EC3204: Programming Languages and Compilers

Lecture 4 — Lexical Analysis (3)
Construction of String Recognizers

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This Lecture: Construction of DFA

Methodology: transform a lexical specification (regular expression) into an equivalent string recognizer (DFA).

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction

cf) The transformations are instances of compilation. Their correctness is defined by the semantic equivalence:

- $\boldsymbol{L(RE) = L(NFA)}$ for Thomson's construction
- $\boldsymbol{L(NFA) = L(DFA)}$ for subset construction

Thompson's construction: RE to NFA

Recall RE from Lec. 2:

$$\begin{array}{rcl} R & \rightarrow & \emptyset \mid \epsilon \mid a \in \Sigma \qquad \text{(base cases)} \\ & \mid & R_1 \mid R_2 \mid R_1 \cdot R_2 \mid R_1^* \mid (R) \qquad \text{(inductive cases)} \end{array}$$

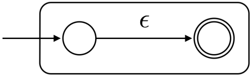
Method: use two kinds of transformation rules

- **Basic rules** for transforming primitive regexs into NFA
- **Inductive rules** for constructing larger NFA from sub-regexs' NFA

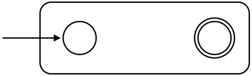
A final NFA will have exactly one start and one accepting state.

Basic Rules

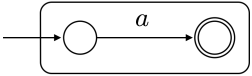
- $R = \epsilon$



- $R = \emptyset$



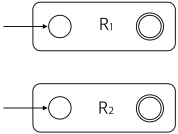
- $R = a \ (\in \Sigma)$



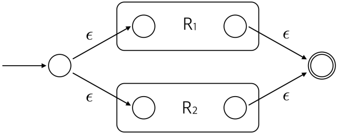
Clearly, $L(NFA) = L(R)$ in every case.

Inductive Rules

- $R = R_1|R_2$:
  1. Compile R_1 and R_2:



  2. Construct $R_1|R_2$ using the intermediate results:

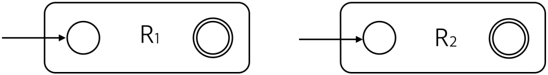


$L(NFA) = L(R_1) \cup L(R_2)$

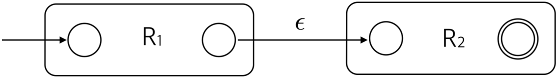
- Any path from the start to the final must path through either NFA_{R_1} or NFA_{R_2}, which accept $L(R_1)$ and $L(R_2)$, respectively.
- Strings (labels) are not changed by ϵ-transitions.

Inductive Rules

- $R = R_1 \cdot R_2$:
  1. Compile R_1 and R_2:



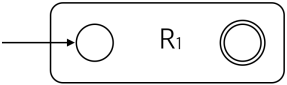
  2. Construct $R_1 \cdot R_2$ using the intermediate results:



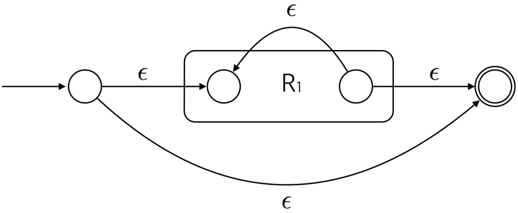
$$
\begin{aligned}
L(NFA) &= \{x\epsilon y \mid x \in L(R_1) \land y \in L(R_2)\} \\
&= \{xy \mid x \in L(R_1) \land y \in L(R_2)\} = L(R_1)L(R_2)
\end{aligned}
$$

Compilation

- $R = R_1^*$:
  1. Compile R_1:



  2. Construct R_1^* using the intermediate results:



$$L(NFA) = \{\epsilon\} \cup (L(R_1))^+ = (L(R_1))^0 \cup (L(R_1))^+ = (L(R_1))^*$$

Exercises

Construct NFAs that accept the languages described by the following regular expressions.

