Theorem 3.1 – Handshaking Theorem.

If G is a graph with m edges and vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ then,

$$\sum_{i=1}^{n} \delta(v_i) = 2m$$

In particular, the sum of the degrees of all the vertices in a graph is even.
$\delta(v_i)$ is the degree of the vertex v_i

Proof:-

Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, each edge gets counted twice, once at each end. Thus the sum of the degrees is equal twice the number of edges.

Note: This theorem applies even if multiple edges and loops are present.

Theorem 3.2.

In any graph, the number of vertices of odd degree is even.

Proof:-

Let V_1 be the set of vertices of even degree and V_2 be the set of vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then,

$$2m = \sum_{v \in V} \delta(v) = \sum_{v \in V_1} \delta(v) + \sum_{v \in V_2} \delta(v)$$

Since $\delta(v)$ is even for $v \in V_1$, $\sum_{v \in V_1} \delta(v)$ is even.
Also, the sum two summation in the previous equation is even by the Handshaking theorem.
Hence, $\sum_{v \in V_2} \delta(v)$ must be even.
Since $\delta(v)$ is odd for $v \in V_2$, there should be even number of vertices in the set V_2. Thus, there are an even number of vertices of odd degree.

Theorem 3.3.

A graph has a path with no repeated edges from *v* to *w* ($v \neq w$) containing all the edges and vertices if and only if it is connected and *v* and *w* are the only vertices having odd degree.

Proof:-

Suppose that a graph has a path *P* with no repeated edges from *v* to *w* containing all the edges and vertices. The graph is surely connected. If we add an edge from *v* to *w*, the graph has a path that visits every edge exactly once and starts and ends in the same vertex. Therefore, the graph has a Euler cycle, namely the path *P* together with the added edge. By Theorem 4.1 every vertex has even degree. Removing the added edge only affects the degrees of *v* and *w*, which are reduced by 1. Thus, in the original graph, *v* and *w* have odd degree and all other vertices have even degree.

Suppose the graph is connected, therefore, there is a path from between every two vertices. Suppose *v* and *w* are the only vertices having odd degree. If we add an edge from *v* to *w* because there already exists a path from *v* to *w* adding a new edge will create a cycle. Adding the edge only affects the edges *v* and *w*, which are increased by 1 making the degrees of *v* and *w* even. Thus, degrees of all the edges are even. By Theorem 4.1 this connected graph has a Euler cycle form *v* to *v* going through *w*. By the definition of the Euler cycle, this cycle (path) has no repeated edges and visits every edge. Removing the added edge from *v* to *w* will break the Euler cycle but it will still be a Euler path. Therefore, by the definition of a Euler path, this graph has a path with no repeated edges from *v* to *w* ($v \neq w$) containing all the edges and vertices.

Theorem 3.4.
If a graph *G* contains a cycle from *v* to *v*, *G* contains a simple cycle from *v* to *v*.

Proof:-
If a given vertex v_i occurs twice in the cycle, we can remove the part of it that goes from v_i and back to v_i. If the resulting cycle still contains repeated vertices, we can repeat the operation until there are no more repeated vertices. This will result in a cycle without repeated vertices therefore, according to the definition, a simple cycle.

Theorem 4.1 – Euler's Theorem.
A connected graph has a Euler cycle if and only if every vertex has even degree.

Proof:-
For the first implication, take $G = (V, E)$ as the Euler graph, with a closed Euler cycle $C = [v_0, v_1, v_2, \ldots., v_{k-1}, v_k]$ with $v_0 = v_k$.

Due to the nature of the Euler cycle, for each $v \in V$, the cycle C enters v through an edge and departs v from another edge of G. Thus, at each stage, the process of coming in and going out contributes 2 to the degree of v. In addition, the cycle C passes through each edge of G exactly once and hence each vertex must be of even degree.

Therefore, if a connected graph has a Euler cycle, then every vertex has even degree.

Suppose every degree is even. We will show that there is a Euler circuit by induction on the number of edges in the graph.

The base case is for a graph G with two vertices with two edges between them. This graph obviously has a Euler cycle.

Now suppose we have a graph G on m > 2 edges. We start at an arbitrary vertex v and follow edges, arbitrarily selecting one after another until we return to v. Call this trail W. We know that we will return to v eventually because every time we encounter a vertex other than v we are listing one edge adjacent to it. There are an even number of edges adjacent to every vertex, so there will always be a suitable unused edge to list next. So, this process will always lead us back to v.

