EC3204: Programming Languages and Compilers

Lecture 13 — Semantic Analysis (4)
Interval Analysis

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Fixed Point Computation May Not Terminate

- We compute fixed points to obtain safe approximations.
- Q. Does this computation always terminate?

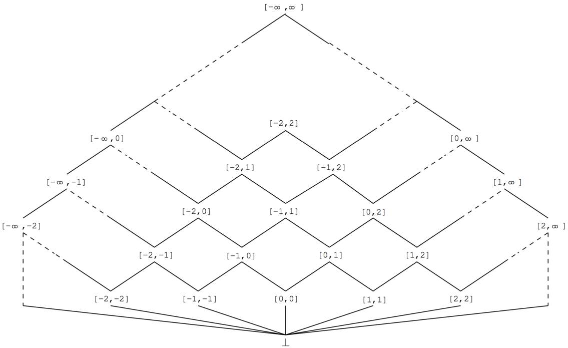
Fixed Point Computation May Not Terminate

- We compute fixed points to obtain safe approximations.
- Q. Does this computation always terminate?
- A. Yes if the abstract domain (lattice) is finite. Otherwise, it may not.
- Unfortunately, many useful domains have infinite heights. To ensure the termination, we need **widening** operators.

- The interval domain $\mathbb{I}$ has an infinite height.

$$\mathbb{I} = \{\bot\} \cup \{[l, u] \mid l, u \in \mathbb{Z} \cup \{-\infty, +\infty\} \land l \le u\}$$

- Abstract values are expressed by lower and upper bounds: $[l, u]$
  - If the abstract value of x is $[1, 3]$ at some program point p, $1 \le x \le 3$ is an invariant at p.



Example: Non-Terminating Fixed Point Computation

Q. What is the resulting abstract state at the loop entry?

```
x = 0;
y = 0;
while (x < 10) {
  x = x+1;
  y = y+1;
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A. You cannot obtain it, because computation does not terminate (i.e., we cannot reach a fixed point).

	0	1	2	...	9	10	11	12	...	k
x	$[0,0]$	$[0,1]$	$[0,2]$	...	$[0,9]$	$[0,10]$	$[0,10]$	$[0,10]$	...	$[0,10]$
y	$[0,0]$	$[0,1]$	$[0,2]$	...	$[0,9]$	$[0,10]$	$[0,11]$	$[0,12]$	...	$[0,k]$

Fixed Point Computation with Widening and Narrowing

Two staged fixed point computations:

1. **Widening**: If the abstract domain does not have the finite-height property, we need a widening operator $\triangledown$ to enforce convergence.
2. **Narrowing**: After finding a post-fixed point using widening, we have a second pass using a narrowing operator $\triangle$.

Example: Fixed Point Computation with Widening

Find a post-fixed point at the loop entry using a widening operator.