- $\mathbf{0 \cdot 1^*}$
- $\mathbf{(0|1) \cdot 0 \cdot 1}$
- $\mathbf{(0|1)^* \cdot 1 \cdot (0|1)}$

Our Context

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction

NFA to DFA

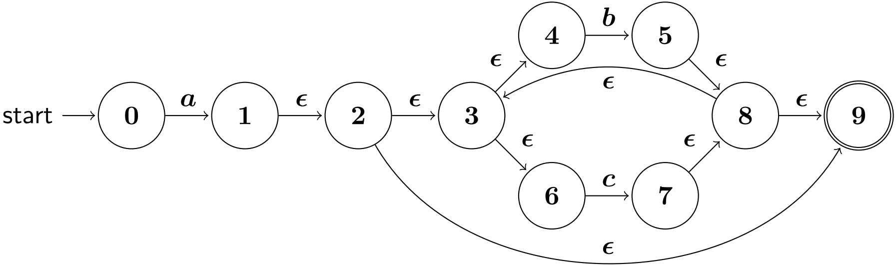
Transform an NFA

$$(N, \Sigma, \delta_N, n_0, N_A)$$

into an equivalent DFA

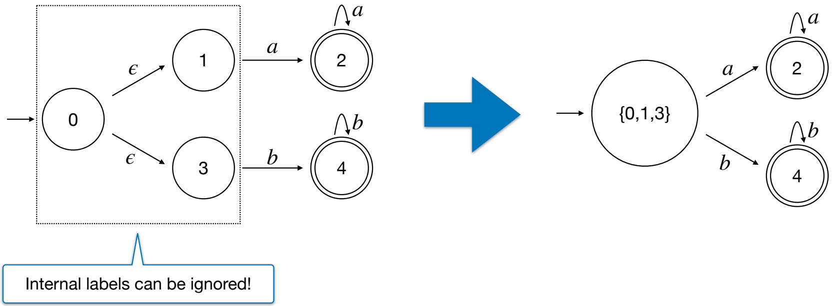
$$(D, \Sigma, \delta_D, d_0, D_A).$$

Running example $(a \cdot (b|c)^*)$:



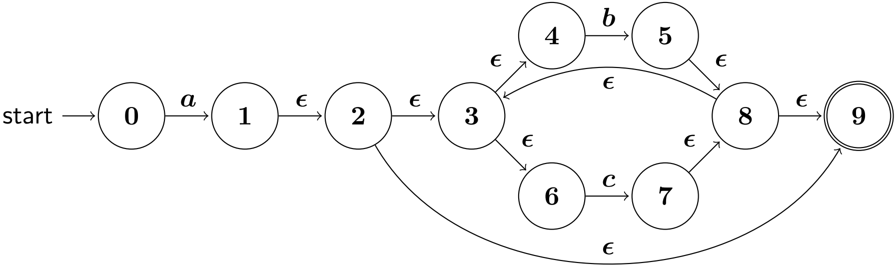
Subset Construction

- Input: an NFA $(N, \Sigma, \delta_N, n_0, N_A)$.
- Output: a DFA $(D, \Sigma, \delta_D, d_0, D_A)$.
- Key Idea: eliminate non-deterministic choices in NFA.
  - How? By merging states whose internal labels do not change strings.



Preliminary: ϵ-Closure

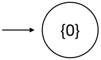
- **ϵ-closure**(I): the set of states reachable from I without consuming any symbols.



$$\begin{aligned}\epsilon\text{-closure}(\{1\}) &= \{1,2,3,4,6,9\} \\ \epsilon\text{-closure}(\{1,5\}) &= \{1,2,3,4,6,9\} \cup \{3,4,5,6,8,9\}\end{aligned}$$

Running Example (1/5)

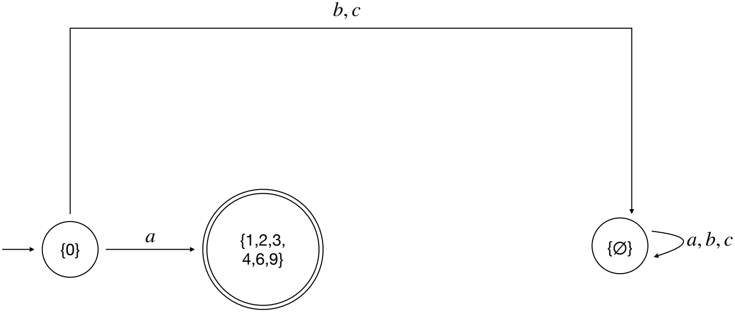
The initial DFA state $d_0 = \epsilon\text{-}\mathbf{closure}(\{0\}) = \{0\}$.