Let E be the edges of W. The graph $G - E$ has components $C_1, C_2, \ldots, C_k$.

These each satisfy the induction hypothesis: connected, less than m edges, and every degree is even. We know that every degree is even in $G - E$, because when we removed W, we removed an even number of edges from those vertices listed in the circuit. By induction, each component has a Euler cycle, call them $E_1, E_2, \ldots, E_k$.

Since G is connected, there is a vertex a_i in each component C_i on both W and E_i. Without loss of generality, assume that as we follow W, the vertices $a_1, a_2, \ldots, a_k$ are encountered in that order.

We describe a Euler circuit in G by starting at v follow W until reaching a_1, follow the entire E_1 ending back at a_1, follow W until reaching a_2, follow the entire E_2, ending back at $a_2$and so on. End by following W until reaching a_k, follow the entire E_k, ending back at a_k, then finish off W, ending at v.

Therefore, if every vertex has even degree, then the graph is a connected graph has a Euler cycle.

A connected graph has a Euler cycle if and only if every vertex has even degree.

Theorem 6.1 – DIRAC'S THEOREM.
If *G* is a simple graph with *n* vertices with $n \geq 3$ such that the degree of every vertex in *G* is at least *n* / 2, then *G* has a Hamilton circuit.

Proof. First, we show that the graph is connected. Suppose G is not connected, so that G has at least two components. Then we could partition $V = V_0 \cup V_1$ into two non-empty pieces so there are no edges between V_0 and V_1. (V_0 and V_1 might not be components themselves, because there might be more than two components; instead V_0 and V_1 are unions of components.) Since $n = |V| = |V_0| + |V_1|$, we must have either $|V_0| \leq n/2$ or $|V_1| \leq n/2$. Say V_0 has size $\leq n/2$ and pick any $v \in V_0$. Then $deg(v) \geq n/2$, but every neighbor of v is contained in V_0 and is not v, so $deg(v) < n/2$; this is a contradiction. So G is connected.

We prove there is a Hamilton circuit by induction. Let p_m be the statement "As long as $m + 1 \leq n$, there is a path visiting $m + 1$ distinct vertices with no repetitions". p_0 is trivial—just take a single vertex.

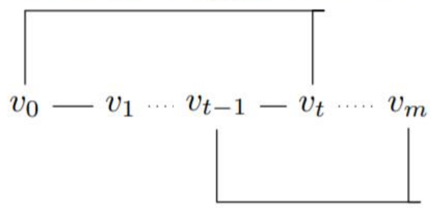
Suppose p_m is true, so we have a path

$$v_0 - v_1 - \cdots - v_m.$$

We want to show that we can extend this to a circuit with one more element. If v_0 is adjacent to any vertex not already in the path, we could just add it before v_0 and be done. Similarly, if v_m were adjacent to any vertex not already in the path, we could add it after v_m and be done.

So we have to consider the hard case. In this case, we have an important additional fact: all neighbors of v_0 and all neighbors of v_m are somewhere in the path.

We want to turn our path into a cycle. If v_0 is adjacent to v_m then we already have a circuit. Suppose not. We want to find the following arrangement:



because then we could break the link between v_{t-1} and v_t and have the circuit

$$v_t - \cdots - v_m - v_{t-1} - \cdots - v_1 - v_0 - v_t.$$

We know that v_0 has $n/2$ neighbors, all of them are in this path, and none are v_m. Let A be the vertices adjacent to v_0, so $|A| \geq n/2$. Let B be all the vertices which are adjacent to v_m, so $|B| \geq n/2$. Every vertex in B belongs on the path, so we can ask about the vertex immediately after it on the path. Let C be the set of vertices which are immediately after some vertex in B in the path. Then $|C| = |B| \geq n/2$. If $A \cap C = \emptyset$—if A and C are disjoint—then $|A \cup C| \geq n/2 + n/2 \geq n$, so $A \cup C$ would have to include all the vertices. But v_0 is in neither A nor C, so $A \cup C$ isn't all the vertices, so there is some vertex $v_t \in A \cap C$, and so $v_t \in A$ while $v_{t-1} \in B$.

