EC3204: Programming Languages and Compilers

Lecture 4 — Lexical Analysis (3)

Construction of String Recognizers

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This Lecture: Construction of DFA

Methodology: transform a lexical specification (regular expression) into an equivalent string recognizer (DFA).

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction

cf) The transformations are instances of compilation. Their correctness is defined by the semantic equivalence:

- $\boldsymbol{L(RE) = L(NFA)}$ for Thomson's construction
- $\boldsymbol{L(NFA) = L(DFA)}$ for subset construction

Thompson's construction: RE to NFA

Recall RE from Lec. 2:

$$\begin{array}{lclr} R & \rightarrow & \emptyset \mid \epsilon \mid a \in \Sigma & \text{(base cases)} \\ & \mid & R_1 \mid R_2 \mid R_1 \cdot R_2 \mid R_1^* \mid (R) & \text{(inductive cases)} \end{array}$$

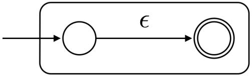
Method: use two kinds of transformation rules

- **Basic rules** for transforming primitive regexs into NFA
- **Inductive rules** for constructing larger NFA from sub-regexs' NFA

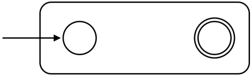
A final NFA will have exactly one start and one accepting state.

Basic Rules

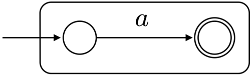
- $R = \epsilon$



- $R = \emptyset$



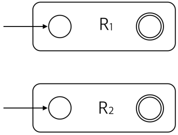
- $R = a \ (\in \Sigma)$



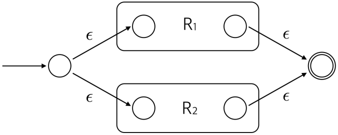
Clearly, $L(NFA) = L(R)$ in every case.

Inductive Rules

- $R = R_1 | R_2$:
  1. Compile R_1 and R_2:



  2. Construct $R_1 | R_2$ using the intermediate results:

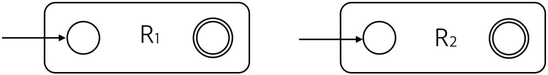


$L(NFA) = L(R_1) \cup L(R_2)$

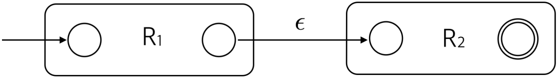
- Any path from the start to the final must path through either NFA_{R_1} or NFA_{R_2}, which accept $L(R_1)$ and $L(R_2)$, respectively.
- Strings (labels) are not changed by ϵ-transitions.

Inductive Rules

- $R = R_1 \cdot R_2$:
  1. Compile R_1 and R_2:



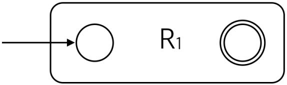
  2. Construct $R_1 \cdot R_2$ using the intermediate results:



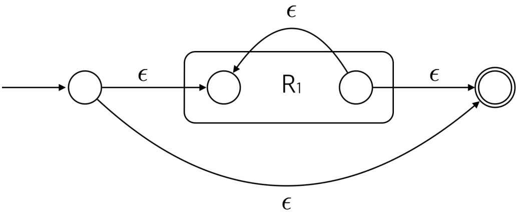
$$\begin{aligned} L(NFA) &= \{x\epsilon y \mid x \in L(R_1) \wedge y \in L(R_2)\} \\ &= \{xy \mid x \in L(R_1) \wedge y \in L(R_2)\} = L(R_1)L(R_2) \end{aligned}$$

Compilation

- $R = R_1^*$:
  1. Compile R_1:



  2. Construct R_1^* using the intermediate results:



$$L(NFA) = \{\epsilon\} \cup (L(R_1))^+ = (L(R_1))^0 \cup (L(R_1))^+ = (L(R_1))^*$$

Exercises

Construct NFAs that accept the languages described by the following regular expressions.