Running Example (2/5)

For the initial state $d_0 = \{0\}$, consider every $x \in \Sigma$ and compute the corresponding next states:

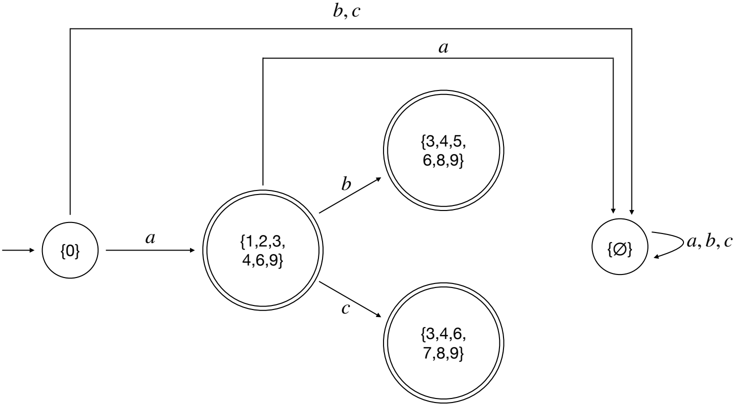
$$\begin{array}{rcl} \epsilon\text{-}\mathbf{closure}(\bigcup_{s\in\{0\}} \delta(s,a)) & = & \{1,2,3,4,6,9\} \\ \epsilon\text{-}\mathbf{closure}(\bigcup_{s\in\{0\}} \delta(s,b)) & = & \emptyset \\ \epsilon\text{-}\mathbf{closure}(\bigcup_{s\in\{0\}} \delta(s,c)) & = & \emptyset \end{array}$$



Running Example (3/5)

For the state $\{1, 2, 3, 4, 6, 9\}$, compute the next states:

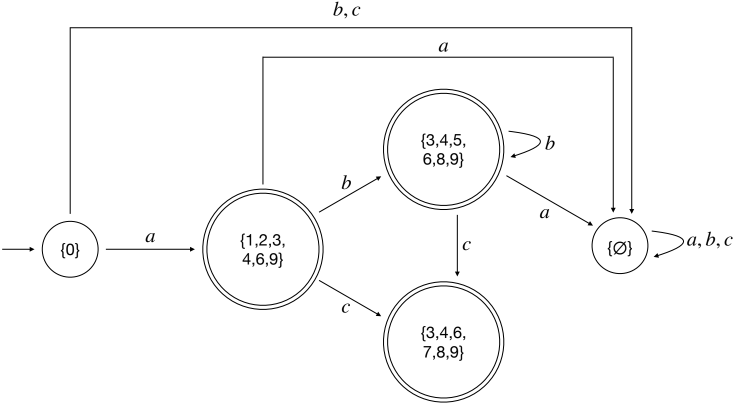
$$\epsilon\text{-closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, a)) = \emptyset$$
$$\epsilon\text{-closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, b)) = \{3, 4, 5, 6, 8, 9\}$$
$$\epsilon\text{-closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, c)) = \{3, 4, 6, 7, 8, 9\}$$



Running Example (4/5)

Compute the next states of $\mathbf{\{3,4,5,6,8,9\}}$:

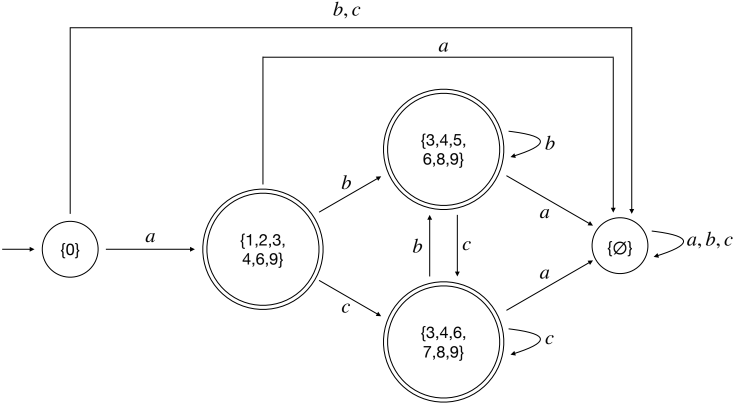
$$\boldsymbol{\epsilon}\textbf{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}}\delta(s,a)) = \emptyset$$
$$\boldsymbol{\epsilon}\textbf{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}}\delta(s,b)) = \{3,4,5,6,8,9\}$$
$$\boldsymbol{\epsilon}\textbf{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}}\delta(s,c)) = \{3,4,6,7,8,9\}$$