Therefore (remember, we're still in the case where we can't just tack an element on at the beginning or end) we have turned our path into the circuit

$$v_t - \cdots - v_m - v_{t-1} - \cdots - v_1 - v_0 - v_t.$$

If $m + 1 = n$, we have included all the vertices, so we have a Hamilton circuit and we're done. If $m + 1 < n$, there must be some vertex not included in our circuit, and since G is connected, there must be some vertex w which isn't in our circuit but is adjacent to something in our circuit, say v_u. So we can rotate our circuit so v_u is the first vertex and then tack on w before it, say

$$w - v_u - v_{u+1} - \cdots - v_m - v_{t-1} - \cdots - v_1 - v_0 - v_t - \cdots - v_{u-1}.$$

This is a path with $m + 2$ elements, so we have shown p_{m+1}.

By induction, we know that for every m, p_{m+1} is true, so in particular, there is a path of length $m + 1$. In particular, we have a path of length n, and, by the argument just given, we can turn this path into a circuit. $\square$

Theorem 6.2 – ORE'S THEOREM.
If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Proof. First we show that G is connected. If not, let v and w be vertices in two different connected components of G, and suppose the components have n_1 and n_2 vertices. Then $\mathrm{d}(v) \leq n_1 - 1$ and $\mathrm{d}(w) \leq n_2 - 1$, so $\mathrm{d}(v) + \mathrm{d}(w) \leq n_1 + n_2 - 2 < n$. But since v and w are not adjacent, this is a contradiction.

Now consider a longest possible path in G: $v_1, v_2, \ldots, v_k$. Suppose, for a contradiction, that $k < n$, so there is some vertex w adjacent to one of $v_2, v_3, \ldots, v_{k-1}$, say to v_i. If v_1 is adjacent to v_k, then $w, v_i, v_{i+1}, \ldots, v_k, v_1, v_2, \ldots v_{i-1}$ is a path of length $k+1$, a contradiction. Hence, v_1 is not adjacent to v_k, and so $\mathrm{d}(v_1) + d(v_k) \geq n$. The neighbors of v_1 are among $\{v_2, v_3, \ldots, v_{k-1}\}$ as are the neighbors of v_k. Consider the vertices

$$W = \{v_{l+1} \mid v_l \text{ is a neighbor of } v_k\}.$$

Then $|N(v_k)| = |W|$ and $W \subseteq \{v_3, v_4, \ldots, v_k\}$ and $N(v_1) \subseteq \{v_2, v_3, \ldots, v_{k-1}\}$, so $W \cup N(v_1) \subseteq \{v_2, v_3, \ldots, v_k\}$, a set with $k - 1 < n$ elements. Since $|N(v_1)| + |W| = |N(v_1)| + |N(v_k)| \geq n$, $N(v_1)$ and W must have a common element, v_j; note that $3 \leq j \leq k-1$. Then this is a cycle of length k:

$$v_1, v_j, v_{j+1}, \ldots, v_k, v_{j-1}, v_{j-2}, \ldots, v_1.$$

We can relabel the vertices for convenience:

$$v_1 = w_1, w_2, \ldots, w_k = v_2, w_1.$$

Now as before, w is adjacent to some w_l, and $w, w_l, w_{l+1}, \ldots, w_k, w_1, w_2, \ldots w_{l-1}$ is a path of length $k+1$, a contradiction. Thus, $k = n$, and, renumbering the vertices for convenience, we have a Hamilton path $v_1, v_2, \ldots, v_n$. If v_1 is adjacent to v_n, there is a Hamilton cycle, as desired.

If v_1 is not adjacent to v_n, the neighbors of v_1 are among $\{v_2, v_3, \ldots, v_{n-1}\}$ as are the neighbors of v_n. Consider the vertices

$$W = \{v_{l+1} \mid v_l \text{ is a neighbor of } v_n\}.$$

Then $|N(v_n)| = |W|$ and $W \subseteq \{v_3, v_4, \ldots, v_n\}$, and $N(v_1) \subseteq \{v_2, v_3, \ldots, v_{n-1}\}$, so $W \cup N(v_1) \subseteq \{v_2, v_3, \ldots, v_n\}$, a set with $n-1 < n$ elements. Since $|N(v_1)| + |W| = |N(v_1)| + |N(v_k)| \geq n$, $N(v_1)$ and W must have a common element, v_i; note that $3 \leq i \leq n-1$. Then this is a cycle of length n:

$$v_1, v_i, v_{i+1}, \ldots, v_k, v_{i-1}, v_{i-2}, \ldots, v_1,$$

and is a Hamilton cycle. ■

Example 6.5. Let G be a simple graph with n vertices and m edges where m is at least 3. If $m = \frac{1}{2}(n-1)(n-2) + 2$, prove that G is Hamiltonian.