- $\mathbf{0 \cdot 1^*}$
- $\mathbf{(0|1) \cdot 0 \cdot 1}$
- $\mathbf{(0|1)^* \cdot 1 \cdot (0|1)}$

Our Context

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction

NFA to DFA

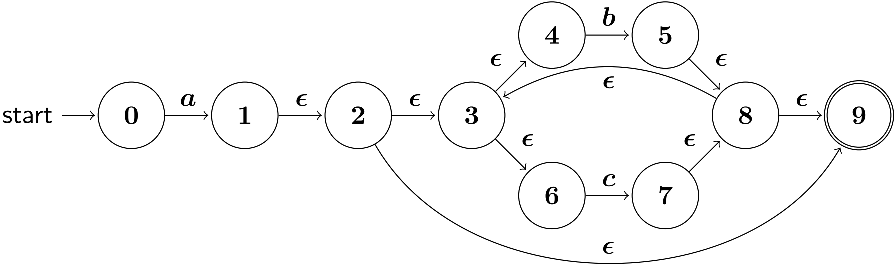
Transform an NFA

$$(N, \Sigma, \delta_N, n_0, N_A)$$

into an equivalent DFA

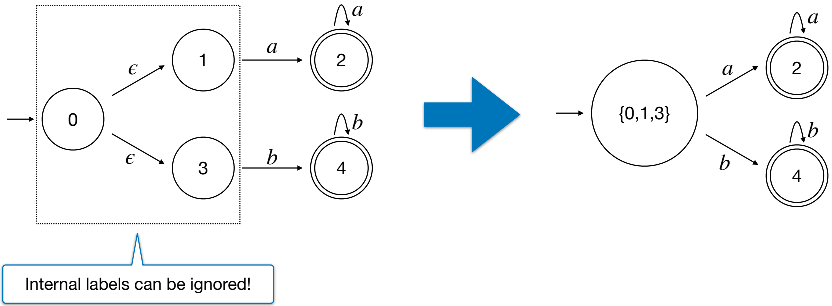
$$(D, \Sigma, \delta_D, d_0, D_A).$$

Running example ($a \cdot (b|c)^*$):



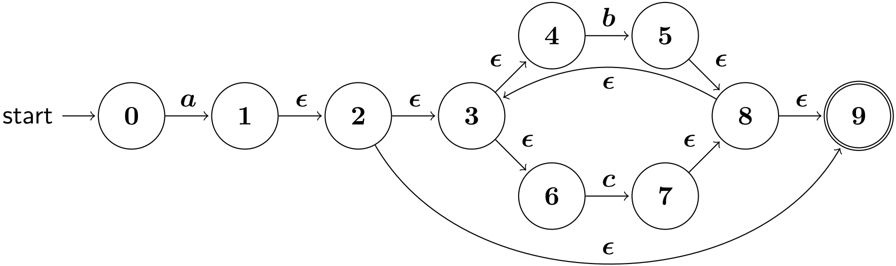
Subset Construction

- Input: an NFA $(N, \Sigma, \delta_N, n_0, N_A)$.
- Output: a DFA $(D, \Sigma, \delta_D, d_0, D_A)$.
- Key Idea: eliminate non-deterministic choices in NFA.
  - How? By merging states whose internal labels do not change strings.



Preliminary: ϵ-Closure

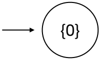
- **ϵ-closure**(I): the set of states reachable from I without consuming any symbols.



$$\begin{array}{rcl} \epsilon\text{-closure}(\{1\}) & = & \{1,2,3,4,6,9\} \\ \epsilon\text{-closure}(\{1,5\}) & = & \{1,2,3,4,6,9\} \cup \{3,4,5,6,8,9\} \end{array}$$

Running Example (1/5)

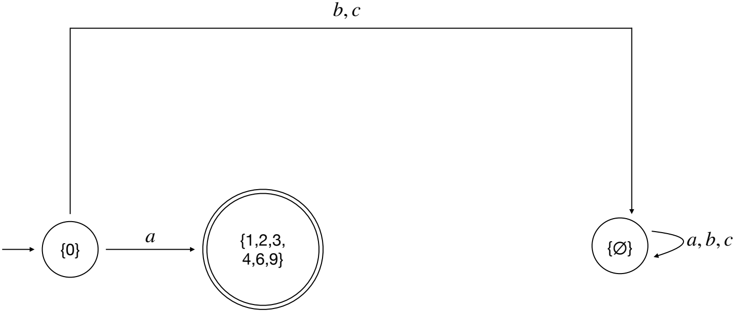
The initial DFA state $d_0 = \epsilon$-**closure**$(\{0\}) = \{0\}$.