```
x = 0;
y = 0;
while (x < 10) {
  x = x+1;
  y = y+1;
}
```

	0	1	2
x	$[0, 0]$	$[0, \infty]$	$[0, \infty]$
y	$[0, 0]$	$[0, \infty]$	$[0, \infty]$

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- With widening:

	0	1	2
x	$[0, 0]$	$[0, \infty]$	$[0, \infty]$
y	$[0, 0]$	$[0, \infty]$	$[0, \infty]$

- With narrowing:

	0	1	2
x	$[0, \infty]$	$[0, 10] (= [0, \infty] \bigtriangleup [0, 10])$	$[0, 10]$
y	$[0, \infty]$	$[0, \infty] (= [0, \infty] \bigtriangleup [0, \infty])$	$[0, \infty]$

Step 1. Interval Domain

Plan: formally define the widening/narrowing operators for the interval domain.

The interval domain is a pair of $(\mathbb{I}, \sqsubseteq)$.

- $\mathbb{I} = \{\bot\} \cup \{[l, u] \mid l, u \in \mathbb{Z} \cup \{-\infty, +\infty\} \wedge l \leq u\}$
- How to define $\sqsubseteq$?
  - $\bot \sqsubseteq i$ for all $i \in \mathbb{I}$
  - $i \sqsubseteq [-\infty, \infty]$ for all $i \in \mathbb{I}$
  - $[1, 3] \sqsubseteq [0, 4]$
  - $[1, 3] \not\sqsubseteq [0, 2]$

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  - $[1, 3] \sqsubseteq [0, 4]$
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$$i_1 \sqsubseteq i_2 \iff \begin{cases} i_1 = \bot \vee \\ i_2 = [-\infty, \infty] \vee \\ (i_1 = [l_1, u_1] \wedge i_2 = [l_2, u_2] \wedge l_1 \geq l_2 \wedge u_1 \leq u_2) \end{cases}$$

Step 2. Abstract Semantics

Abstract semantics for the arithmetic expressions:

$$
\begin{array}{rcl}
\widehat{\mathcal{A}}[\![\, a \,]\!] & : & \widehat{\textsf{State}} \to \mathbb{I} \\
\widehat{\mathcal{A}}[\![\, n \,]\!](\hat{s}) & = & \alpha(\{n\}) \\
\widehat{\mathcal{A}}[\![\, x \,]\!](\hat{s}) & = & \hat{s}(x) \\
\widehat{\mathcal{A}}[\![\, a_1 + a_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{A}}[\![\, a_1 \,]\!](\hat{s}) \hat{+} \widehat{\mathcal{A}}[\![\, a_2 \,]\!](\hat{s}) \\
\widehat{\mathcal{A}}[\![\, a_1 \star a_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{A}}[\![\, a_1 \,]\!](\hat{s}) \hat{\star} \widehat{\mathcal{A}}[\![\, a_2 \,]\!](\hat{s}) \\
\widehat{\mathcal{A}}[\![\, a_1 - a_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{A}}[\![\, a_1 \,]\!](\hat{s}) \hat{-} \widehat{\mathcal{A}}[\![\, a_2 \,]\!](\hat{s})
\end{array}
$$

where $\alpha : \mathbb{Z} \to \mathbb{I}$ abstracts the integer constants:

$$\alpha(z) = [z, z]$$

Step 2. Abstract Semantics

Abstract arithmetic operators:

$$
\begin{aligned}
\bot \mathbin{\widehat{+}} i &= \\
i \mathbin{\widehat{+}} \bot &= \\
[l_1, u_1] \mathbin{\widehat{+}} [l_2, u_2] &= \\
\\
\bot \mathbin{\widehat{-}} i &= \\
i \mathbin{\widehat{-}} \bot &= \\
[l_1, u_1] \mathbin{\widehat{-}} [l_2, u_2] &= \\
\\
\bot \mathbin{\widehat{\star}} i &= \\
i \mathbin{\widehat{\star}} \bot &= \\
[l_1, u_1] \mathbin{\widehat{\star}} [l_2, u_2] &=
\end{aligned}
$$

Step 2. Abstract Semantics

Abstract arithmetic operators:

$$
\begin{array}{rcl}
\bot \mathbin{\widehat{+}} i & = & \bot \\
i \mathbin{\widehat{+}} \bot & = & \bot \\
[l_1, u_1] \mathbin{\widehat{+}} [l_2, u_2] & = & [l_1 + l_2, u_1 + u_2] \\
\bot \mathbin{\widehat{-}} i & = & \bot \\
i \mathbin{\widehat{-}} \bot & = & \bot \\
[l_1, u_1] \mathbin{\widehat{-}} [l_2, u_2] & = & [l_1 - u_2, u_1 - l_2] \\
\bot \mathbin{\widehat{\star}} i & = & \bot \\
i \mathbin{\widehat{\star}} \bot & = & \bot \\
[l_1, u_1] \mathbin{\widehat{\star}} [l_2, u_2] & = & [\min(l_1 \star l_2, l_1 \star u_2, u_1 \star l_2, u_1 \star u_2), \\
& & \max(l_1 \star l_2, l_1 \star u_2, u_1 \star l_2, u_1 \star u_2)]
\end{array}
$$

Step 2. Abstract Semantics

Abstract semantics for the boolean expressions:

$$
\begin{array}{rcl}
\widehat{\mathcal{B}}[\![\, b \,]\!] & : & \widehat{\mathsf{State}} \rightarrow \widehat{\mathsf{T}} \\