Running Example (5/5)

Compute the next states of $\mathbf{\{3,4,6,7,8,9\}}$:

$$\epsilon\text{-}\mathbf{closure}\Big(\bigcup_{s \in \{3,4,6,7,8,9\}} \delta(s,a)\Big) = \emptyset$$
$$\epsilon\text{-}\mathbf{closure}\Big(\bigcup_{s \in \{3,4,6,7,8,9\}} \delta(s,b)\Big) = \{3,4,5,6,8,9\}$$
$$\epsilon\text{-}\mathbf{closure}\Big(\bigcup_{s \in \{3,4,6,7,8,9\}} \delta(s,c)\Big) = \{3,4,6,7,8,9\}$$



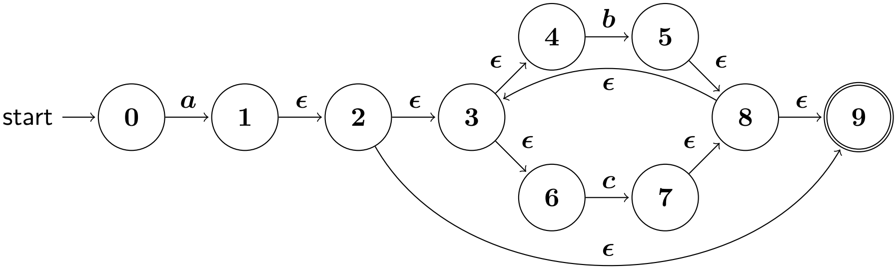
Subset Construction Algorithm

Algorithm 1 Subset Construction

Input: An NFA $(N, \Sigma, \delta_N, n_0, N_A)$
Output: An equivalent DFA $(D, \Sigma, \delta_D, d_0, D_A)$

$d_0 \leftarrow \epsilon$-**closure**$(\{n_0\})$
$D \leftarrow \{d_0\}$ ▷ D: a set of DFA states
$W \leftarrow \{d_0\}$ ▷ W: a set of DFA states to process
while $W \neq \emptyset$ **do**
    remove q from W
    for $c \in \Sigma$ **do** ▷ consider each input symbol
        $t \leftarrow \epsilon$-**closure**$(\bigcup_{s \in q} \delta(s, c))$
        $D \leftarrow D \cup \{t\}$
        $\delta_D(q, c) \leftarrow t$ ▷ update the transition table
        if t was newly added to D **then**
            $W \leftarrow W \cup \{t\}$
$D_A \leftarrow \{q \mid q \in D, q \cap N_A \neq \emptyset\}$
return $(D, \Sigma, \delta_D, d_0, D_A)$

Running Example (1/5)



The initial state $d_0 = \epsilon\text{-}\textbf{closure}(\{0\}) = \{0\}$. Initialize D and W:

$$D = \{\{0\}\}, \qquad W = \{\{0\}\}$$

Running Example (2/5)

Choose $q = \{0\}$ from W. For all $c \in \Sigma$, update δ_D:

	a	b	c
$\{0\}$	$\{1,2,3,4,6,9\}$	$\emptyset$	$\emptyset$

Update D and W:

$$D = \{\{0\}, \{1,2,3,4,6,9\}\}, \qquad W = \{\{1,2,3,4,6,9\}\}$$

Running Example (3/5)

Choose $q = \{1, 2, 3, 4, 6, 9\}$ from W. For all $c \in \Sigma$, update δ_D:

	a	b	c
$\{0\}$	$\{1, 2, 3, 4, 6, 9\}$	$\emptyset$	$\emptyset$
$\{1, 2, 3, 4, 6, 9\}$	$\emptyset$	$\{3, 4, 5, 6, 8, 9\}$	$\{3, 4, 6, 7, 8, 9\}$

