I-YEAR B.E./B.TECH. (COMMON TO ALL BRANCHES)
MATRICES AND CALCULUS (MA3151) QUESTION BANK

UNIT – I MATRICES

PART-A

1	**State Cayley Hamilton theorem. Also write uses of Cayley Hamilton theorem.**
	Statement: Every square matrix satisfies its own characteristic equation. The most important applications of Cayley–Hamilton theorem which are related to the calculation of inverse and powers of square matrices.
2	**If A $= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, then find $2A^2 - 12A + 10I$** **(NOV/ DEC 2019)**
	Solution: The Characteristic equation of A is $\lvert A - \lambda I \rvert = 0$ (i.e.) $\lambda^2 - S_1\lambda + S_2 = 0$ Where, S_1 = Sum of the main diagonal elements $= 1 + 5 = 6$ $S_2 = \lvert A \rvert = \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = 5 - 0 = 5$ Therefore, the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$ By Cayley Hamilton theorem ("Every square matrix satisfies its own characteristic equation ") $A^2 - 6A + 5I = 0$ Multiply by 2 on both sides, Hence, $2(A^2 - 6A + 5I) = 2(0)$ $2A^2 - 12A + 10I = 0$
3	**Show that $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.**
	Solution: $AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & -\cos\theta\sin\theta+\sin\theta\cos\theta \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $A^TA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\cos\theta\sin\theta \\ \cos\theta\sin\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\therefore AA^T = A^TA = I \Rightarrow$ A is orthogonal.
4	**Find the constants 'a' &'b' such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 & –2 as its eigen values. (APR/MAY 2018)**
	Solution: Sum of the main diagonal elements = sum of the eigen values $\Rightarrow a + b = 3 - 2 = 1 - - - - - (1)$ Product of the eigen values = $\lvert A \rvert$ $\Rightarrow (3)(-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix}$ $i.e. -6 = ab - 4$ $\Rightarrow ab = -6 + 4 \Rightarrow ab = -2$ $\therefore b = -2/a$ Sub in (1) $a + b = 1 \Rightarrow a + \left(-\frac{2}{a}\right) = 1 \Rightarrow \frac{a^2-2}{a} = 1 \Rightarrow a^2 - 2 = a \quad i.e.\ a^2 - a - 2 = 0$ By Solving, $a = -1, 2 \Rightarrow b = 2, -1$

5	**If the product of eigen values of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 1 & k & 2 \end{bmatrix}$ is -8, then find the value of k?**
	Solution: The product of the eigen values $= \lvert A \rvert$ $-8 = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 1 & k & 2 \end{vmatrix}$ $\Rightarrow 1(10-2k) - 2(6-2) - 1(3k-5) = -8$ $\Rightarrow 10 - 2k - 8 - 3k + 5 = -8$ $\Rightarrow 7 - 5k = -8$ $\Rightarrow -5k = -8 - 7 \Rightarrow -5k = -15$ Hence, $k = 3$
6	**If $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$, find the corresponding eigen value.**
	Solution: Let X = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be the eigen vector of the matrix corresponding to the eigen value λ. The eigen vectors are obtained from the equation $AX = \lambda X$ $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\begin{pmatrix} -2+4+3 \\ 2+2+6 \\ -1-4+0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix}$ Two matices are equal when their corresponding elements are equal. $-2+4+3 = \lambda$ $\therefore \lambda = 5.$
7	**If trace and determinant of 2 × 2 is – 2 and –35 respectively, then find the eigen values of matrix?**
	Solution: Let λ_1 and λ_2 are two eigen value of the given matrix A sum of the eigen values = Trace of A $\Rightarrow \lambda_1 + \lambda_2 = -2$ $\lambda_1 = -2 - \lambda_2$…….. (1) Product of the eigen values= Determinant of A $\Rightarrow \lambda_1\lambda_2 = -35$……(2) Sub (1) in equation (2) $(-2-\lambda_2)\lambda_2 = -35$ $(-2-\lambda_2)\lambda_2 + 35 = 0 \Rightarrow -2\lambda_2 - \lambda_2^2 + 35 = 0$ $\lambda_2^2 + 2\lambda_2 - 35 = 0$ $(\lambda_2 - 5)(\lambda_2 + 7) = 0$ Therefore, $\lambda_2 = 5$ and -7 To find λ_1 substitute λ_2 values in (1) If the $\lambda_2 = 5$ then $\lambda_1 = -2 - 5 = -7$ If the $\lambda_2 = -7$ then $\lambda_1 = -2 + 7 = 5$ Hence the eigen values are 5 and -7.

8	**If** $\mathbf{A}=\begin{bmatrix}2&1&0\\0&3&4\\0&0&4\end{bmatrix}$ **then find the eigen values of A^{-1} and A^2-2I.** **(APR/MAY 2018)**
	Solution: In a triangular matrix, the main diagonal values are the eigen values of the matrix. $\therefore$ 2, 3, 4 are the eigen values of A. Hence the eigen values of A^{-1} = $\frac{1}{2},\frac{1}{3},\frac{1}{4}$. The eigen values of A^2 are $(2)^2$, $(3)^2$, $(4)^2$ = 4, 9, 16. The eigen values of A^2-2I are $4-2, 9-2, 16-2 = 2, 7, 14$.
9	**Determine whether the given matrix** $\boldsymbol{A}=\begin{bmatrix}2&0&0\\2&6&0\\3&2&1\end{bmatrix}$ **is Diagonalizable?**
	Solution: The given matrix $A=\begin{bmatrix}2&0&0\\2&6&0\\3&2&1\end{bmatrix}$ is triangular matrix $\therefore$ the eigenvalues are $\lambda_1 = 2, \lambda_2 = 6$ and $\lambda_3 = 1$. since eigenvectors corresponding to distinct eigenvalues are linearly independent, 'A' has three linearly independent eigenvectors and it is therefore diagonalizable.
10	**If** $\mathbf{A}=\begin{bmatrix}2&0&0\\0&3&0\\0&0&1\end{bmatrix}$**, then find the eigen values of adjoint of A.** **(APR/MAY 2019)**
	Solution: Given matrix is triangular therefore its eigenvalues are 2,3 and 1. We know that, eigenvector of adjoint of A = eigenvector of A × $\lvert A\rvert$ $\lvert A\rvert$ = product of the eigen values = (2)(3)(1) = 6 Eigen values of A^{-1} are $\frac{1}{2},\frac{1}{3},1$ $\therefore$ Eigen values of adj A are $\frac{1}{2}(6), \frac{1}{3}(6), (1)(6) = 3, 2, 6$
11	**Write down the quadratic form corresponding to the matrix** $\mathbf{A}=\begin{bmatrix}0&5&-1\\5&1&6\\-1&6&2\end{bmatrix}$
	Solution: The matrix of the quadratic form of A is given by $X^TAX=(x_1\ x_2\ x_3)\begin{bmatrix}0&5&-1\\5&1&6\\-1&6&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0x_1^2+x_2^2+2x_3^2+10x_1x_2+12x_2x_3-2x_1x_3$
12	**Determine the nature of the following quadratic form** $\boldsymbol{f(x_1,x_2,x_3)=x_1^2+2x_2^2-9x_3^2}$**.**
	Solution: The matrix of the quadratic form is $A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-9\end{bmatrix}$ The eigen values of the matrix are 1, 2, -9 Therefore, the quadratic form is indefinite.

13	**Discuss the nature of the quadratic form $2x^2+3y^2+2z^2+2xy$ without reducing it to canonical form.**
	Solution: The matrix of the quadratic form is $A=\begin{bmatrix}2&1&0\\1&3&0\\0&0&2\end{bmatrix}$ The principal sub determinants are $D_1=\lvert a_{11}\rvert=2\ (+ve)$ $D_2=\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}=\begin{vmatrix}2&1\\1&3\end{vmatrix}=6-1=5(+ve)$ $D_3=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=\begin{vmatrix}2&1&0\\1&3&0\\0&0&2\end{vmatrix}=2(6-0)-1(2-0)+0=10(+ve)$ Since $D_1>0, D_2>0, D_3>0$ Therefore, the quadratic form is positive definite.
14	**Find the values of λ so that the QF $\lambda(x_1^2+x_2^2+x_3^2)+2x_1x_2-2x_2x_3+2x_3x_1$ may be positive definite.**
	Solution: The matrix of the quadratic form is $A=\begin{vmatrix}\lambda&1&1\\1&\lambda&-1\\1&-1&\lambda\end{vmatrix}$ The principal sub determinants are $D_1=\lambda$ $D_2=\begin{vmatrix}\lambda&1\\1&\lambda\end{vmatrix}=\lambda^2-1=(\lambda+1)(\lambda-1)$ $D_3=\begin{vmatrix}\lambda&1&1\\1&\lambda&-1\\1&-1&\lambda\end{vmatrix}=\lambda(\lambda^2-1)-1(\lambda+1)+1(-1-\lambda)$ $=\lambda^3-3\lambda-2=(\lambda+1)^2(\lambda-2)$ $\lambda=-1$ and $\lambda=2$ For positive definite $D_1>0, D_2>0, D_3>0$ Since λ values must be positive Therefore, $\lambda>2$ the quadratic form is positive definite.
15	**Find the matrix whose eigen values are 1, 3 and eigen vectors are $(0,-1)^T$ and $(1,0)^T$.**
	Solution: Since the given eigen vectors $\begin{pmatrix}0\\-1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}$ are orthogonal By orthogonal reduction $A=NDN^T$ The normalized model matrix of the given eigenvector $N=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $\therefore A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $=\begin{bmatrix}0&3\\-1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $\therefore A=\begin{bmatrix}3&0\\0&1\end{bmatrix}$
16	**Find the nature of the conic $8x^2-4xy+5y^2=36$ by reducing the quadratic form**

	$8x^2 - 4xy + 5y^2$ **to the form** $4X^2 + 9Y^2$**.**
	Solution: The matrix of the given quadratic form is $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$ The Characteristic equation of A is $\lvert A - \lambda I \rvert = 0$ (i.e.) $\lambda^2 - S_1\lambda + S_2 = 0$ Where, S_1 = Sum of the main diagonal elements $= 8 + 5 = 13$ $S_2 = \lvert A \rvert = \begin{vmatrix} 8 & -2 \\ -2 & 5 \end{vmatrix} = 40 - 4 = 36$ Therefore, the characteristic equation is $\lambda^2 - 13\lambda + 36 = 0$ $\Rightarrow \quad (\lambda - 4)(\lambda - 9) = 0$ $\Rightarrow \quad \lambda = 4$ or $\lambda = 9$ Hence, the eigenvalues are 4 and 9. $\therefore$ The eigen values of A are $\lambda = 4, 9$ the canonical form is of the form $4X^2 + 9Y^2$ Given conic becomes $4X^2 + 9Y^2 = 36$ i.e. $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ which is an ellipse.
17	**Write down the matrix equation of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is in standard position to the coordinate axis .**
	Solution: The equation of an ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is in quadratic form. Hence $\frac{x^2}{16} + \frac{y^2}{9} = 1$ $\Longrightarrow [x \;\; y] \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1.$
18	**If λ is the eigen value of the matrix A, then prove that λ^2 is the eigen value of A^2. (APR/MAY2019)**
	Solution: If X is the eigen vector of the matrix A corresponding to the eigen value λ, then AX = λX. Pre multiply by A $\Rightarrow A^2 X = A(\lambda X)$ $= \lambda(AX)$ $= \lambda(\lambda X)$ $= \lambda^2 X$ Hence, λ^2 is the eigen value of A^2.
19	**Prove that the matrices A and A^T have the same Eigen values. (NOV/DEC2019)**
	Solution: We know that, $\lvert A \rvert = \lvert A^T \rvert$**.** $\therefore \lvert A - \lambda I \rvert = \lvert (A - \lambda I)^T \rvert$

$$= \left|\left(A^T - \lambda I^T\right)\right|$$

$$= |A^T - \lambda I)| \qquad (\because I = I^T)$$

$\therefore$ A and A^T have same characteristic polynomial $\Rightarrow$ A and A^T have same characteristic equation. Hence, A and A^T have same eigen values.

20 **Show that the eigen values of $-3A^{-1}$ are the same as those of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$**

Solution: The characteristic equation of A is $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda+1)(\lambda-3) = 0$$

$\therefore$ $\lambda = -1, 3$ are the eigen values of A.

Now the eigen values of A^{-1} are $\frac{1}{-1}, \frac{1}{3}$ and hence the eigen values of $-3A^{-1}$ are 3, –1.

PART – B

1 **(i) Verify Caley Hamilton theorem for $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$. Also find adj(A) and A^{-1}.**

(APR / MAY 18)

Solution: Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 1 + 3 - 4 = 0$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= (-12 - 12) + (-4 + 6) + (3 - 1) = -24 + 2 + 2 = -20$$

$$S_3 = |A| = 1(-12 - 12) - 1(-4 - 6) + 3(-4 + 6)$$

$$= -24 + 10 + 6 = -8$$

$\therefore$ $\lambda^3 - 0\lambda^2 - 20\lambda + 8 = 0$

By Caley Hamilton theorem, $A^3 - 20A + 8I = 0$.....(1)

<u>Verification:</u>

$$A^2 = A \times A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4-8+24 & -4-24+48 & -12+24+48 \\ 10+22-12 & 10+66-24 & 30-66-24 \\ 2+2-44 & 2+6-88 & 6-6-88 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix}$$

Therefore,

$$A^3 - 20A + 8I = \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - 20 \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} - \begin{bmatrix} 20 & 20 & 60 \\ 20 & 60 & -60 \\ -40 & -80 & -80 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the Cayley-Hamilton theorem is verified.

<u>To find A^{-1}:</u>

Pre multiply (1) by A^{-1} on both sides of Equation (1)

$$\Rightarrow A^2 - 20I + 8A^{-1} = 0$$

$$\Leftrightarrow 8A^{-1} = -A^2 + 20A$$

$$8A^{-1} = -(A^2 - 20A)$$

$$\therefore \qquad A^{-1} = \frac{-1}{8}(A^2 - 20A)$$

$$A^{-1} = \frac{1}{8}(-A^2 + 20A) = \frac{-1}{8}\left(\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} - 20 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{-1}{8}\left(\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right)$$

$$= \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Hence, $A^{-1} = \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$.

<u>To find $Adj(A)$:</u>

We know that $Adj(A) = |A|\, A^{-1}$

$$Adj(A) = 8 \times \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\therefore Adj(A) = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

(ii) Verify Cayley-Hamilton theorem for the matrix A= $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$**, hence find A^4 and A^{-1}.**

Solution: Given $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 1 + 2 + 1 = 4$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$$= (2-6) + (1-7) + (2-12) = -4 - 6 - 10 = -20$$

$$S_3 = |A| = 1(2-6) - 3(4-3) + 7(8-2)$$

$$= -4 - 3 + 42 = 35$$

$\therefore \quad \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

By Caley Hamilton theorem, $A^3 - 4A^2 - 20A - 35 = 0$.....(1)

<u>Verification:</u>

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Substituting A^3 , A^2 & A in (1)

$$A^3 - 4A^2 - 20A - 35I$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - \begin{bmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{bmatrix} - \begin{bmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the Cayley-Hamilton theorem is verified.

<u>To find A^4:</u>

Pre multiply by A on both sides of Equation (1)

$$\Rightarrow A(A^3 - 4A^2 - 20A - 35I) = 0$$

$$\Rightarrow A^4 - 4A^3 - 20A^2 - 35A = 0$$

$$\Leftrightarrow A^4 = 4A^3 + 20A^2 + 35A$$

$$A^4 = 4\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + 20\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} + 35\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 540 & 608 & 928 \\ 560 & 652 & 832 \\ 240 & 304 & 444 \end{bmatrix} + \begin{bmatrix} 400 & 460 & 460 \\ 300 & 440 & 740 \\ 200 & 180 & 280 \end{bmatrix} + 35 \begin{bmatrix} 35 & 105 & 245 \\ 140 & 70 & 105 \\ 35 & 70 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 975 & 1173 & 1633 \\ 1000 & 1162 & 1677 \\ 475 & 554 & 759 \end{bmatrix}$$

<u>To find A^{-1}:</u>

Pre multiply (1) by A^{-1} on both sides of Equation (1)

$$\Rightarrow A^2 - 4A - 20I - 35A^{-1} = 0$$

$$\Leftrightarrow 35A^{-1} = A^2 - 4A - 20I$$

$$\therefore \qquad A^{-1} = \frac{-1}{35}(A^2 - 4A - 20I)$$

$$A^{-1} = \frac{-1}{35}(A^2 - 4A - 20I)$$

$$= \frac{-1}{35}\left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{-1}{35}\left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right)$$

$$= \frac{-1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

Therefore,

Hence, $A^{-1} = \frac{-1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$.

2 **(i) Use Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ to express as a linear polynomial in $A^6 - 5A^5 + 8A^4 - 2A^3 + 9A^2 - 31A - 36I$.**

Solution: Given $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 2 + 1 + 2 = 5$

$S_2 =$ sum of minors of main diagonal of A

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0) = 2 + 3 + 2 = 7$$

$$S_3 = |A| = 2(2-0) - 1(0+0) + 1(0-1)$$

$$= 4 - 0 - 1 = 3$$

$\therefore$ The Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Caley Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = 0$.....(1)

<u>To find $A^6 - 5A^5 + 8A^4 - 8A^3 + 9A^2 - 31A + 36I$:</u>

$$\begin{array}{r|l}
 & A^3 + A \\
\hline
A^3 - 5A^2 + 7A - 3I & A^6 - 5A^5 + 8A^4 - 8A^3 + 9A^2 - 31A - 36I \\
 & A^6 - 5A^6 + 7A^4 - 3A^3 \\
\hline
 & A^4 - 5A^3 + 9A^2 \\
 & A^4 - 5A^3 + 7A^2 - 31A \\
\hline
 & 2A^2 + 31A + 36I \\
\hline
\end{array}$$

By division algorithm, $p(x) = g(x) \times q(x) + r(x)$

$$A^6 - 5A^5 + 8A^4 - 8A^3 - 9A^2 - 31A - 36I$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^2 + A) + (2A^2 + 31A + 36I)$$

$$= (0)(A^5 + A) + (2A^2 + 31A + 36I) \qquad \text{from Equ(1)}$$

$$= 0 + (2A^2 + 31A + 36I)$$

$$= (2A^2 + 31A + 36I)$$

<u>To find $A^2 + A + I$:</u>

$$A^2 = A \times A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$2A^2 + 31A + 36I = 2\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 31\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 36\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 & 8 \\ 0 & 2 & 0 \\ 8 & 8 & 10 \end{bmatrix} + \begin{bmatrix} 62 & 31 & 31 \\ 0 & 31 & 0 \\ 31 & 31 & 62 \end{bmatrix} + \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 108 & 39 & 39 \\ 0 & 69 & 0 \\ 39 & 39 & 108 \end{bmatrix}$$

Hence, $2A^2 + 31A + 36I = \begin{bmatrix} 108 & 39 & 39 \\ 0 & 69 & 0 \\ 39 & 39 & 108 \end{bmatrix}$.

(ii) Using Cayley-Hamilton theorem, find the matrix represented by

$\boldsymbol{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I}$ **when** $\boldsymbol{A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}}$.

Solution: Given $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 2 + 1 + 2 = 5$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2 - 0) + (4 - 1) + (2 - 0) = 2 + 3 + 2 = 7$$

$$S_3 = |A| = 2(2 - 0) - 1(0 + 0) + 1(0 - 1)$$

$$= 4 - 0 - 1 = 3$$

$\therefore$ The Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Caley Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = 0$.....(1)

<u>To find $\boldsymbol{A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I}$:</u>

$A^5 + A$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$A^3 - 5A^2 + 7A - 3I$ | $A^8 - 5A^7 + 7A^6 - 3A^5$

$A^4 - 5A^3 + 8A^2 - 2A$

$A^4 - 5A^3 + 7A^2 - 3A$

$A^2 + A + I$

By division algorithm, $p(x) = g(x) \times q(x) + r(x)$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 + 2A - I$

$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$

$= (0)(A^5 + A) + (A^2 + A + I)$ from Equ(1)

$= 0 + (A^2 + A + I)$

$= (A^2 + A + I)$

<u>To find $A^2 + A + I$:</u>

$$A^2 = A \times A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+1 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Hence, $A^2 + A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$

3 **(i) Find the eigen values and eigen vectors of the matrix** $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

<u>Step 1: To find Eigen values</u>

Since the given matrix is diagonal matrix.

$\therefore$ The eigen values are diagonal elements.

Hence, the Eigen values are 2, 2 and 2. (all three eigen values are equal)

<u>Step 2: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(2-\lambda)x_1 + x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$0x_1 + (2-\lambda)x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$0\,x_1 + 0x_2 + (2-\lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$$0x_1 + 1x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$0x_1 + 0x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$0\,x_1 + 0x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\Rightarrow x_2 = 0 \text{ and } x_3 = 0$$

and also x_1 takes any values.

$\therefore$ The corresponding Eigenvector is $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = X_2 = X_3$

(ii) Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 0 + 0 + 0 = 0$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1) = -1 - 1 - 1 = -3$$

$$S_3 = |A| = 0(0-1) - 1(0-1) + 1(1-0)$$

$$0 + 1 + 1 = 2$$

$\therefore$ Characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$

<u>Step 2: To find Eigen values</u>

Solve $\lambda^3 - 3\lambda - 2 = 0$

By inspection $\lambda = 2$ is a one of the root, other roots can be find using synthetic division.

2	1	0	-3	- 2
	0	2	4	2
	1	2	1	0

$\therefore$ we have $\lambda^2 + 2\lambda + 1 = 0$

$$\Rightarrow \quad (\lambda + 1)(\lambda + 1) = 0$$

$$\Rightarrow \quad \lambda = -1 \text{ or } \lambda = -1$$

Hence, the Eigen values are -1, -1 and 2 (Two equal eigen values)

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$-\lambda x_1 + x_2 + x_3 = 0 \text{ .....(1)}$$

$$x_1 - \lambda x_2 + x_3 = 0 \text{ .....(2)}$$

$$x_1 + x_2 - \lambda x_3 = 0 \text{ .....(3)}$$

Case (i): When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$$-2x_1 + x_2 + x_3 = 0 \text{ .....(4)}$$

$$x_1 - 2x_2 + x_3 = 0 \text{ .....(5)}$$

$$x_1 + x_2 - 2x_3 = 0 \text{ .....(6)}$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
1	1	-2	1
-2	1	1	-2

we get $\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$

$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = -1$, the simultaneous equations (1), (2), (3) becomes

$$x_1 + x_2 + x_3 = 0 \text{ .....(7)}$$

$$x_1 + x_2 + x_3 = 0 \text{ .....(8)}$$

$$x_1 + x_2 + x_3 = 0 \text{ .....(9)}$$

The equations (7) ,(8) and (9) are all identical.

We have only one equation $x_1 + x_2 + x_3 = 0$ to solve for x_1, x_2, x_3. Assign arbitrary values for two variables and solve for third.

Choose $x_3 = 0$, then $x_1 + x_2 = 0$

$$\Rightarrow x_1 = -x_2$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-1}$$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Case (iii): To find third eigen vector corresponding to $\lambda = -1$, let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since A is symmetric matrix whose eigen vectors are pair wise orthogonal. Therefore $X_2^T X_3 = 0$.

$$X_2^T X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \quad a - b = 0 \text{ .....(10)}$$

$$\Rightarrow \quad a = b$$

And also $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must satisfy $x_1 + x_2 + x_3 = 0$

$$\therefore a + b + c = 0 \text{ .....(11)}$$

For solving equation (10) and (11)

Choose $a = 1 \Rightarrow \quad b = 1$ and $c = -2$

$\therefore$ the third Eigenvector corresponding to $\lambda = 4$ is also $X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Hence, the Eigen values are 2, -1 and -1.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

4 **(i) The Eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values 2,3,6 are(1,0,-1)^T, (1,1,1)^T and (1,-2,1)^T respectively. Find the matrix A.**

Solution: We know that eigen vectors of symmetric matrix are orthogonal in pairs

and $N^T AN = D$ where N is normalized modal matrix, D is a diagonal matrix of eigen values.

Since N is orthogonal, $N^T = N^{-1}$

Therefore, $N^{-1}AN = D$

$NN^{-1}ANN^{-1} = NDN^{-1}$, pre multiply by N and post multiply by N^{-1}

$$A = NDN^{-1}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \times \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}} \\ \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & \frac{-2}{2}+\frac{3}{3}+\frac{6}{6} \\ 0+\frac{3}{3}-\frac{12}{6} & 0+\frac{3}{3}+\frac{24}{6} & 0+\frac{3}{3}-\frac{12}{6} \\ \frac{-2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & \frac{2}{2}+\frac{3}{3}+\frac{6}{6} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Therefore the matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.

(ii) Prove that the eigen vectors of the real symmetric matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are orthogonal in pairs

Solution: Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 1 + 5 + 1 = 7$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= (5-1) + (1-9) + (5-1) = 4 - 8 + 4 = 0$$

$$S_3 = |A| = 1(5-1) - 1(1-3) + 3(1-15)$$

$$= 4 + 2 - 42 = -36$$

$\therefore$ Characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$

<u>Step 2: To find Eigen values</u>

Solve $\lambda^3 - 7\lambda^2 + 36 = 0$

By inspection $\lambda = -2$ is a one of the root, other roots can be find using synthetic division.

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

$\therefore$ we have $(\lambda^2 - 9\lambda + 18) = 0$

$$\Rightarrow \quad (\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ or } \lambda = 6$$

Hence, the Eigen values are -2, 3 and 6. (Two equal eigen values)

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(1 - \lambda)x_1 + x_2 + 3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$x_1 + (5 - \lambda)x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$3x_1 + x_2 + (1 - \lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case (i): When $\lambda = -2$, the simultaneous equations (1), (2), (3) becomes

$$3x_1 + x_2 + 3x_3 = 0 \quad \ldots\ldots(4)$$

$$x_1 + 7x_2 + x_3 = 0 \quad \ldots\ldots(5)$$

$$3x_1 + x_2 + 3x_3 = 0 \quad \ldots\ldots(6)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
1	3	3	1
7	1	1	7

we get $\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$

$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$ (i.e) $\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations (1), (2), (3) becomes

$$-2x_1 + x_2 + 3x_3 = 0 \quad \ldots\ldots(7)$$

$$x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots(8)$$

$$3x_1 + x_2 - 2x_3 = 0 \quad \ldots\ldots(9)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
1	3	-2	1
2	1	1	2

we get $\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$

$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 6$, the simultaneous equations (1), (2), (3) becomes

$$-5x_1 + x_2 + 3x_3 = 0 \quad \ldots\ldots(10)$$

$$x_1 - 1x_2 + x_3 = 0 \quad \ldots\ldots(11)$$

$$3x_1 + x_2 - 5x_3 = 0 \quad \ldots\ldots(12)$$

Solving (10) and (11) by cross multiplication method,

x_1	x_2	x_3	
1	3	-5	1
-1	1	1	-1

we get $\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$

$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$

$\therefore$ The corresponding Eigenvector is $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Hence, the eigenvalues are -2, 3 and 6 and the corresponding eigenvectors are $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,$X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

<u>Step 3: To verify orthogonality:</u>

A real symmetric matrix with eigen vectors X_1, X_2, X_3 are pair wise orthogonal, then

$X_1^T X_2 = 0, X_2^T X_3 = 0,\ X_3^T X_1 = 0$

$$X_1^T X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow X_1^T X_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = (-1) + 0 + 1 = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow X_2^T X_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 - 2 + 1 = 0$$

$$X_3^T X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow X_3^T X_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

Hence the eigen vector are orthogonal in pairs.

5 **(i) Diagonalize the matrix** $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **by means of orthogonal transformation.**

Solution: Given $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Step 1: To find the characteristic equation

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 6 + 3 + 3 = 12$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4) = 8 + 14 + 14 = 36$$

$$S_3 = |A| = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 48 - 8 - 8 = 32$$

$\therefore$ Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

Step 2: To find Eigen values

Solve $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

By inspection $\lambda = 2$ is a one of the root, other roots can be find using synthetic division.

2	1	-12	36	-32
	0	2	−20	32
	1	−10	−16	0

$\therefore$ we have $\lambda^2 + 10\lambda - 16 = 0$

$\Rightarrow \quad (\lambda - 8)(\lambda - 2) = 0$

$\Rightarrow \quad \lambda = 8$ or $\lambda = 2$

Hence, the Eigen values are 2, 2 and 8 (Two equal eigen values)

Step 3: To find the Eigenvectors:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$-2x_1 + (3-\lambda)x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$2\,x_1 - x_2 + (3-\lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case (i): When $\lambda = 8$, the simultaneous equations (1), (2), (3) becomes

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$2\,x_1 - x_2 - 5x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
-2	2	-2	-2
-5	-1	-2	-5

we get $\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$

$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$ (i.e) $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$-2x_1 + x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$2\,x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The equations (7), (8) and (9) are all identical.

We have only one equation $2\,x_1 - x_2 + x_3 = 0$ to solve for x_1, x_2, x_3. Assign arbitrary values for two variables and solve for third.

Choose $x_3 = 0$, then $2\,x_1 - x_2 = 0$

$$\Rightarrow 2x_1 = x_2$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Case (iii): To find third eigenvector corresponding to $\lambda = 2$, let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since A is symmetric matrix with eigen vectors are pair wise orthogonal. Therefore $X_2^T X_3 = 0$.

$$X_2^T X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \quad a + 2b = 0 \text{ .....(10)}$$

$$\Rightarrow \quad a = -2b$$

And also $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must satisfy $2\,x_1 - x_2 + x_3 = 0$

$$\therefore 2a - b + c = 0 \text{ .....(11)}$$

For solving equation (10) and (11) by cross multiplication method,

a	b	c	
2	0	1	2
-1	1	2	-1

we get $\frac{a}{2-0} = \frac{b}{0-1} = \frac{c}{1-4}$ (i.e) $\frac{a}{2} = \frac{b}{-1} = \frac{c}{5}$

$\therefore$ the corresponding eigenvector is $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ The eigenvector corresponding to $\lambda = 2$ is also $X_3 = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$

Hence, the eigenvalues are 1, 4 and 4 corresponding eigenvectors are $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$

The normalized eigen vectors are $\begin{bmatrix} \frac{2}{\sqrt{2^2+(-1)^2+1^2}} \\ \frac{-1}{\sqrt{2^2+(-1)^2+1^2}} \\ \frac{1}{\sqrt{2^2+(-1)^2+1^2}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{1^2+2^2+0^2}} \\ \frac{-1}{\sqrt{1^2+2^2+0^2}} \\ \frac{1}{\sqrt{1^2+2^2+0^2}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{1^2+(-1)^2+(-5)^2}} \\ \frac{-1}{\sqrt{1^2+(-1)^2+(-5)^2}} \\ \frac{-2}{\sqrt{1^2+(-1)^2+(-5)^2}} \end{bmatrix}$

$\therefore$ The normalized eigen vectors are $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{-1}{\sqrt{30}} \\ \frac{-5}{\sqrt{30}} \end{bmatrix}$

The normalized model matrix $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{30}} & \frac{-1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{\sqrt{6}} + \frac{-2}{\sqrt{6}} + \frac{2}{\sqrt{6}} & \frac{6}{\sqrt{5}} - \frac{2}{\sqrt{5}} - \frac{0}{\sqrt{5}} & \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{5}{\sqrt{3}} \\ \frac{-4}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{5}} + \frac{6}{\sqrt{5}} + \frac{0}{\sqrt{5}} & \frac{12}{\sqrt{30}} + \frac{2}{\sqrt{30}} - \frac{10}{\sqrt{3}} \\ \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{3}{\sqrt{6}} & \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{5}} + \frac{0}{\sqrt{5}} & \frac{-4}{\sqrt{30}} + \frac{1}{\sqrt{30}} - \frac{15}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{30}} \\ \frac{-8}{\sqrt{6}} & \frac{4}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{8}{\sqrt{6}} & 0 & \frac{-10}{\sqrt{30}} \end{bmatrix}$$

$$N^T AN = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{30}} \\ \frac{-8}{\sqrt{6}} & \frac{4}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{8}{\sqrt{6}} & 0 & \frac{-10}{\sqrt{30}} \end{bmatrix}$$

$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ which is a diagonal matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable, hence find M such that $M^{-1}AM$ is a diagonal matrix. Then obtain the matrix $B = A^2 + 5A + 3I$

Solution: Given $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 2 + 1 + 2 = 6$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= (2-2) + (6-0) + (3+2) = 0 + 6 + 5 = 11$$

$$S_3 = |A| = 3(2-2) - 1(-4-0) - 1(-2-0)$$

$$= 0 + 4 + 2 = 6$$

$\therefore$ Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

<u>Step 2: To find Eigen values</u>

Solve $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By inspection $\lambda = 1$ is a one of the root, other roots can be find using synthetic division.