Proof:-
Let u and v be any two non-adjacent vertices and n_1 and n_2 be their degrees respectively.
When we eliminate the vertices u, v from graph G, we get a subgraph with $n-2$ vertices.
This subgraph is a simple graph and if it has q edges then $q \leq \frac{1}{2}(n-2)(n-3)$
Since u and v are non-adjacent, $m = q + n_1 + n_2$.
That implies,

$$n_1 + n_2 = m - q \geq \frac{1}{2}(n-1)(n-2) + 2 - \frac{1}{2}(n-2)(n-3) = n$$

If u and v are two any non-adjacent vertices , $deg(u) + deg(v) \geq n$.
Thus, by the Ore's theorem the graph has a Hamilton circuit and therefore, the graph is Hamiltonian.
The converse of the result just proved is not always true.

Theorem 8.1.
If A is the adjacency matrix of a simple graph, the $(i,j)^{th}$ entry of A^n is equal to the number of paths of length l from vertex i to vertex j, $n = 1, 2, \ldots$.

Proof:-
We shall use mathematical induction on l to prove this theorem. Result is trivial for $l = 0$, $l = 1$.
Let l = 2.
Since the adjacency matrix is symmetric, it's square also a symmetric matrix. Note that the $(i,j)^{th}$ entry of A_G^2 is equal to the number of places in which both i^{th} and j^{th} rows of A_G have ones, hence which is equal to the number of vertices those are adjacent to both i^{th} and j^{th} vertices. Thus, the $(i,j)^{th}$entry of $A_G^2$is equal to the number of different paths of length 2 between $(i,j)^{th}$ and $(i,j)^{th}$ vertices.
While the $(i,j)^{th}$ entry of A_G^2 is equal to the number of 1's in the i^{th} row, and hence equal to the degree of the i^{th}vertex. Hence, the result is true for $l = 2$.

Now assume the result is true for p.
Note that the $(i,j)^{th}$ entry, $(A_G^{p+1})_{ij}$ of A_G^{p+1} is equal to $\sum_{r=1}^{n} (A_G^p)_{ir}(A_G)_{rj}$.
Every path with length (p+1) from the vertex v_i to v_j consists of paths from v_i to v_t of length p together an edge $v_i v_j$. Since there are $(A_G^p)_{ir}$ number of such paths of length p and a_{rj} such edges for each vertex v_r, the total number of paths of length (p + 1) from the vertex vi to vj is equal to $\sum_{r=1}^{n} (A_G^p)_{ir}(A_G)_{rj}$.

Theorem 9.1.

Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal.

Proof Suppose that G_1 and G_2 are isomorphic. Then there is a one-to-one, onto function f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2.

Let $v_1, \ldots, v_n$ be an ordering of the vertices of G_1. Let A_1 be the adjacency matrix of G_1 relative to the ordering $v_1, \ldots, v_n$, and let A_2 be the adjacency matrix of G_2 relative to the ordering $f(v_1), \ldots, f(v_n)$. Suppose that the entry in row i, column j, $i \neq j$, of A_1 is equal to k. Then there are k edges, say $e_1, \ldots, e_k$, incident on v_i and v_j. Therefore, there are exactly k edges $g(e_1), \ldots, g(e_k)$ incident on $f(v_i)$ and $f(v_j)$ in G_2. Thus the entry in row i, column j in A_2, which counts the number of edges incident on $f(v_i)$ and $f(v_j)$, is also equal to k. A similar argument shows that the entries on the diagonals in A_1 and A_2 are also equal. Therefore, $A_1 = A_2$.

The converse is similar and is left as an exercise (see Exercise 25).

Example 9.2

Let e and n are nonnegative integers. Show that the properties "has e edges" and "has n vertices" are invariants.

<u>Proof:-</u>

By Definition 8.6.1, if graphs G_1 and G_2 are isomorphic, there are one-to-one, onto functions from the edges (respectively, vertices) of G_1 to the edges (respectively, vertices) of G_2. Thus, if G_1 and G_2 are isomorphic, then G_1 and G_2 have the same number of edges and the same number of vertices. Therefore, if e and n are nonnegative integers, the properties "has e edges" and "has n vertices" are invariants.

Example 9.3
Show that if k is a positive integer, "has a vertex of degree k" is an invariant.