Update D and W:

$$\begin{array}{rcl} D & = & \{\{0\}, \{1, 2, 3, 4, 6, 9\}, \{3, 4, 5, 6, 8, 9\}, \{3, 4, 6, 7, 8, 9\}\} \\ W & = & \{\{3, 4, 5, 6, 8, 9\}, \{3, 4, 6, 7, 8, 9\}\} \end{array}$$

Running Example (4/5)

Choose $q = \{3,4,5,6,8,9\}$ from W. For all $c \in \Sigma$, update δ_D:

	a	b	c
$\{0\}$	$\{1,2,3,4,6,9\}$	$\emptyset$	$\emptyset$
$\{1,2,3,4,6,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$
$\{3,4,5,6,8,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$

D and W:

$$\begin{array}{rcl} D & = & \{\{0\},\{1,2,3,4,6,9\},\{3,4,5,6,8,9\},\{3,4,6,7,8,9\}\} \\ W & = & \{\{3,4,6,7,8,9\}\} \end{array}$$

Running Example (5/5)

Choose $q = \{3,4,6,7,8,9\}$ from W. For all $c \in \Sigma$, update δ_D:

	a	b	c
$\{0\}$	$\{1,2,3,4,6,9\}$	$\emptyset$	$\emptyset$
$\{1,2,3,4,6,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$
$\{3,4,5,6,8,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$
$\{3,4,6,7,8,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$

D and W:

$$\begin{aligned} D &= \{\{0\},\{1,2,3,4,6,9\},\{3,4,5,6,8,9\},\{3,4,6,7,8,9\}\} \\ W &= \emptyset \end{aligned}$$

The while loop terminates. The accepting states:

$$D_A = \{\{1,2,3,4,6,9\},\{3,4,5,6,8,9\},\{3,4,6,7,8,9\}\}$$

Algorithm for computing ϵ-Closures

- The definition

  ϵ-**closure**(I) is the set of states reachable from I without consuming any symbols.

  is neither formal nor constructive. Let's define it precisely!
- A formal definition:
  $T = \epsilon$-**closure**(I) is the smallest set such that

$$I \cup \bigcup_{s \in T} \delta(s, \epsilon) \subseteq T.$$

- Alternatively, T is the smallest solution of the equation

$$F(X) \subseteq (X)$$

  where

$$F(X) = I \cup \bigcup_{s \in X} \delta(s, \epsilon).$$

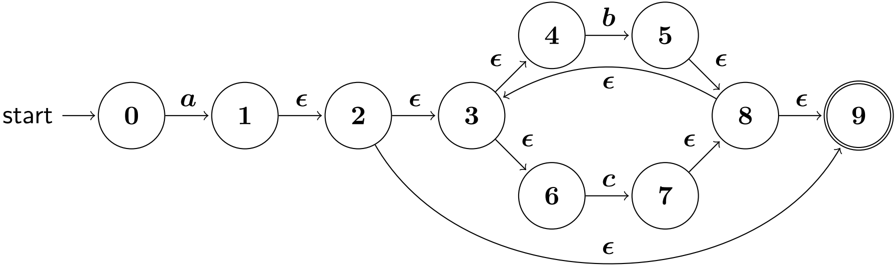
  Such a solution is called the least fixed point of F.

Fixed Point Iteration

The least fixed point of a function can be computed by the fixed point iteration.

$$
\begin{array}{l}
T = \emptyset \\
\textbf{repeat} \\
\quad T' = T \\
\quad T = T' \cup F(T') \\
\textbf{until } T = T' \\
\textbf{return } T
\end{array}
$$

Example



ϵ-**closure**$(\{1\})$:

Iteration	T'	T
1	$\emptyset$	$\{1\}$
2	$\{1\}$	$\{1, 2\}$
3	$\{1, 2\}$	$\{1, 2, 3, 9\}$
4	$\{1, 2, 3, 9\}$	$\{1, 2, 3, 4, 6, 9\}$
5	$\{1, 2, 3, 4, 6, 9\}$	$\{1, 2, 3, 4, 6, 9\}$

Summary

Construction of string recognizers (DFA)

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction

Next class: syntax analysis