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$\therefore$ we have $(\lambda^2 - 5\lambda + 6) = 0$

$$\Rightarrow \quad (\lambda - 2)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = 2 \text{ or } \lambda = 3$$

Hence, the Eigen values are 1, 2 and 3.

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{3} - \lambda & \mathbf{1} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{1} - \lambda & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(3 - \lambda)x_1 + x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$-2x_1 + (1 - \lambda)x_2 + 2x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$0\,x_1 + 1x_2 + (2 - \lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case (i): When $\lambda = 1$, the simultaneous equations (1), (2), (3) becomes

$$2x_1 + x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$-2x_1 + 0x_2 + 2x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$0\,x_1 + 1x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
1	-1	2	1
0	2	-2	0

we get $\frac{x_1}{2-0} = \frac{x_2}{2-4} = \frac{x_3}{0+2}$

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{2}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$x_1 + x_2 - x_3 = 0$ ……(7)

$-2x_1 - x_2 + 2x_3 = 0$ ……(8)

$0\,x_1 + 1x_2 + 0x_3 = 0$ ……(9)

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
1	-1	1	1
-1	2	-2	-1

we get $\frac{x_1}{2-1} = \frac{x_2}{2-2} = \frac{x_3}{-1+2}$

$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$

$\therefore$ The corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 3$, the simultaneous equations (1), (2), (3) becomes

$0x_1 + x_2 - x_3 = 0$ ……(7)

$-2x_1 - 2x_2 + 2x_3 = 0$ ……(8)

$0\,x_1 + 1x_2 - x_3 = 0$ ……(9)

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
1	-1	0	1
-2	2	-2	-2

we get $\frac{x_1}{2-2} = \frac{x_2}{2-0} = \frac{x_3}{0+2}$

$\frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2}$ (i.e) $\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Hence, the Eigen values are 1, 2 and 3.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Since the given eigen vectors are linearly independent, therefore the matrix A is linearly independent.

<u>Step 4: To find the matrix $M^{-1}AM$:</u>

Let the matrix M is model matrix formed by the Eigen vector of a matrix.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{|M|} adj(M) = \frac{1}{1}\begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore M^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3+2+0 & -1-1+1 & 1-2+2 \\ 6-2+0 & 2+1-1 & -2+1-2 \\ -3+0+0 & -1+0+1 & 1+0+2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 4 & 2 & -2 \\ -3 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+1+1 & -1+0+1 & 0-1+1 \\ 4-2-2 & 4+0-2 & 0+2-2 \\ -3-1+3 & -3+0+3 & 0+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Hence $A = M^1 AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

<u>Step 4: To find the matrix $\boldsymbol{B = A^2 + 5A + 3I}$:</u>

Since the diagonal matrix of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A^2 + 5A + 3I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$\therefore A^2 + 5A + 3I = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 27 \end{bmatrix}.$$

6 **(i)** **Compute the eigen values and eigen vectors of the following system:**

$$\boldsymbol{10x_1 + 2x_2 + x_3 = \lambda x_1}$$
$$\boldsymbol{2x_1 + 10x_2 + x_3 = \lambda x_2}$$
$$\boldsymbol{2x_1 + x_2 + 10x_3 = \lambda x_3}$$

Solution: Given system of linear equation can be written in the form of the matrix.

$$\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The system of linear equation is of the form $AX = \lambda X$.

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 10 + 10 + 10 = 30$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 10 & 1 \\ 1 & 10 \end{vmatrix} + \begin{vmatrix} 10 & 1 \\ 2 & 10 \end{vmatrix} + \begin{vmatrix} 10 & 2 \\ 2 & 10 \end{vmatrix}$$

$$= (100-1)+(100-2)+(100-4) = 99+98+96 = 293$$

$$S_3 = |A| = 10(100-1) - 2(20-2) + 1(2-20)$$

$$= 990 - 36 - 18 = 936$$

$\therefore$ Characteristic equation is $\lambda^3 - 30\lambda^2 + 293\lambda - 936 = 0$

<u>Step 2: To find Eigen values</u>

Solve $\lambda^3 - 30\lambda^2 + 293\lambda - 936 = 0$

By inspection $\lambda = 8$ is a one of the root, other roots can be find using synthetic division.

8	1	-30	293	-936
	0	8	-176	936
	1	-22	117	0

$\therefore$ we have $(\lambda^2 - 22\lambda + 177) = 0$

$$\Rightarrow \quad (\lambda - 9)(\lambda - 13) = 0$$

$$\Rightarrow \quad \lambda = 9 \text{ or } \lambda = 13$$

Hence, the Eigen values are 8, 9 and 13.

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10-\lambda & 2 & 1 \\ 2 & 10-\lambda & 1 \\ 2 & 1 & 10-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(10-\lambda)x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2x_1 + (10-\lambda)x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$2\,x_1 + x_2 + (10-\lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case (i): When $\lambda = 8$, the simultaneous equations (1), (2), (3) becomes

$$2x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$2x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$2\,x_1 + x_2 + 2x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (5) and (6) by cross multiplication method,

x_1	x_2	x_3	
2	1	2	2
1	2	2	1

we get $\frac{x_1}{4-1} = \frac{x_2}{2-4} = \frac{x_3}{2-4}$

$\frac{x_1}{3} = \frac{x_2}{-2} = \frac{x_3}{-2}$ (i.e) $\frac{x_1}{-3} = \frac{x_2}{2} = \frac{x_3}{2}$

$\therefore$ The corresponding Eigenvector is $X_1 = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$

Case (ii): When $\lambda = 9$, the simultaneous equations (1), (2), (3) becomes

$$x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$2x_1 + x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$2\,x_1 + x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
2	1	1	2
1	1	2	1

we get $\frac{x_1}{2-1} = \frac{x_2}{2-1} = \frac{x_3}{1-4}$

$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-3}$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

Case (iii): When $\lambda = 13$, the simultaneous equations (1), (2), (3) becomes

$$-3x_1 + 2x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$2x_1 - 3x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$2\,x_1 + x_2 - 3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Solving (10) and (11) by cross multiplication method,

x_1	x_2	x_3	
2	1	-3	2
-3	1	2	-3

we get $\frac{x_1}{2+3} = \frac{x_2}{2+3} = \frac{x_3}{9-4}$

$$\frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{5}$$

$\therefore$ The corresponding Eigenvector is $X_3 = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The corresponding Eigen vectors are $\begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(ii) Verify that the eigen vector of the real symmetric matrix $\boldsymbol{A} = \begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{5} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$ are orthogonal to each other.

Solution: Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 3 + 5 + 3 = 11$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15 - 1) + (9 - 1) + (15 - 1) = 14 + 8 + 14 = 36$$

$$S_3 = |A| = 3(15 - 1) + 1(-3 + 1) + 1(1 - 5)$$

$$= 42 - 2 - 4 = 36$$

$\therefore$ Characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

Step 2: To find Eigen values

Solve $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

By inspection $\lambda = 1$ is a one of the root, other roots can be find using synthetic division.

2	1	-11	36	-36
	0	2	-18	36
	1	-9	18	0

$\therefore$ we have $(\lambda^2 - 9\lambda + 18) = 0$

$$\Rightarrow \quad (\lambda - 3)(\lambda - 6) = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ or } \lambda = 6$$

Hence, the Eigen values are 2, 3 and 6. (Two equal eigen values)

Step 3: To find the Eigenvectors:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(3 - \lambda)x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$-x_1 + (5 - \lambda)x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$x_1 - x_2 + (3 - \lambda)x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case (i): When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$$x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$-x_1 + 3x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
-1	1	1	-1
3	-1	-1	3

we get $\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$

$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$ (i.e) $\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations (1), (2), (3) becomes

$$0x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$-x_1 + 2x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$x_1 - x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
-1	1	0	-1
2	-1	-1	2

we get $\frac{x_1}{1-2} = \frac{x_2}{-1+0} = \frac{x_3}{0-1}$

$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 6$, the simultaneous equations (1), (2), (3) becomes

$$-3x_1 - x_2 + x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$-x_1 - x_2 - x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$x_1 - x_2 - 3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
-1	1	-3	-1
-1	-1	-1	-1

we get $\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$

$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence, the Eigen values are 2, 3 and 6.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

<u>Step 3: To verify orthogonality:</u>

A real symmetric matrix with eigen vectors X_1, X_2, X_3 are pair wise orthogonal, then

$X_1^T X_2 = 0, X_2^T X_3 = 0, X_3^T X_1 = 0.$

$$X_1^T X_2 = [-1 \quad 0 \quad 1] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= -1(1) + 0(1) + 1(1) = 0$$

$$X_2^T X_3 = [1 \quad 1 \quad 1] \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= 1(1) + 1(-2) + 1(1) = 0$$

$$X_3^T X_1 = [1 \quad -2 \quad 1] \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1(-1) - 2(0) + 1(1) = 0$$

Hence the eigen vector are pairwise orthogonal.

7 **Reduce the quadratic form $10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$ into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.**

Solution: Given quadratic form is $10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$

The symmetric matrix $A = \begin{bmatrix} coeff(x^2) & \frac{1}{2}coeff(xy) & \frac{1}{2}coeff(xz) \\ \frac{1}{2}coeff(yx) & coeff(y^2) & \frac{1}{2}coeff(yz) \\ \frac{1}{2}coeff(zx) & \frac{1}{2}coeff(zy) & coeff(z^2) \end{bmatrix} = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

The matrix of the quadratic form is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

<u>Step 1: To find the characteristic equation</u>

The characteristic equation of A is $|A - \lambda I| = 0$

i.e. $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 =$ sum of main diagonal values of A $= 10 + 2 + 5 = 17$

$S_2 =$ sum of minors of leading diagonal of A

$$= \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 1 + 25 + 16 = 42$$

$$S_3 = |A| = 10(10-9) + 2(-10+15) - 5(-6+10) = 10 + 10 - 20 = 0$$

$\therefore \lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 42) = 0$

$\lambda(\lambda - 3)(\lambda - 14) = 0$

$\therefore$ The eigen values of A are $\lambda = 0, 3, 14$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(10-\lambda)x_1 - 2x_2 - 5x_3 = 0 \quad \ldots\ldots(1)$$

$$-2x_1 + (2-\lambda)\lambda x_2 + 3x_3 = 0 \quad \ldots\ldots(2)$$

$$-5x_1 + 3x_2 + (5-\lambda)x_3 = 0 \quad \ldots\ldots(3)$$

Case (i): When $\lambda = 0$, the simultaneous equations (1), (2), (3) becomes

$$10x_1 - 2x_2 - 5x_3 = 0 \quad \ldots\ldots(4)$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \quad \ldots\ldots(5)$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \quad \ldots\ldots(6)$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
-2	-5	10	-2
2	3	-2	2

we get $\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$

$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations (1), (2), (3) becomes

$$7x_1 - 2x_2 - 5x_3 = 0 \quad \ldots\ldots(7)$$

$$-2x_1 - x_2 + 3x_3 = 0 \quad \ldots\ldots(8)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \quad \ldots\ldots(9)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
-2	-5	7	-2
-1	3	-2	-1

we get $\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{11} \quad \text{(i.e)} \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 14$, the simultaneous equations (1), (2), (3) becomes

$$-4x_1 - 2x_2 - 5x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
-2	-5	-4	-2
-12	3	-2	-12

we get $$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44} \quad \text{(i.e)} \frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

$\therefore$ the corresponding Eigenvector is $X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

Hence, the Eigen values are 0, 3 and 14.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.

Now X_1, X_2, X_3 are pairwise orthogonal.

Consider the modal matrix $M = \begin{pmatrix} 1 & 1 & -3 \\ -5 & 1 & 1 \\ 4 & 1 & 2 \end{pmatrix}$

Then the normalized modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$

Now $A\times N = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix} \times \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} \\ -\frac{5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{\sqrt{3}} & -\frac{42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{pmatrix}$

$$N^T \times A\times N = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{3}{\sqrt{3}} & -\frac{42}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{14}{\sqrt{14}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{28}{\sqrt{14}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $X^T AX = Y^T (D) Y$ {by diagonalisation}

$$= (y_1 \ y_2 \ y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 14y_3^2$$

Since the sign of the eigen values are positive and zero, the given quadratic form is positive semi definite in nature.

8 **Reduce the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to canonical form through orthogonal transformation. Also sind its nature.** **(APR / MAY 18)**

Solution: Given quadratic form is $\boldsymbol{2x^2 + 5y^2 + 3z^2 + 4xy}$

The symmetric matrix $A = \begin{bmatrix} coeff(x^2) & \frac{1}{2}coeff(xy) & \frac{1}{2}coeff(xz) \\ \frac{1}{2}coeff(yx) & coeff(y^2) & \frac{1}{2}coeff(yz) \\ \frac{1}{2}coeff(zx) & \frac{1}{2}coeff(zy) & coeff(z^2) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 2 + 5 + 3 = 10$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= (15 - 0) + (6 - 0) + (10 - 4) = 15 + 6 + 6 = 27$$

$$S_3 = |A| = 2(15 - 0) - 2(6 - 0) + 0$$

$$= 30 - 12 + 0 = 18$$

$\therefore$ Characteristic equation is $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

<u>Step 2: To find Eigen values</u>

Solve $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

By inspection $\lambda = 1$ is a one of the root, other roots can be find using synthetic division.

1	1	-10	27	-18
	0	1	-9	18
	1	-9	18	0

$\therefore$ we have $(\lambda^2 - 9\lambda + 18) = 0$

$$\Rightarrow \quad (\lambda - 6)(\lambda - 3) = 0$$

$$\Rightarrow \quad \lambda = 6 \text{ or } \lambda = 3$$

Hence, the Eigen values are 1, 3 and 6.

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - \lambda & 2 & 0 \\ 2 & 5 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(2-\lambda)x_1 + 2x_2 + 0x_3 = 0 \quad \ldots\ldots(1)$$

$$2x_1 + (5-\lambda)x_2 + 0x_3 = 0 \quad \ldots\ldots(2)$$

$$0x_1 + 0x_2 + (3-\lambda)x_3 = 0 \quad \ldots\ldots(3)$$

Case (i): When $\lambda = 1$, the simultaneous equations (1), (2), (3) becomes

$$x_1 + 2x_2 + 0x_3 = 0 \quad \ldots\ldots(4)$$

$$2x_1 + 4x_2 + 0x_3 = 0 \quad \ldots\ldots(5)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \quad \ldots\ldots(6)$$

Solving (5) and (6) by cross multiplication method,

x_1	x_2	x_3	
4	0	2	4
0	2	0	0

we get $\frac{x_1}{8-0} = \frac{x_2}{0-4} = \frac{x_3}{0-0}$

$\frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{0}$ (i.e) $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Case (ii): When $\lambda = 3$, the simultaneous equations (1), (2), (3) becomes

$$-x_1 + 2x_2 + 0x_3 = 0 \quad \ldots\ldots(7)$$

$$2x_1 + 2x_2 + 0x_3 = 0 \quad \ldots\ldots(8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \quad \ldots\ldots(9)$$

Solving (7) and (8) by cross multiplication method,

x_1	x_2	x_3	
2	0	-1	0
2	0	2	2

we get $\frac{x_1}{0+0} = \frac{x_2}{0-0} = \frac{x_3}{-2-0}$

$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 6$, the simultaneous equations (1), (2), (3) becomes

$$-4x_1 + 2x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$2x_1 - x_2 + 0x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

$$0x_1 + 0x_2 - 3x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Solving (10) and (11) are identical. Therefore the system has infinite number of soultion.

We have only one equation $2x_1 - x_2 + 0x_3 = 0$ to solve for x_1, x_2, x_3. Assign arbitrary values for two variables and solve for third.

Choose $x_3 = 0$, then $2\,x_1 - x_2 = 0$

$$\Rightarrow 2x_1 = x_2$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{2}$$

$\therefore$ The Eigenvector corresponding to $\lambda = 6$ is also $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Hence, the Eigen values are 1, 3 and 6.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

The normalized eigen vectors are $\begin{bmatrix} \frac{2}{\sqrt{2^2+(-1)^2+0^2}} \\ \frac{-1}{\sqrt{2^2+(-1)^2+0^2}} \\ \frac{0}{\sqrt{2^2+(-1)^2+0^2}} \end{bmatrix}, \begin{bmatrix} \frac{0}{\sqrt{0^2+0^2+1^2}} \\ \frac{0}{\sqrt{0^2+0^2+1^2}} \\ \frac{1}{\sqrt{0^2+0^2+1^2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{1^2+2+0^2}} \\ \frac{2}{\sqrt{1^2+2+0^2}} \\ \frac{0}{\sqrt{1^2+2+0^2}} \end{bmatrix}$

$\therefore$ The normalized eigen vectors are $\begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$

The normalized model matrix $N = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix}$$

$$AN = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{6}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{12}{\sqrt{5}} \\ 0 & 3 & 0 \end{bmatrix}$$

$$N^T AN = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{6}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{12}{\sqrt{5}} \\ 0 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$ which is a diagonal matrix.

Thus A has been diagonalized by N through the orthogonal transformation.

Consider, the linear transformation $X = NY$

$$Q = X^T AX = (NY)^T A(NY)$$

$$= Y^T(N^T AN)Y$$

$$= Y^T DY$$

$= y_1^2 + 3y_2^2 + 6y_1^2.$

9 **Reduce the quadratic form $2xy + 2yz + 2xz$ to canonical form by orthogonal reduction. Also find its nature.**

Solution: Given quadratic form is $2xy + 2yz + 2xz$

The symmetric matrix $A = \begin{bmatrix} coeff(x^2) & \frac{1}{2}coeff(xy) & \frac{1}{2}coeff(xz) \\ \frac{1}{2}coeff(yx) & coeff(y^2) & \frac{1}{2}coeff(yz) \\ \frac{1}{2}coeff(zx) & \frac{1}{2}coeff(zy) & coeff(z^2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Step 1: To find the characteristic equation

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 0 + 0 + 0 = 0$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1) = -1 - 1 - 1 = -3$$

$$S_3 = |A| = 0(0-1) - 1(0-1) + 1(1-0)$$

$$0 + 1 + 1 = 2$$

$\therefore$ Characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$

Step 2: To find Eigen values

Solve $\lambda^3 - 3\lambda - 2 = 0$

By inspection $\lambda = 2$ is a one of the root, other roots can be find using synthetic division.

2	1	0	-3	-2
	0	2	4	2
	1	2	1	0

$\therefore$ we have $(\lambda^2 + 2\lambda + 1) = 0$

$\Rightarrow \quad (\lambda + 1)(\lambda + 1) = 0$

$\Rightarrow \quad \lambda = -1$ or $\lambda = -1$

Hence, the Eigen values are -1, -1 and 2 (Two equal eigen values)

Step 3: To find the Eigenvectors:

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0-\lambda & 1 & 1\\1 & 0-\lambda & 1\\1 & 1 & 0-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

by matrix multiplication

$$-\lambda x_1 + x_2 + x_3 = 0 \text{ .....(1)}$$

$$x_1 - \lambda x_2 + x_3 = 0 \text{ .....(2)}$$

$$x_1 + x_2 - \lambda x_3 = 0 \text{ .....(3)}$$

Case (i): When $\lambda = 2$, the simultaneous equations (1), (2), (3) becomes

$$-2x_1 + x_2 + x_3 = 0 \text{ .....(4)}$$

$$x_1 - 2x_2 + x_3 = 0 \text{ .....(5)}$$

$$x_1 + x_2 - 2x_3 = 0 \text{ .....(6)}$$

Solving (4) and (5) by cross multiplication method,

x_1	x_2	x_3	
1	1	-2	1
-2	1	1	-2

we get $\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$

$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$ (i.e) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ the corresponding Eigenvector is $X_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Case (ii): When $\lambda = -1$, the simultaneous equations (1), (2), (3) becomes

$$x_1 + x_2 + x_3 = 0 \text{ .....(7)}$$

$$x_1 + x_2 + x_3 = 0 \text{ .....(8)}$$

$$x_1 + x_2 + x_3 = 0 \text{ .....(9)}$$

The equations (7) ,(8) and (9) are all identical.

We have only one equation $x_1 + x_2 + x_3 = 0$ to solve for x_1, x_2, x_3. Assign arbitrary values for two variables and solve for third.

Choose $x_3 = 0$, then $x_1 + x_2 = 0$

$$\Rightarrow x_1 = -x_2$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{-1}$$

$\therefore$ the corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Case (iii): To find second eigen vector corresponding to $\lambda = -1$, let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since A is symmetric matrix with eigen vectors are pair wise orthogonal. Therefore $X_2^T X_3 = 0$.

$$X_2^T X_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \quad a - b = 0 \text{ .....(10)}$$

$$\Rightarrow \quad a = b$$

And also $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ must satisfy $x_1 + x_2 + x_3 = 0$

$$\therefore a + b + c = 0 \text{ .....(11)}$$

For solving equation (10) and (11)

Choose $a = 1 \Rightarrow \quad b = 1$ and $c = -2$

$\therefore$ the third Eigenvector corresponding to $\lambda = 4$ is also $X_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Hence, the Eigen values are 2, -1 and -1.

The corresponding Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

The normalized eigen vectors are $\begin{bmatrix} \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2+1^2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{1^2+(-1)^2+0^2}} \\ \frac{-1}{\sqrt{1^2+(-1)^2+0^2}} \\ \frac{0}{\sqrt{1^2+(-1)^2+0^2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2+(-2)^2}} \\ \frac{1}{\sqrt{1^2+1^2+(-2)^2}} \\ \frac{-2}{\sqrt{1^2+1^2+(-2)^2}} \end{bmatrix}$

$\therefore$ The normalized eigen vectors are $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{0}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-2}{\sqrt{3}} \end{bmatrix}$

The normalized model matrix is $N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$ and $N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$

$$AN = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ which is a diagonal matrix.

10 **(i)** **An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point P: (x_1, x_2) goes over into the point Q: (y_1, y_2) given by**

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AX = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ in components, } \begin{matrix} y_1 = 5x_1 + 3x_2 \\ y_2 = 3x_1 + 5x_2 \end{matrix}$$

Find the principal directions, that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Solution:

Given $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

<u>Step 1: To find the characteristic equation</u>

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^2 - S_1\lambda + S_2 = 0$

Where, S_1 = Sum of the main diagonal elements $= 5 + 5 = 10$

$$S_2 = |A| = \begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} = 25 - 9 = 16$$

Therefore, the characteristic equation is

$$\lambda^2 - 10\lambda + 16 = 0$$

<u>Step 2: To find the Eigen values</u>

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0$$

$$\lambda = 8, \lambda = 2$$

Hence, the Eigen values are 2 and 8.

<u>Step 3: To find the Eigenvectors:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value λ.

To find the Eigenvectors, solve $(A - \lambda I)X = 0$.

$$\left(\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(5 - \lambda)x_1 + 3x_2 = 0 \quad \text{.....(1)}$$

$$3x_1 + (5 - \lambda)x_2 = 0 \quad \text{.....(2)}$$

Case 1. If $\lambda = 2$, then Eqn. (1) & (2) becomes

$$3x_1 + 3x_2 = 0 \quad \text{.....(3)}$$

$$3x_1 + 3x_2 = 0 \quad \text{.....(4)}$$

The equations (3) and (4) are identical. Therefore it is enough to solve any one.

$$3x_1 + 3x_2 = 0$$

$$\Rightarrow \quad 3x_1 = -3x_2$$

$\frac{x_1}{-1} = \frac{x_2}{1}$

$\therefore$ The corresponding Eigenvector is $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Case 2. If $\lambda = 8$, then Eqn. (1) & (2) becomes

$$-3x_1 + 3x_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$3x_1 - 3x_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The equations (5) and (6) are identical. Therefore it is enough to solve any one.

$$3x_1 - 3x_2 = 0$$

$$\Rightarrow \quad 3x_1 = 3x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$\therefore$ The corresponding Eigenvector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

These vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ make 45° and 135° angles with the positive x_1-direction. The eigenvalues show that in the principal directions the membrane is stretched by factors 8 and 2, respectively.

Accordingly, if we choose the principal directions as directions of a new Cartesian XY-coordinate system and if we set $x_1 = r\cos\phi$, $y_1 = r\sin\phi$ then a boundary point of the unstretched circular membrane has co-ordinates $\cos\phi$, $\sin\phi$.

Hence, after the stretch we have

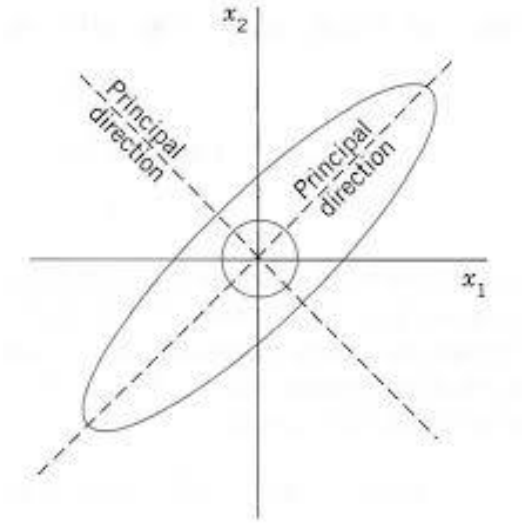
$$z_1 = 8\cos\phi \text{ and } z_2 = 2\sin\phi$$

$$\Rightarrow \cos\phi = \frac{z_1}{8} \text{ and } \sin\phi = \frac{z_2}{2}$$

Since $\sin^2\phi + \cos^2\phi = 1$, this shows that the deformed boundary is an ellipse

$$\frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1$$

<u>Shape of the deformed membrane</u>



(ii) Verify Caley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence compute A^{-1}.

Solution:

The Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = Sum of the main diagonal elements $= 2 + 2 + 2 = 6$

S_2 = sum of minors of main diagonal of A

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-1) + (4-1) = 9$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1 = 4$$

$\therefore \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

By Caley Hamilton theorem, $A^3 - 6A^2 + 9A - 4I = 0$.....(1)

<u>Verification:</u>

$$A^2 = A \times A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Therefore,

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the Cayley-Hamilton theorem is verified.

<u>To find A^{-1}:</u>

Pre multiply (1) by A^{-1} on both sides of Equation (1)

$$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Leftrightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$\therefore \qquad A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I) = \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

	Hence, $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.
11	**An elastic membrane in the x_1x_2-plane with boundary $x_1^2 + x_2^2 = 1$ is stretched so that point $P(x_1, x_2)$ goes over into point $Q(y_1, y_2)$ such that $Y = AX$ with $A = \frac{3}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find the principal directions (Eigen vector) and corresponding factors of extension or contraction of the elastic deformation (eigenvalues). Sketch the shape of the deformed membrane.**
	Solution: Given $A = \frac{3}{2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ Let $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ <u>Step 1: To find the characteristic equation</u> The Characteristic equation of B is $\lvert B - \lambda I \rvert = 0$ (i.e.) $\lambda^2 - S_1\lambda + S_2 = 0$ Where, $S_1 =$ Sum of the main diagonal elements $= 2 + 2 = 4$ $S_2 = \lvert B \rvert = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$ Therefore, the characteristic equation is $\lambda^2 - 4\lambda + 3 = 0$ <u>Step 2: To find the Eigen values</u> $\lambda^2 - 4\lambda + 3 = 0$ $(\lambda - 1)(\lambda - 3) = 0$ $\lambda = 1, \lambda = 3$ Hence, the Eigen values of B are 1, 3. $\therefore$ Eigen values of B are $\frac{3}{2}(1), \frac{3}{2}(3) \Rightarrow 1.5, 4.5$ <u>Step 3: To find the Eigenvectors of B:</u> Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector of B corresponding to the eigen value λ. To find the Eigenvectors, solve $(B - \lambda I)X = 0$. $\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

by matrix multiplication

$$(2-\lambda)x_1 + x_2 = 0 \text{ .....(1)}$$

$$x_1 + (2-\lambda)x_2 = 0 \text{ .....(2)}$$

Case 1. If λ = 1, then Eqn. (1) & (2) becomes

$$x_1 + x_2 = 0 \text{ .....(3)}$$

$$x_1 + x_2 = 0 \text{ .....(4)}$$

The above equations (3) and (4) are identical. Therefore it is enough to solve one.

$$x_1 + x_2 = 0$$

$$\Rightarrow \quad x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$\therefore$ The corresponding Eigenvector is $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Case 2. If λ = 3, then Eqn. (1) & (2) becomes

$$-x_1 + x_2 = 0 \text{.....(5)}$$

$$x_1 - x_2 = 0 \text{.....(6)}$$

The equations (5)and (6) are identical. Therefore it is enough to solve one.

$$x_1 - x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

The Eigenvector corresponding to λ = 3 is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigen vector of B are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the eigen vector of a matrix A and kA are same for corresponding eigen values.

	Therefor the eigen vector of A are also $\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ These vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$make $45°$ and $135°$ angles with the positive x_1-direction. The eigenvalues show that in the principal directions the membrane is stretched by factors 4.5 and 1.5, respectively; Accordingly, if we choose the principal directions as directions of a new Cartesian XY-coordinate system and if we set $x = r\cos\phi$, $y = r sin\,\phi$ then a boundary point of the unstretched circular membrane has coordinates $\cos\phi$, $\sin\phi$. Hence, after the stretch we have $z_1 = \frac{9}{2}\cos\phi$ and $z_2 = \frac{3}{2} sin\phi$ Since $\sin^2\phi + \cos^2\phi = 1$, this shows that the deformed boundary is an ellipse $$\frac{z_1^2}{\left(\frac{9}{2}\right)^2} + \frac{z_2^2}{\left(\frac{3}{2}\right)^2} = 1.$$

UNIT II – DIFFERENTIAL CALCULUS

PART – A

1 **Find the domain of the function $f(x)=\sqrt{3-x}-\sqrt{2+x}$** **(NOV/DEC 2018)**

Solution:

The given $f(x)=\sqrt{3-x}-\sqrt{2+x}$ is real valued function.

The domain of $f(x)$ consist of all the values of x such that $3-x\geq 0$ and $x+2\geq 0$.

i.e $x\leq 3$ and $x\geq -2$, so the domain of $f(x)$ is $[-2,3]$.

2 **Find the domain, range and sketch the graph of the function $f(x)=\sqrt{4-x^2}$**

Solution:

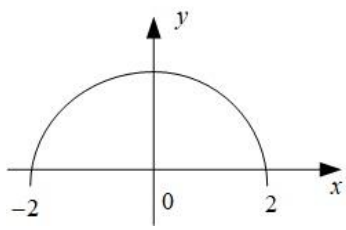
Given $y=\sqrt{4-x^2}$

Squaring on both sides,

$\therefore y^2=4-x^2$

$\Rightarrow x^2+y^2=2^2$

Which is the circle equation with Centre at $(0,0)$ and radius is 2.



Since $y=\sqrt{4-x^2}\geq 0$, the desired graph is the upper half of the circle.

The domain is $-2\leq x\leq 2$ and the range is $0\leq y\leq 2$.

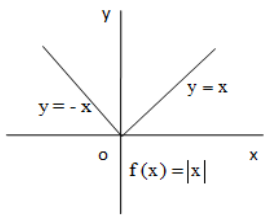
3 **Sketch the graph of the absolute function $f(x)=|x|=\begin{cases}-x, & x<0\\ x, & x\geq 0\end{cases}$**

Solution:

The given function is $f(x)=|x|=\begin{cases}-x, & x<0\\ x, & x\geq 0\end{cases}$

The x intercepts of the function is zero.