Proof:-

Suppose G_1 and G_2 are isomorphic graphs and f (respectively, g) is a one-to-one, onto function from the vertices (respectively, edges) of G_1 onto the vertices (respectively, edges) of G_2. Suppose that G_1 has a vertex v of degree k. Then there are k edges $e_1, \ldots, e_k$ incident on v. By Definition 8.6.1, $g(e_1), \ldots, g(e_k)$ are incident on $f(v)$. Because g is one-to-one, $\delta(f(v)) \geq k$.

Let E be an edge that is incident on $f(v)$ in G_2. Since g is onto, there is an edge e in G_1 with $g(e) = E$. Since $g(e)$ is incident on $f(v)$ in G_2, by Definition 8.6.1, e is incident on v in G_1. Since $e_1, \ldots, e_k$ are the only edges in G_1 incident on v, $e = e_i$ for some $i \in \{1, \ldots, k\}$. Now $g(e_i) = g(e) = E$. Thus $\delta(f(v)) = k$, so G_2 has a vertex, namely $f(v)$, of degree k. ◀

Theorem 10.1 – EULER'S FORMULA.
Let G be a connected planar simple graph with e edges and v vertices. Let f be the number of regions or faces in a planar representation of G. Then $f = e - v + 2$.

Proof:-
We prove the theorem by induction on e, the number of edges of G.
If $e = 0$, then $v = 1$ as G is connected. So, f =1. Thus, v - e + f = 1-0+1 = 2. So, the theorem is true for $e = 0$.
Assume that the theorem is true for all connected planar graphs with k or fewer edges. Let G be a connected planar graph with k+1 edges.
If G has a vertex of degree 1, say x, then let G' = G-x.
It is easy to see that G' is a connected planar graph with k edges. Again, number of vertices of G' is one less than the number of vertices of G and number of faces of G' is equal to the number of vertices of G. So, by induction hypothesis, (n-1)-k + f = 2, where n and f are the number of vertices and faces of G, respectively. So, n-(k+1)-f=2. So, the theorem is true for G in this case.
Next assume that G has no vertex of degree 1. Since, G is connected, $\delta(G) \geq 2$.
So, G has a cycle. Let e be any edge in some cycle in G. Let G' = G - e. Now G' is a connected planar graph and has same number of vertices as G and one less edge than G. The cycle containing the edge e determines a face in every planar embedding of G. Now, that cycle will be missing in G'. So, G' has f - 1 faces and k edges. So, by induction hypothesis, n – k + f - 1 = 2.
This implies n - (k + 1) + f = 2.
So, the theorem is true for G in this case as well.
So, by induction principle, the theorem is true.

Corollary 10.1.

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then,

$$e \leq 3v - 6.$$

Lemma 5.8.4. *In a planar embedding of a connected graph, each edge is traversed once by each of two different faces, or is traversed exactly twice by one face.*

Lemma 5.8.5. *In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.*

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 5.8.4, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 5.8.5, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{5.5}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (5.5) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

Another Proof:-

Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 3. So we have $2e \geq 3f$. Then $\frac{2}{3}e \geq f$.

Euler's formula says that $v - e + f = 2$, so $f = e - v + 2$. Combining this with $\frac{2}{3}e \geq f$, we get

$$e - v + 2 \leq \frac{2}{3}e$$

So $\frac{e}{3} - v + 2 \geq 0$. So $\frac{e}{3} \leq v - 2$. Therefore $e \leq 3v - 6$.

Corollary 10.2.
If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof: This is clearly true if G has one or two vertices.

If G has three vertices, we know that $e \leq 3v-6$. So $2e \leq 6v-12$.

By the handshaking theorem, $2e$ is the sum of the degrees of the vertices. Suppose that the degree of each vertex was at least 6. Then we would have $2e \geq 6v$. But this contradicts the fact that $2e \leq 6v-12$.

Corollary 10.3.
If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 4 because we have no circuits of length 3. So we have $2e \geq 4f$. Then $\frac{1}{2}e \geq f$.

Euler's formula says that $v - e + f = 2$. or $e - v + 2 = f$. Combining this with $\frac{1}{2}e \geq f$, we get

$$e - v + 2 \leq \frac{1}{2}e$$

So $\frac{e}{2} - v + 2 \leq 0$. So $\frac{e}{2} \leq v - 2$. Therefore $e \leq 2v - 4$.

Example 11.1 What is the chromatic number of K_n?
Chromatic number of K_n is n. The colors of all vertices in K_n are distinct because there is an edge between every two vertices.

Example 11.2 What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers?
2

Example 11.3 What is the chromatic number of the graph C_n, where $n \geq 3$?
2 if n is even, and 3 if n is odd.