Running Example (2/5)

For the initial state $d_0 = \{0\}$, consider every $x \in \Sigma$ and compute the corresponding next states:

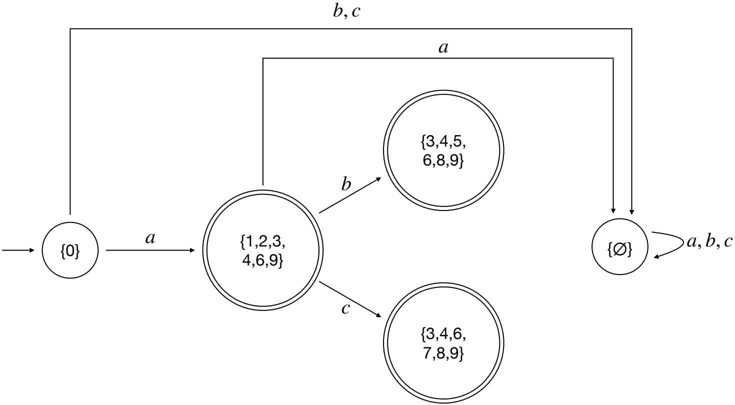
$$
\begin{array}{rcl}
\epsilon\text{-}\mathbf{closure}(\bigcup_{s \in \{0\}} \delta(s, a)) & = & \{1, 2, 3, 4, 6, 9\} \\
\epsilon\text{-}\mathbf{closure}(\bigcup_{s \in \{0\}} \delta(s, b)) & = & \emptyset \\
\epsilon\text{-}\mathbf{closure}(\bigcup_{s \in \{0\}} \delta(s, c)) & = & \emptyset
\end{array}
$$



Running Example (3/5)

For the state $\mathbf{\{1, 2, 3, 4, 6, 9\}}$, compute the next states:

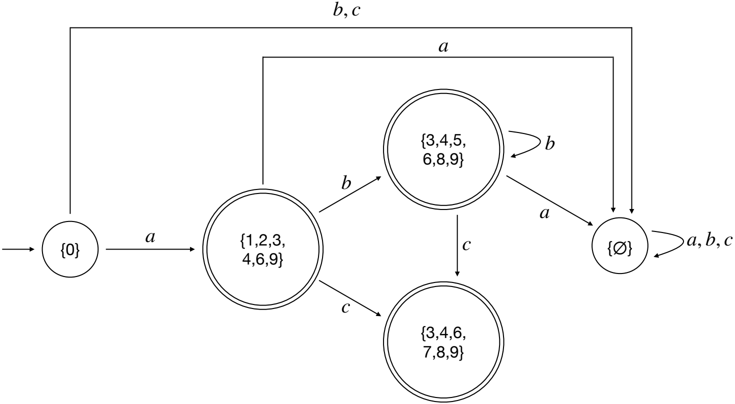
$$\epsilon\text{-}\textbf{closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, a)) = \emptyset$$
$$\epsilon\text{-}\textbf{closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, b)) = \{3, 4, 5, 6, 8, 9\}$$
$$\epsilon\text{-}\textbf{closure}(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta(s, c)) = \{3, 4, 6, 7, 8, 9\}$$



Running Example (4/5)

Compute the next states of $\{3,4,5,6,8,9\}$:

$$\epsilon\text{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}} \delta(s,a)) = \emptyset$$
$$\epsilon\text{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}} \delta(s,b)) = \{3,4,5,6,8,9\}$$
$$\epsilon\text{-closure}(\bigcup_{s\in\{3,4,5,6,8,9\}} \delta(s,c)) = \{3,4,6,7,8,9\}$$



Running Example (5/5)