Draw a line of $y=-x$ for $x<0$ and $y=x$, for $x\geq 0$



4 **If $\lim\limits_{x\to 1}\dfrac{f(x)-8}{x-1}=10$, then find $\lim\limits_{x\to 1}f(x)$** **(NOV/ DEC 2020)**

Solution:

To find $\lim\limits_{x\to 1}f(x)$

Add and Subtract 8 to $\lim\limits_{x\to 1}f(x)$

Let $\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}[f(x)-8+8]$

$\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}[f(x)-8+8]=\lim\limits_{x\to 1}[f(x)-8]+\lim\limits_{x\to 1}8$ $\boxed{\because \underset{x\to 1}{\text{Lt}}\, 8=8}$

Multiply and divided by (x-1) in the first term

$$=\lim_{x\to 1}\left[\frac{f(x)-8}{x-1}.(x-1)\right]+8$$

$$=\left[\lim_{x\to 1}\frac{f(x)-8}{x-1}\right].\left[\lim_{x\to 1}(x-1)\right]+8$$

$=10(0)+8=0+8=8$ $\therefore \lim\limits_{x\to 1}f(x)=8$

5	**Evaluate $\lim_{t\to 1}\frac{t^4-1}{t^3-1}$** **(NOV/ DEC 2018)**
	Solution: $\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\frac{(1)^4-1}{(1)^3-1}=\frac{1-1}{1-1}=\frac{0}{0}$ (Indeterminate form) By using L'Hospital Rule, $\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\lim_{t\to 1}\frac{\frac{d}{dt}(t^4-1)}{\frac{d}{dt}(t^3-1)}=\lim_{t\to 1}\frac{4t^3}{3t^2}=\frac{4(1)^3}{3(1)^2}=\frac{4}{3}$
6	**Check whether $\lim_{x\to -3}\frac{3x+9}{\lvert x+3\rvert}$ exist.** **(APR/ MAY 2019)**
	Solution: Compute LHL & RHL **LHL :** $\lim_{x\to -3^-}\frac{3x+9}{\lvert x+3\rvert}=\lim_{x\to -3^-}\frac{3x+9}{-(x+3)}=\frac{3(x+3)}{-(x+3)}=-3$ **RHL :** $\lim_{x\to -3^+}\frac{3x+9}{\lvert x+3\rvert}=\lim_{x\to -3^+}\frac{3x+9}{(x+3)}=\frac{3(x+3)}{(x+3)}=3$ $\therefore$ LHL $\neq$ RHL, Hence limit does not exist.
7	**If $f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4\\ 8-2x, & \text{if } x<4\end{cases}$, determine whether $\lim_{x\to 4} f(x)$ exists.**
	Solution: Given $f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4\\ 8-2x, & \text{if } x<4\end{cases}$ Compute Left Hand Limit **(LHL)** & Right Hand Limit **(RHL)** **LHL :** $\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}(8-2x)=8-2(4)=0$ **RHL :** $\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\sqrt{x-4}=\sqrt{4-4}=0$ LHL = RHL $\therefore$ The limit exists, and $\lim_{x\to 4} f(x)=0$
8	**Show that the function $f(x)=4-\sqrt{4-x^2}$ is continuous on $[-2,2]$**
	Solution: If $-2<a<2$, then $\lim_{x\to a} f(x)=f(a)$ $\therefore f(x)$ is continuous at $-2<a<2$ Compute RHL & LHL at the end points **RHL :** $\lim_{x\to(-2)^+} f(x)=\lim_{x\to(-2)^+} 4-\sqrt{4-x^2}=4-\sqrt{4-(-2)^2}=4-\sqrt{4-4}=4-0=4=f(-2)$ **LHL :** $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 4-\sqrt{4-x^2}=4-\sqrt{4-(+2)^2}=4-\sqrt{4-4}=4-0=4=f(2)$ $\therefore f(x)$ is continuous on $[-2,2]$

9	**For what values of *a* and *b*, the function is continuous if** $f(x)=\begin{cases} a+bx & if \quad x<1 \\ 4 & if \quad x=1 \\ b-ax & if \quad x>1 \end{cases}$
	Solution: The given function is $f(x)=\begin{cases} a+bx & if \quad x<1 \\ 4 & if \quad x=1 \\ b-ax & if \quad x>1 \end{cases}$ Now **LHL :** $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(a+bx)=a+b$ **RHL :** $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(b-ax)=b-a$ also $f(1)=4$ Since $f(x)$ is continuous at $x=1$ $\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)=f(1)$ $\Rightarrow a+b=4 \dashrightarrow (1) \quad \& \quad b-a=4 \dashrightarrow (2)$ Solve the equations(1) & (2), Adding equations (1) & (2), $2b=8 \Rightarrow b=\frac{8}{2}=4$ Substitute $b=4$ in Equation (1) $a+4=4 \Rightarrow a=0$ The values are $a=0, b=4$
10	**If $f(x)=xe^x$ then find $f''(x)$** **(NOV/DEC 2019)**
	Solution: Given $f(x)=xe^x$ $f'(x)=x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x)$ $f'(x)=x(e^x)+e^x(1)$ $f''(x)=\frac{d}{dx}f'(x)=\frac{d}{dx}\left[x(e^x)+e^x\right]=\frac{d}{dx}\left[xe^x\right]+\frac{d}{dx}\left[e^x\right]$ $f''(x)=\left[x\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x)\right]+e^x=\left[x(e^x)+e^x(1)\right]+e^x$ $f''(x)=xe^x+2e^x=e^x(x+2).$
11	**If $xe^y=x-y$, then find $\frac{dy}{dx}$** **(NOV/DEC 2020)**
	Solution: Given $xe^y=x-y$ Differentiate w.r.to x $\frac{d}{dx}\left[xe^y=x-y\right]$ $\frac{d}{dx}\left(xe^y\right)=\frac{d}{dx}(x-y)$ $\left[x\frac{d}{dx}\left(e^y\right)+e^y\frac{d}{dx}(x)\right]=\frac{d}{dx}(x)-\frac{d}{dx}(y)$

	$xe^y \frac{dy}{dx} + e^y(1) = 1 - \frac{dy}{dx}$ $xe^y \frac{dy}{dx} + \frac{dy}{dx} = 1 - e^y$ $(xe^y + 1)\frac{dy}{dx} = 1 - e^y$ $\therefore \frac{dy}{dx} = \frac{1-e^y}{1+xe^y}$
12	**If $f(x) = \sqrt{x}$, then find the derivative of $f(x)$. Also state the domain of $f'(x)$**
	Solution: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ $= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ $\left(\because f(x) = \sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}\right)$ Multiply and divided by conjugate factor of the numerator, $= \lim_{h\to 0}\left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) = \lim_{h\to 0}\left(\frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\sqrt{x+h} + \sqrt{x}}\right)$ $= \lim_{h\to 0}\left(\frac{(\cancel{x}+h) - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}\right) = \lim_{h\to 0}\left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}\right)$ $= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ $f'(x)$ exists only if $x > 0$, the domain of $f'(x)$ is $(0, \infty)$.
13	**Find $\frac{dy}{dx}$ if $\sin(x+y) = y^2 \cos x$**
	Solution: Given $\sin(x+y) = y^2 \cos x$ Diff. w.r.to x, $\frac{d}{dx}\sin(x+y) = \frac{d}{dx}\left(y^2 \cos x\right)$ $\frac{d}{dx}\sin(x+y) = \left(y^2 \frac{d}{dx}\cos x + \cos x \frac{d}{dx} y^2\right)$ $\cos(x+y)\left(1 + \frac{dy}{dx}\right) = y^2(-\sin x) + (\cos x)\left(2y\frac{dy}{dx}\right)$ $\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2 \sin x + 2y\frac{dy}{dx}\cos x$ $\cos(x+y) + y^2 \sin x = \left[2y\cos x - \cos(x+y)\right]\frac{dy}{dx}$ $\therefore \frac{dy}{dx} = \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}$.

14	**Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$**
	Solution: Given $x = at^2$, $y = 2at$ $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$ $\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{2}\cancel{a}}{\cancel{2}\cancel{a}t} = \frac{1}{t}$
15	**Find the critical points of $y = 5x^3 - 6x$** **(APR/MAY 2019)**
	Solution: Given $y = 5x^3 - 6x$ $\frac{dy}{dx} = 15x^2 - 6$ $\frac{dy}{dx} = 0 \Rightarrow 15x^2 - 6 = 0$ $\Rightarrow 15x^2 = 6$ $\Rightarrow x^2 = \frac{6}{15}$ Take square root on both sides, $\sqrt{x^2} = \sqrt{\frac{6}{15}} = \sqrt{\frac{\cancel{3}(2)}{\cancel{3}(5)}} = \pm\sqrt{\frac{2}{5}}$ $\therefore x = \pm\sqrt{\frac{2}{5}}$ The critical points are $-\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}$
16	**Find an equation of the tangent line to the curve $y = \frac{e^x}{1+x^2}$ at $\left(1, \frac{e}{2}\right)$**
	Solution: Given $y = \frac{e^x}{1+x^2}$ By using Quotient rule, $\therefore \frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2} = \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$ $m = \left[\frac{dy}{dx}\right]_{\left(x=1, y=\frac{e}{2}\right)} = \frac{e^1(1-1)^2}{(1+1^2)^2} = \frac{0}{4} = 0$ The equation of the tangent is, $y - y_1 = m(x - x_1)$ $y - \frac{e}{2} = 0(x-1)$ $y - \frac{e}{2} = 0 \qquad \boxed{y = \frac{e}{2}}$

17	**Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.**
	Solution: Given $y = x^4 - 6x^2 + 4$ $\frac{dy}{dx} = 4x^3 - 12x$ The Horizontal tangents occur when the slope is zero. $i.e\ \frac{dy}{dx} = 0$ $\Rightarrow \frac{dy}{dx} = 4x^3 - 12x = 0$ $4x\left(x^2 - 3\right) = 0$ $x\left(x^2 - 3\right) = \frac{0}{4} = 0 \Rightarrow x = 0 \,\&\, x^2 - 3 = 0 \Rightarrow x = 0 \,\&\, x^2 = 3$ $x = 0,\ \sqrt{x^2} = \pm\sqrt{3}$ $x = 0,\ x = -\sqrt{3}, +\sqrt{3}$ Now find the value of y using the given equation $y = x^4 - 6x^2 + 4$ When $x = 0 \Rightarrow y = (0)^4 - 6(0)^2 + 4 = 4\ ,(0,4)$ When $x = -\sqrt{3} \Rightarrow y = \left(-\sqrt{3}\right)^4 - 6\left(-\sqrt{3}\right)^2 + 4 = 9 - 18 + 4 = -5\ ,\left(-\sqrt{3}, -5\right)$ When $x = \sqrt{3} \Rightarrow y = \left(\sqrt{3}\right)^4 - 6\left(\sqrt{3}\right)^2 + 4 = 9 - 18 + 4 = -5\ ,\left(\sqrt{3}, -5\right)$ The points are $(0,4), \left(-\sqrt{3}, -5\right), \left(\sqrt{3}, -5\right)$
18	**Find the equation of the tangent line to the curve $y = x^4 + 2x^2 - x$ at the point (1,2) (NOV/DEC 2020)**
	Solution: Given $y = x^4 + 2x^2 - x$ and $x_1 = 1, y_1 = 2$ $\therefore\ \frac{dy}{dx} = 4x^3 + 4x - 1$ $m = \left[\frac{dy}{dx}\right]_{(1,2)} = 4(1)^3 + 4(1) - 1 = 7$ The equation of tangent line is $(y - y_1) = m(x - x_1)$ $y - 2 = 7(x - 1) \Rightarrow y - 2 = 7x - 7 \Rightarrow y = 7x - 7 + 2$ $y = 7x - 5$
19	**State Extreme value theorem.**
	Statement: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in $[a,b]$.
20	**State Fermat's theorem.**
	Statement: If a function $f(x)$ has a local <u>extremum</u> at some point and is <u>differentiable</u> there, then the function's derivative at that point must be zero. In other words, If $f(x)$ has a local maximum or local minimum at *c*, and if $f'(c)$ exists, then $f'(c) = 0$

PART – B

1 **(i) A function $f(x)$ is defined by $f(x)=\begin{cases}1-x & \text{if } x\le -1\\ x^2 & \text{if } x>-1\end{cases}$. Evaluate $f(-2)$, $f(-1)$ and $f(0)$ and sketch the graph.**

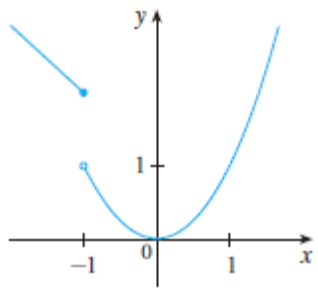
Solution:

Given $f(x)=\begin{cases}1-x & \text{if } x\le -1\\ x^2 & \text{if } x>-1\end{cases}$

Since $-2<-1$, $f(x)=1-x \quad \therefore\ f(-2)=1-(-2)=3$

$-1\le -1$, $f(x)=1-x \quad \therefore\ f(-1)=1-(-1)=2$

$0>-1$, $f(x)=x^2 \quad \therefore\ f(0)=0$



We observe that if $x\le -1$ then $f(x)=1-x$, so the part of the graph of $f(x)$ that lies to the left of the vertical line $x=-1$ must coincide with the line $y=1-x$. If $x>-1$, then $f(x)=x^2$, so the part of the graph of $f(x)$ that lies to the right of the line $x=-1$ must coincide with the graph of $y=x^2$, which is a parabola. This enables us to sketch the graph.

(ii) Does the limit of $f(x)$ exists at x tends to 1 if $f(x)=\begin{cases}x^2+1 & \text{if } x<1\\ (x-2)^2 & \text{if } x\ge 1\end{cases}$

Solution:

Given $f(x)=\begin{cases}x^2+1 & \text{if } x<1\\ (x-2)^2 & \text{if } x\ge 1\end{cases}$

If $x<1$, $f(x)=x^2+1$, LHL: $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}(x^2+1)=(1^2+1)=2$

If $x\ge 1$, $f(x)=(x-2)^2$, RHL: $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(x-2)^2=(1-2)^2=(-1)^2=1$

$\therefore$ LHL $\ne$ RHL $(i.e.)$ $\lim\limits_{x\to 1^-} f(x)\ne \lim\limits_{x\to 1^+} f(x)$, hence $\lim\limits_{x\to 1} f(x)$ does not exists.

(iii) Show that $\lim\limits_{x\to 0} x^3\sin\left(\frac{1}{x}\right)=0$

Solution:

Let $g(x)=x^3\sin\left(\frac{1}{x}\right)$

Since $\lim\limits_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exists

$\therefore$ we cannot use $\lim_{x\to 0}\left(x^3\sin\left(\frac{1}{x}\right)\right)=\lim_{x\to 0}x^3 \,.\, \lim_{x\to 0}\sin\left(\frac{1}{x}\right)$

Since $-1\le \sin\left(\frac{1}{x}\right)\le 1 \quad \Rightarrow \; -x^3\le x^3\sin\left(\frac{1}{x}\right)\le x^3$

Since $\lim_{x\to 0}x^3=0$ and $\lim_{x\to 0}\left(-x^3\right)=0$

$\therefore \; \lim_{x\to 0}x^3\sin\left(\frac{1}{x}\right)=0$ (By Squeeze theorem)

2 **(i) Find the values of *a* and *b* that make $f(x)$ continuous on $(-\infty,\infty)$**

$$f(x)=\begin{cases}\dfrac{x^3-8}{x-2} & \text{if } x<2\\ ax^2-bx+3 & \text{if } 2\le x<3\\ 2x-a+b & \text{if } x\ge 3\end{cases}$$

(NOV/DEC 2018)

Solution: The given function is $f(x)=\begin{cases}\dfrac{x^3-8}{x-2} & \text{if } x<2\\ ax^2-bx+3 & \text{if } 2\le x<3\\ 2x-a+b & \text{if } x\ge 3\end{cases}$

Since $f(x)$ is continuous on $(-\infty,\infty)$, it is continuous at the points $x=2$, $x=3$ also.

(i.e.) $\lim_{x\to 2^-}f(x)=\lim_{x\to 2^+}f(x)$ and $\lim_{x\to 3^-}f(x)=\lim_{x\to 3^+}f(x)$

Now

$$\text{LHL}: \lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}\frac{x^3-8}{x-2}$$

$$=\lim_{x\to 2^-}\frac{(x-2)\left(x^2+2x+4\right)}{(x-2)}$$

$$=\lim_{x\to 2^-}\left(x^2+2x+4\right)=2^2+2(2)+4=4+4+4=12$$

and $\text{RHL}: \lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}\left(ax^2-bx+3\right)\; a\left(2^2\right)-b(2)+3=4a-2b+3$

Hence $\lim_{x\to 2^+}f(x)=\lim_{x\to 2^-}f(x)$

$$4a-2b+3=12$$

$$4a-2b=9 \text{ ---------- (1)}$$

Also

$$\lim_{x\to 3^-}f(x)=\lim_{x\to 3^+}f(x)$$

$$\text{LHL}: \lim_{x\to 3^-}f(x)=\lim_{x\to 3^-}\left(ax^2-bx+3\right)=a\left(3^2\right)-b(3)+3=9a-3b+3$$

and RHL: $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x-a+b) = 2(3)-a+b = 6-a+b$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$9a-3b+3 = 6-a+b$

$10a-4b = 3$ ---------- (2)

Solving equations (1) and (2) (i.e.) (2) – 2(1):

$10a-4b=3$

$-8a+4b=-18$

$2a=-15 \Rightarrow \boxed{a=-\frac{15}{2}}$

Substitute $a=-\frac{15}{2}$ in equation (1)

$4\left(\frac{-15}{2}\right)-2b=9 \Rightarrow \boxed{b=-\frac{39}{2}}$

(ii) For what value of the constant b, is the function $f(x)$ continuous on $(-\infty,\infty)$ if

$$f(x)=\begin{cases} bx^2+2x, & x<2 \\ x^3-bx, & x\ge 2 \end{cases}$$

(APR/MAY 2019)

Solution:

The given $f(x)$ is a polynomial function, it is continuous on $(-\infty,2)$ and $(2,\infty)$.

Now,

LHL: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(bx^2+2x) = b(2)^2+2(2) = 4b+4$

and RHL: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^3-bx) = 2^3-b(2) = 8-2b$

$f(x)$ is continuous at $x=2$ provided LHL=RHL

(i.e.) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$4b+4=8-2b$

$6b=4 \Rightarrow \boxed{b=\frac{2}{3}}$

$\therefore$ $f(x)$ is continuous on $(-\infty,\infty)$ for $b=\frac{2}{3}$

3 **(i) The equation of motion of a particle is $s=t^3-3t$, where s is in meters and t is in seconds. Find (a) the velocity and acceleration as functions of t, (b) the acceleration after 2 seconds and (c) the acceleration when the velocity is zero.**

Solution:

Given $s=t^3-3t$

(a) The velocity and acceleration are given by $v(t)=\frac{ds}{dt}=\frac{d}{dt}(t^3-3t)=3t^2-3$

and $a(t)=\frac{dv}{dt}=\frac{d}{dt}\left(3t^2-3\right)=6t$

(b) Since $a(t)=6t$, the acceleration after 2 seconds is $a(2)=6(2)=12\,cm/s^2$

(c) The acceleration when the velocity is zero, $v(t)=0 \Rightarrow 3t^2-3=0$

$t^2-1=0 \Rightarrow \quad t^2=1 \quad \Rightarrow t=\pm 1 \qquad \therefore a(1)=6\,cm/s^2 \qquad \left(\because a(t)=6t\right)$

(ii) Show that the function $f(x)=|x-6|$ is not differentiable at 6. Find a formula for first derivative of *f*(*x*) and sketch its graph. (NOV/DEC 2020)

Solution:

Given $f(x)=|x-6|=\begin{cases} x-6 & \text{if } x-6\ge 0 \\ -(x-6) & \text{if } x-6<0 \end{cases} \quad =\begin{cases} x-6 & \text{if } x\ge 6 \\ -(x-6) & \text{if } x<6 \end{cases}$

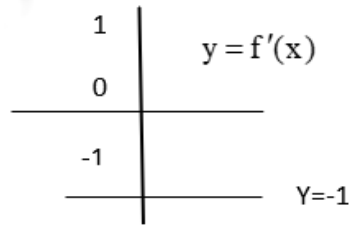
If $x\ge 6$, $f(x)=x-6$ RHD: $f'(6^+)=\lim\limits_{x\to 6^+}\frac{f(x)-f(6)}{x-6}=\lim\limits_{x\to 6^+}\frac{(x-6)-0}{x-6}=1$

If $x<6$, $f(x)=-(x-6)$

LHD: $f'(6^-)=\lim\limits_{x\to 6^-}\frac{f(x)-f(6)}{x-6}=\lim\limits_{x\to 6^+}\frac{-(x-6)-0}{x-6}=-1$

$\therefore\ f'(6^+)\ne f'(6^-)$, it follows that $f'(6)=\lim\limits_{x\to 6}\frac{f(x)-(6)}{x-6}$ does not exist and hence *f*(*x*) is not differentiable at $x=6$.

The first derivative of *f*(*x*) is given by $f'(x)=\begin{cases} 1 & \text{if } x-6\ge 0 \\ -1 & \text{if } x-6<0 \end{cases}$



4 **(i) If $f(x)=\frac{1-x}{2+x}$ then, find the equation for $f'(x)$ using the concept of derivatives. (NOV/DEC 2019)**

Solution: Differentiation of *f*(*x*) is $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Given $f(x)=\frac{1-x}{2+x} \Rightarrow f(x+h)=\frac{1-(x+h)}{2+(x+h)}=\frac{1-x-h}{2+x+h}$

$$\therefore f'(x)=\lim_{h\to 0}\frac{\left(\frac{1-x-h}{2+x+h}\right)-\left(\frac{1-x}{2+x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(1-x-h)\ (2+x) - (1-x)\ (2+x+h)}{(2+x+h)(2+x)}}{h}$$

$$= \lim_{h\to 0} \frac{(1-x-h)\ (2+x) - (1-x)\ (2+x+h)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{2-2x-2h+x-x^2-xh\ -\left(2+x+h-2x-x^2-xh\right)}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{2-2x-2h+x-x^2-xh\ -2-x-h+2x+x^2+xh}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2+x+h)(2+x)}$$

$$= \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)}$$

$$\boxed{f'(x) = \frac{-3}{(2+x)^2}}$$

(ii) Find $\dfrac{dy}{dx}$ if $y = x^2e^{2x}\left(x^2+1\right)^4$ **(APR/MAY 2019)**

Solution:

Given $y = x^2e^{2x}\left(x^2+1\right)^4$ $\qquad \left(\because (uvw)' = uvw' + uwv' + vwu'\right)$

Diff. w r to x on both sides, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2e^{2x}\left(x^2+1\right)^4\right]$$

$$= e^{2x}\left(x^2+1\right)^4 \frac{d}{dx}\left(x^2\right) + x^2\left(x^2+1\right)^4 \frac{d}{dx}\left(e^{2x}\right) + x^2e^{2x}\frac{d}{dx}\left(\left(x^2+1\right)^4\right)$$

$$= e^{2x}\left(x^2+1\right)^4 2x + x^2\left(x^2+1\right)^4 2e^{2x} + x^2e^{2x}4\left(x^2+1\right)^3 2x$$

$$= 2xe^{2x}\left(x^2+1\right)^4 + 2x^2e^{2x}\left(x^2+1\right)^4 + 8x^3e^{2x}\left(x^2+1\right)^3$$

$$= 2xe^{2x}\left(x^2+1\right)^3\left[\left(x^2+1\right) + x\left(x^2+1\right) + 4x^2\right]$$

$$= 2xe^{2x}\left(x^2+1\right)^3\left[x^2+1+x^3+x+4x^2\right]$$

$$= 2xe^{2x}\left(x^2+1\right)^3\left[x^3+5x^2+x+1\right]$$

$$\therefore \frac{dy}{dx} = 2xe^{2x}\left(x^2+1\right)^3\left[x^3+5x^2+x+1\right]$$

5 **(i) Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$** **(NOV/DEC 2018)**

Solution:

Given that $f(x) = \cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$

Putting $u=\dfrac{b+a\cos x}{a+b\cos x}$, then given function becomes

$$f(x)=\cos^{-1}u$$

$$f'(x)=\frac{d}{du}\left(\cos^{-1}u\right)\frac{du}{dx}$$

$$=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$=-\frac{1}{\sqrt{1-\left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\frac{d}{dx}\left(\frac{b+a\cos x}{a+b\cos x}\right)$$

$$=-\frac{a+b\cos x}{\sqrt{(a+b\cos x)^2-(b+a\cos x)^2}}\times$$

$$\left[\frac{(a+b\cos x)\dfrac{d}{dx}(b+a\cos x)-(b+a\cos x)\dfrac{d}{dx}(a+b\cos x)}{(a+b\cos x)^2}\right]$$

$$=-\frac{a+b\cos x}{\sqrt{\left(a^2+b^2\cos^2 x+2ab\cos x\right)-\left(b^2+a^2\cos^2 x+2ab\cos x\right)}}\times$$

$$\left[\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}\right]$$

$$=-\frac{a+b\cos x}{\sqrt{a^2-b^2-\left(a^2-b^2\right)\cos^2 x}}\times$$

$$\left[\frac{-a^2\sin x-ab\sin x\cos x+b^2\sin x+ab\sin x\cos x}{(a+b\cos x)^2}\right]$$

$$=-\frac{a+b\cos x}{\sqrt{\left(a^2-b^2\right)\left(1-\cos^2 x\right)}}\times\left[-\frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)^2}\right]$$

$$=\frac{\left(a^2-b^2\right)\sin x}{(a+b\cos x)\sqrt{\sin^2 x\left(a^2-b^2\right)}}$$

$$=\frac{\left(a^2-b^2\right)\sin x}{\sin x(a+b\cos x)\sqrt{a^2-b^2}}$$

$$=\frac{\sqrt{a^2-b^2}}{a+b\cos x}$$

(ii) Find y' if $\tan(x-y)=\dfrac{y}{1+x^3}$ **(NOV/DEC 2018)**

Solution:

Given $\tan(x-y)=\dfrac{y}{1+x^3}$

Differentiate with respect to x,

$$\frac{d}{dx}\left(\tan(x-y)\right)=\frac{d}{dx}\left(\frac{y}{1+x^3}\right)$$

$$\sec^2(x-y)\left(1-\frac{dy}{dx}\right)=\frac{\left(1+x^3\right)\frac{dy}{dx}-y(0+3x^2)}{\left(1+x^3\right)^2}$$

$$\left(1+\tan^2(x-y)\right)\left(1-\frac{dy}{dx}\right)=\frac{\left(1+x^3\right)\frac{dy}{dx}-3x^2y}{\left(1+x^3\right)^2}$$

$$\left(1+\left(\frac{y}{1+x^3}\right)^2\right)\left(1-\frac{dy}{dx}\right)=\frac{\left(1+x^3\right)\frac{dy}{dx}-3x^2y}{\left(1+x^3\right)^2}$$

$$\left(\frac{\left(1+x^3\right)^2+y^2}{\left(1+x^3\right)^2}\right)\left(1-\frac{dy}{dx}\right)=\frac{\left(1+x^3\right)\frac{dy}{dx}-3x^2y}{\left(1+x^3\right)^2}$$

$$\left(\left(1+x^3\right)^2+y^2\right)\left(1-\frac{dy}{dx}\right)=\left(1+x^3\right)\frac{dy}{dx}-3x^2y$$

$$\left(1+x^3\right)^2+y^2-\left(\left(1+x^3\right)^2+y^2\right)\frac{dy}{dx}=\left(1+x^3\right)\frac{dy}{dx}-3x^2y$$

$$3x^2y+\left(1+x^3\right)^2+y^2=\left(1+x^3\right)\frac{dy}{dx}+\left(\left(1+x^3\right)^2+y^2\right)\frac{dy}{dx}$$

$$\left(\left(1+x^3\right)+\left(1+x^3\right)^2+y^2\right)\frac{dy}{dx}=\left(1+x^3\right)^2+3x^2y+y^2$$

$$\frac{dy}{dx}=\frac{\left(1+x^3\right)^2+3x^2y+y^2}{\left(1+x^3\right)+\left(1+x^3\right)^2+y^2}$$

6 **(i) Find the derivative of $f(x)=\tanh^{-1}\left[\tan\frac{x}{2}\right]$** **(NOV/DEC 2019)**

Solution:

$$f(x)=\tanh^{-1}\left[\tan\frac{x}{2}\right]$$

Let $u=\tan\frac{x}{2}$ then $f(x)=\tanh^{-1}[u]$ and also $\frac{du}{dx}=\sec^2\left(\frac{x}{2}\right).\frac{1}{2}$

$$f'(x)=\frac{d}{du}\left[\tanh^{-1}u\right].\frac{du}{dx} \qquad \left[\because \frac{d}{dx}\left[\tanh^{-1}(x)\right]=\frac{1}{1-x^2}\ \&\ \frac{d}{dx}\left[\tan(x)\right]=\sec^2 x\right]$$

$$f'(x)=\frac{1}{1-u^2}\cdot\frac{du}{dx}$$

$$=\frac{1}{1-\left(\tan\frac{x}{2}\right)^2}.\sec^2\frac{x}{2}.\frac{1}{2} \qquad \left[\because \cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}\right]$$

$$=\frac{1}{2}\left[\frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right] \qquad \left[\because \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\cos x}\right]$$

$$=\frac{1}{2}\sec x$$

(ii) If $f(x)=\dfrac{1-\sec x}{\tan x}$, then find the first derivative of $f(x)$

Solution:

Given $f(x)=\dfrac{1-\sec x}{\tan x}$

Differentiate with respect to 'x' on both sides, we get

$$f'(x)=\frac{d}{dx}\left(\frac{1-\sec x}{\tan x}\right)$$

$$=\frac{(\tan x)\frac{d}{dx}(1-\sec x)-(1-\sec x)\frac{d}{dx}(\tan x)}{(\tan x)^2}$$

$$=\frac{(\tan x)(0-\sec x\tan x)-(1-\sec x)(\sec^2 x)}{\tan^2 x}$$

$$=\frac{-\sec x\tan^2 x-(1-\sec x)(1+\tan^2 x)}{\tan^2 x}$$

$$=\frac{-\sec x\tan^2 x-(1-\sec x+\tan^2 x-\sec x\tan^2 x)}{\tan^2 x}$$

$$=\frac{\cancel{-\sec x\tan^2 x}-1+\sec x-\tan^2 x+\cancel{\sec x\tan^2 x}}{\tan^2 x}$$

$$=\frac{\sec x-(1+\tan^2 x)}{\tan^2 x}$$

$$=\frac{\sec x-\sec^2 x}{\tan^2 x}$$

	$= \dfrac{\sec x(1-\sec x)}{\tan^2 x}$ $= \dfrac{\sec x(1-\sec x)}{\sec^2 x - 1}$ $\quad (\because 1+\tan^2 x = \sec^2 x)$ $= \dfrac{-\sec x(\sec x - 1)}{(\sec x - 1)(\sec x + 1)}$ $= \dfrac{-\sec x}{\sec x + 1}$
	(iii) Differentiate $f(x) = \log_{10}(2+\sin x)$
	Solution: $f'(x) = \dfrac{d}{dx}\left(\log_{10}(2+\sin x)\right)$ $\quad \left(\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x\log b}\right)$ $= \dfrac{1}{(2+\sin x)\log 10}\dfrac{d}{dx}(2+\sin x) = \dfrac{1}{(2+\sin x)\log 10}(0+\cos x) = \dfrac{\cos x}{(2+\sin x)\log 10}$
	(iv) Find $\dfrac{dy}{dx}$ if $y = \sqrt{x}\log x$
	Solution: Given $y = \sqrt{x}\log x$ $\dfrac{dy}{dx} = \sqrt{x}\dfrac{d}{dx}(\log x) + \log x\dfrac{d}{dx}\left(x^{\frac{1}{2}}\right)$ $\quad \left(\because \dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx} \ \& \sqrt{x} = x^{\frac{1}{2}}\right)$ $= \sqrt{x}\dfrac{1}{x} + \log x\dfrac{1}{2}(x)^{\frac{1}{2}-1}$ $\quad \left(\because \dfrac{d}{dx}(\log x) = \dfrac{1}{x} \ \& \dfrac{d}{dx}(x^n) = nx^{n-1}\right)$ $= \sqrt{x}\dfrac{1}{x} + \log x\dfrac{1}{2}(x)^{-\frac{1}{2}}$ $= \dfrac{1}{\sqrt{x}} + \log x\dfrac{1}{2\sqrt{x}}$
7	**(i) Use the intermediate value theorem to show that there is a root of the equation $\sqrt[3]{x} = 1 - x$ in the interval (0,1) (NOV/DEC 2020)**
	Solution: Given $\sqrt[3]{x} = 1 - x$ Let $f(x) = \sqrt[3]{x} - 1 + x$ The function $f(x)$ is continuous on [0,1]

$\Rightarrow f(0) = \sqrt[3]{0} - 1 + 0 = -1$

$f(1) = \sqrt[3]{1} - 1 + 1 = 1 - 1 + 1 = 1$

Since 0 < x < 1, there is a number c in (0,1) such that $f(c) = 0$ (by intermediate value theorem)

$\therefore$ there is a root of the equation $f(x) = 0$ (i.e) there is a root of the equation $\sqrt[3]{x} = 1 - x$ in the interval (0,1)

(ii) Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval

(i) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3],$

(ii) $f(x) = 2\cos x - \sin 2x, \quad \left[0, \frac{\pi}{2}\right]$

Solution (a):

The given function $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is continuous on [−2, 3].

