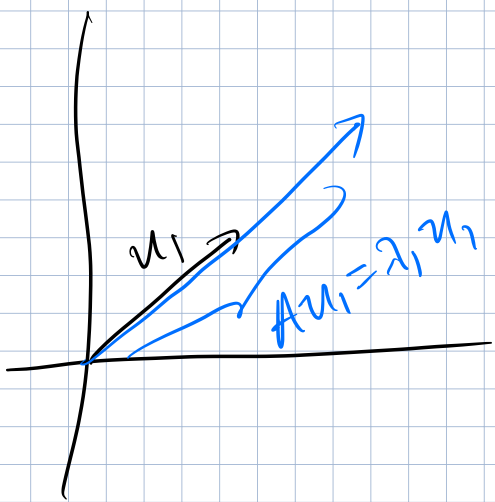
$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$$

$$= \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_p \end{bmatrix}$$

U holds the eigenvectors

$$U = [u_1 \cdots u_p]$$

$$\underline{Au_1} = U\Lambda U^T u_1 = U\Lambda \begin{bmatrix} — u_1^T — \\ — u_2^T — \\ \vdots \\ — u_p^T — \end{bmatrix} u_1$$



$$= U\Lambda \begin{bmatrix} u_1^T u_1 \\ u_2^T u_1 \\ \vdots \\ u_p^T u_1 \end{bmatrix} = U\Lambda \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= U \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_p \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= U \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 u_1 + 0u_2 + 0u_3 + \cdots + 0u_p$$

$$= \underline{\lambda_1 u_1}$$

"dot product"
=
If "inner product"

(13) $\langle x, y \rangle = x^T y = 0 \iff x$ & y are orthogonal

(14) trace is cyclic:

$$tr(ABC) = tr(BCA) = tr(CAB) = tr(ABC)$$

$$\neq tr(BAC) \text{ in general}$$

$H_0: \beta_1 = 0$

$H_1: \beta_1 \neq 0$

$t = |-4| > t^*_{1-\alpha/2, 18}$

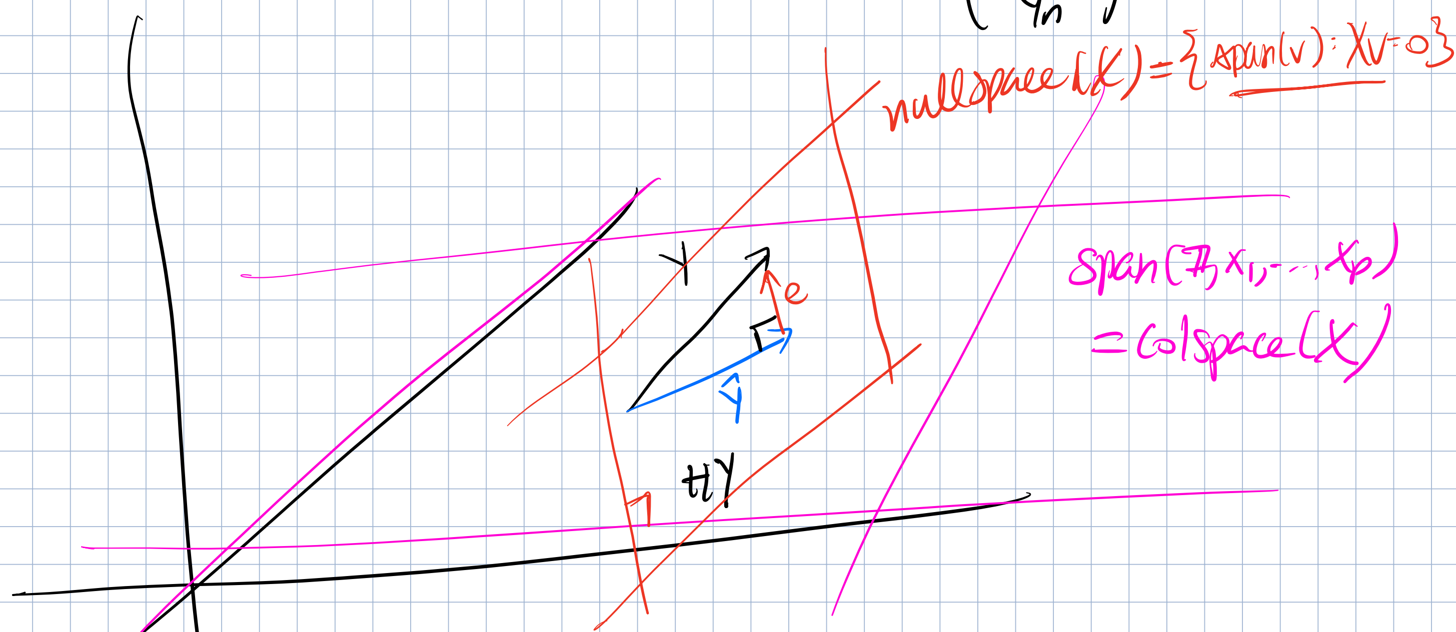
or

$H_0: \beta_1 \geq 0$

$H_1: \beta_1 < 0$

$t = |-4| > t^*_{1-\alpha, 18}$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} \in \mathbb{R}^n$$



Since we assumed

$$\hat{y} = \underline{\hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 \cdots + \hat{\beta}_{p-1} x_{p-1}}$$

linear combination of $(1, x_1, \ldots, x_{p-1})$

Why does e live in the nullspace of X?

$$Xv = 0$$

$$\Sigma_i e_i = 0$$
$$\Sigma_i e_i x_{1i} = 0 \quad \cdots \quad \Sigma_i e_i x_{(p-1)i} = 0$$
$$\Sigma_i e_i x_{2i} = 0$$

$$\underset{p \times n}{X^T} \underset{n \times 1}{e} \overset{?}{=} 0$$

$e = (I - H)Y$

$$X^T(I-H)Y = (X^T - X^T H)Y$$
$$= (X^T - \cancel{X^T X (X^T X)^{-1}} X^T)Y$$
$$= (X^T - X^T)Y = 0Y = 0.$$

① $\Sigma_i e_i = 0 = \langle e, \mathbb{1} \rangle$

② $\Sigma_i e_i x_i = 0 = \langle e, x \rangle$