Compute the next states of $\mathbf{\{3,4,6,7,8,9\}}$:

$$\epsilon\text{-}\mathbf{closure}(\bigcup_{s\in\{3,4,6,7,8,9\}} \delta(s,a)) = \emptyset$$
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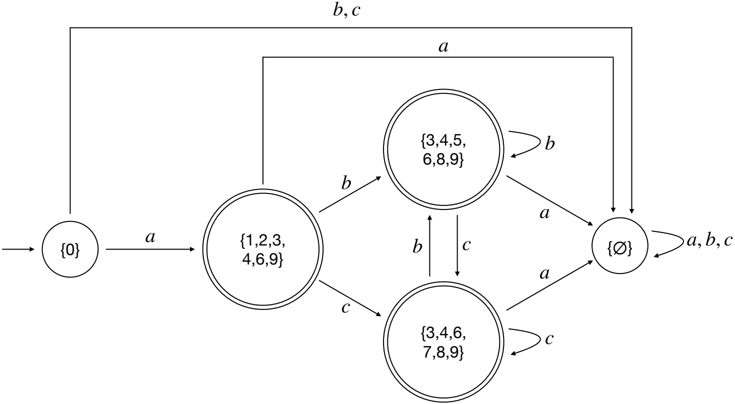


Subset Construction Algorithm

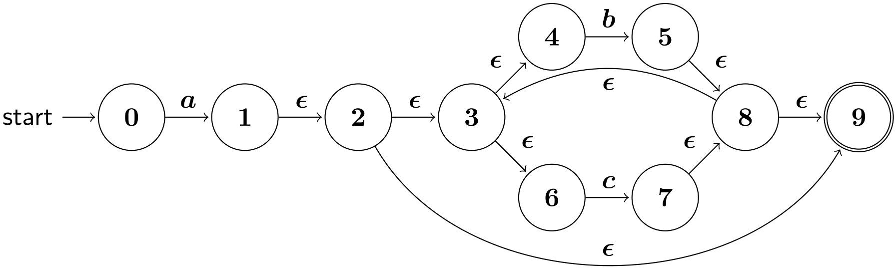
Algorithm 1 Subset Construction

Input: An NFA $(N, \Sigma, \delta_N, n_0, N_A)$
Output: An equivalent DFA $(D, \Sigma, \delta_D, d_0, D_A)$

```
1:  d_0 ← ε-closure({n_0})
2:  D ← {d_0}                                  ▷ D: a set of DFA states
3:  W ← {d_0}           ▷ W (workset): a set of DFA states to process
4:  while W ≠ ∅ do
5:      pick and remove q from W
6:      for x ∈ Σ do                           ▷ consider each input symbol
7:          t ← ε-closure(⋃_{s∈q} δ_N(s, x))
8:          D ← D ∪ {t}
9:          δ_D(q, c) ← t
10:         if t has not been added to W before then
11:             W ← W ∪ {t}
12: D_A ← {q | q ∈ D, q ∩ N_A ≠ ∅}
13: return (D, Σ, δ_D, d_0, D_A)
```

- Note (small optimization): At line 10, if $t = \emptyset$, we can skip line 11, and instead update $\delta_D(\emptyset, x)$ as $\emptyset$ for all $x \in \Sigma$.

Running Example (1/5)



The initial state $d_0 = \epsilon$-**closure**$(\{0\}) = \{0\}$. Initialize D and W:

$$D = \{\{0\}\}, \qquad W = \{\{0\}\}$$

Running Example (2/5)

Choose $q = \{0\}$ from W.

- When $x = a$:
  - ϵ-**closure**$(\bigcup_{s\in\{0\}} \delta_N(s, a)) = \{1, 2, 3, 4, 6, 9\}$
  - $D = \{\{0\}, \{1, 2, 3, 4, 6, 9\}\}$
  - $W = \{\{1, 2, 3, 4, 6, 9\}\}$
- When $x = b$:
  - ϵ-**closure**$(\bigcup_{s\in\{0\}} \delta_N(s, b)) = \emptyset$
  - $D = \{\emptyset, \{0\}, \{1, 2, 3, 4, 6, 9\}\}$
  - $W = \{\{1, 2, 3, 4, 6, 9\}\}$
- When $x = c$:
  - ϵ-**closure**$(\bigcup_{s\in\{0\}} \delta_N(s, c)) = \emptyset$
  - $D = \{\emptyset, \{0\}, \{1, 2, 3, 4, 6, 9\}\}$
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Running Example (3/5)

Choose $q = \{1,2,3,4,6,9\}$ from W.