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

Put $f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$$12x(x^2 - x - 2) = 0$$

$$x(x^2 - x - 2) = 0$$

$$x = 0,\ x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow (x-2) = 0\ \&\ (x+1) = 0$$

$$x = 2\ \&\ x = -1$$

$\Rightarrow x = 0, -1, 2$.

Hence $x = 0, -1, 2$ are the critical points.

To find the absolute maximum and absolute minimum values of f(x), we find the function value at the critical points and at the end points of the given interval.

$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 1 = 1$

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = -4$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = -31$

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 33$

$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 28$

Comparing these values, the absolute minimum value is $f(2) = -31$ and the absolute maximum value is $f(-2) = 33$.

Solution (b):

Given $f(x) = 2\cos x - \sin 2x$

Put $f'(x) = -2\sin x - 2\cos 2x$

$f'(x) = 0 \Rightarrow -2(\sin x + \cos 2x) = 0$

$\sin x + \cos 2x = 0$

$\sin x + (1 - 2\sin^2 x) = 0$ $\quad (\because \cos 2x = 1 - 2\sin^2 x)$

$-2\sin^2 x + \sin x + 1 = 0$

Put $\sin x = t$

$-2t^2 + t + 1 = 0$

$-2t^2 + 2t - t + 1 = 0$

$-2t(t-1) - (t-1) = 0$

$(t-1)(-2t-1) = 0$

$t = 1, -2t - 1 = 0$

$-2t = 1$

$t = 1, \quad t = -\frac{1}{2}$

$\sin x = 1, \sin x = -\frac{1}{2}$

$x = \frac{\pi}{2}, x = \frac{7\pi}{6} \notin \left[0, \frac{\pi}{2}\right]$

$\therefore x = \frac{\pi}{2}$ is the critical point and $f\left(\frac{\pi}{2}\right) = 0$

Also at the end points $f(0) = 2, f\left(\frac{\pi}{2}\right) = 0$

Hence the absolute minimum value is $f\left(\frac{\pi}{2}\right) = 0$, and the absolute maximum value is $f(0) = 2$.

8 **For the function $f(x) = 2x^3 + 3x^2 - 36x$** **(APR/MAY 2019)& (NOV/DEC 2019)**

(i) Find the intervals on which it is increasing and decreasing.

(ii) Find the local maxima and minima values of $f(x)$.

(iii) Find the intervals of concavity and the inflection points.

Solution:

Given $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

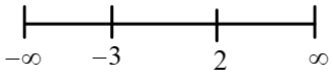
Put $f'(x)=0 \Rightarrow 6x^2+6x-36=0 \Rightarrow 6(x+3)(x-2)=0$

$$\Rightarrow x+3=0,\ x-2=0$$

$$\Rightarrow x=-3,\ x=2$$

Therefore $x=-3,2$ are the critical points.

i) The intervals on which the function is increasing or decreasing:



Interval	$f'(x)=6x^2+6x-36$			$f(x)$
	x value	$f'(x)$ value	Sign	
$x<-3$	$x=-4$	$f'(-4)=36$	+	Increasing
$-3<x<2$	$x=0$	$f'(0)=-36$	-	Decreasing
$x>2$	$x=3$	$f'(3)=36$	+	Increasing

ii) Local maxima and minima value of f :

The function $f(x)$ changes from increasing to decreasing at $x=-3,$ the function has local maximum at $x=-3.$ Local maximum value is

$$f(-3)=2(-3)^3+3(-3)^2-36(-3)$$

$$=2(-27)+3(9)+108$$

$$=-54+27+108$$

$$f(-3)=81$$

And the function $f(x)$ changes from decreasing to increasing at $x=2,$ the function has local minimum at $x=2.$ Local minimum value is

$$f(2)=2(2)^3+3(2)^2-36(2)$$

$$=2(8)+3(4)-72$$

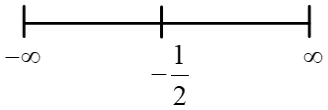
$$=16+12-72$$

$$f(2)=-44$$

iii) The intervals of concavity and the points of inflection (Concavity Test):

$$f''(x)=0 \Rightarrow 12x+6=0$$

$$\Rightarrow \boxed{x=\frac{-1}{2}}$$



Interval	$f''(x)=12x+6$			Concavity
	x value	$f''(x)$	Sign	
$x<-\frac{1}{2}$	$x=-1$	– 6	-	Downward
$x>-\frac{1}{2}$	$x=1$	18	+	upward

Since the curve changes from concave downward to upward at $x=-\frac{1}{2}$.

Thus, the point of inflection at $x=-\frac{1}{2}$ is $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$

$$f\left(\frac{-1}{2}\right)=2\left(\frac{-1}{2}\right)^3+3\left(\frac{-1}{2}\right)^2-36\left(\frac{-1}{2}\right)$$

$$=2\left(\frac{-1}{8}\right)+3\left(\frac{1}{4}\right)+18$$

$$=\frac{-1}{4}+\frac{3}{4}+18$$

$$f\left(-\frac{1}{2}\right)=\frac{37}{2}$$

$\therefore$ The point of inflection is $\left(-\frac{1}{2},\frac{37}{2}\right)$.

9 **For the function $f(x)=2+2x^2-x^4$, find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and the inflection points.**

Solution:

Given the function $f(x)=2+2x^2-x^4$

Diff. w r to x on both sides, we get

$$f'(x)=4x-4x^3$$

$$=4x\left(1-x^2\right) \qquad [\because a^2-b^2=(a+b)(a-b)$$

$$=4x(1+x)(1-x) \qquad \left(1-x^2\right) \text{ (i.e) } \left(1^2-x^2\right)=(1+x)(1-x)\ ]$$

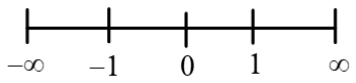
Again, Diff. w r to x on both sides, we get $f''(x)=4-12x^2$

Put $f'(x)=0 \Rightarrow 4x(1+x)(1-x)=0$

$$\Rightarrow x=0, \quad 1+x=0, \quad 1-x=0$$
$$\Rightarrow x=0, \quad x=-1, \quad x=1$$

$\therefore x=0,-1,1$ are the critical points.

(i) The intervals on which the function is increasing or decreasing:



Interval	$f'(x)=4x(1+x)(1-x)$			$f(x)$
	x value	$f'(x)$ value	Sign	
$x<-1$	$x=-2$	$f'(-2)=24$	+	Increasing
$-1<x<0$	$x=-0.5$	$f'(-0.5)=-1.5$	-	Decreasing
$0<x<1$	$x=0.5$	$f'(0.5)=1.5$	+	Increasing
$x>1$	$x=2$	$f'(2)=-24$	-	Decreasing

(ii) Local maxima and minima value of *f*:

$f(x)=2+2x^2-x^4$ changes from increasing to decreasing at $x=-1$, So the function has local maximum at $x=-1$. Local maximum value is

$$f(-1)=2+2(-1)^2-(-1)^4$$
$$=2+2(1)-1$$
$$f(-1)=3$$

$f(x)=2+2x^2-x^4$ changes from decreasing to increasing at $x=0$, So the function has local minimum at $x=0$. Local minimum value is

$$f(0)=2+2(0)^2-(0)^4$$
$$f(0)=2$$

$f(x)=2+2x^2-x^4$ changes from increasing to decreasing at $x=1$, So the function has local maximum at $x=1$. Local maximum value is

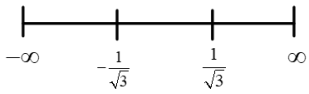
$$f(1)=2+2(1)^2-(1)^4$$
$$=2+2(1)-1$$
$$f(1)=3$$

(iii) The intervals of concavity and the points of inflection:

Concavity Test:

Put $f''(x)=0 \Rightarrow 4-12x^2=0 \Rightarrow 12x^2=4$

$$x^2=\frac{4}{12}=\frac{1}{3} \Rightarrow x=\pm\frac{1}{\sqrt{3}}$$



Interval	$f''(x)=4-12x^2$			Concavity
	x value	$f''(x)$	Sign	
$x<-\frac{1}{\sqrt{3}}$	$x=-1$	-8	-	downward
$-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$	$x=0$	4	+	upward
$x>\frac{1}{\sqrt{3}}$	$x=1$	-8	-	downward

Since the curve changes from concave downward to upward at $x=-\frac{1}{\sqrt{3}}$ and the curve changes from concave upward to downward at $x=\frac{1}{\sqrt{3}}$

Thus, the points of inflection are $\left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right)$ and $\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right)$

We have $f(x)=2+2x^2-x^4$

$$f\left(-\frac{1}{\sqrt{3}}\right)=2+2\left(-\frac{1}{\sqrt{3}}\right)^2-\left(-\frac{1}{\sqrt{3}}\right)^4=\frac{23}{9}$$

$$f\left(\frac{1}{\sqrt{3}}\right)=2+2\left(\frac{1}{\sqrt{3}}\right)^2-\left(\frac{1}{\sqrt{3}}\right)^4=\frac{23}{9}$$

Thus, the points of inflection are $\left(-\frac{1}{\sqrt{3}},\frac{23}{9}\right)$ and $\left(\frac{1}{\sqrt{3}},\frac{23}{9}\right)$.

10 **Find the local maximum value and local minimum value, the interval of concavity and the inflection points of a function $\mathbf{f(x)=x^3-3x^2-12x}$, Also sketch the graph of that satisfies all the above conditions. (NOV/DEC2020)**

Solution:

Given $f(x)=x^3-3x^2-12x$

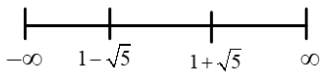
$f'(x)=3x^2-6x-12$

and $f''(x)=6x-6$

Put $f'(x)=0 \Rightarrow 3x^2-6x-12=0$ $\left(x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$

$$x=\frac{6\pm\sqrt{36-(-144)}}{6}=\frac{6\pm\sqrt{180}}{6}=\frac{6\pm\sqrt{36x5}}{6}=1\pm\sqrt{5}$$

Therefore the critical points are $1\pm\sqrt{5}$



Interval	$f'(x)=3x^2-6x-12$		f(x)
	x	sign	
$x<1-\sqrt{5}$	–1.5	+	Increasing
$1-\sqrt{5}<x<1+\sqrt{5}$	2	-	Decreasing
$x>1+\sqrt{5}$	4	+	Increasing

(i) $f'(x)$ changes from +ve to -ve at $1-\sqrt{5}$. It has local maximum.

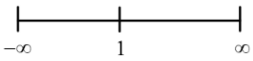
Local maximum value = $f(1-\sqrt{5})=(1-\sqrt{5})^3-3(1-\sqrt{5})^2-12(1-\sqrt{5})=-14+10\sqrt{5}$

(ii) $f'(x)$ changes from -ve to +ve at $1+\sqrt{5}$ .It has local minimum.

Local minimum value = $f(1+\sqrt{5})=(1+\sqrt{5})^3-3(1+\sqrt{5})^2-12(1+\sqrt{5})=-14-10\sqrt{5}$

For concavity put $f''(x)=0$

$$6x-6=0$$
$$\Rightarrow x-1=0$$
$$\Rightarrow x=1$$



interval	$f''(x)$		Concavity
	x value	sign	
$x<1$	0	-	Downward

$x > 1$	2	+	Upward

The point of inflection is $(1, f(1))$

$$f(x) = x^3 - 3x^2 - 12x$$
$$f(1) = (1)^3 - 3(1)^2 - 12(1) = -14$$

The point of inflection is $(1, -14)$.

I-YEAR B.E./B.TECH. (COMMON TO ALL BRANCHES)
MATRICES AND CALCULUS (MA3151) QUESTION BANK

UNIT III – FUNCTIONS OF SEVERAL VARIA BLES

PART – A

1	**If $u = xy - 2yz + z^2$, then find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$**
	Solution: Given $u = xy - 2yz + z^2$ $u_x = (1)y - 0 + 0$ $\quad u_y = x(1) - 2(1)z + 0$ $\quad u_z = 0 - 2y(1) + 2z$ $u_x = y$ $\quad u_y = x - 2z$ $\quad u_z = -2y + 2z$ $\therefore u_x + u_y + u_z = y + (x - 2z) + (-2y + 2z)$ $= y + x \cancel{-2z} - 2y + \cancel{2z}$ $= x - y$
2	**If $z = x^2 - 3xy^2$, then find $3x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$**
	Solution: Given $z = x^2 - 3xy^2$ $z_x = 2x - 3(1)y^2 = 2x - 3y^2$ $\quad z_{xx} = 2(1) - 0 = 2$ $z_y = 0 - 3x(2y) = -6xy$ $\quad z_{yy} = -6x(1) = -6x$ $3x\, z_{xx} = 3x(2) = 6x$ $\therefore 3x\, z_{xx} + z_{yy} = 6x - 6x = 0$
3	**If $\mathbf{u(x,y) = x^y + y^x}$, then find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$** **(NOV/DEC 2020)**
	Solution: Given $u(x, y) = x^y + y^x$ Here $u(x, y) = u = x^y + y^x$ $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^y + y^x) = \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial x}(y^x) = yx^{y-1} + y^x \log y$ $\quad \left(\because \frac{d}{dx}(a^x) = a^x \log a,\ but\ \frac{d}{dx}(x^a) = ax^{a-1}\right)$ $\frac{\partial u}{\partial x} = yx^{y-1} + y^x \log y$ $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^y + y^x) = \frac{\partial}{\partial y}(x^y) + \frac{\partial}{\partial y}(y^x) = x^y \log x + xy^{x-1}$ $\frac{\partial u}{\partial y} = x^y \log x + xy^{x-1}$
4	**Verify the Euler's theorem for the function $u = x^2 + y^2 + 2xy$** **(NOV/DEC 2019)**
	Solution: Given $u(x, y) = x^2 + y^2 + 2xy$ To find the degree of the given function, replace x by tx and y by ty. $u(tx, ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)$

	$u(tx,ty)=t^2x^2+t^2y^2+2t^2xy=t^2\left(x^2+y^2+2xy\right)$ $i.e$ t^2u $\because$ u is homogeneous function of 2nd degree $(i.e)$ n = 2 By Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ $\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$ $(i.e)\, x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\left(x^2+y^2+2xy\right)$ Now, $u=x^2+y^2+2xy$ $\frac{\partial u}{\partial x}=2x+2y$ $x\frac{\partial u}{\partial x}=x\left(2x+2y\right)=2x^2+2xy$ $----(1)$ $\frac{\partial u}{\partial y}=2y+2x$ $y\frac{\partial u}{\partial y}=y\left(2y+2x\right)=2y^2+2xy$ $----(2)$ Adding (1) and (2) $L.H.S=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ $=2x^2+2xy+2y^2+2xy=2x^2+2y^2+4xy$ $=2(x^2+y^2+2xy)=R.H.S$ Hence Euler's theorem is verified.
5	**If $z=xf\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$ using Euler's theorem** **(NOV / DEC 2020)**
	Solution: Given $z=xf\left(\frac{y}{x}\right)$. i.e., $z(x,y)=xf\left(\frac{y}{x}\right)$ To find the degree of the given function, replace x by tx and y by ty. $z(tx,ty)=txf\left(\frac{ty}{tx}\right)=txf\left(\frac{y}{x}\right)=t.z(x,y)$ $z(tx,ty)=t^1z(x,y)$ $\therefore z(x,y)$ is a homogeneous function of degree 1 $(i.e)$ n = 1. By Euler's theorem , $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1(z)=z$
6	**If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{2}\cot u$**

Solution: $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)\Rightarrow \cos u=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}=f\ (x,\ y)$

$$f(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

To find the degree of the given function, replace x by tx and y by ty.

$$f(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=\frac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)}=\frac{\sqrt{t}(x+y)}{\left(\sqrt{x}+\sqrt{y}\right)}=t^{1/2}f(x,y)$$

$f=\cos u$ is a homogeneous function of degree $n=\dfrac{1}{2}$ in x and y.

Therefore, by Euler's theorem we get,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

$$x\frac{\partial}{\partial x}(\cos u)+y\frac{\partial}{\partial y}(\cos u)=\frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x}+y(-\sin u)\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\cos u$$

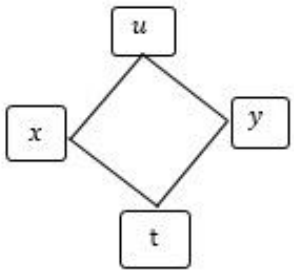
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{2}\left(\frac{\cos u}{\sin u}\right)=\frac{-1}{2}\cot u$$

7 **If $u=x^3+y^3$ and $x=at^2,\ y=2at$, then find $\dfrac{du}{dt}$.** **(APR/MAY 2019)**

Solution:

Given $u=x^3+y^3$ and $x=at^2,\ y=2at$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}$$



$u=x^3+y^3$		$x=at^2$	$y=2at$
$\dfrac{\partial u}{\partial x}=3x^2$	$\dfrac{\partial u}{\partial y}=3y^2$	$\dfrac{dx}{dt}=2at$	$\dfrac{dy}{dt}=2a$

$$\frac{du}{dt}=(3x^2)(2at)+(3y^2)(2a)$$

$$=\left[3(\mathrm{at}^2)^2(2at)\right]+\left[(3)(2\,\mathrm{at})^2(2a)\right]\qquad\left(\because x=at^2,\ y=2at\right)$$

$$=6a^3t^5+24a^3t^2=6a^3t^2(\mathrm{t}^3+4)$$

8 **If $x^3+y^3=6xy$, using implicit function method find $\dfrac{dy}{dx}$**

Solution:

$x^3+y^3-6xy=0$

Let $f(x,y)=x^3+y^3-6xy$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 - 6xy)$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(6xy) = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - 6y\frac{\partial}{\partial x}(x) = 3x^2 + 0 - 6y(1)$$

$$f_x = 3x^2 - 6y$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 - 6xy)$$

$$= \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(6xy) = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - 6x\frac{\partial}{\partial y}(y) = 0 + 3y^2 - 6x(1)$$

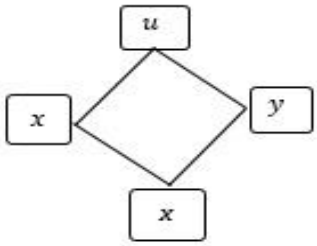
$$f_y = 3y^2 - 6x$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(3x^2 - 6y)}{3y^2 - 6x} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{\cancel{3}\left(2y - x^2\right)}{\cancel{3}\left(y^2 - 2x\right)} = \frac{2y - x^2}{y^2 - 2x}$$

9 **Find $\frac{du}{dx}$, if $u = x^2y^3$ when $2\sin x - 3y = 0$**

Solution:

$$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} \qquad (by\ total\ derivative)$$



$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$$2\sin x - 3y = 0$$

$$differentiate\ with\ respect\ to\ 'x'$$

$$2\cos x - 3\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\cos x}{3}$$

$u = x^2y^3$ $diff.\ with\ respect\ to\ x\ partially,$ $\frac{\partial u}{\partial x} = 2xy^3;$	$diff.\ with\ respect\ to\ y\ partially,$ $\frac{\partial u}{\partial y} = 3x^2y^2$

$$\frac{du}{dx} = 2xy^3 + 3x^2y^2.\frac{2\cos x}{3} = 2xy^3 + 2x^2y^2\cos x$$

$$\therefore \frac{du}{dx} = 2xy^2(y + x\cos x)$$

10 **State the properties of Jacobian.** **(NOV/DEC 2018)**

Solution:

(i) If u and v are functions of r and s, r and s are functions of x and y then, $\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}$

(ii) If u and v are functions of x and y then, $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$ (i.e) $JJ' = 1$

(iii) If u, v, w are functionally dependent functions of three independent variable x, y, z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

11 **If $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x and y with respect to u and v** **(APR/MAY 2019)**

Solution:

Given $x = u^2 - v^2$ & $y = 2uv$

$$\frac{\partial x}{\partial u} = x_u = 2u \qquad \frac{\partial y}{\partial u} = y_u = 2v$$

$$\frac{\partial x}{\partial v} = x_v = -2v \qquad \frac{\partial y}{\partial v} = y_v = 2u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 2u(2u) - 2v(-2v) = 4u^2 + 4v^2 = 4(u^2 + v^2)$$

12 **If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$** **(JAN 2018)**

Solution:

Given $x = uv$ & $y = \frac{u}{v}$

$$x_u = v \qquad y_u = \frac{1}{v}$$

$$x_v = u \qquad y_v = -\frac{u}{v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right) = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}$$

13 **If $x = r\cos\theta,\ y = r\sin\theta$ *then find* J'** **(JAN 2018) & (NOV/DEC 2019)**

Solution: $x = r\cos\theta \Rightarrow x_r = \cos\theta;\ x_\theta = r(-\sin\theta)$ and $y = r\sin\theta \Rightarrow y_r = \sin\theta;\ y_\theta = r\cos\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r(1) = r$$

$$\therefore JJ' = 1 \Rightarrow J' = \frac{1}{J} \Rightarrow \frac{1}{r}$$

14 **Find Taylor's series expansion of x^y near the point (1,1) up to first degree terms.**

Solution:

The Taylor's series expansion near the point (1,1) is

$$f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \ldots$$

$f(x,y) = x^y$	$f(1,1) = (1)^1 = 1$
$f_x(x,y) = yx^{y-1}$	$f_x(1,1) = 1(1)^{1-1} = 1(1) = 1$
$f_y(x,y) = x^y \log x$	$f_y(1,1) = (1)^1 \log(1) = (1)(0) = 0$

$\because (1)^0 = 1$

$\because \log(1) = 0$

$$f(x,y) = 1 + (x-1)(1) + (y-1)(0) + \ldots$$

$=1+x-1+...$

$f(x,y)=x+...$

15 **Obtain Taylor's series expansion of e^{x+y} in powers of x and y up to first degree terms.**

Solution:

Let $f(x,y)=e^{x+y}$

The Taylor's series expansion is

$$f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+\ldots$$

$f(x,y)=e^{x+y}$	$f(0,0)=e^{0+0}=e^0=1$
$f_x(x,y)=e^{x+y}$	$f_x(0,0)=e^{0+0}=e^0=1$
$f_y(x,y)=e^{x+y}$	$f_y(0,0)=e^{0+0}=e^0=1$

$$\therefore f(x,y)=f(0,0)+\left[(x-0)f_x(0,0)+(y-0)f_y(0,0)\right]+\ldots$$

$$=1+x(1)+y(1)+\ldots$$

$$f(x,y)=1+x+y+\ldots$$

16 **State the conditions for maxima and minima of $f(x,y)$.**

Solution:

If $f_x(a,b)=0,\ f_y(a,b)=0$ then,

(i) $f(x,y)$ has maximum value at (a,b) if $\left[f_{xx}(a,b)\right]\left[f_{yy}(a,b)\right]-\left[f_{xy}(a,b)\right]^2>0$ and $f_{xx}(a,b)<0$

(ii) $f(x,y)$ has minimum value at (a,b) if $\left[f_{xx}(a,b)\right]\left[f_{yy}(a,b)\right]-\left[f_{xy}(a,b)\right]^2>0$ and $f_{xx}(a,b)>0$

(iii) $f(x,y)$ neither a maximum nor a minimum at (a,b) if $\left[f_{xx}(a,b)\right]\left[f_{yy}(a,b)\right]-\left[f_{xy}(a,b)\right]^2<0$

Such a point is called a saddle point.

17 **Find the maxima and minima of $f(x,y)=x^2+y^2+6x+4y+12$**

Solution: Given $f(x,y)=x^2+y^2+6x+4y+12$

$f_x=2x+6;\ f_{xx}=2;\ f_{xy}=0$

$f_y=2y+4;\ f_{yy}=2$

The stationary point is

$f_x=0\Rightarrow 2x+6=0;2x=-6;x=-\frac{6}{2};\boxed{\therefore x=-3}$

$f_y=0\Rightarrow 2y+4=0;2y=-4;y=-\frac{4}{2};\boxed{\therefore y=-2}$

The stationary point is $(-3,-2)$

$\left[f_{xx}(a,b)\right]\left[f_{yy}(a,b)\right]-\left[f_{xy}(a,b)\right]^2=(2)(2)-0^2=4>0$ and $f_{xx}(-3,-2)=2>0$

	$\therefore f$ is minimum at $(-3,-2)$ and the minimum value is $f(-3,-2)=(-3)^2+(-2)^2+6(-3)+4(-2)+12$ $=9+4-18-8+12=25-26=-1$
18	**Find the extreme point of** $\boldsymbol{f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}}$
	Solution: Given $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$ $\frac{\partial f}{\partial x}=2x-\frac{2}{x^2};\frac{\partial f}{\partial x}=0\Rightarrow 2x-\frac{2}{x^2}=0;\quad 2x=\frac{2}{x^2};\quad x=\frac{1}{x^2};\ x^3=1;$ (Take cube root on both sides) $(x^3)^{1/3}=1^{1/3}\ ;\ x=1$ $\frac{\partial f}{\partial y}=2y-\frac{2}{y^2};\frac{\partial f}{\partial y}=0\Rightarrow 2y-\frac{2}{y^2}=0;\quad 2y=\frac{2}{y^2};\quad y=\frac{1}{y^2};\ y^3=1;$ (Take cube root on both sides) $(y^3)^{1/3}=1^{1/3}\ ;\ y=1$ $\therefore$ The extreme point is $(1,1)$
19	**Formulate the optimization function to find the dimensions of a rectangular box open at the top is to have a maximum capacity whose surface area is 648 square centimeters.**
	Solution: Let x, y, z be the length, breadth and height of the box . Volume = length $\times$ breadth $\times$ height $= xyz$ Let $f(x,y,z)=xyz$ $\therefore$ Total surface area of the box $= xy+2yz+2zx=648$ (given) Let $g(x,y,z)=xy+2yz+2zx-648$ Optimize F = Objective $+\lambda$ (Constraint) $F(x,y,z)=f(x,y,z)+\lambda g(x,y,z)$ Hence $F(x,y,z)=xyz+\lambda(xy+2yz+2zx-648)$
20	**Formulate the optimization function to find the volume of the largest rectangular solid which can be inscribed in an ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
	Solution: Let the sides of the rectangular box be $2x,\ 2y,\ 2z$ The largest rectangular parallelopiped inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ will have its corners on the ellipsoid and its sides parallel to the coordinate plane. Volume $V=2x\times 2y\times 2z$ $=8xyz$ Let $f=8xyz$

$$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

Optimize $F = \text{Objective } + \lambda$ (Constraint)

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$F(x,y,z) = (8xyz) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

PART – B

1 **For the function $z = \tan^{-1}\left(\frac{x}{y}\right) - xy$, verify whether the statement $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ is correct or not.** **(NOV/DEC 2019)**

Solution:

Given $z = \tan^{-1}\left(\frac{x}{y}\right) - xy$

Differentiate z with respect to x partially,

$$z_x = \frac{\partial}{\partial x}\tan^{-1}\left(\frac{x}{y}\right) - \frac{\partial}{\partial x}xy$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) - (1)\,y \qquad \left[\because \frac{\partial}{\partial x}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right) - y$$

$$= \frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right) - y$$

$$= \frac{1}{\frac{y^2+x^2}{y^{\cancel{2}}}}\left(\frac{1}{\cancel{y}}\right) - y$$

$$= \frac{y}{x^2+y^2} - y$$

Differentiate z_x with respect to y partially,

$$z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}(y)$$

$$= \frac{\left(x^2+y^2\right)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2+y^2)}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{\left(x^2+y^2\right)(1) - y(0+2y)}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{x^2+y^2-2y^2}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{x^2+y^2-2y^2-\left(x^2+y^2\right)^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^2-y^2-\left(x^4+y^4+2x^2y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^2-y^2-x^4-y^4-2x^2y^2}{\left(x^2+y^2\right)^2} \quad \text{----------} \quad (1)$$

Differentiate z with respect to y partially,

$$z_y = \frac{\partial}{\partial y}\tan^{-1}\left(\frac{x}{y}\right) - \frac{\partial}{\partial y}xy$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) - x(1) \qquad \left[\because \frac{\partial}{\partial x}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right) - x$$

$$= \frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(\frac{-x}{y^2}\right) - x$$

$$= \frac{1}{\frac{y^2+x^2}{\cancel{y^2}}}\left(\frac{-x}{\cancel{y^2}}\right) - x$$

$$= \frac{-x}{x^2+y^2} - x$$

Differentiate z_y with respect to x partially,

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial x}(x)$$

$$= \frac{\left(x^2+y^2\right)\frac{\partial}{\partial x}(-x) - (-x)\frac{\partial}{\partial x}(x^2+y^2)}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{\left(x^2+y^2\right)(-1) - (-x)(2x+0)}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{-x^2-y^2+2x^2}{\left(x^2+y^2\right)^2} - 1$$

$$= \frac{-x^2-y^2+2x^2-\left(x^2+y^2\right)^2}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^2-y^2-\left(x^4+y^4+2x^2y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^2-y^2-x^4-y^4-2x^2y^2}{\left(x^2+y^2\right)^2} \text{ ----------(2)}$$

From (1) & (2) $z_{xy} = z_{yx}$ $(i.e.)$ $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

2 **If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$** **(JAN 2018)**

Solution:

Given $u = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\left(x^2+y^2+z^2\right)^{\frac{1}{2}}} = \left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$

Differentiate u with respect to x partially,

$$u_x = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}-1}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(2x+0+0\right)$$

$$= \frac{-1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(\cancel{2}x\right)$$

$$= -x\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}$$

Differentiate u with respect to y partially,

$$u_y = \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}-1}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(0+2y+0\right)$$

$$= \frac{-1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(\cancel{2}y\right)$$

$$= -y\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}$$

Differentiate u with respect to z partially,

$$u_z = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}-1}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)$$

$$= \frac{-1}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(0+0+2z\right)$$

$$= \frac{-1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\left(\cancel{2}z\right)$$

$$= -z\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}$$

Differentiate u_x with respect to *x* partially,

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\right)$$

$$= -\left[x\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}} + \left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\frac{\partial}{\partial x}(x)\right]$$

$$= -\left[x\left(\frac{-3}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}-1}\right)\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) + \left(x^2+y^2+z^2\right)^{\frac{-3}{2}}(1)\right]$$

$$= -\left[x\left(\frac{-3}{2}\left(x^2+y^2+z^2\right)^{\frac{-3}{2}-1}\right)\left(2x+0+0\right) + \left(x^2+y^2+z^2\right)^{\frac{-3}{2}}(1)\right]$$

$$= -\left[x\left(\frac{-3}{\cancel{2}}\left(x^2+y^2+z^2\right)^{\frac{-5}{2}}\right)\left(\cancel{2}x\right) + \left(x^2+y^2+z^2\right)^{\frac{-3}{2}}\right]$$

$$u_{xx} = 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}}\ x^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly,

$$u_{yy} = 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}}\ y^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$u_{zz} = 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}} z^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(1)+(2)+(3) $\Rightarrow$

$$\begin{aligned}
u_{xx}+u_{yy}+u_{zz} &= 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}} x^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \\
&\quad + 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}} y^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \\
&\quad + 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}} z^2 - \left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \\
&= 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}}\left(x^2+y^2+z^2\right) - 3\left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \\
&= 3\left(x^2+y^2+z^2\right)^{\frac{-5}{2}+1} - 3\left(x^2+y^2+z^2\right)^{\frac{-3}{2}} \\
&= 3\left(x^2+y^2+z^2\right)^{\frac{-3}{2}} - 3\left(x^2+y^2+z^2\right)^{\frac{-3}{2}}
\end{aligned}$$

$$\boxed{u_{xx}+u_{yy}+u_{zz}=0}$$

3 **If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then find $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$.** **(JAN 2019)**