\widehat{\mathcal{B}}[\![\, \texttt{true} \,]\!](\hat{s}) & = & \widehat{true} \\
\widehat{\mathcal{B}}[\![\, \texttt{false} \,]\!](\hat{s}) & = & \widehat{false} \\
\widehat{\mathcal{B}}[\![\, a_1 = a_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{A}}[\![\, a_1 \,]\!](\hat{s}) \mathrel{\widehat{=}} \widehat{\mathcal{A}}[\![\, a_2 \,]\!](\hat{s}) \\
\widehat{\mathcal{B}}[\![\, a_1 \leq a_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{A}}[\![\, a_1 \,]\!](\hat{s}) \mathrel{\widehat{\leq}} \widehat{\mathcal{A}}[\![\, a_2 \,]\!](\hat{s}) \\
\widehat{\mathcal{B}}[\![\, \neg b \,]\!](\hat{s}) & = & \widehat{\neg}\widehat{\mathcal{B}}[\![\, b \,]\!](\hat{s}) \\
\widehat{\mathcal{B}}[\![\, b_1 \wedge b_2 \,]\!](\hat{s}) & = & \widehat{\mathcal{B}}[\![\, b_1 \,]\!](\hat{s}) \mathbin{\widehat{\wedge}} \widehat{\mathcal{B}}[\![\, b_2 \,]\!](\hat{s})
\end{array}
$$

Step 2: Abstract Semantics (Cont'd)

$$
\begin{array}{rcl}
\widehat{\mathcal{C}}[\![\, c \,]\!] & : & \widehat{\mathbf{State}} \to \widehat{\mathbf{State}} \\
\widehat{\mathcal{C}}[\![\, x := a \,]\!] & = & \lambda \widehat{s}.\widehat{s}[x \mapsto \widehat{\mathcal{A}}[\![\, a \,]\!](\widehat{s})] \\
\widehat{\mathcal{C}}[\![\, \texttt{skip} \,]\!] & = & \mathbf{id} \\
\widehat{\mathcal{C}}[\![\, c_1; c_2 \,]\!] & = & \widehat{\mathcal{C}}[\![\, c_2 \,]\!] \circ \widehat{\mathcal{C}}[\![\, c_1 \,]\!] \\
\widehat{\mathcal{C}}[\![\, \texttt{if}\ b\ c_1\ c_2 \,]\!] & = & \begin{cases} \bot & \cdots \widehat{\mathcal{B}}[\![\, b \,]\!](\widehat{s}) = \bot \\ \widehat{\mathcal{C}}[\![\, c_1 \,]\!](\widehat{s}) & \cdots \widehat{\mathcal{B}}[\![\, b \,]\!](\widehat{s}) = \widehat{true} \\ \widehat{\mathcal{C}}[\![\, c_2 \,]\!](\widehat{s}) & \cdots \widehat{\mathcal{B}}[\![\, b \,]\!](\widehat{s}) = \widehat{false} \\ \widehat{\mathcal{C}}[\![\, c_1 \,]\!](\mathbf{filter}(b)(\widehat{s})) & \cdots \widehat{\mathcal{B}}[\![\, b \,]\!](\widehat{s}) = \top \\ \sqcup\ \widehat{\mathcal{C}}[\![\, c_2 \,]\!](\mathbf{filter}(\neg b)(\widehat{s})) & \end{cases} \\
\widehat{\mathcal{C}}[\![\, \texttt{while}\ b\ c \,]\!] & = & \lambda \widehat{s}.\mathbf{filter}(\neg b)(\mathit{fix}(\lambda \widehat{x}.\widehat{s} \sqcup \widehat{\mathcal{C}}[\![\, c \,]\!](\mathbf{filter}(b)(\widehat{x}))))
\end{array}
$$

Step 2: Abstract Semantics

filter$(p)(\hat{s})$ returns the abstract state $\hat{s}'$ that can make p *true*. Let $\hat{s}(x) = [l, u]$.

$$\textbf{filter}(x < n)(\hat{s}) = \begin{cases} \lambda y \in \mathbb{X}.\bot & \text{if } l \geq n \\ \hat{s}[x \mapsto [l, n-1]] & \text{if } l < n \leq u \\ \hat{s} & \text{if } u < n \end{cases}$$

$$\textbf{filter}(x \leq n)(\hat{s}) = \begin{cases} \lambda y \in \mathbb{X}.\bot & \text{if } l > n \\ \hat{s}[x \mapsto [l, n]] & \text{if } l \leq n < u \\ \hat{s} & \text{if } u \leq n \end{cases}$$

Other cases can be defined in similar ways.

Widening and Narrowing

During analyzing `while`-loop, replace $\bigsqcup$ with $\triangledown$ and $\triangle$ in sequence (possibly after some iterations).

A simple widening operator:

$$\begin{array}{rcl} [a,b] \triangledown \perp & = & [a,b] \\ \perp \triangledown [c,d] & = & [c,d] \\ [a,b] \triangledown [c,d] & = & [(c < a ? -\infty : a), (b < d ? \infty : b)] \end{array}$$

A simple narrowing operator:

$$\begin{array}{rcl} [a,b] \triangle \perp & = & \perp \\ \perp \triangle [a,b] & = & \perp \\ [a,b] \triangle [c,d] & = & [(a = -\infty ? c : a), (b = +\infty ? d : b)] \end{array}$$

Summary

- Fixed point computations may not terminate.
- Widening ensures convergence and narrowing helps to regain precision.