- When $x = a$:
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  - $W = \emptyset$
- When $x = b$:
  - ϵ-**closure**$(\bigcup_{s\in\{1,2,3,4,6,9\}} \delta_N(s,b)) = \{3,4,5,6,8,9\}$
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  - $W = \{\{3,4,6,7,8,9\}\}$
- When $x = b$:
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Running Example (5/5)

Choose $q = \{3,4,6,7,8,9\}$ from W.

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  - $D = \{\emptyset, \{0\}, \{1,2,3,4,6,9\}, \{3,4,5,6,8,9\}, \{3,4,6,7,8,9\}\}$
  - $W = \emptyset$
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  - $D = \{\emptyset, \{0\}, \{1,2,3,4,6,9\}, \{3,4,5,6,8,9\}, \{3,4,6,7,8,9\}\}$
  - $W = \emptyset$

Running Example: Termination

The while-loop terminates:

$$D = \{\emptyset, \{0\}, \{1,2,3,4,6,9\}, \{3,4,5,6,8,9\}, \{3,4,6,7,8,9\}\}$$

Since $N_A = \{9\}$, the accepting states of DFA is:

$$D_A = \{\{1,2,3,4,6,9\}, \{3,4,5,6,8,9\}, \{3,4,6,7,8,9\}\}$$

The final transition table can be obtained by incorporating δ_D computed so far:

	a	b	c
$\{0\}$	$\{1,2,3,4,6,9\}$	$\emptyset$	$\emptyset$
$\{1,2,3,4,6,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$
$\{3,4,5,6,8,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$
$\{3,4,6,7,8,9\}$	$\emptyset$	$\{3,4,5,6,8,9\}$	$\{3,4,6,7,8,9\}$

Algorithm for Computing ϵ-Closures

- The definition "ϵ-**closure**(I) is the set of states reachable from I without consuming any symbols." is neither formal nor constructive.

Algorithm for Computing ϵ-Closures

- The definition "ϵ-**closure**(I) is the set of states reachable from I without consuming any symbols." is neither formal nor constructive.
- A formal definition:
  $T = \epsilon$-**closure**(I) is the smallest set such that

$$I \cup \bigcup_{s \in T} \delta(s, \epsilon) \subseteq T.$$

- Equivalently, T is the smallest solution X of the equation

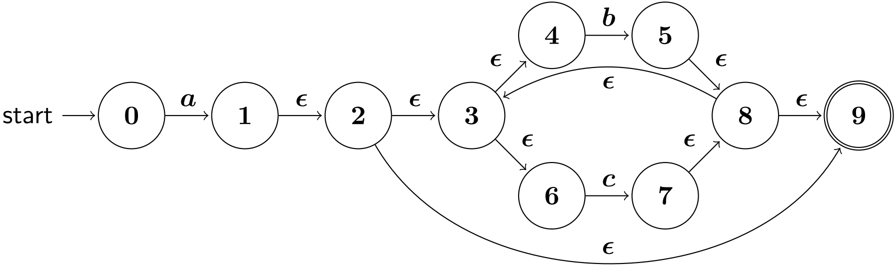
$$F(X) \subseteq X$$

  where

$$F(X) = I \cup \bigcup_{s \in X} \delta(s, \epsilon).$$

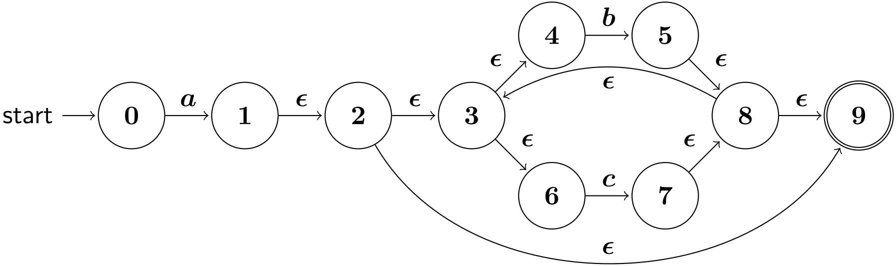
  Such a solution is called the least fixed point of F.
  In words: we want the smallest ("least") and stabilized ("fixed point") solution X that does not change no matter how we apply F on X.