Solution:

Let $u = f(r, s)$

$r = \frac{y-x}{xy}$	$s = \frac{z-x}{xz}$
$r_x = \frac{\partial}{\partial x}\left(\frac{y-x}{xy}\right)$	$s_x = \frac{\partial}{\partial x}\left(\frac{z-x}{xz}\right)$
$= \frac{xy\frac{\partial}{\partial x}(y-x) - (y-x)\frac{\partial}{\partial x}(xy)}{(xy)^2}$	$= \frac{xz\frac{\partial}{\partial x}(z-x) - (z-x)\frac{\partial}{\partial x}(xz)}{(xz)^2}$
$= \frac{xy(0-1)-(y-x)(1)y}{(xy)^2}$	$= \frac{xz(0-1)-(z-x)(1)z}{(xz)^2}$
$= \frac{\cancel{-xy} - y^2 + \cancel{xy}}{(xy)^2}$	$= \frac{\cancel{-xz} - z^2 + \cancel{xz}}{(xz)^2}$
$= \frac{-y^2}{(xy)^2}$	$= \frac{-z^2}{(xz)^2}$
$= \frac{-\cancel{y^2}}{x^2\cancel{y^2}}$	$= \frac{-\cancel{z^2}}{x^2\cancel{z^2}}$
$= \frac{-1}{x^2}$	$= \frac{-1}{x^2}$

$r_y = \frac{\partial}{\partial y}\left(\frac{y-x}{xy}\right)$ $= \frac{xy\frac{\partial}{\partial y}(y-x)-(y-x)\frac{\partial}{\partial y}(xy)}{(xy)^2}$ $= \frac{xy(1-0)-(y-x)x(1)}{(xy)^2}$ $= \frac{\cancel{xy}-\cancel{xy}+x^2}{(xy)^2}$ $= \frac{x^2}{(xy)^2}$ $= \frac{\cancel{x^2}}{\cancel{x^2}y^2}$ $= \frac{1}{y^2}$	$s_y = \frac{\partial}{\partial y}\left(\frac{z-x}{xz}\right) = 0$
$r_z = \frac{\partial}{\partial z}\left(\frac{y-x}{xy}\right) = 0$	$s_z = \frac{\partial}{\partial z}\left(\frac{z-x}{xz}\right)$ $= \frac{xz\frac{\partial}{\partial z}(z-x)-(z-x)\frac{\partial}{\partial z}(xz)}{(xz)^2}$ $= \frac{xz(1-0)-(z-x)x(1)}{(xz)^2}$ $= \frac{\cancel{xz}-\cancel{xz}+x^2}{(xz)^2}$ $= \frac{x^2}{(xz)^2}$ $= \frac{\cancel{x^2}}{\cancel{x^2}z^2}$ $= \frac{1}{z^2}$

To find:

$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$:

$u_x = u_r.r_x + u_s.s_x$	$u_y = u_r.\ r_y + u_s.\ s_y$	$u_z = u_r.\ r_z + u_s.\ s_z$

$u_x = u_r\left(\frac{-1}{x^2}\right) + u_s\left(\frac{-1}{x^2}\right)$ $= \left(\frac{-1}{x^2}\right)(u_r + u_s)$	$u_y = u_r\left(\frac{1}{y^2}\right) + u_s(0)$ $= u_r\left(\frac{1}{y^2}\right)$	$u_z = u_r(0) + u_s\left(\frac{1}{z^2}\right)$ $= u_s\left(\frac{1}{z^2}\right)$

$$\Rightarrow x^2u_x + y^2u_y + z^2u_z = \cancel{x^2}\left(\frac{-1}{\cancel{x^2}}\right)(u_r + u_s) + \cancel{y^2}\,u_r\left(\frac{1}{\cancel{y^2}}\right) + \cancel{z^2}\,u_s\left(\frac{1}{\cancel{z^2}}\right)$$

$$= -u_r - u_s + u_r + u_s$$

$$= 0$$

4 **If $u = (x-y)\,f\left(\frac{y}{x}\right)$, then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$**

Solution:

Let $u(x,y) = (x-y)\,f\left(\frac{y}{x}\right)$

Replace x by tx and y by ty

$$u(tx,ty) = (tx - ty)\,f\left(\frac{ty}{tx}\right)$$

$$= t(x-y)\,f\left(\frac{y}{x}\right) = t\,u(x,y)$$

$\therefore$ u is homogeneous function in x and y with degree 1. i.e., n = 1.

By Euler's Extension theorem we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)$$

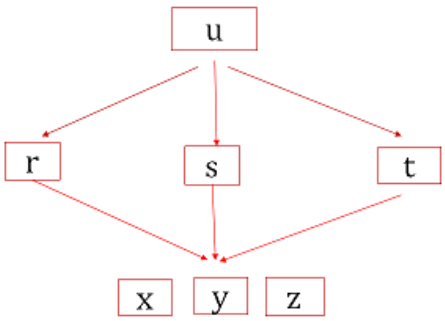
$$= 1(1-1) = 0$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

5 **If $u = f(2x-3y, 3y-4z, 4z-2x)$, then find $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$.** **(JAN 2019)**

Solution: Let $u = f(r,s,t)$

$r = 2x - 3y$	$s = 3y - 4z$	$t = 4z - 2x$
$\frac{\partial r}{\partial x} = 2$	$\frac{\partial s}{\partial x} = 0$	$\frac{\partial t}{\partial x} = -2$
$\frac{\partial r}{\partial y} = -3$	$\frac{\partial s}{\partial y} = 3$	$\frac{\partial t}{\partial y} = 0$
$\frac{\partial r}{\partial z} = 0$	$\frac{\partial s}{\partial z} = -4$	$\frac{\partial t}{\partial z} = 4$



$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}.(2)+\frac{\partial u}{\partial s}.(0)+\frac{\partial u}{\partial t}.(-2)$$

$$\frac{\partial u}{\partial x}=2\left(\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}\right)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}.\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial r}.(-3)+\frac{\partial u}{\partial s}.(3)+\frac{\partial u}{\partial t}.(0)$$

$$\frac{\partial u}{\partial y}=3\left(-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-4)+\frac{\partial u}{\partial t}(4)$$

$$=-4\frac{\partial u}{\partial s}+4\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z}=4\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}\right)$$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=\frac{1}{\cancel{2}}\left[\cancel{2}\left(\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}\right)\right]+\frac{1}{\cancel{3}}\left[\cancel{3}\left(-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right)\right]+\frac{1}{\cancel{4}}\left[\cancel{4}\left(\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}\right)\right]$$

$$=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}-\frac{\partial u}{\partial s}=0$$

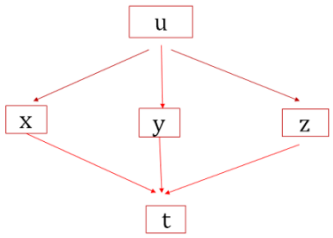
$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0$$

6 **Find $\frac{du}{dt}$, if $u=xy+yz+zx$ where $x=t$, $y=e^t$, $z=t^2$.**

Solution:

Given $u=xy+yz+zx$ where $x=t,\ y=e^t,\ z=t^2$

$\frac{\partial u}{\partial x}=y+z$	$\frac{\partial u}{\partial y}=x+z$	$\frac{\partial u}{\partial z}=y+x$	$\frac{dx}{dt}=1$	$\frac{dy}{dt}=e^t$	$\frac{dz}{dt}=2t$



$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}+\frac{\partial u}{\partial z}.\frac{dz}{dt}$$

$$\frac{du}{dt}=(y+z)(1)+(x+z)(e^t)+(y+x)(2t)$$

$$=\left(e^t+t^2\right)+\left(t+t^2\right)\left(e^t\right)+\left(e^t+t\right)(2t)$$
$$=e^t+t^2+te^t+t^2e^t+2te^t+2t^2$$
$$=3t^2+e^t+3te^t+t^2e^t$$
$$=3t^2+e^t\left(1+3t+t^2\right)$$

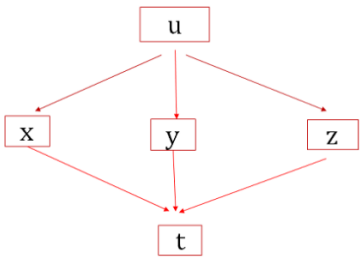
7 **Find $\frac{du}{dt}$, if $u=x-y+z$ where $x=e^{2t}$, $y=e^{2t}\cos 3t$, $z=e^{2t}\sin 3t$**

Solution:

Given $\mathrm{u}=\mathrm{x}-\mathrm{y}+\mathrm{z}$ where $x=e^{2t}$, $y=e^{2t}\cos 3t$, $z=e^{2t}\sin 3t$

$u=x-y+z$			$x=e^{2t}$	$y=e^{2t}\cos 3t$	$z=e^{2t}\sin 3t$
$\frac{\partial u}{\partial x}=1$	$\frac{\partial u}{\partial y}=-1$	$\frac{\partial u}{\partial z}=1$	$\frac{dx}{dt}=2e^{2t}$	$\frac{dy}{dt}=2e^{2t}\cos 3t-3e^{2t}\sin 3t$	$\frac{dz}{dt}=2e^{2t}\sin 3t+3e^{2t}\cos 3t$

$$\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}+\frac{\partial u}{\partial z}.\frac{dz}{dt}$$
$$\frac{du}{dt}=(1).2e^{2t}+(-1).\left(2e^{2t}\cos 3t-3e^{2t}\sin 3t\right)$$
$$+(1).\left(2e^{2t}\sin 3t+3e^{2t}\cos 3t\right)$$
$$=e^{2t}(2-2\cos 3t+3\sin 3t+2\sin 3t+3\cos 3t)$$
$$=e^{2t}(2+5\sin 3t+\cos 3t)$$



8 **If $z=f(x,y)$, where $x=u^2-v^2$, $y=2uv$, then prove that**

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=4\left(u^2+v^2\right)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

Solution:

Given $x=u^2-v^2$, $y=2uv$

$x=u^2-v^2$	$y=2uv$
$\frac{\partial x}{\partial u}=2u$, $\frac{\partial x}{\partial v}=-2v$	$\frac{\partial y}{\partial u}=2v$, $\frac{\partial y}{\partial v}=2u$

Diff $z=f(x,y)$ with respect to u

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}\quad=\frac{\partial z}{\partial x}(2u)+\frac{\partial z}{\partial y}(2v)$$
$$\frac{\partial z}{\partial u}=2u\frac{\partial z}{\partial x}+2v\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial u}(z) = 2u\frac{\partial}{\partial x}(z) + 2v\frac{\partial}{\partial y}(z)$$

$$\frac{\partial}{\partial u} = 2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\right)\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4uv\frac{\partial^2 z}{\partial x\partial y} + 4uv\frac{\partial^2 z}{\partial x\partial y} + 4v^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 8uv\frac{\partial^2 z}{\partial x\partial y} + 4v^2\frac{\partial^2 z}{\partial y^2} \quad ----------(1)$$

Diff $z = f(x,y)$ with respect to v

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}.(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\frac{\partial z}{\partial v} = (-2v)\frac{\partial z}{\partial x} + (2u)\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial v}(z) = (-2v)\frac{\partial}{\partial x}(z) + (2u)\frac{\partial}{\partial y}(z)$$

$$\frac{\partial}{\partial v} = (-2v)\frac{\partial}{\partial x} + (2u)\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left[(-2v)\frac{\partial}{\partial x} + (2u)\frac{\partial}{\partial y}\right]\left[(-2v)\frac{\partial z}{\partial x} + (2u)\frac{\partial z}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 4uv\frac{\partial^2 z}{\partial x\partial y} - 4uv\frac{\partial^2 z}{\partial x\partial y} + 4u^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial x\partial y} + 4u^2\frac{\partial^2 z}{\partial y^2} \quad ----------(2)$$

(1) + (2) $\Rightarrow$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + \cancel{8uv\frac{\partial^2 z}{\partial x\partial y}} + 4v^2\frac{\partial^2 z}{\partial y^2} + 4v^2\frac{\partial^2 z}{\partial x^2} - \cancel{8uv\frac{\partial^2 z}{\partial x\partial y}} + 4u^2\frac{\partial^2 z}{\partial y^2}$$

$$= 4u^2\frac{\partial^2 z}{\partial x^2} + 4v^2\frac{\partial^2 z}{\partial x^2} + 4u^2\frac{\partial^2 z}{\partial y^2} + 4v^2\frac{\partial^2 z}{\partial y^2}$$

$$= 4(u^2+v^2)\frac{\partial^2 z}{\partial x^2} + 4(u^2+v^2)\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

9 **If $z = f(x,y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ then show that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$**

Given $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$\frac{\partial x}{\partial u} = e^u$ $\qquad$ $\frac{\partial y}{\partial u} = -e^{-u}$

$\frac{\partial x}{\partial v} = -e^{-v}$ $\qquad$ $\frac{\partial y}{\partial v} = -e^v$

Now $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ & $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.e^u + \frac{\partial z}{\partial y}.(-e^{-u}) = \frac{\partial z}{\partial x}.e^u - \frac{\partial z}{\partial y}.e^{-u} \qquad ----(1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.(-e^{-v}) + \frac{\partial z}{\partial y}.(-e^v) = -\frac{\partial z}{\partial x}.e^{-v} - \frac{\partial z}{\partial y}.e^v \qquad ----(2)$$

Subtracting (1) and (2),

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.e^u - \frac{\partial z}{\partial y}.e^{-u} - \left(-\frac{\partial z}{\partial x}.e^{-v} - \frac{\partial z}{\partial y}.e^v\right)$$

$$= \frac{\partial z}{\partial x}.e^u - \frac{\partial z}{\partial y}.e^{-u} + \frac{\partial z}{\partial x}.e^{-v} + \frac{\partial z}{\partial y}.e^v$$

$$= \frac{\partial z}{\partial x}(e^u + e^{-v}) + \frac{\partial z}{\partial y}(e^v - e^{-u})$$

$$= \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v)$$

$$= \frac{\partial z}{\partial x}(x) - \frac{\partial z}{\partial y}(y) \qquad \left[\because x = e^u + e^{-v} \ \& \ y = e^{-u} - e^v\right]$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

10 **If $u = 2xy, v = x^2 - y^2$ and $x = r\cos\theta, y = r\sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$**

Solution : Given $u = 2xy, v = x^2 - y^2$, and $x = r\cos\theta, y = r\sin\theta$

$u_x = \frac{\partial u}{\partial x} = 2y$	$u_y = \frac{\partial u}{\partial y} = 2x$	$v_x = \frac{\partial v}{\partial x} = 2x$	$v_y = \frac{\partial v}{\partial y} = -2y$

By the properties of Jacobian, $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}.\frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=\begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}=-4y^2-4x^2=-4(x^2+y^2)$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)}=-4r^2 \qquad [\because x^2+y^2=(r\cos\theta)^2+(r\sin\theta)^2=r^2\cos^2\theta+r^2\sin^2\theta$$

$$=r^2(\cos^2\theta+\sin^2\theta)=r^2(1)=r^2]$$

Since $x=r\cos\theta, y=r\sin\theta$

$x_r=\frac{\partial x}{\partial r}=\cos\theta$	$x_\theta=\frac{\partial x}{\partial \theta}=-r\sin\theta$	$y_r=\frac{\partial y}{\partial r}=\sin\theta$	$y_\theta=\frac{\partial y}{\partial \theta}=r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}=\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}=\left(r\cos^2\theta\right)-\left(-r\sin^2\theta\right)=r\left(\sin^2\theta+\cos^2\theta\right)=r(1)=r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}=\left(-4r^2\right)r=-4r^3$$

11 **Find the Jacobian of A, B, C with respect to x_1, x_2, x_3 if**

$$A=\frac{x_2x_3}{x_1},\quad B=\frac{x_1x_3}{x_2},\quad C=\frac{x_1x_2}{x_3}$$

Solution:

$A=\frac{x_2x_3}{x_1}$	$A_{x_1}=\frac{-x_2x_3}{x_1^2}$	$A_{x_2}=\frac{x_3}{x_1}$	$A_{x_3}=\frac{x_2}{x_1}$
$B=\frac{x_1x_3}{x_2}$	$B_{x_1}=\frac{x_3}{x_2}$	$B_{x_2}=\frac{-x_1x_3}{x_2^2}$	$B_{x_3}=\frac{x_1}{x_2}$
$C=\frac{x_1x_2}{x_3}$	$C_{x_1}=\frac{x_2}{x_3}$	$C_{x_2}=\frac{x_1}{x_3}$	$C_{x_3}=\frac{-x_1x_2}{x_3^2}$

$$\frac{\partial(A,B,C)}{\partial(x_1,x_2,x_3)}=\begin{vmatrix} A_{x_1} & A_{x_2} & A_{x_3} \\ B_{x_1} & B_{x_2} & B_{x_3} \\ C_{x_1} & C_{x_2} & C_{x_3} \end{vmatrix}=\begin{vmatrix} \frac{-x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$=\frac{-x_2x_3}{x_1^2}\begin{vmatrix} \frac{-x_1x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_1}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}-\frac{x_3}{x_1}\begin{vmatrix} \frac{x_3}{x_2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{-x_1x_2}{x_3^2} \end{vmatrix}+\frac{x_2}{x_1}\begin{vmatrix} \frac{x_3}{x_2} & \frac{-x_1x_3}{x_2^2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} \end{vmatrix}$$

$$=\frac{-x_2x_3}{x_1^2}\left[\left(\frac{-x_1x_3}{x_2^2}\cdot\frac{-x_1x_2}{x_3^2}\right)-\left(\frac{x_1}{x_3}\cdot\frac{x_1}{x_2}\right)\right]-\frac{x_3}{x_1}\left[\left(\frac{x_3}{x_2}\cdot\frac{-x_1x_2}{x_3^2}\right)-\left(\frac{x_2}{x_3}\cdot\frac{x_1}{x_2}\right)\right]+\frac{x_2}{x_1}\left[\left(\frac{x_3}{x_2}\cdot\frac{x_1}{x_3}\right)-\left(\frac{x_2}{x_3}\cdot\frac{-x_1x_3}{x_2^2}\right)\right]$$

$$=\frac{-x_2x_3}{x_1^2}\left[\left(\frac{x_1^2x_2x_3}{x_2^2x_3^2}\right)-\left(\frac{x_1^2}{x_2x_3}\right)\right]-\frac{x_3}{x_1}\left[\left(\frac{-x_1x_2x_3}{x_2x_3^2}\right)-\left(\frac{x_1x_2}{x_2x_3}\right)\right]+\frac{x_2}{x_1}\left[\left(\frac{x_1x_3}{x_2x_3}\right)+\left(\frac{x_1x_2x_3}{x_2^2x_3}\right)\right]$$

$$=\left[\left(\frac{-x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\right)+\left(\frac{x_1^2x_2x_3}{x_1^2x_2x_3}\right)\right]+\left[\left(\frac{x_1x_2x_3^2}{x_1x_2x_3^2}\right)+\left(\frac{x_1x_2x_3}{x_1x_2x_3}\right)\right]+\left[\left(\frac{x_1x_2x_3}{x_1x_2x_3}\right)+\left(\frac{x_1x_2^2x_3}{x_1x_2^2x_3}\right)\right]$$

$$=-1+1+1+1+1+1=4$$

12 **If $u=x+y+z,\quad uv=y+z,\quad uvw=z$, find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$**

Solution:

$u=x+y+z,\qquad uv=y+z,\qquad z=uvw$

$x=u-(y+z),\qquad y=uv-z,$

$x=u-uv,\qquad y=uv-uvw,$

$x_u=\dfrac{\partial x}{\partial u}=1-v\qquad y_u=\dfrac{\partial y}{\partial u}=v-vw\qquad z_u=\dfrac{\partial z}{\partial u}=vw$

$x_v=\dfrac{\partial x}{\partial v}=-u\qquad y_v=\dfrac{\partial y}{\partial v}=u-uw\qquad z_v=\dfrac{\partial z}{\partial v}=uw$

$x_w=\dfrac{\partial x}{\partial w}=0,\qquad y_w=\dfrac{\partial y}{\partial v}=-uv\qquad z_w=\dfrac{\partial z}{\partial w}=uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} x_u & x_v & x_w\\ y_u & y_v & y_w\\ z_u & z_v & z_w\end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0\\ v-vw & u-uw & -uv\\ vw & uw & uv\end{vmatrix}$$

$$=(1-v)\left[(u-uw)(uv)+(uv)(uw)\right]-(-u)\left[uv(v-vw)+(uv)vw\right]$$

$$=(1-v)\left[u^2v-\cancel{u^2wv}+\cancel{u^2wv}\right]-(-u)\left[uv^2-\cancel{uv^2w}+\cancel{uv^2w}\right]$$

$$=(1-v)(u^2v)+u(uv^2)$$

$$=u^2v-u^2v^2+u^2v^2$$

$$=u^2v$$

13 **If $u=\dfrac{1}{x},\quad v=\dfrac{x^2}{y},\quad w=x+y+zy^2$, find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$**

Solution:

$u=\dfrac{1}{x}\qquad v=\dfrac{x^2}{y}\qquad w=x+y+zy^2,$

$u_x=\dfrac{-1}{x^2}\qquad v_x=\dfrac{2x}{y}\qquad w_x=1$

$u_y = 0 \qquad v_y = \dfrac{-x^2}{y^2} \qquad w_y = 1+2yz$

$u_z = 0 \qquad v_z = 0 \qquad w_z = y^2$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} \dfrac{-1}{x^2} & 0 & 0 \\ \dfrac{2x}{y} & \dfrac{-x^2}{y^2} & 0 \\ 1 & 1+2yz & y^2 \end{vmatrix}$$

$$= \frac{-1}{x^2}\left[\left(\frac{-x^2}{\cancel{y^2}}\right)\times \cancel{y^2} - 0\right] - 0 + 0$$

$$= \frac{-1}{\cancel{x^2}}(\cancel{-x^2}) = -1(-1) = 1$$

14 **If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$, prove that they are functionally dependent and also determine the functional relationship between u, v, w**

Solution:

$u = xy + yz + zx \qquad u_x = y+z \quad u_y = x+z \quad u_z = y+x$

$v = x^2 + y^2 + z^2 \qquad v_x = 2x \quad v_y = 2y \quad v_z = 2z$

$w = x + y + z \qquad w_x = 1 \quad w_y = 1 \quad w_z = 1$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (y+z)\begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix} - (x+z)\begin{vmatrix} 2x & 2z \\ 1 & 1 \end{vmatrix} + (y+x)\begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix}$$

$$= (y+z)[2y-2z] - (x+z)[2x-2z] + (y+x)[2x-2y]$$

$$= 2(y+z)[y-z] - 2(x+z)[x-z] + 2(x+y)[x-y]$$

$$= 2(y^2-z^2) - 2(x^2-z^2) + 2(x^2-y^2)$$

$$= 2y^2 - 2z^2 - 2x^2 + 2z^2 + 2x^2 - 2y^2$$

$$= \cancel{2y^2} \cancel{-2z^2} \cancel{-2x^2} + \cancel{2z^2} + \cancel{2x^2} \cancel{-2y^2}$$

$$= 0$$

$\therefore$ u, v *and* w are functionally dependent.

$$w^2 = (x+y+z)^2$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$
$$= (x^2 + y^2 + z^2) + 2(xy + yz + zx)$$
$$= v + 2u$$

(i.e.,) $w^2 = v + 2u$.

15 **If $\mathbf{u = 3x + 2y - z, v = x - 2y + z}$ and $\mathbf{w = x(x + 2y - z)}$, check whether u, v, w are functionally dependent and also find the relation. (NOV/ DEC 2020)**

Solution: Consider $u = 3x + 2y - z, v = x - 2y + z$ and $w = x(x + 2y - z)$

$u = 3x + 2y - z, \quad v = x - 2y + z \qquad w = x^2 + 2xy - xz$

$u_x = \frac{\partial u}{\partial x} = 3 \qquad v_x = \frac{\partial v}{\partial x} = 1 \qquad w_x = \frac{\partial w}{\partial x} = 2x + 2y - z$

$u_y = \frac{\partial u}{\partial y} = 2 \qquad v_y = \frac{\partial v}{\partial y} = -2 \qquad w_y = \frac{\partial w}{\partial y} = 2x$

$u_z = \frac{\partial u}{\partial z} = -1 \qquad v_z = \frac{\partial v}{\partial z} = 1 \qquad w_z = \frac{\partial w}{\partial z} = -x$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$
$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$$
$$= 3\begin{vmatrix} -2 & 1 \\ 2x & -x \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2x+2y-z & -x \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ 2x+2y-z & 2x \end{vmatrix}$$
$$= 3(2x - 2x) - 2\left[-x - (2x + 2y - z)\right] + (-1)\left[2x - (-2)(2x + 2y - z)\right]$$
$$= 3(0) - 2(-x - 2x - 2y + z) - (2x + 2(2x + 2y - z))$$
$$= 0 - 2(-3x - 2y + z) - (2x + 4x + 4y - 2z))$$
$$= -2(-3x - 2y + z) - (6x + 4y - 2z)$$
$$= 6x + 4y - 2z - 6x - 4y + 2z$$
$$= \cancel{6x} + \cancel{4y} - \cancel{2z} - \cancel{6x} - \cancel{4y} + \cancel{2z}$$
$$= 0$$

Hence they are functionally dependent

$u = 3x + 2y - z \qquad ----(1)$

$v = x - 2y + z \qquad ----(2)$

$w = x(x + 2y - z) \quad ----(3)$

Squaring & subtracting (1) and (2) we get

$$u^2 - v^2 = (3x+2y-z)^2 - (x-2y+z)^2 \qquad \left[\because (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac\right]$$

$$= \left[(3x)^2 + (2y)^2 + z^2 + 2(3x)(2y) + 2(2y)(-z) + 2(3x)(-z)\right]$$

$$-\left[(x)^2 + (2y)^2 + z^2 + 2(x)(-2y) + +2(-2y)(z) + 2(x)(z)\right]$$

$$= 9x^2 + \cancel{4y^2} + \cancel{z^2} + 12xy - \cancel{4yz} - 6xz - x^2 - \cancel{4y^2} - \cancel{z^2} + 4xy + \cancel{4yz} - 2xz$$

$$= 8x^2 + 16xy - 8xz$$

$$= 8x(x+2y-z) = 8w$$

$$\therefore u^2 - v^2 = 8w$$

16 **Obtain the Taylor's series expansion of $e^x \sin y$ in powers of x and y up to third degree terms.**

(NOV/DEC 2019)

Solution:

Let $f(x,y) = e^x \sin y$

Function	**Values at (0,0)**	
$f(x,y) = e^x \sin y$	$f(0,0) = e^0 \sin(0) = 1(0) = 0$	$\left[\because e^0 = 1 \,\&\, \sin 0 = 0\right]$
$f_x(x,y) = e^x \sin y$	$f_x(0,0) = e^0 \sin(0) = 1(0) = 0$	$\left[\because e^0 = 1 \,\&\, \cos 0 = 1\right]$
$f_y(x,y) = e^x \cos y$	$f_y(0,0) = e^0 \cos(0) = (1)(1) = 1$	
$f_{xy}(x,y) = e^x \cos y$	$f_{xy}(0,0) = e^0 \cos(0) = (1)(1) = 1$	
$f_{xx}(x,y) = e^x \sin y$	$f_{xx}(0,0) = e^0 \sin(0) = (1)(0) = 0$	
$f_{yy}(x,y) = -e^x \sin y$	$f_{yy}(0,0) = -e^0 \sin(0) = -(1)(0) = 0$	
$f_{xxx}(x,y) = e^x \sin y$	$f_{xxx}(0,0) = e^0 \sin(0) = (1)(0) = 0$	
$f_{xxy}(x,y) = e^x \cos y$	$f_{xxy}(0,0) = e^0 \cos(0) = (1)(1) = 1$	
$f_{xyy}(x,y) = -e^x \sin y$	$f_{xyy}(0,0) = -e^0 \sin(0) = -(1)(0) = 0$	
$f_{yyy}(x,y) = -e^x \cos y$	$f_{yyy}(0,0) = -e^0 \cos(0) = -(1)(1) = -1$	

The Taylor's series expansion at (a,b) is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$$+\frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b)+3(x-a)^2(y-b) f_{xxy}(a,b)+3(x-a)(y-b)^2 f_{xyy}(a,b)+(y-b)^3 f_{yyy}(a,b)\right]+\ldots$$

The Taylor's series expansion at (0,0) is

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[x\ f_x(0,0)+y\ f_y(0,0)\right]$$

$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0)+2xy\ f_{xy}(0,0)+y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0)+3x^2y\ f_{xxy}(0,0)+3xy^2 f_{xyy}(0,0)+y^3 f_{yyy}(0,0)\right]+\ldots$$

$$f(x,y)=0+\frac{1}{1!}\left[x(0)+y(1)\right]+\frac{1}{2!}\left[x^2(0)+2xy(1)+y^2(0)\right]$$

$$+\frac{1}{3!}\left[x^3(0)+3x^2y(1)+3y^2x(0)+y^3(-1)\right]+\ldots$$

$$f(x,y)=y+\frac{1}{2!}[2xy]+\frac{1}{6}\left[3x^2y-y^3\right]+\ldots$$

$$f(x,y)=y+xy+\frac{1}{2}x^2y-\frac{1}{6}y^3+\ldots$$

17 **Expand $e^x \log(1+y)$ in powers of x and y up to third degree using Taylor's series.**

Solution:

Let $f(x,y)=e^x\log(1+y)$

Function	Values at (0,0)	
$f(x,y)=e^x\log(1+y)$	$f(0,0)=e^0\log(1+0)=(1)(0)=0$	$\left[\because e^0=1\ \&\ \log 1=0\right]$
$f_x(x,y)=e^x\log(1+y)$	$f_x(0,0)=e^0\log(1+0)=(1)(0)=0$	
$f_y(x,y)=e^x(1+y)^{-1}$	$f_y(0,0)=e^0(1+0)^{-1}=(1)(1)=1$	$\left[\because \frac{\partial}{\partial y}\log(1+y)=\frac{1}{1+y}=(1+y)^{-1}\right]$
$f_{xx}(x,y)=e^x\log(1+y)$	$f_{xx}(0,0)=e^0\log(1+0)=(1)(0)=0$	
$f_{xy}(x,y)=e^x(1+y)^{-1}$	$f_{xy}(0,0)=e^0(1+0)^{-1}=(1)(1)=1$	
$f_{yy}(x,y)=-e^x(1+y)^{-2}$	$f_{yy}(0,0)=-e^0(1+0)^{-2}=-(1)(1)^{-2}=-1$	$\left[\because \frac{\partial}{\partial y}(1+y)^{-1}=-(1+y)^{-2}\right]$
$f_{xxx}(x,y)=e^x\log(1+y)$	$f_{xxx}(0,0)=e^0\log(1+0)=(1)(0)=0$	
$f_{xxy}(x,y)=e^x(1+y)^{-1}$	$f_{xxy}(0,0)=e^0(1+0)^{-1}=(1)(1)=1$	
$f_{xyy}(x,y)=-e^x(1+y)^{-2}$	$f_{xyy}(0,0)=-e^0(1+0)^{-2}=-(1)(1)=-1$	
$f_{yyy}(x,y)=2e^x(1+y)^{-3}$	$f_{yyy}(0,0)=2e^0(1+0)^{-3}=2(1)(1)=2$	$\left[\because \frac{\partial}{\partial y}(1+y)^{-2}=-2(1+y)^{-3}\right]$