Why Smallest Solution?



- Recall ϵ-**closure**$(\{1\}) = \{1, 2, 3, 4, 6, 9\}$. Is this a unique solution that satisfies $F(X) \subseteq X$?

Why Smallest Solution?



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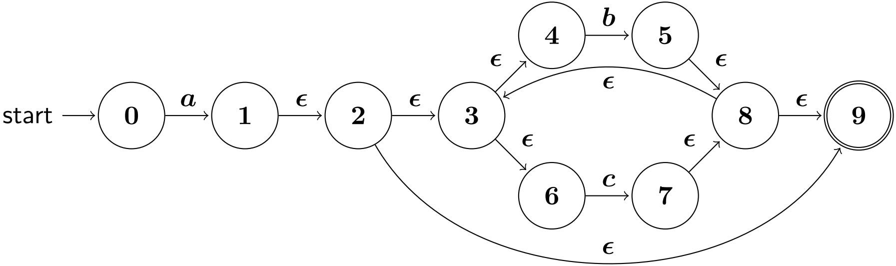

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- $X = \{1, 2, 3, 4, 6, 7, 8, 9\}$ is also the solution that satisfies $F(X) \subseteq X$. So what is the problem? We may accept an invalid lexeme c!

cf) In programming language theories, we are mostly interested in computing the least fixed point F, denoted $\mathit{fix}\,F$ (typically indicates the most precise solution).

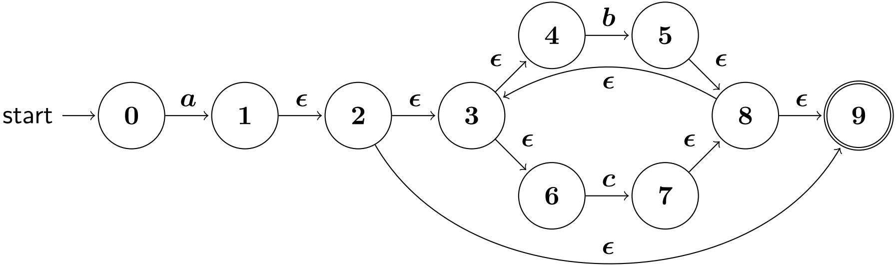
Fixed Point Iteration

The least fixed point of a function can be computed by the **fixed point iteration**.

$$
\begin{aligned}
&\boldsymbol{T = \emptyset} \\
&\textbf{repeat} \\
&\quad \boldsymbol{T' = T} \\
&\quad \boldsymbol{T = T' \cup F(T')} \\
&\textbf{until } \boldsymbol{T = T'} \\
&\textbf{return } \boldsymbol{T}
\end{aligned}
$$

In words: starting from $\emptyset$, iteratievly apply $\boldsymbol{F}$ until $\boldsymbol{T}$ remains unchanged.

Example



ϵ-**closure**$(\{1\})$:

Iteration	T'	T
1	$\emptyset$	$\{1\}$
2	$\{1\}$	$\{1, 2\}$
3	$\{1, 2\}$	$\{1, 2, 3, 9\}$
4	$\{1, 2, 3, 9\}$	$\{1, 2, 3, 4, 6, 9\}$
5	$\{1, 2, 3, 4, 6, 9\}$	$\{1, 2, 3, 4, 6, 9\}$

Summary

Construction of string recognizers (DFA)

- RE to NFA: Thompson's construction
- NFA to DFA: subset construction
  - Key idea: eliminate non-deterministic transitions in NFA.
  - More specifically, to make every transition unique, we simulate all possibilities at once for each input symbol, where all possibilities for each input symbol are computed using ϵ-**closure**.

Next class: functional programming in OCaml. Bring your laptop after installing OCaml!