The Taylor's series expansion at (0,0) is

$$f(x,y)=f(0,0)+\frac{1}{1!}\left[x\ f_x(0,0)+y\ f_y(0,0)\right]+\frac{1}{2!}\left[x^2 f_{xx}(0,0)+2xy\ f_{xy}(0,0)+y^2\ f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3\ f_{xxx}(0,0)+3x^2y\ f_{xxy}(0,0)+3xy^2\ f_{xyy}(0,0)+y^3\ f_{yyy}(0,0)\right]+\ldots$$

$$f(x,y)=0+\frac{1}{1!}\left[(x)(0)+(y)(1)\right]+\frac{1}{2!}\left[(x^2)(0)+2(xy)(1)+(y^2)(-1)\right]$$

$$+\frac{1}{3!}\left[(x^3)(0)+3(x^2y)(1)+3(xy^2)(-1)+(y^3)(2)\right]$$

$$f(x,y)=y+\frac{2xy-y^2}{2}+\frac{3x^2y-3xy^2+2y^3}{6}+\cdots$$

$$f(x,y)=y+\frac{2xy}{2}-\frac{y^2}{2}+\frac{3x^2y}{6}-\frac{3xy^2}{6}+\frac{2y^3}{6}+\cdots$$

$$f(x,y)=y+xy-\frac{y^2}{2}+\frac{x^2y}{2}-\frac{xy^2}{2}+\frac{y^3}{3}+\cdots$$

18 **Find the Taylor's series expansion of $f(x,y)=x^2y^2+2x^2y+3xy^2$ in powers of (x+2) and (y – 1) up to second degree term.** **(NOV/ DEC 2020)**

Solution:

The Taylor's series expansion in power of (x + 2) and (y – 1),

$x+2=0\Rightarrow x=-2$ and $y-1=0\Rightarrow y=1$

Given $f(x,y)=x^2y^2+2x^2y+3xy^2$

Function	**Values at (-2,1)**
$f(x,y)=x^2y^2+2x^2y+3xy^2$	$f(-2,1)=(-2)^2(1)^2+2(-2)^2(1)+3(-2)(1)^2=4+8-6=6$
$f_x(x,y)=2xy^2+4xy+3y^2$	$f_x(-2,1)=2(-2)(-1)^2+4(-2)(1)+3(1)^2=-4+-8+3=-9$
$f_y(x,y)=2x^2y+2x^2+6xy$	$f_y(-2,1)=2(-2)^2(1)+2(-2)^2+6(-2)(1)=8+8-12=4$
$f_{xx}(x,y)=2y^2+4y$	$f_{xx}(-2,1)=2(1)^2+4(1)=2+4=6$
$f_{xy}(x,y)=4xy+4x+6y$	$f_{xy}(-2,1)=4(-2)(1)+4(-2)+6(1)=-8+-8+6=-10$
$f_{yy}(x,y)=2x^2+6x$	$f_{yy}(-2,1)=2(-2)^2+6(-2)=8-12=-4$

The Taylor's series expansion at (-2,1) is

$$f(x,y)=f(-2,1)+\frac{1}{1!}\left[(x-(-2)\ f_x(-2,1)+(y-1)\ f_y(-2,1)\right]$$

$$+\frac{1}{2!}\left[\left(x-(-2)\right)^2 f_{xx}(-2,1)+2\left(x-(-2)\right)(y-1)\ f_{xy}(-2,1)+(y-1)^2\ f_{yy}(-2,1)\right]$$

$+\ldots$

$$f(x,y)=f(-2,1)+\frac{1}{1!}\left[(x+2)\,f_x(-2,1)+(y-1)\,f_y(-2,1)\right]$$

$$+\frac{1}{2!}\left[(x+2)^2\ f_{xx}(-2,1)+2(x+2)(y-1)\,f_{xy}(-2,1)+(y-1)^2\ f_{yy}(-2,1)\right]+\ldots$$

$$\therefore\ f(x,y)=6+\frac{1}{1!}\left[(x+2)\ (-9)+(y-1)(4)\right]$$

$$+\frac{1}{2!}\left[(x+2)^2(6)+2(x+2)(y-1)(-10)+(y-1)^2(-4)\right]+\ldots..$$

$$f(x,y)=6-9(x+2)+4(y-1)+\frac{1}{2}[6(x+2)^2-20(x+2)(y-1)-4(y-1)^2]+\ldots..$$

$$f(x,y)=6-9(x+2)+4(y-1)+3(x+2)^2-10(x+2)(y-1)-2(y-1)^2]+\ldots..$$

19 **Expand the function $\sin(xy)$ in powers of $(x-1)$ and $\left(y-\frac{\pi}{2}\right)$ as a Taylor series.**

Solution:

Let $f(x,y)=\sin(xy)$

The Taylor's series expansion in powers of (x – 1) and $\left(y-\frac{\pi}{2}\right)$,

$x-1=0\Rightarrow x=1$ and $y-\frac{\pi}{2}=0\Rightarrow y=\frac{\pi}{2}$

The Taylor's series expansion at $\left(1,\frac{\pi}{2}\right)$ is

$$f(x,y)=f\left(1,\frac{\pi}{2}\right)+\frac{1}{1!}\left[(x-1)\,f_x\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2\ f_{xx}\left(1,\frac{\pi}{2}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2\ f_{yy}\left(1,\frac{\pi}{2}\right)\right]+\ldots.$$

Function	Values at $\left(1,\frac{\pi}{2}\right)$
$f(x,y)=\sin(xy)$	$f\left(1,\frac{\pi}{2}\right)=\sin\left[(1)\left(\frac{\pi}{2}\right)\right]=1$
$f_x(x,y)=y\cos(xy)$	$f_x\left(1,\frac{\pi}{2}\right)=\frac{\pi}{2}\cos\left[(1)\left(\frac{\pi}{2}\right)\right]=\frac{\pi}{2}(0)=0$
$f_y=x\cos(xy)$	$f_y\left(1,\frac{\pi}{2}\right)=(1)\cos\left[(1)\left(\frac{\pi}{2}\right)\right]=(1)(0)=0$
$f_{xx}=-y^2\sin(xy)$	$f_{xx}\left(1,\frac{\pi}{2}\right)=-\left(\frac{\pi}{2}\right)^2\sin\left[(1)\left(\frac{\pi}{2}\right)\right]=-\left(\frac{\pi}{2}\right)^2(1)=-\frac{\pi^2}{4}$
$f_{xy}=-xy\sin(xy)+\cos(xy)$	$f_{xy}\left(1,\frac{\pi}{2}\right)=-(1)\left(\frac{\pi}{2}\right)\sin\left[(1)\left(\frac{\pi}{2}\right)\right]+\cos\left[(1)\left(\frac{\pi}{2}\right)\right]$ $=-\frac{\pi}{2}\sin\frac{\pi}{2}+\cos\frac{\pi}{2}=-\frac{\pi}{2}(1)+0=-\frac{\pi}{2}$
$f_{yy}=-x^2\sin(xy)$	$f_{yy}=-1$ $f_{yy}\left(1,\frac{\pi}{2}\right)=-(1)^2\sin\left[(1)\left(\frac{\pi}{2}\right)\right]=-(1)^2(1)=-1$

$$f(x,y)=1+\frac{1}{1!}\left[(x-1)(0)+\left(y-\frac{\pi}{2}\right)(0)\right]+\frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2(-1)\right]+....$$

$$f(x,y)=1+\frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2-\pi(x-1)\left(y-\frac{\pi}{2}\right)-\left(y-\frac{\pi}{2}\right)^2\right]+....$$

20 **Find the extreme values of $f(x,y)=x^3+y^3-3x-12y+20$**

Solution:

Given $f(x,y)=x^3+y^3-3x-12y+20$

$$f_x(x,y)=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+y^3-3x-12y+20\right)=3x^2+0-3-0+0=3x^2-3$$

$$f_{xx}(x,y)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(3x^2-3\right)=6x-0=6x$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+y^3-3x-12y+20\right)=0+3y^2+0-12+0=3y^2-12$$

$$f_{yy}(x,y)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(3y^2-12\right)=6y-0=6y$$

$$f_{xy}(x,y)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(3y^2-12\right)=0-0=0$$

To find stationary points:

$f_x(x,y)=0$	$f_y(x,y)=0$
$3x^2-3=0$	$3y^2-12=0$

$3\left(x^2-1\right)=0$	$3\left(y^2-4\right)=0$
$\left(x^2-1\right)=\frac{0}{3}=0$	$\left(y^2-4\right)=\frac{0}{3}=0$
$x^2-1=0$	$y^2-4=0$
$x^2=1$	$y^2=4$
Taking square root on both side	Taking square root on both side
$\sqrt{x^2}=\pm\sqrt{1}$	$\sqrt{y^2}=\pm\sqrt{4}$
$x=\pm1$	$y=\pm2$
$x=-1,1$	$y=-2,2$

The Stationary points are $(-1,-2),(-1,2),(1,-2),(1,2)$

Stationary points	$f_{xx}=6x$	$\Delta=f_{xx}f_{yy}-\left(f_{xy}\right)^2$	Extreme Point	Extreme Values $f(x,y)=x^3+y^3-3x-12y+20$
$(-1,-2)$	$f_{xx}=6(-1)$ $=-6<0$	$(6x)(6y)-(0)^2$ $=(-6)(-12)-(0)^2$ $=72>0$	Max point	$f(-1,-2)$ $=(-1)^3+(-2)^3-3(-1)-12(-2)+20$ $=-1-8+3+24+20$ $=38$
$(-1,2)$	$f_{xx}=6(-1)$ $=-6<0$	$(6x)(6y)-(0)^2$ $=(-6)(12)-(0)^2$ $=-72<0$	Saddle point	-
$(1,-2)$	$f_{xx}=6(1)$ $=6>0$	$(6x)(6y)-(0)^2$ $=(6)(-12)-(0)^2$ $=-72<0$	Saddle point	-
$(1,2)$	$f_{xx}=6(1)$ $=6>0$	$(6x)(6y)-(0)^2$ $=(6)(12)-(0)^2$ $=72>0$	Min point	$f(1,2)$ $=(1)^3+(2)^3-3(1)-12(2)+20$ $=1+8-3-24+20$ $=2$

21 **Examine the extreme values of $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$.** **(APR/MAY 2019)**

Solution:

Given $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$

$$f_x(x,y)=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(x^3+3xy^2-15x^2-15y^2+72x\right)$$

$$=3x^2+3y^2-30x-0+72=3x^2+3y^2-30x+72$$

$$f_{xx}(x,y)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(3x^2+3y^2-30x+72\right)=6x+0-30+0=6x-30$$

$$f_y(x,y)=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(x^3+3xy^2-15x^2-15y^2+72x\right)=0+6xy+0-30y+0=6xy-30y$$

$$f_{yy}(x,y)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}(6xy-30y)=6x-30$$

$$f_{xy}(x,y)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(6xy-30y)=6y-0=6y$$

To find stationary points:

$$f_x(x,y)=0$$
$$3\left(x^2+y^2-10x+14\right)=0$$
$$x^2+y^2-10x+14=0 \quad ----(1)$$

$$f_y(x,y)=0$$
$$6xy-30y=0$$
$$6y(x-5)=0$$
$$6y=0,\ x-5=0$$
$$y=0,\ x=5$$

Sub $y=0$ in (1)

$$x^2+0-10x+24=0$$
$$x^2-10x+24=0$$
$$x^2-6x-4x+24=0$$
$$x(x-6)-4(x-6)=0$$
$$(x-6)(x-4)=0$$
$$x-6=0,\ x-4=0$$
$$x=6,\ x=4$$

When $y=0\Rightarrow x=6,\ x=4$

Put $x=5$ in (1)

$$(5)^2+y^2-10(5)+24=0$$
$$25+y^2-50+24=0$$
$$y^2-1=0$$
$$y^2=1$$

Taking square root on both side

$$\sqrt{y^2}=\pm\sqrt{1}$$
$$y=\pm 1$$
$$y=1,\ -1$$

When $x=5\Rightarrow y=1,\ y=-1$

The Stationary points are $(4,0),(6,0),(5,-1),(5,1)$

Stationary points	$f_{xx}=6x-30$	$\Delta=f_{xx}f_{yy}-\left(f_{xy}\right)^2$	Extreme Point	Extreme Values $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$
$(4,0)$	$f_{xx}=6(4)-30$ $=-6<0$	$(6x-30)(6x-30)-(6y)^2$ $=(-6)(-6)-0$ $=36>0$	Max point	$f(4,0)$ $=(4)^3+3(4)(0)^2-15(4)^2-15(0)^2+72(4)$ $=64+0-240-0+288$ $=112$
$(6,0)$	$f_{xx}=6(6)-30$ $=6>0$	$(6x-30)(6x-30)-(6y)^2$ $=(6)(6)-0$ $=36>0$	Min point	$f(6,0)$ $=(6)^3+3(6)(0)^2-15(6)^2-15(0)^2+72(6)$ $=216+0-540-0+434$ $=108$
$(5,-1)$	$f_{xx}=6(5)-30$ $=0$	$(6x-30)(6x-30)-(6y)^2$ $=(0)(0)-36$ $=-36<0$	Saddle point	-

$(5,1)$	$f_{xx}=6(5)-30$ $=0$	$(6x-30)(6x-30)-(6y)^2$ $=(0)(0)-36$ $=-36<0$	Saddle point	-

22 **A thin closed rectangular box is to have one edge equal to twice the other and constant volume 72m$^{3.}$ Find the least surface area of the box. (NOV/DEC 2019)**

Solution:

Let x, y, z be the length, breadth and height of the box in which have one edge equal to twice the other that is $z=2y$

Volume $xyz = xy(2y) = 2xy^2$

$2xy^2 = 72 \Rightarrow xy^2 = 36 \Rightarrow xy^2 - 36 = 0$

Surface area of the box $= 2xy + 2yz + 2zx$

$$= 2xy + 2y(2y) + 2(2y)x$$
$$= 2xy + 4y^2 + 4xy$$
$$= 6xy + 4y^2$$

Thus $F(x,y,z) = 6xy + 4y^2 + \lambda(xy^2 - 36)$ where λ is Lagrangian Multiplier

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(6xy + 4y^2 + \lambda\left(xy^2 - 36\right)\right) = 6y + \lambda y^2$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(6xy + 4y^2 + \lambda\left(xy^2 - 36\right)\right) = 6x + 8y + 2\lambda xy$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\left(6xy + 4y^2 + \lambda\left(xy^2 - 36\right)\right) = 0$$

To find the stationary point:

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$6y + \lambda y^2 = 0$	$6x + 8y + 2\lambda xy = 0$	
$\lambda y^2 = -6y$	$2\lambda xy = -6x - 8y$	
$\lambda = \frac{-6y}{y^2}$	$\lambda = \frac{-6x-8y}{2xy} = \frac{-6x}{2xy} - \frac{8y}{2xy}$	
$\lambda = \frac{-6}{y} ----(1)$	$\lambda = \frac{-3}{y} - \frac{4}{x} ---(2)$	

Equating λ in $(1)\&(2)$, we get

$$\frac{-6}{y} = \frac{-3}{y} - \frac{4}{x} \Rightarrow \frac{-6}{y} + \frac{3}{y} = \frac{-4}{x} \Rightarrow \frac{-6+3}{y} = \frac{-4}{x} \Rightarrow \frac{-3}{y} = \frac{-4}{x} \Rightarrow \frac{x}{y} = \frac{4}{3}$$

$x = 4, y = 2, z = 2y = 2(3) = 6$

The minimum surface area is $f(x,y) = 6xy + 4y^2$ at $(4,3)$ is

$f(4,3) = 6(4)(3) + 4(3^2) = 72 + 36 = 108$.

23 **Find the dimension of the rectangular box open at the top of maximum capacity 432 cc. (NOV/DEC 2020)**

Solution: Let x, y, z be the length,breadth and height of the box.

Since the rectangular box is of open at the top, Surface area on the top is zero.

$\therefore$ Total surface area of the box $= xy + 2yz + 2zx$

$Let\ f(x,y,z) = xy + 2yz + 2zx$

volume = length×breadth×height

$= xyz = 432$ (given capacity = volume=432)

$Let\ g(x,y,z) = xyz - 432 = 0$

To find the dimension with the constraint capacity 432 cc.

Optimize F = Objective + λ(Constraint)

$$F(x,y,z) = xy + 2yz + 2zx + \lambda(xyz - 432)$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(xy + 2yz + 2zx + \lambda(xyz - 432)\right) = y + 2z + \lambda(yz)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(xy + 2yz + 2zx + \lambda(xyz - 432)\right) = x + 2z + \lambda(xz)$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\left(xy + 2yz + 2zx + \lambda(xyz - 432)\right) = 2y + 2x + \lambda(xy)$$

To find the stationary point:

$\frac{\partial F}{\partial x} = 0$	$\frac{\partial F}{\partial y} = 0$	$\frac{\partial F}{\partial z} = 0$
$\Rightarrow y + 2z + \lambda(yz) = 0$	$\Rightarrow x + 2z + \lambda(xz) = 0$	$\Rightarrow 2y + 2x + \lambda(xy) = 0$
$\Rightarrow y + 2z = -\lambda yz$	$\Rightarrow x + 2z = -\lambda xz$	$\Rightarrow 2y + 2x = -\lambda xy$
$\Rightarrow \frac{y + 2z}{yz} = -\lambda$	$\Rightarrow \frac{x + 2z}{xz} = -\lambda$	$\Rightarrow \frac{2y + 2x}{xy} = -\lambda$
$\frac{\cancel{y}}{\cancel{y}z} + \frac{2\cancel{z}}{y\cancel{z}} = -\lambda$	$\Rightarrow \frac{\cancel{x}}{\cancel{x}z} + \frac{2\cancel{z}}{x\cancel{z}} = -\lambda$	$\Rightarrow \frac{2\cancel{y}}{x\cancel{y}} + \frac{2\cancel{x}}{\cancel{x}y} = -\lambda$
$\Rightarrow \frac{1}{z} + \frac{2}{y} = -\lambda \quad \ldots(1)$	$\Rightarrow \frac{1}{z} + \frac{2}{x} = -\lambda \quad \ldots(2)$	$\Rightarrow \frac{2}{x} + \frac{2}{y} = -\lambda \quad \ldots(3)$

	Equating (1) and (2) $\cancel{\frac{1}{z}}+\frac{2}{y}=\cancel{\frac{1}{z}}+\frac{2}{x};$ $\frac{2}{y}=\frac{2}{x}$ $2x=2y\Rightarrow x=y$ Equating (2) and (3) $\frac{1}{z}+\cancel{\frac{2}{x}}=\cancel{\frac{2}{x}}+\frac{2}{y}$ $\frac{1}{z}=\frac{2}{y}\Rightarrow y=2z$ *Hence* $x=y=2z$ since volume $=xyz=432$.....(4) sub $x=2z; y=2z$ in (4) $(2z)(2z)z=432$ $4z^3=432$ $z^3=\frac{432}{4}=108$ $z^3=108$ $\therefore\ z=(108)^{\frac{1}{3}}\ ,\ \ x=2(108)^{\frac{1}{3}},\ \ y=2(108)^{\frac{1}{3}}$
24	**Find the length of the shortest line from the point $\left(0,0,\frac{25}{9}\right)$ to the surface $z=xy$**
	Solution: Let (x,y,z) be any point on the the surface $z=xy$ We know that the distance between two points is $d^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$ The distance from $(0,0,\frac{25}{9})$ *to* (x,y,z) is $\therefore d^2=(x-0)^2+(y-0)^2+\left(z-\frac{25}{9}\right)^2$ Now, Optimize $d^2=(x-0)^2+(y-0)^2+\left(z-\frac{25}{9}\right)^2=f(x,y,z);$ subject to $g(x,y,z)=z-xy=0;$ Let Optimize F = Objective + λ(Constraint) $F(x,y,z)=x^2+y^2+\left(z-\frac{25}{9}\right)^2+\lambda(z-xy)$

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2+\left(z-\frac{25}{9}\right)^2+\lambda\left(z-xy\right)\right)=2x-\lambda y$$

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left(x^2+y^2+\left(z-\frac{25}{9}\right)^2+\lambda\left(z-xy\right)\right)=2y-\lambda x$$

$$\frac{\partial F}{\partial z}=\frac{\partial}{\partial z}\left(x^2+y^2+\left(z-\frac{25}{9}\right)^2+\lambda\left(z-xy\right)\right)=2\left(z-\frac{25}{9}\right)+\lambda$$

To find the stationary point:

$$\frac{\partial F}{\partial x}=0\Rightarrow 2x-\lambda y=0$$
$$\Rightarrow \lambda y=2x$$
$$\Rightarrow \lambda=\frac{2x}{y}\text{.....(1)}$$

$$\frac{\partial F}{\partial y}=0\Rightarrow 2y-\lambda x=0$$
$$\Rightarrow \lambda x=2y$$
$$\Rightarrow \lambda=\frac{2y}{x}\text{.....(2)}$$

$$\frac{\partial F}{\partial z}=0\Rightarrow 2\left(z-\frac{25}{9}\right)+\lambda=0$$
$$\Rightarrow \lambda=-2\left[z-\frac{25}{9}\right]\text{.....(3)}$$

From (1) and (2), $\frac{2x}{y}=\frac{2y}{x}$; $x^2=y^2 \Rightarrow x=\pm y$

From (2) and (3), $\frac{2y}{x}=-2\left[z-\frac{25}{9}\right]$

If $x=y$

$$\frac{\cancel{2y}}{-\cancel{2(y)}}=\left[z-\frac{25}{9}\right]\Rightarrow -1=z-\frac{25}{9}\Rightarrow z=-1+\frac{25}{9}=\frac{-9}{9}+\frac{25}{9}\Rightarrow z=\frac{16}{9}$$

$\therefore\ z=xy$

$\Rightarrow z=(x)(x)=x^2 \qquad (\because\ x=y)$

$\frac{16}{9}=x^2\Rightarrow x=\pm\frac{4}{3}$

if $x=-y\Rightarrow z=(-y)(y)=-y^2$ (which is not possible)

$\therefore$ Minimum Distance is

$$\sqrt{x^2+y^2+\left(z-\frac{25}{9}\right)^2}=\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{4}{3}\right)^2+\left(\frac{16}{9}-\frac{25}{9}\right)^2}$$
$$=\sqrt{\frac{16}{9}+\frac{16}{9}+1}=\sqrt{\frac{16}{9}+\frac{16}{9}+\frac{9}{9}}=\sqrt{\frac{41}{9}}=\frac{\sqrt{41}}{3}$$

25 **Find the shortest and longest distances from the point (1,2,-1) to the sphere $x^2+y^2+z^2=24$.**

(APR/MAY 2019)

Solution:

Let (x,y,z) be any point on the sphere $x^2+y^2+z^2=24$.

The distance from $(1,2,-1)$ *to* (x,y,z) is

$$d=\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$$

$$d^2=(x-1)^2+(y-2)^2+(z+1)^2$$

Now, Optimize $d^2=(x-1)^2+(y-2)^2+(z+1)^2 = f(x,y,z)$

subject to $g(x,y,z)=x^2+y^2+z^2-24=0$

Let Optimize $\mathrm{F}=\text{Objective}+\lambda(\text{Constraint})$

$$F=(x-1)^2+(y-2)^2+(z+1)^2+\lambda(\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2-24)$$

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}\left((x-1)^2+(y-2)^2+(z+1)^2+\lambda(\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2-24)\right)=2(x-1)+2\lambda x$$

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\left((x-1)^2+(y-2)^2+(z+1)^2+\lambda(\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2-24)\right)=2(y-2)+2\lambda y$$

$$\frac{\partial F}{\partial z}=\frac{\partial}{\partial z}\left((x-1)^2+(y-2)^2+(z+1)^2+\lambda(\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2-24)\right)=2(z+1)+2\lambda z$$

To find the stationary point:

$\frac{\partial F}{\partial x}=0\Rightarrow 2(x-1)+2\lambda x=0$ $\Rightarrow 2(x-1)=-2\lambda x$ $\Rightarrow \frac{(x-1)}{x}=-\lambda$ $\Rightarrow \frac{\cancel{x}}{\cancel{x}}-\frac{1}{x}=-\lambda$ $\Rightarrow 1-\frac{1}{x}=-\lambda$.....(1)	$\frac{\partial F}{\partial y}=0\Rightarrow 2(y-2)+2\lambda y=0.$ $\Rightarrow 2(y-2)=-2\lambda y$ $\Rightarrow \frac{(y-2)}{y}=-\lambda$ $\Rightarrow \frac{\cancel{y}}{\cancel{y}}-\frac{2}{y}=-\lambda$ $\Rightarrow 1-\frac{2}{y}=-\lambda$.....(2)	$\frac{\partial F}{\partial z}=0\Rightarrow 2(z+1)+2\lambda z=0$ $\Rightarrow 2(z+1)=-2\lambda z$ $\Rightarrow \frac{(z+1)}{z}=-\lambda$ $\Rightarrow \frac{\cancel{z}}{\cancel{z}}+\frac{1}{z}=-\lambda$ $\Rightarrow 1+\frac{1}{z}=-\lambda$.....(3)

From (1) and (2), $1-\frac{1}{x}=1-\frac{2}{y}\Rightarrow \frac{1}{x}=\frac{2}{y}\Rightarrow y=2x$.....(4)

From (2) and (3), $1-\frac{2}{\mathrm{y}}=1+\frac{1}{\mathrm{z}}\Rightarrow -\frac{2}{\mathrm{y}}=\frac{1}{\mathrm{z}}\Rightarrow -2\mathrm{z}=\mathrm{y}$.....(5)

From (4) and (5)

$$2x=y=-2z$$

$$\Rightarrow y=2x \text{ and } 2x=-2z\Rightarrow z=-x$$

Now substitute all the values in $x^2+y^2+z^2 = 24$

$$x^2+(2x)^2+(-x)^2 = 24$$

$$x^2+4x^2+x^2 = 24$$

$$6x^2 = 24$$

$$x^2=4$$

$$x=\pm 2$$

	if $x = 2$, then $(x, y, z) = (2, 4, -2)$ if $x = -2$, then $(x, y, z) = (-2, -4, 2)$ at (2, 4, -2), $d = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2}$ $d = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6} = 2.449$ at (-2, -4, 2), $d = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$ $d = \sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$ =7.3484 Therefore, the shortest and longest distances are 2.449 and 7.348 respectively.

UNIT IV – INTEGRAL CALCULUS

PART – A

1 State the properties of definite integrals.

Solution:

(i) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$$

(ii) $$\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$$

2 Set up an expression for $\int_{1}^{3} e^x dx$ as a limit of sums.

Solution:

Let $f(x) = e^x,\ a=1,\ b=3,\ \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n},$

$x_i = a + i\Delta x \Rightarrow 1 + \frac{2i}{n}$ (By Riemann Sum method)

$$\int_{1}^{3} e^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right) \qquad \left[\ f(x) = e^x\ \right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} e^{\left(1+\frac{2i}{n}\right)}$$

3 Evaluate the following integrals using fundamental theorem of calculus $\int_{1}^{8} \sqrt[3]{x}\,dx$

Solution:

Let $f(x) = x^{1/3},\ a=1, b=8$

Integrate on both sides

$$\int f(x)\,dx = \int x^{1/3} dx$$

$$= \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \frac{x^{\frac{1+3}{3}}}{\frac{1+3}{3}} = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3}{4}x^{\frac{4}{3}} \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$\therefore\ F(x) = \frac{3}{4}x^{\frac{4}{3}} \qquad \left[\because \int f(x)\,dx = F(x)\right]$$

By using Fundamental theorem of calculus $\left[\int_{a}^{b} f(x)\,dx = F(b) - F(a)\right]$

$$\int_{1}^{8} \sqrt[3]{x}\,dx = F(8) - F(1)$$

$$= \frac{3}{4}(8)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}} = \frac{3}{4}\left[\left(8^{1/3}\right)^4 - 1^{\frac{4}{3}}\right] = \frac{3}{4}\left[\left(\left[2^3\right]^{1/3}\right)^4 - 1\right] = \frac{3}{4}\left[2^4 - 1\right] = \frac{3}{4}[16-1] = \frac{3}{4}(15) = \frac{45}{4}$$

4	**Evaluate $\int\left(10x^4-2\sec^2 x\right)dx$**
	Solution: $\int\left(10x^4-2\sec^2 x\right)dx=\int 10x^4dx-\int 2\sec^2 xdx$ $=10\int x^4dx-2\int\sec^2 xdx$ $=10\left(\frac{x^{4+1}}{4+1}\right)-2\tan x+c$ $\left[\because \int x^n dx=\frac{x^{n+1}}{n+1}+c \ \& \ \int\sec^2 xdx=\tan x+c\right]$ $=10\left(\frac{x^5}{5}\right)-2\tan x+c$ $\therefore\int\left(10x^4-2\sec^2 x\right)dx=2x^5-2\tan x+c$
5	**Evaluate $\int_1^2\left(-3x^{1/2}+\frac{1}{x^2}\right)dx$**
	Solution: $\int_1^2\left(-3x^{1/2}+\frac{1}{x^2}\right)dx=\int_1^2\left(-3x^{1/2}\right)dx+\int_1^2\frac{1}{x^2}dx$ $=-3\int_1^2\left(x^{1/2}\right)dx+\int_1^2\left(x^{-2}\right)dx$ $\left[\int x^n dx=\frac{x^{n+1}}{n+1}+c\right]$ $=-3\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2+\left[\frac{x^{-2+1}}{-2+1}\right]_1^2=-3\left[\frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}}\right]_1^2+\left[\frac{x^{-2+1}}{-2+1}\right]_1^2$ $=-3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^2+\left[\frac{x^{-1}}{-1}\right]_1^2$ $=(-\cancel{3})\left(\frac{2}{\cancel{3}}\right)\left[x^{\frac{3}{2}}\right]_1^2-\left[x^{-1}\right]_1^2$ $=(-2)\left[x^{\frac{3}{2}}\right]_1^2-\left[\frac{1}{x}\right]_1^2$ $=(-2)\left[\left(2^{1/2}\right)^3-(1)^{\frac{3}{2}}\right]-\left[\frac{1}{2}-\frac{1}{1}\right]$ $=(-2)\left[2\sqrt{2}-1\right]-\left[\frac{1-2}{2}\right]=(-2)\left[2\sqrt{2}-1\right]-\left[\frac{-1}{2}\right]$ $=\left[-4\sqrt{2}+2\right]-\left[\frac{-1}{2}\right]=-4\sqrt{2}+2+\frac{1}{2}=-4\sqrt{2}+\frac{4+1}{2}$ $\therefore\int_1^2\left(-3x^{1/2}+\frac{1}{x^2}\right)dx=-4\sqrt{2}+\frac{5}{2}$

6	**Evaluate by substitution method $\int\left(2x\sqrt{1+x^2}\right)dx$**
	Solution: Given $I=\int\left(2x\sqrt{1+x^2}\right)dx$ Let $u=1+x^2$ $\Rightarrow \frac{du}{dx}=\frac{d}{dx}\left(1+x^2\right)=0+2x$ $\Rightarrow \frac{du}{dx}=2x\Rightarrow du=2xdx$ $\therefore \int\left(2x\sqrt{1+x^2}\right)dx=\int\left(\sqrt{1+x^2}\right)2xdx=\int\sqrt{u}du=\int(u)^{\frac{1}{2}}du$ $=\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}=\frac{u^{\frac{1+2}{2}}}{\frac{1+2}{2}}=\frac{u^{\frac{3}{2}}}{\frac{3}{2}}$ $\left[\because \int x^n dx=\frac{x^{n+1}}{n+1}+c\right]$ $=\frac{2}{3}u^{\frac{3}{2}}$ $\therefore\ I=\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}+c$
7	**Find $\int\left(\frac{\cos\theta}{\sqrt{\sin\theta}}\right)d\theta$ by substitution method.**
	Solution: Let $I=\int\left(\frac{\cos\theta}{\sqrt{\sin\theta}}\right)d\theta$ Put $u=\sin\theta\Rightarrow\frac{du}{d\theta}=\frac{d}{d\theta}(\sin\theta)\Rightarrow\frac{du}{d\theta}=\cos\theta\Rightarrow du=\cos\theta\, d\theta$ $\therefore I=\int\frac{du}{\sqrt{u}}=\int(u)^{\frac{-1}{2}}du=\frac{u^{\frac{-1}{2}+1}}{\frac{-1}{2}+1}=\frac{u^{\frac{-1+2}{2}}}{\frac{-1+2}{2}}=\frac{u^{\frac{1}{2}}}{\frac{1}{2}}=2\sqrt{u}+c=2\sqrt{\sin\theta}+c$ $\left[\because \int x^n dx=\frac{x^{n+1}}{n+1}+c\right]$
8	**If f is continuous and $\int_0^4 f(x)dx=10$, then find $\int_0^2 f(2x)dx$ (NOV/DEC 2018)**
	Solution: Given f is continuous, let $t=2x\Rightarrow\frac{dt}{dx}=2\Rightarrow dt=2dx\Rightarrow dx=\frac{dt}{2}$ when $x=2$ $\Rightarrow t=4$ when $x=0$ $\Rightarrow t=0$ $\therefore\int_0^2 f(2x)dx=\int_0^4 f(t)\frac{dt}{2}=\frac{1}{2}\int_0^4 f(t)dt=\frac{1}{2}(10)=5$ $\left[\because\int_0^4 f(x)dx=10\right]$
9	**Evaluate $\int\tan^{-1}x\,dx$**
	Solution:

$u=\tan^{-1}x$	$dv=dx$
$\frac{du}{dx}=\frac{1}{1+x^2}$	$\int dv=\int dx$
$du=\frac{dx}{1+x^2}$	$v=x$

We know that $\left(\int \frac{f'(x)}{f(x)}dx=\log f(x)\right)$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x-\int x\frac{dx}{1+x^2} \qquad \left[\because \int u\,dv = uv-\int v\,du\right]$$

$$= x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx$$

Multiply and divided by 2 to the last term

$$\left[\frac{d}{dx}\left(1+x^2\right)=2x\right]$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x-\frac{1}{2}\log\left(1+x^2\right)+c$$

10 **Find the Integral of $x\sin x$ using integration by parts.**

Solution:

Let $\int x\sin x\,dx=\int u\,dv$

$u=x$	$dv=\sin x\,dx$
$\frac{du}{dx}=1$ $du=dx$	$\int dv=\int \sin x\,dx$ $v=-\cos x$

$$\int x\sin x\,dx = x(-\cos x)-\int(-\cos x)dx \qquad \left(\because \int u\,dv = uv-\int v\,du\right)$$

$$=-x\cos x+\int\cos x\,dx$$

$$=-x\cos x+\sin x+c$$

$$=\sin x-x\cos x+c$$

11 **Evaluate $\int_0^{\pi/2}\sin^6 x\ \cos^5 x\,dx$**

Solution:

Let $I=\int_0^{\pi/2}\sin^6 x\ \cos^5 x\,dx$

$$\int_0^{\pi/2}\sin^m x\cos^n x\,dx=\frac{\left[(m-1)(m-3)\ldots(5)(3)(1)\right]\left[(n-1)(n-3)\ldots(6)(4)(2)\right]}{(m+n)(m+n-2)(m+n-4)\ldots(5)(3)(1)}$$

when m is an even and n is an odd integer. Here $m=6, n=5$

$$I=\frac{[(6-1)(6-3)(6-5)][(5-1)(5-3)]}{(6+5)(6+5-2)(6+5-4)(6+5-6)(6+5-8)(6+5-10)}$$

$$=\frac{[(5)(3)(1)][(4)(2)]}{(11)(9)(7)(5)(3)(1)}=\frac{8}{693}$$

12 **Use the hyperbolic substitution to show that $\int\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\left(\frac{x}{a}\right)+c$**

Solution:

Let $I=\int\frac{dx}{\sqrt{x^2+a^2}}$

Put $x=a\sinh t\Rightarrow\frac{dx}{dt}=\frac{d}{dt}(a\sinh t)=a\frac{d}{dt}(\sinh t)=a\cosh t\Rightarrow dx=a\cosh t dt$

$$I=\int\frac{a\cosh t dt}{\sqrt{(a\sinh t)^2+a^2}}=\int\frac{a\cosh t dt}{\sqrt{a^2\sinh^2 t+a^2}}=\int\frac{a\cosh t dt}{\sqrt{a^2\left(\sinh^2 t+1\right)}}$$

$$=\int\frac{\cancel{a}\cosh t dt}{\cancel{a}\sqrt{\sinh^2 t+1}}$$

$$=\int\frac{\cosh t dt}{\sqrt{\cosh^2 t}}\qquad\left(\because\cosh^2 t-\sinh^2 t=1\Rightarrow\cosh^2 t=1+\sinh^2 t\right)$$

$$=\int\frac{\cancel{\cosh t}\,dt}{\cancel{\cosh t}}$$

$$=\int dt=t+c\qquad\left(x=a\sinh t\Rightarrow\sinh t=\frac{x}{a},t=\sinh^{-1}\left(\frac{x}{a}\right)\right)$$

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\sinh^{-1}\left(\frac{x}{a}\right)+c$$

13 **Evaluate** $\int\frac{x-9}{(x+5)(x-2)}dx$

Solution:

Let $I=\int\frac{x-9}{(x+5)(x-2)}dx$

$$\frac{x-9}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2}----(1)$$

$$\frac{x-9}{\cancel{(x+5)(x-2)}}=\frac{A(x-2)+B(x+5)}{\cancel{(x+5)(x-2)}}$$

$$x-9=A(x-2)+B(x+5)$$

When $x=2\Rightarrow 2-9=A(2-2)+B(2+5)\ \Rightarrow -7=A(0)+B(7)\ \Rightarrow -7=7B\ \Rightarrow\boxed{B=-1}$

When $x=-5\Rightarrow -5-9=A(-5-2)+B(-5+5)\Rightarrow -14=A(-7)+B(0)\Rightarrow -14=-7A\Rightarrow\boxed{A=2}$

Substitute A and B values in (1)

$$\frac{x-9}{(x+5)(x-2)}=\frac{2}{x+5}+\left(\frac{-1}{x-2}\right)=\frac{2}{x+5}-\frac{1}{x-2}$$

Taking integral

$$\int\frac{x-9}{(x+5)(x-2)}dx=\int\left(\frac{2}{x+5}-\frac{1}{x-2}\right)dx$$

$$=\int\frac{2}{x+5}dx-\int\frac{1}{x-2}dx$$

$$=2\int\frac{1}{x+5}dx-\int\frac{1}{x-2}dx\qquad\left[\int\frac{f'(x)}{f(x)}dx=\log f(x)\right]$$

	$= 2\log(x+5) - \log(x-2) + c$ $\left[b\log a = \log a^b\right]$ $= \log(x+5)^2 - \log(x-2) + c$ $\left[\log a - \log b = \log\left[\frac{a}{b}\right]\right]$ $\therefore I = \log\left[\frac{(x+5)^2}{(x-2)}\right]$
14	**Evaluate $\int\left(\frac{x}{x-6}\right)dx$**
	Solution: Let $I = \int \frac{x}{x-6}dx = \int \frac{x-6+6}{x-6}dx = \int \frac{\cancel{x-6}}{\cancel{x-6}}dx + \int \frac{6}{x-6}dx$ $= \int dx + 6\int \frac{1}{x-6}dx$ $\left[\int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$ $= x + 6\log(x-6) + c$ $\left[b\log a = \log a^b\right]$ $= x + \log(x-6)^6 + c$
15	**Determine whether the integral $\int_1^\infty \frac{\ln x}{x}dx$ is convergent or divergent. (NOV/DEC 2020)**
	Solution: $\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\log x}{x}dx$ (we know that $\ln x = \log x$) $= \lim_{t\to\infty}\int_1^t (\log x)\,d(\log x) = \lim_{t\to\infty}\left(\frac{(\log x)^2}{2}\right)_1^t = \frac{1}{2}\lim_{t\to\infty}\left((\log t)^2 - (\log 1)^2\right)$ $\left(\because d(\log x) = \frac{dx}{x}\right)$ $= \frac{1}{2}\lim_{t\to\infty}\left((\log t)^2 - 0\right)$ $(\because \log 1 = 0)$ $= \frac{1}{2}(\log\infty)^2 = \frac{1}{2}(\infty) = \infty$ $(\because \log\infty = \infty)$ Hence the given integral is divergent
16	**Determine whether the integral $\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ is convergent or divergent.**
	Solution: Let $I = \int_3^\infty \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty}\int_3^t \frac{dx}{(x-2)^{3/2}} = \lim_{t\to\infty}\int_3^t (x-2)^{-3/2}dx$ $= \lim_{t\to\infty}\left(\frac{(x-2)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1}\right)_3^t = \lim_{t\to\infty}\left(\frac{(x-2)^{\frac{-3+2}{2}}}{\frac{-3+2}{2}}\right)_3^t = \lim_{t\to\infty}\left(\frac{(x-2)^{\frac{-1}{2}}}{\frac{-1}{2}}\right)_3^t = \lim_{t\to\infty}\left(\frac{-2}{(x-2)^{\frac{1}{2}}}\right)_3^t$ $= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{x-2}}\right)_3^t = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} - \frac{-2}{\sqrt{3-2}}\right) = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}\right) = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}} + 2\right)$

$$= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}}\right) + \lim_{t\to\infty}(2) = \frac{-2}{\sqrt{\infty-2}} + 2 = \frac{-2}{\infty} + 2 = 0 + 2 = 2$$

Hence the given integral is convergent.

17 **An aquarium 5 m long, 10 m wide and 3 m deep is filled with seawater of density 1030 kg/m³ to a depth of 2.5 m, then find the hydrostatic pressure and force at the bottom of the pool.**

Solution:

The hydrostatic pressure at the bottom of the pool is:

Pressure = density × gravity × depth

= 1030 kg/m³ × 9.8 m/s² × 2.5 m = 25,235 pa = 25.235 N/m^{-2}

The hydrostatic force at the bottom of the pool is:

Force = pressure × area

= 25,235 × 5 × 10 = 1,261,750 N

18 **A rod has a length of 40 cm. The rod's density changes linearly along its length from 20 g/cm to 60 g/cm. Find the center of mass of the rod.**

The equation of straight line passing through the points $\rho(0)=20,\ \rho(40)=60$ is

$\dfrac{\rho-\rho(0)}{\rho(40)-\rho(0)} = \dfrac{x-0}{40-0} \Rightarrow \dfrac{\rho-20}{60-20} = \dfrac{x}{40} \Rightarrow \rho = x+20,$ where ρ is measured in g/cm and x is measured in cm.

Calculate the mass m and the first moment M_0 of the rod.

$$m = \int_a^b \rho(x)\,dx = \int_0^{40}(x+20)\,dx = \left[\frac{x^2}{2}+20x\right]_0^{40}$$

$$= \left(\frac{40^2}{2}+20(40)\right) - \left(\frac{0^2}{2}+20(0)\right) = \frac{1600}{2}+800 = 800+800 = 1600$$

$$M_0 = \int_a^b x\rho(x)\,dx = \int_0^{40} x(x+20)\,dx = \int_0^{40}\left(x^2+20x\right)dx$$

$$= \left[\frac{x^3}{3}+\frac{\overset{10}{\cancel{20}}\,x^2}{\cancel{2}}\right]_0^{40} = \left[\left(\frac{40^3}{3}+10(40)^2\right)-\left(\frac{0}{3}+10(0)^2\right)\right] = \frac{64000}{3}+16000 \approx 37333$$

Hence, the center of mass $G(\bar{x})$ is located at the point is

$$\bar{x} = \frac{M_0}{m} = \frac{37333}{1600} \approx 23.3cm$$

19 **Find the center of mass of a semicircular plate of radius r**

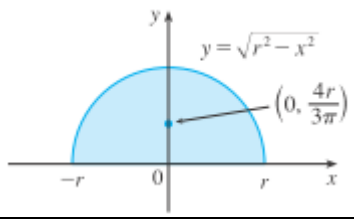
Solution:

The equation of circle with radius r is $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2 \Rightarrow y=\pm\sqrt{r^2-x^2}$

Let $f(x)=\sqrt{r^2-x^2}$ where x ranges from $-r$ to r.

The center of mass of $f(x)$ is $(\bar{x},\bar{y})$

The semicircular plate is symmetric about y- axis therefore the center of mass $\bar{y}$ lie on y- axis and $\bar{x}=0$.



$$\bar{y}=\frac{1}{A}\int_{-r}^{r}\frac{1}{2}\left[f(x)\right]^2 dx \text{ where } A=\text{area of semicircle}=\frac{1}{2}\pi r^2$$

$$\bar{y}=\frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r}\frac{1}{2}\left[\sqrt{r^2-x^2}\right]^2 dx$$

$$=\frac{\cancel{2}}{\pi r^2}\frac{1}{\cancel{2}}\int_{-r}^{r}\left(r^2-x^2\right)dx$$

$$=\frac{1}{\pi r^2}\int_{-r}^{r}\left(r^2-x^2\right)dx \qquad \text{(since } x^2 \text{ is a even function } x\text{)}$$

$$=\frac{2}{\pi r^2}\int_{0}^{r}\left(r^2-x^2\right)dx$$

$$=\frac{2}{\pi r^2}\left(xr^2-\frac{x^3}{3}\right)_0^r$$

$$=\frac{2}{\pi r^2}\left(r^2(r)-\frac{r^3}{3}\right)-\left(r^2(0)-\frac{0^3}{3}\right)$$

$$=\frac{2}{\pi r^2}\left(r^3-\frac{r^3}{3}\right)=\frac{2}{\pi r^2}\left(\frac{3r^3}{3}-\frac{r^3}{3}\right)=\frac{2}{\pi r^2}\left(\frac{2r^3}{3}\right)=\frac{4r}{3\pi}$$

$\therefore$ The center of mass $(\bar{x},\bar{y})$ is $\left(0,\frac{4r}{3\pi}\right)$

20 **Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (-1,1), (2,-1) and (3,2) respectively.**

Solution:

Let the mass be $m_1=3, m_2=4, m_3=8$ at the points

$(x_1,y_1)=(-1,1),(x_2,y_2)=(2,-1)$ and $(x_3,y_3)=(3,2)$ respectively.

Moment of the system about the y-axis is

$$M_y=\sum_{i=1}^{n}m_ix_i=m_1x_1+m_2x_2+m_3x_3 \qquad (\because n=3)$$

$$=3(-1)+4(2)+8(3)=29$$

Moment of the system about the x-axis is

$$M_x=\sum_{i=1}^{n}m_iy_i=m_1y_1+m_2y_2+m_3y_3 \qquad (\because n=3)$$

$$=3(1)+4(-1)+8(2)=15$$

The center of mass $(\bar{x},\bar{y})$ in terms of the moments is

$\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$ where $m=\sum m_i=m_1+m_2+m_3=3+4+8=15$

$\bar{x}=\frac{29}{15}$ and $\bar{y}=\frac{15}{15}=1$

$\therefore$ The center of mass is $(\bar{x},\bar{y})=\left(\frac{29}{15},1\right)$

PART – B

1 **Evaluate $\int_0^3 \left(x^3 - 6x\right) dx$ by using Riemann sum with n sub intervals. (N/D 2019)**

Solution:

Let $f(x) = x^3 - 6x$, here $a = 0, b = 3$ $\quad \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

$x_0 = 0 + \frac{(0)3}{n} = 0,\ x_1 = 0 + \frac{(1)3}{n} = \frac{3}{n},\ x_2 = 0 + \frac{(2)3}{n} = \frac{6}{n},\ \ldots\ x_i = 0 + \frac{i3}{n} = \frac{3i}{n}$ $\qquad (\because\ x_i = a + i\Delta x)$

By using right end points

$$\int_0^3 \left(x^3 - 6x\right) dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$\left[\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2;\quad \sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n^2(n+1)^2}{4}\right) - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n^2\ n^2\left(1+\frac{1}{n}\right)^2}{4}\right) - \frac{54}{n^2}\left(\frac{n\ n\left(1+\frac{1}{n}\right)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{\cancel{n^4}}\left(\frac{\cancel{n^4}\left(1+\frac{1}{n}\right)^2}{4}\right) - \frac{54}{\cancel{n^2}}\left(\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{54}{2}\left(1+\frac{1}{n}\right)\right]$$

	$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2\right] - \lim_{n\to\infty}\left[27\left(1+\frac{1}{n}\right)\right]$ $= \frac{81}{4}\left(1+\frac{1}{\infty}\right)^2 - 27\left(1+\frac{1}{\infty}\right)$ $= \frac{81}{4}(1+0)^2 - 27(1+0)$ $= \frac{81}{4} - 27$ $= 20.25 - 27$ $= -6.75$
2	**Evaluate the following integral by using Riemann sum by taking right end points as sample points. Hence verify it by using fundamental theorem of calculus** $\int_1^4 \left(x^2+2x-5\right)dx$
	Solution: Riemann Sum Method $f(x) = x^2+2x-5,\ a=1,\ b=4,$ $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n},\ x_i = a + i\Delta x \Rightarrow x_i = 1+\frac{3i}{n}$ $\int_1^4 \left(x^2+2x-5\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ $= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\frac{3}{n}$ $= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]$ $= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[1+\frac{9i^2}{n^2}+\frac{6i}{n}+2+\frac{6i}{n}-5\right]$ $= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{9i^2}{n^2}+\frac{12i}{n}-2\right]$ $= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}\right)+\sum_{i=1}^{n}\left(\frac{12i}{n}\right)-\sum_{i=1}^{n}(2)\right]$ $= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 + \frac{12}{n}\sum_{i=1}^{n} i - 2\sum_{i=1}^{n}(1)\right]$ $= \lim_{n\to\infty}\left[\frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{6}{n}\sum_{i=1}^{n} 1\right]$ $= \lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{36}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{6}{\cancel{n}}\cancel{n}\right]$ $\left[\sum_{i=1}^{n} i^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6};\quad \sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}\right]$ $= \lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{(n^2+n)(2n+1)}{6}\right)+\frac{36}{n^2}\left(\frac{n^2+n}{2}\right)-6\right]$

$$= \lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{2n^3+n^2+2n^2+n}{6}\right)+\frac{36}{n^2}\left(\frac{n^2+n}{2}\right)-6\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{6}\left(\frac{2n^3+3n^2+n}{n^3}\right)+18\left(\frac{n^2+n}{n^2}\right)-6\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{6}\frac{\cancel{n^3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{\cancel{n^3}}+18\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{\cancel{n^2}}-6\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+18\left(1+\frac{1}{n}\right)-6\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right]+\lim_{n\to\infty}\left[18\left(1+\frac{1}{n}\right)\right]-\lim_{n\to\infty}[6]$$

$$= \frac{27}{6}\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)+18\lim_{n\to\infty}\left(1+\frac{1}{n}\right)-6\lim_{n\to\infty}1$$

$$= \frac{27}{6}\left(2+\frac{3}{\infty}+\frac{1}{\infty}\right)+18\left(1+\frac{1}{\infty}\right)-6(1)$$

$$= \frac{27}{6}(2+0+0)+(18)(1+0)-6$$

$$= \frac{27}{6}\times 2+18-6$$

$$= 9+18-6$$

$$= 21$$

Verification by Fundamental theorem of calculus:

$f(x) = x^2+2x-5$

Integrate on both sides

$$\int f(x)\,dx = \int\left(x^2+2x-5\right)dx = \int x^2dx+\int 2xdx-\int 5dx = \frac{x^{2+1}}{2+1}+2\frac{x^{1+1}}{1+1}-5(x) = \frac{x^3}{3}+2\frac{x^2}{2}-5x$$

$$F(x) = \frac{x^3}{3}+x^2-5x$$

$a=1, b=4,\ \int_a^b f(x)\,dx = F(b)-F(a)$

$$= F(4)-F(1)$$

$$= \left(\frac{4^3}{3}+4^2-5(4)\right)-\left(\frac{1^3}{3}+1^2-5(1)\right)$$

$$= \left(\frac{64}{3}+16-20\right)-\left(\frac{1}{3}+1-5\right)$$

$$= \frac{64}{3}+16-20-\frac{1}{3}-1+5$$

$$= \frac{64}{3}-\frac{1}{3} = \frac{64-1}{3} = \frac{63}{3}$$

$$\int_1^4 \left(x^2+2x-5\right)dx = 21$$

3 **Evaluate $\int e^x \sin x\ dx$ by using integration by parts. (N/D 2019)**

Solution:

$u=\sin x$	$dv=e^x dx$
$\frac{du}{dx}=\frac{d}{dx}(\sin x)=\cos x$	$\int dv=\int e^x dx$
$du=\cos x dx$	$v=e^x$

Using integration by parts $\int udv = uv-\int vdu$ we get

$$\int e^x \sin x dx = (\sin x)e^x - \int e^x \cos x dx$$

$$= e^x \sin x - \int e^x \cos x dx \text{ -----}(1)$$

To find $\int e^x \cos x dx$

$u=\cos x$	$dv=e^x dx$
$\frac{du}{dx}=\frac{d}{dx}(\cos x)=-\sin x$	$\int dv=\int e^x dx$
$du=-\sin x dx$	$v=e^x$

$$\int e^x \cos x dx = e^x \cos x - \int e^x(-\sin x)dx$$

$$= e^x \cos x + \int e^x \sin x dx \text{ -------- (2)}$$

Substitute (2) in (1)

$$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$\int e^x \sin x\, dx + \int e^x \sin x\, dx = e^x(\sin x - \cos x)+c$$

$$2\int e^x \sin x\, dx = e^x(\sin x - \cos x)+c$$

$$\int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x)+c$$

4 **Evaluate $\int \frac{xe^{2x}}{(1+2x)^2}dx$ by using integration by parts.**

Solution:

Let $I=\int \frac{xe^{2x}}{(1+2x)^2}dx$

Let $f(x)=\frac{x}{(1+2x)^2}$

Split in to partial fraction

$$\frac{x}{(1+2x)^2}=\frac{A}{(1+2x)}+\frac{B}{(1+2x)^2}\text{----------}(1)$$

$$x=A(1+2x)+B$$

$x=\frac{-1}{2}$ $\qquad$ $x=0$

$\frac{-1}{2}=A(0)+B$ $\qquad$ $0=A+B$

$A=-B$

$\boxed{B=\frac{-1}{2}}$ $\qquad$ $\boxed{A=\frac{1}{2}}$

$$Eqn.(1)\Rightarrow \frac{x}{(1+2x)^2}=\frac{\frac{1}{2}}{(1+2x)}+\frac{-\frac{1}{2}}{(1+2x)^2}$$

$$I=\int\frac{xe^{2x}}{(1+2x)^2}dx=\int e^{2x}\frac{\frac{1}{2}}{(1+2x)}dx+\int e^{2x}\frac{-\frac{1}{2}}{(1+2x)^2}dx$$

$$I=\frac{1}{2}\int\frac{e^{2x}}{(1+2x)}dx-\frac{1}{2}\int\frac{e^{2x}}{(1+2x)^2}dx=I_1+I_2 \quad ----------(2)$$

Evaluation of I_1 using Integration By Parts,

$$I_1=\frac{1}{2}\int\frac{e^{2x}}{(1+2x)}dx$$

$$put\ \ u=\frac{1}{(1+2x)}\ \Rightarrow du=\frac{2dx}{-(1+2x)^2}$$

$$\int dv=\int e^{2x}dx\Rightarrow v=\frac{e^{2x}}{2}$$

$$I_1=\frac{1}{2}\int\frac{e^{2x}}{(1+2x)}dx=uv-\int v\ du=\frac{1}{2}\left[\frac{1}{(1+2x)}\left(\frac{e^{2x}}{2}\right)-\int\left(\frac{e^{2x}}{\cancel{2}}\right)\left(\frac{-\cancel{2}}{(1+2x)^2}\right)dx\right]$$

$$I_1=\frac{1}{(1+2x)}\left(\frac{e^{2x}}{4}\right)+\frac{1}{2}\int\left(\frac{e^{2x}}{(1+2x)^2}\right)dx----------(3)$$

Substitute Eqn.(3) in Eqn.(2)

$$I=I_1+I_2$$

$$=\frac{1}{2}\int\frac{e^{2x}}{(1+2x)}dx-\frac{1}{2}\int\frac{e^{2x}}{(1+2x)^2}dx$$

$$=\frac{1}{(1+2x)}\left(\frac{e^{2x}}{4}\right)+\frac{1}{2}\int\left(\frac{e^{2x}}{(1+2x)^2}\right)dx-\frac{1}{2}\int\frac{e^{2x}}{(1+2x)^2}dx$$

$$\therefore I=\frac{e^{2x}}{4(1+2x)}+c$$

5 **Prove the reduction formula $\int_0^{\pi/2}\sin^n x\,dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$. Hence by using it evaluate $\int_0^{\pi/2}\sin^7 x\,dx$**

Solution:

$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$

$u = \sin^{n-1} x$	$dv = \sin x dx$
$\frac{du}{dx} = \frac{d}{dx}\left(\sin^{n-1} x\right) = (n-1)\sin^{n-2} x \cos x$	$\int dv = \int \sin x dx$
$du = (n-1)\sin^{n-2} x \cos x dx$	$v = -\cos x$

We know that $\int \text{udv} = \text{uv} - \int \text{vdu}$ apply for $\int \sin^n x dx$

$$\int \sin^n x\, dx = \sin^{n-1} x(-\cos x) - \int(-\cos x)(n-1)\sin^{n-2} x \cos x dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x dx \qquad \left(\because \cos^2 x = 1 - \sin^2 x\right)$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\left(1 - \sin^2 x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int\left(\sin^{n-2} x - \sin^{n-2} x \sin^2 x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int\left(\sin^{n-2} x - \sin^{n-2+2} x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int\left(\sin^{n-2} x - \sin^n x\right) dx$$

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$$

$$\int \sin^n x dx + (n-1)\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

$$\int \sin^n x dx + n\int \sin^n x dx - \int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

$$n\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx$$

$$\int \sin^n x dx = \frac{-1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x dx$$

Now,

$$\int_0^{\pi/2} \sin^n x dx = \left[\frac{-1}{n}\cos x \sin^{n-1} x\right]_0^{\pi/2} + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \left[\frac{-1}{n}\cos\left(\frac{\pi}{2}\right)\sin^{n-1}\left(\frac{\pi}{2}\right)\right] - \left[\frac{-1}{n}\cos(0)\sin^{n-1}(0)\right] + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$= \left[\frac{-1}{n}(0)\sin^{n-1}\left(\frac{\pi}{2}\right)\right] + \left[\frac{1}{n}(1)(0)\right] + \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)}{n}\int_0^{\pi/2} \sin^{n-2} x dx \qquad \left(\because \cos\frac{\pi}{2} = 0,\ \sin 0 = 0\right)$$

Put $n = 7$

$$\int_0^{\pi/2} \sin^7 x dx = \frac{(7-1)}{7}\int_0^{\pi/2} \sin^{7-2} x dx$$

$$= \frac{6}{7}\int_0^{\pi/2} \sin^5 x dx$$

$$= \frac{6}{7}\frac{5-1}{5}\int_0^{\pi/2} \sin^{5-2} x dx$$

$$= \frac{6}{7}\frac{4}{5}\int_0^{\pi/2} \sin^3 x dx$$

$$= \frac{6}{7}\frac{4}{5}\frac{3-1}{3}\int_0^{\pi/2} \sin^{3-2} x dx$$

$$= \frac{6}{7}\frac{4}{5}\frac{2}{3}\int_0^{\pi/2} \sin x dx$$

$$= \frac{16}{35}[-\cos x]_0^{\pi/2}$$

$$= \frac{16}{35}\left[-\cos\left(\frac{\pi}{2}\right)+\cos(0)\right] \qquad \left(\because \cos\frac{\pi}{2}=0,\ \cos 0 = 1\right)$$

$$= \frac{16}{35}[-0+1]$$

$$\int_0^{\pi/2} \sin^7 x\, dx = \frac{16}{35}$$

6 **Prove that $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$ where m and n are positive integrals.**

(N/D 2020)

Solution: Let $I = \int_{-\pi}^{\pi} \sin mx \sin nx dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] dx \qquad \left[\because \sin A \sin B = \frac{\cos(A-B)-\cos(A+B)}{2}\right]$$

$$I = \frac{1}{2}\left[\int_{-\pi}^{\pi}[\cos(m-n)x]dx - \int_{-\pi}^{\pi}[\cos(m+n)x]dx\right] \text{----------(1)}$$

In Eqn.(1) *when* $m \neq n$

$$I = \frac{1}{2}\left[\int_{-\pi}^{\pi}[\cos(m-n)x]dx - \int_{-\pi}^{\pi}[\cos(m+n)x]dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(m-n)x}{m-n}\right)_{-\pi}^{\pi} - \left(\frac{\sin(m+n)x}{m+n}\right)_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m-n)(-\pi)}{m-n}\right) - \left(\frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m+n)(-\pi)}{m+n}\right)\right]$$

$$(\because \sin(-\theta) = -\sin\theta)$$

$$= \frac{1}{2}\left[\left(\frac{\sin(m-n)\pi}{m-n} + \frac{\sin(m-n)(\pi)}{m-n}\right) - \left(\frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m+n)(\pi)}{m+n}\right)\right]$$

$$(\because \sin\pi = 0)$$

$$= \frac{1}{2}\left[\left(\frac{0}{m-n} + \frac{0}{m-n}\right) - \left(\frac{0}{m+n} + \frac{0}{m+n}\right)\right]$$

$$= \frac{1}{2}[(0+0)-(0+0)]$$

$$I = 0$$

In Eqn.(1) *when* $m=n$

$$I=\frac{1}{2}\left[\int_{-\pi}^{\pi}\left[\cos(n-n)x\right]dx-\int_{-\pi}^{\pi}\left[\cos(n+n)x\right]dx\right]$$

$$=\frac{1}{2}\left[\int_{-\pi}^{\pi}\left[\cos(0)x\right]dx-\int_{-\pi}^{\pi}\left[\cos(2n)x\right]dx\right]$$

$$=\frac{1}{2}\left[\int_{-\pi}^{\pi}(1)dx-\int_{-\pi}^{\pi}\left[\cos(2nx)\right]dx\right]$$

$$=\frac{1}{2}\left[\left[x\right]_{-\pi}^{\pi}-\left[\frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}\right]$$

$$=\frac{1}{2}\left[\left[\pi-(-\pi)\right]-\left[\frac{\sin 2n\pi}{2n}-\frac{\sin 2n(-\pi)}{2n}\right]\right]$$

$$=\frac{1}{2}\left[\left[\pi+\pi\right]-\left[\frac{\sin 2n\pi}{2n}+\frac{\sin 2n\pi}{2n}\right]\right] \qquad (\because \sin 2n\pi=0)$$

$$=\frac{1}{2}\left[\left[2\pi\right]-\left[\frac{0}{2n}+\frac{0}{2n}\right]\right]$$

$$=\frac{1}{2}\left[2\pi-0\right]$$

$$=\frac{1}{\cancel{2}}\left[\cancel{2}\pi\right]$$

$$I=\pi$$

$$\therefore \int_{-\pi}^{\pi}\sin mx\sin nx dx=\begin{cases}0 & m\neq n\\ \pi & m=n\end{cases}$$

7 **Evaluate** $\displaystyle\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$ **(A/M 2019)**

Solution:

Let $t=\cos x$	when $x=0\Rightarrow t=\cos(0)=1$
$\frac{dt}{dx}=\frac{d}{dx}(\cos x)=-\sin x$	when $x=\frac{\pi}{2}\Rightarrow t=\cos\left(\frac{\pi}{2}\right)=0$
$dt=-\sin x dx$	
$\sin x dx=-dt$	

$$\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx=\int_1^0\frac{t(-dt)}{t^2+3t+2}=-\int_1^0\frac{t\,dt}{t^2+3t+2}=\int_0^1\frac{t\,dt}{t^2+3t+2} \quad \text{-------- (1)}$$

$$\left[\because \int_a^b f(x)dx=-\int_b^a f(x)dx\right]$$

Consider

$$\frac{t}{t^2+3t+2}=\frac{t}{(t+1)(t+2)}=\frac{A}{t+1}+\frac{B}{t+2} \quad \text{---------- (2)}$$

$$\frac{t}{\cancel{(t+1)(t+2)}}=\frac{A(t+2)+B(t+1)}{\cancel{(t+1)(t+2)}}$$

$t = A(t+2) + B(t+1)$

put $t=-2 \Rightarrow -2 = A(-2+2)+B(-2+1) \Rightarrow -2 = A(0)+B(-1) \Rightarrow -2 = 0-B \Rightarrow \boxed{B=2}$

put $t=-1 \Rightarrow -1 = A(-1+2)+B(-1+1) \Rightarrow -1 = A(1)+B(0) \Rightarrow -1 = A+0 \Rightarrow \boxed{A=-1}$

substitute A and B values in (2)

$$\frac{t}{(t+1)(t+2)} = \frac{-1}{t+1} + \frac{2}{t+2}$$

$$\therefore (1) \Rightarrow \int_0^1 \frac{t}{(t+1)(t+2)}dt = \int_0^1\left(\frac{-1}{t+1}+\frac{2}{t+2}\right)dt = \int_0^1\left(\frac{2}{t+2}\right)dt - \int_0^1\left(\frac{1}{t+1}\right)dt$$

$$= 2\int_0^1\left(\frac{1}{t+2}\right)dt - \int_0^1\left(\frac{1}{t+1}\right)dt$$

$$= 2\left[\log(t+2)\right]_0^1 - \left[\log(t+1)\right]_0^1 \qquad \left[\because \int \frac{1}{t}dt = \log t\right]$$

$$= 2\left[\log(1+2)-\log(0+2)\right]-\left[\log(1+1)-\log(0+1)\right]$$

$$= 2\left[\log 3 - \log 2\right] - \left[\log 2 - \log 1\right]$$

$$= 2\log 3 - 2\log 2 - \log 2 + 0 \qquad \left[\because \log 1 = 0\right]$$

$$= 2\log 3 - 3\log 2$$

$$= \log 3^2 - \log 2^3 \qquad \left(\because b\log a = \log a^b\right)$$

$$= \log 9 - \log 8 \qquad \left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$$

$$= \log\left(\frac{9}{8}\right)$$

8 **Evaluate** $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Solution:

Let $I = \int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx$ -------(1)

$$= \int_0^{\pi/2}\left(\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\right)dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2}\left(\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx \text{-------}(2) \qquad \left[\because \sin\left(\frac{\pi}{2}-x\right) = \cos x,\ \cos\left(\frac{\pi}{2}-x\right) = \sin x\right]$$

Adding (1) and (2), we get

$$I + I = \int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx + \int_0^{\pi/2}\left(\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx$$

$$= \int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right)dx$$

$$2I = \int_0^{\pi/2} \left(\frac{\cancel{\sqrt{\sin x} + \sqrt{\cos x}}}{\cancel{\sqrt{\sin x} + \sqrt{\cos x}}} \right) dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx = \frac{\pi}{4}$$

9 **Evaluate** $\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$ **(N/D 2018)**

Solution: Given $\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}}$

Let $x = \frac{1}{3}\sec\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{3}\sec\theta\tan\theta \Rightarrow dx = \frac{1}{3}\sec\theta\tan\theta \;\; d\theta$

When $x = \frac{\sqrt{2}}{3} \Rightarrow \frac{\sqrt{2}}{3} = \frac{1}{3}\sec\theta \Rightarrow \sqrt{2} = \sec\theta \Rightarrow \theta = \sec^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

When $x = \frac{2}{3} \Rightarrow \frac{2}{3} = \frac{1}{3}\sec\theta \Rightarrow 2 = \sec\theta \Rightarrow \theta = \sec^{-1}(2) = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

$$\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{3}\sec\theta\tan\theta}{\left(\frac{1}{3}\sec\theta\right)^5\sqrt{9\left(\frac{1}{3}\sec\theta\right)^2-1}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{3}\sec\theta\tan\theta}{\left(\frac{1}{3^5}\sec^5\theta\right)\sqrt{\cancel{9}\left(\frac{1}{\cancel{9}}\sec^2\theta\right)-1}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3^5\sec\theta\tan\theta}{\left(3\sec^5\theta\right)\sqrt{\sec^2\theta-1}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3^{5-1}\sec\theta\tan\theta}{\left(\sec^5\theta\right)\sqrt{\tan^2\theta}} d\theta \qquad \left(\because \; \tan^2\theta = \sec^2\theta - 1\right)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3^4\sec\theta\tan\theta}{\sec^5\theta\tan\theta} d\theta$$

$$= 3^4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\sec^4\theta} = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4\theta d\theta = 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\cos^2\theta\right)^2 d\theta$$

$$=81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \qquad \left(\because \ \cos^2\theta=\frac{1+\cos 2\theta}{2}\right)$$

$$=81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{(1+\cos 2\theta)^2}{4}d\theta$$

$$=\frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$=\frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(1+2\cos 2\theta+\left(\frac{1+\cos 4\theta}{2}\right)\right)d\theta \qquad \left(\because \ \cos^2 2\theta=\frac{1+\cos 4\theta}{2}\right)$$

$$=\frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{2+4\cos 2\theta+1+\cos 4\theta}{2}\right)d\theta$$

$$=\frac{81}{8}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(3+4\cos 2\theta+\cos 4\theta\right)d\theta$$

$$=\frac{81}{8}\left[\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(3)\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(4\cos 2\theta)\,d\theta+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(\cos 4\theta)\,d\theta\right]$$

$$=\frac{81}{8}\left[\left[3\theta\right]_{\pi/4}^{\pi/3}+4\left[\frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/3}+\left[\frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/3}\right]$$

$$=\frac{81}{8}\left[\left[\frac{3\pi}{3}-\frac{3\pi}{4}\right]+2\left[\sin\left(\frac{2\pi}{3}\right)-\sin\left(\frac{2\pi}{4}\right)\right]+\frac{1}{4}\left[\sin\left(\frac{4\pi}{3}\right)-\sin\left(\frac{4\pi}{4}\right)\right]\right]$$

$$=\frac{81}{8}\left[\left[\pi-\frac{3\pi}{4}\right]+2\left[\sin\left(\pi-\frac{\pi}{3}\right)-\sin\left(\frac{\pi}{2}\right)\right]+\frac{1}{4}\left[\sin\left(\pi+\frac{\pi}{3}\right)-\sin(\pi)\right]\right]$$

$$\left(\because \sin\pi=0 \ \& \ \sin\frac{\pi}{2}=1\right)$$

$$=\frac{81}{8}\left[\left[\frac{4\pi-3\pi}{4}\right]+2\left[\sin\left(\frac{\pi}{3}\right)-1\right]+\frac{1}{4}\left[-\sin\left(\frac{\pi}{3}\right)-0\right]\right] \qquad \left(\because \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}\right)$$

$$=\frac{81}{8}\left[\frac{\pi}{4}+2\left(\frac{\sqrt{3}}{2}\right)-2-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{81}{8}\left[\frac{\pi}{4}+\left(2-\frac{1}{4}\right)\frac{\sqrt{3}}{2}-2\right]=\frac{81}{8}\left[\frac{\pi}{4}+\left(\frac{8-1}{4}\right)\frac{\sqrt{3}}{2}-2\right]$$

$$\boxed{\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}}\frac{dx}{x^5\sqrt{9x^2-1}}=\frac{81}{8}\left(\frac{\pi}{4}+\frac{7\sqrt{3}}{8}-2\right)}$$

10	**Evaluate the integral 1) $\int\left(x^3\sqrt{x^2+1}\right)dx$ and 2) $\int_0^1 \frac{1}{\left(1+\sqrt{x}\right)^4}dx$** **(N/D 2020)**

Solution:

(i) Given $\int\left(x^3\sqrt{x^2+1}\right)dx$

put $u=x^2+1 \Rightarrow x^2=u-1 \Rightarrow 2x\frac{dx}{du}=1-0=1 \Rightarrow 2xdx=du \Rightarrow xdx=\frac{du}{2}$

$$\int\left(x^3\sqrt{x^2+1}\right)dx=\int\left(x^2\sqrt{x^2+1}\right)xdx$$

$$=\int(u-1)\sqrt{u}\,\frac{du}{2}$$

$$=\int(u-1)u^{\frac{1}{2}}\,\frac{du}{2}$$

$$=\frac{1}{2}\int\left(u^1u^{\frac{1}{2}}-u^{\frac{1}{2}}\right)du$$

$$=\frac{1}{2}\int\left(u^{1+\frac{1}{2}}-u^{\frac{1}{2}}\right)du$$

$$=\frac{1}{2}\int\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$$

$$=\frac{1}{2}\left[\int\left(u^{\frac{3}{2}}\right)du-\int\left(u^{\frac{1}{2}}\right)du\right] \qquad \left(\int x^n dx=\frac{x^{n+1}}{n+1}+c\right)$$

$$=\frac{1}{2}\left[\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1}-\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$$

$$=\frac{1}{2}\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]=\frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]+c \qquad \text{replace } u=x^2+1$$

$$=\frac{1}{2}\left[\frac{2}{5}\left(x^2+1\right)^{\frac{5}{2}}-\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}\right]+c$$

(ii) Given $\int_0^1 \frac{1}{\left(1+\sqrt{x}\right)^4}dx$

put $u=1+\sqrt{x} \quad \Rightarrow \sqrt{x}=u-1 \quad \Rightarrow u=1+\sqrt{x}$

$$\frac{du}{dx}=0+\frac{1}{2\sqrt{x}} \qquad \left(\because \frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{\frac{-1}{2}}=\frac{1}{2x^{\frac{1}{2}}}=\frac{1}{2\sqrt{x}}\right)$$

$\Rightarrow 2\sqrt{x}du=dx \Rightarrow 2(u-1)du=dx$

when $x=0 \Rightarrow u=1+\sqrt{x}=1+\sqrt{0}=1$

when $x=1 \Rightarrow u=1+\sqrt{x}=1+\sqrt{1}=2$

$$\int_0^1 \frac{1}{\left(1+\sqrt{x}\right)^4}dx = \int_1^2 \frac{1}{u^4}2\left(u-1\right)du$$

$$= 2\int_1^2 \left(\frac{u-1}{u^4}\right)du$$

$$= 2\int_1^2 \left(\frac{u}{u^4}-\frac{1}{u^4}\right)du$$

$$= 2\int_1^2 \left(\frac{1}{u^3}-\frac{1}{u^4}\right)du$$

$$= 2\int_1^2 \left(u^{-3}-u^{-4}\right)du$$

$$= 2\left[\int_1^2 \left(u^{-3}\right)du - \int_1^2 \left(u^{-4}\right)du\right] \qquad \left(\int x^n dx = \frac{x^{n+1}}{n+1}+c\right)$$

$$= 2\left[\left(\frac{u^{-3+1}}{-3+1}\right)_1^2 - \left(\frac{u^{-4+1}}{-4+1}\right)_1^2\right]$$

$$= 2\left[\left(\frac{u^{-2}}{-2}\right)_1^2 - \left(\frac{u^{-3}}{-3}\right)_1^2\right]$$

$$= 2\left[\frac{-1}{2}\left(\frac{1}{u^2}\right)_1^2 + \frac{1}{3}\left(\frac{1}{u^3}\right)_1^2\right]$$

$$= 2\left[\frac{-1}{2}\left(\frac{1}{2^2}-\frac{1}{1^2}\right) + \frac{1}{3}\left(\frac{1}{2^3}-\frac{1}{1^3}\right)\right]$$

$$= 2\left[\frac{-1}{2}\left(\frac{1}{4}-1\right) + \frac{1}{3}\left(\frac{1}{8}-1\right)\right]$$

$$= 2\left[\frac{-1}{2}\left(\frac{1-4}{4}\right) + \frac{1}{3}\left(\frac{1-8}{8}\right)\right]$$

$$= 2\left[\frac{-1}{2}\left(\frac{-3}{4}\right) + \frac{1}{3}\left(\frac{-7}{8}\right)\right]$$

$$= 2\left[\frac{3}{8}-\frac{7}{24}\right]$$

$$= 2\left[\frac{3(3)-7}{24}\right]$$

$$= 2\left[\frac{9-7}{24}\right]$$

$$= 2\left[\frac{2}{24}\right] = \frac{4}{24} = \frac{1}{6}$$

11 **Evaluate the integral** $\int \frac{x^2-2x-1}{\left(x-1\right)^2\left(x^2+1\right)}dx$ **(N/D 2020)**

Solution:

Let $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{(x^2+1)}$ ----(1)

$$\frac{x^2-2x-1}{\cancel{(x-1)^2(x^2+1)}}=\frac{A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2}{\cancel{(x-1)^2(x^2+1)}}$$

$$x^2-2x-1=A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-1)^2$$

$$=A(x^3+x-x^2-1)+B(x^2+1)+(Cx+D)(x^2-2x+1)$$

$$=Ax^3+Ax-Ax^2-A+Bx^2+B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

$$0x^3+x^2-2x-1=(A+C)x^3+(-A+B-2C+D)x^2+(A+C-2D)x-A+B+D$$

Equating the coefficient of x^3 on both sides

$A+C=0\Rightarrow A=-C$ ...(2)

Equating the coefficient of x^2 on both sides

$-A+B-2C+D=1$ ...(3)

Equating the coefficient of x on both sides

$A+C-2D=-2$ ...(4)

Equating the constant on both sides

$-A+B+D=-1$ ...(5)

Substitute (2) in (4)

$-C+C-2D=-2\Rightarrow -2D=-2\Rightarrow D=\dfrac{-2}{-2}\Rightarrow \boxed{D=1}$

$(3)\Rightarrow -A+B-2C+D=1$

$(5)\Rightarrow -A+B+D \quad =-1$

$\quad\quad + \quad - \quad - \quad\quad\quad\quad +$

$-2C=2$

$\Rightarrow C=\dfrac{2}{-2}=-1 \Rightarrow \boxed{C=-1}$

Substitute $C=-1$ in (2)

$A=-(-1)\Rightarrow \boxed{A=1}$

Substitute $A=1, D=1$ in (5)

$-(1)+B+1=-1\Rightarrow \boxed{B=-1}$

Sub in (1) $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)}=\dfrac{1}{x-1}+\dfrac{-1}{(x-1)^2}+\dfrac{-1x+1}{x^2+1}$

Let $I=\displaystyle\int\frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx=\int\left(\frac{1}{x-1}+\frac{-1}{(x-1)^2}+\frac{-1x+1}{x^2+1}\right)dx$

$$=\int\left(\frac{1}{x-1}\right)dx-\int\frac{1}{(x-1)^2}dx+\int\left(\frac{-1x+1}{x^2+1}\right)dx$$

$$= \int\left(\frac{1}{x-1}\right)dx - \int\frac{1}{(x-1)^2}dx + \int\left(\frac{1}{x^2+1}\right)dx - \int\left(\frac{x}{x^2+1}\right)dx$$

Multiply and divided by 2 to the last term

$$I = \int\left(\frac{1}{x-1}\right)dx - \int(x-1)^{-2}dx + \int\left(\frac{1}{x^2+1}\right)dx - \frac{1}{2}\int\left(\frac{2x}{x^2+1}\right)dx$$

$$= \log(x-1) - \frac{(x-1)^{-2+1}}{-2+1} + \tan^{-1}x - \frac{1}{2}\log\left(x^2+1\right) + c \qquad \left(\because \int\frac{f'(x)}{f(x)}dx = \log f(x) + C\right)$$

$$= \log(x-1) - \frac{(x-1)^{-1}}{-1} + \tan^{-1}x - \frac{1}{2}\log\left(x^2+1\right) + c$$

$$= \log(x-1) + \frac{1}{x-1} + \tan^{-1}x - \frac{1}{2}\log\left(x^2+1\right) + c$$

12 **Evaluate $\int\frac{2x+1}{x^3+2x^2-x-2}dx$ by partial fraction method.**

Solution:

Given $\int\frac{2x+1}{x^3+2x^2-x-2}dx$

Consider $\frac{2x+1}{x^3+2x^2-x-2} = \frac{2x+1}{x^2(x+2)-1(x+2)} = \frac{2x+1}{(x^2-1)(x+2)} = \frac{2x+1}{(x-1)(x+1)(x+2)}$

$$\left[\because a^2 - b^2 = (a+b)(a-b)\right]$$

$$\frac{2x+1}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} \quad \ldots\ldots\ldots\ldots (1)$$

$$\frac{2x+1}{\cancel{(x-1)(x+1)(x+2)}} = \frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{\cancel{(x-1)(x+1)(x+2)}}$$

$$2x+1 = A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1) \quad \ldots\ldots\ldots\ldots (2)$$

Put $x=1$ in (2)

$$2(1)+1 = A(1+1)(1+2)+B(1-1)(1+2)+C(1-1)(1+1)$$

$$2+1 = A(2)(3)+B(0)+C(0)$$

$$3 = 6A \Rightarrow A = \frac{3}{6} \Rightarrow \boxed{A = \frac{1}{2}}$$

Put $x=-1$ in (2)

$$2(-1)+1 = A(-1+1)(-1+2)+B(-1-1)(-1+2)+C(-1-1)(-1+1)$$

$$-2+1 = A(0)+B(-2)+C(0)$$

$$-1 = -2B \Rightarrow B = \frac{-1}{-2} \Rightarrow \boxed{B = \frac{1}{2}}$$

Put $x=-2$ in (2)

$$2(-2)+1 = A(-2+1)(-2+2)+B(-2-1)(-2+2)+C(-2-1)(-2+1)$$

$$-4+1 = A(0)+B(0)+C(-3)(-1)$$

$$-3 = 3C \Rightarrow C = \frac{-3}{3} \Rightarrow \boxed{C = -1}$$

Substitue $A, B \& C$ values in (1)

$$\frac{2x+1}{(x-1)(x+1)(x+2)} = \frac{1/2}{x-1} + \frac{1/2}{x+1} + \frac{-1}{x+2}$$

$$\frac{2x+1}{(x-1)(x+1)(x+2)} = \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x+2}$$

Taking integral on both sides,

$$\int \frac{2x+1}{(x-1)(x+1)(x+2)}dx = \int\left[\frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) - \frac{1}{x+2}\right]dx$$

$$= \int \frac{1}{2}\left(\frac{1}{x-1}\right)dx + \int \frac{1}{2}\left(\frac{1}{x+1}\right)dx - \int \frac{1}{x+2}dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx \qquad \left(\int \frac{f'(x)}{f(x)}dx = \log f(x)\right)$$

$$= \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) - \log(x+2) + c \qquad \left[b\log a = \log a^b\right]$$

$$= \log(x-1)^{1/2} + \log(x+1)^{1/2} - \log(x+2) + c \qquad \left[\log a + \log b = \log(ab)\right]$$

$$= \log\left((x-1)^{1/2}(x+1)^{1/2}\right) - \log(x+2) + c \qquad \left[\log a - \log b = \log\left(\frac{a}{b}\right)\right]$$

$$= \log\left(\frac{(x-1)^{1/2}(x+1)^{1/2}}{(x+2)}\right) + c \qquad \left[a^m\ b^m = (ab)^m\right]$$

$$= \log\left(\frac{\left[(x-1)(x+1)\right]^{1/2}}{(x+2)}\right) + c \qquad \left[(a+b)(a-b) = a^2 - b^2\right]$$

$$= \log\left(\frac{(x^2-1)^{1/2}}{(x+2)}\right) + c$$

$$\therefore \int \frac{2x+1}{(x-1)(x+1)(x+2)}dx = \log\left(\frac{\sqrt{x^2-1}}{(x+2)}\right) + c$$

13 **Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx$** **(A/M 2019)**

Solution:

Given $\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx$ is an integration of irrational function.

$$\text{Numerator} = A\frac{d}{dx}(\text{Denominator}) + B$$

$$2x+5 = A\frac{d}{dx}(x^2-2x+10) + B$$

$$2x+5 = A\left(\frac{d}{dx}(x^2) - \frac{d}{dx}(2x) + \frac{d}{dx}(10)\right) + B$$

$$2x+5 = A(2x-2+0) + B$$

$$2x+5 = A(2x-2) + B$$

$$2x+5 = 2Ax - 2A + B \text{ ..... (1)}$$

Equate the coefficient of x

$2 = 2A \Rightarrow A = \frac{2}{2} \Rightarrow \boxed{A=1}$

Equate the constant term

$5 = -2A + B$ substitute $A = 1$ we get

$5 = -2(1) + B \Rightarrow 5 = -2 + B \Rightarrow 5 + 2 = B \Rightarrow \boxed{B=7}$

Substitute the values of A & B in (1)

$2x+5 = 1(2x-2)+7$

$2x+5 = (2x-2)+7$

$$\int \frac{2x+5}{\sqrt{x^2-2x+10}}dx = \int\left(\frac{(2x-2)+7}{\sqrt{x^2-2x+10}}\right)dx$$

$$= \int \frac{2x-2}{\sqrt{x^2-2x+10}}dx + \int \frac{7}{\sqrt{x^2-2x+10}}dx \text{ ---------- (2)}$$

$Let\ I_1 = \int \frac{2x-2}{\sqrt{x^2-2x+10}}dx$ and $I_2 = \int \frac{7}{\sqrt{x^2-2x+10}}dx$

Consider $I_1 = \int \frac{2x-2}{\sqrt{x^2-2x+10}}dx$

Let $t^2 = x^2-2x+10 \Rightarrow t = \sqrt{x^2-2x+10}$

$$2t\frac{dt}{dx} = \frac{d}{dx}\left(x^2-2x+10\right) = 2x-2 \Rightarrow 2tdt = (2x-2)dx$$

$$I_1 = \int \frac{2t}{\sqrt{t^2}}dt = 2\int \frac{\cancel{t}}{\cancel{t}}dt = 2\int dt = 2t = 2\sqrt{x^2-2x+10}$$

Consider $I_2 = \int \frac{7}{\sqrt{x^2-2x+10}}dx$

WKT, $x^2-2ax+a^2 = (x-a)^2$

$\Rightarrow x^2-2ax = (x-a)^2 - a^2$

$\therefore x^2-2x = (x-1)^2 - 1^2 \qquad (\because a=1)$

$$I_2 = \int \frac{7}{\sqrt{x^2-2x+10}}dx = \int \frac{7}{\sqrt{(x-1)^2-1+10}}dx$$

$$= \int \frac{7}{\sqrt{(x-1)^2+9}}dx \qquad \left[\because \int \frac{1}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 7\int \frac{1}{\sqrt{(x-1)^2+3^2}}dx = 7\sinh^{-1}\left(\frac{x-1}{3}\right)$$

Substitute (I_1) & (I_2) in (2) we get

$$\therefore \int \frac{2x+5}{\sqrt{x^2-2x+10}}dx = 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right) + c$$

14	**Evaluate $\int \sqrt{a^2-x^2}dx$ by using substitution rule.** **(N/D 2019)**

Solution: Consider $I=\int\sqrt{a^2-x^2}\,dx$

Take $x=a\sin\theta \Rightarrow \frac{dx}{d\theta}=\frac{d}{d\theta}(a\sin\theta)=a\cos\theta \Rightarrow dx=a\cos\theta d\theta$

$$\int\sqrt{a^2-x^2}\,dx=\int\left(\sqrt{a^2-a^2\sin^2\theta}\right)a\cos\theta\,d\theta$$

$$=a\int\left(\sqrt{a^2\left(1-\sin^2\theta\right)}\right)\cos\theta\,d\theta$$

$$=a\int\left(a\sqrt{1-\sin^2\theta}\right)\cos\theta\,d\theta \qquad \left(\cos\theta=\sqrt{1-\sin^2\theta}\right)$$

$$=a^2\int(\cos\theta)\cos\theta\,d\theta$$

$$=a^2\int\cos^2\theta\,d\theta \qquad \left(\cos^2\theta=\frac{1+\cos 2\theta}{2}\right)$$

$$=a^2\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta$$

$$=\frac{a^2}{2}\left(\int d\theta+\int\cos 2\theta\,d\theta\right)$$

$$=\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right) \qquad (\sin 2\theta=2\sin\theta\cos\theta)$$

$$=\frac{a^2}{2}\left(\theta+\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right)$$

$$=\frac{a^2}{2}(\theta+\sin\theta\cos\theta)$$

$$\boxed{\begin{aligned}&x=a\sin\theta\Rightarrow\sin\theta=\frac{x}{a}\Rightarrow\theta=\sin^{-1}\left(\frac{x}{a}\right)\\&\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-\left(\frac{x}{a}\right)^2}=\sqrt{\frac{a^2-x^2}{a^2}}=\frac{1}{a}\sqrt{a^2-x^2}\end{aligned}}$$

$$=\frac{a^2}{2}(\theta)+\frac{a^2}{2}(\sin\theta\cos\theta)$$

$$=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{\cancel{a^2}}{2}\left(\frac{x}{\cancel{a}}\right)\left(\frac{\sqrt{a^2-x^2}}{\cancel{a}}\right)$$

$$\therefore\ \int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c$$

15 **Find the values of p for which the integral $\int_0^1 x^p\ln(x)\,dx$ is converges and evaluate the integral for those values of p** **(N/D 2020)**

Solution:

Given $\int_0^1 x^p\ln x\,dx=\int_0^1 x^p\log x\,dx=\lim_{t\to 0}\int_t^1 x^p\log x\,dx \qquad (\because \ln x=\log x)$

Case (i) when $p=-1$

$$\lim_{t\to 0}\int_t^1 x^p \log x dx = \lim_{t\to 0}\int_t^1 x^{-1}\log x dx$$

$$= \lim_{t\to 0}\int_t^1 \frac{\log x}{x} dx$$

$$= \lim_{t\to 0}\int_t^1 \log x d(\log x)$$

$$= \lim_{t\to 0}\left[\frac{(\log x)^2}{2}\right]_t^1 \qquad (\log 1 = 0)$$

$$= \lim_{t\to 0}\left[\frac{(\log 1)^2}{2} - \frac{(\log t)^2}{2}\right]$$

$$= \lim_{t\to 0}\left[\frac{(0)^2}{2} - \frac{(\log t)^2}{2}\right]$$

$$= \lim_{t\to 0}\left[-\frac{(\log t)^2}{2}\right] = -\frac{(\log 0)^2}{2} = -\frac{\infty}{2} = -\infty \qquad (\because \log 0 = \infty)$$

Case (ii) when $p \neq -1$

$u = \log x$	$dv = x^p dx$
$\frac{du}{dx} = \frac{1}{x}$	$\int dv = \int x^p dx$
$du = \frac{dx}{x}$	$v = \frac{x^{p+1}}{p+1}$

$$\int x^p \log(x) dx = \log x\left(\frac{x^{p+1}}{p+1}\right) - \int\left(\frac{x^{p+1}}{p+1}\right)\frac{dx}{x} \qquad \left(\because \int u dv = uv - \int v du\right)$$

$$= \frac{x^{p+1}}{p+1}\log x - \int\left(\frac{x^{p+1}x^{-1}}{p+1}\right)dx$$

$$= \frac{x^{p+1}}{p+1}\log x - \int\left(\frac{x^{p+1-1}}{p+1}\right)dx$$

$$= \frac{x^{p+1}}{p+1}\log x - \left(\frac{1}{p+1}\right)\int x^p dx$$

$$= \frac{x^{p+1}}{p+1}\log x - \left(\frac{1}{p+1}\right)\left(\frac{x^{p+1}}{p+1}\right)$$

$$= \frac{x^{p+1}}{p+1}\log x - \frac{x^{p+1}}{(p+1)^2}$$

$$\lim_{t\to 0}\int_t^1 x^p \log(x) dx = \lim_{t\to 0}\left(\frac{x^{p+1}}{p+1}\log x - \frac{x^{p+1}}{(p+1)^2}\right)_t^1$$

$$= \lim_{t\to 0}\left[\left(\frac{(1)^{p+1}}{p+1}\log(1) - \frac{(1)^{p+1}}{(p+1)^2}\right) - \left(\frac{(t)^{p+1}}{p+1}\log(t) - \frac{(t)^{p+1}}{(p+1)^2}\right)\right]$$

$$=\lim_{t\to 0}\left(\frac{1}{p+1}(0)-\frac{1}{(p+1)^2}-\frac{(t)^{p+1}}{p+1}\log(t)+\frac{(t)^{p+1}}{(p+1)^2}\right)$$

$$=\lim_{t\to 0}\left(-\frac{1}{(p+1)^2}-\frac{(t)^{p+1}}{p+1}\log(t)+\frac{(t)^{p+1}}{(p+1)^2}\right)$$

When $p<-1$ then $p+1<0$ and $t^{p+1}\to\infty$ as $t\to 0$

$$\lim_{t\to 0}\int_t^1 x^p\log(x)\,dx=-\frac{1}{(p+1)^2}-\frac{\infty}{p+1}\log(0)+\frac{\infty}{(p+1)^2}=-\infty$$

Hence the integral is divergent

When $p>-1$ then $p+1>0$ and $t^{p+1}\to 0$ as $t\to 0$

$$\lim_{t\to 0}\int_t^1 x^p\log(x)\,dx=-\frac{1}{(p+1)^2}-\frac{0}{p+1}\log(t)+\frac{0}{(p+1)^2}=-\frac{1}{(p+1)^2}-0+0=\frac{-1}{(p+1)^2}$$

Hence the integral converges.

$\therefore \int_0^1 x^p\ln(x)\,dx$ is converges to $\frac{-1}{(p+1)^2}$ if $p>-1$ and divergent if $p\le -1$.

16 **Evaluate $\int_{-\infty}^{\infty} x^3e^{-x^4}dx$ and hence discuss their convergence.**

Solution:

Let $I=\int_{-\infty}^{\infty} x^3e^{-x^4}dx$

$$=\int_{-\infty}^{0} x^3e^{-x^4}dx+\int_0^{\infty} x^3e^{-x^4}dx$$

$$=\lim_{t\to-\infty}\int_t^0 x^3e^{-x^4}dx+\lim_{t\to\infty}\int_0^t x^3e^{-x^4}dx$$

Let $I_1=\lim_{t\to-\infty}\int_t^0 x^3e^{-x^4}dx$ and $I_2=\lim_{t\to\infty}\int_0^t x^3e^{-x^4}dx$

Take $u=x^4$

$$\Rightarrow \frac{du}{dx}=\frac{d}{dx}\left(x^4\right)=4x^{4-1}=4x^3\Rightarrow du=4x^3dx\Rightarrow \frac{du}{4}=x^3dx$$

I_1	$u=x^4$
$x=0$	$u=0$
$x=t$	$u=t^4$

I_2	$u=x^4$
$x=t$	$u=t^4$
$x=0$	$u=0$

$$=\lim_{t\to-\infty}\int_{t^4}^{0} e^{-u}\frac{du}{4}+\lim_{t\to\infty}\int_0^{t^4} e^{-u}\frac{du}{4}$$

$$=I_1+I_2 \qquad \text{---------- (1)}$$

Now $I_1=\lim_{t\to-\infty}\int_{t^4}^{0} e^{-u}\frac{du}{4}=\frac{1}{4}\lim_{t\to-\infty}\int_{t^4}^{0} e^{-u}du=\frac{1}{4}\lim_{t\to-\infty}\left[\frac{e^{-u}}{-1}\right]_{t^4}^{0}=\frac{1}{4}\lim_{t\to-\infty}\left[\frac{e^{-0}}{-1}-\frac{e^{-t^4}}{-1}\right]=\frac{1}{4}\lim_{t\to-\infty}\left[\frac{-1}{e^0}+\frac{1}{e^{t^4}}\right]$

$$=\frac{1}{4}\left[-1+\frac{1}{e^{\infty}}\right]=\frac{1}{4}[-1+0]=\frac{-1}{4}\text{.....(2)} \qquad \left(\because \frac{1}{e^{\infty}}=0\right)$$

$$I_2 = \lim_{t\to\infty}\int_0^{t^4} e^{-u}\,\frac{du}{4} = \frac{1}{4}\lim_{t\to\infty}\int_0^{t^4} e^{-u}du = \frac{1}{4}\lim_{t\to\infty}\left[\frac{e^{-u}}{-1}\right]_0^{t^4} = \frac{1}{4}\lim_{t\to\infty}\left[\frac{e^{-t^4}}{-1}-\frac{e^{-0}}{-1}\right] = \frac{1}{4}\lim_{t\to\infty}\left[\frac{-1}{e^{t^4}}+\frac{1}{e^0}\right]$$

$$= \frac{1}{4}\left[\frac{-1}{e^{\infty}}+1\right] = \frac{1}{4}[0+1] = \frac{1}{4}\text{.....(3)} \qquad \left(\because \frac{1}{e^{\infty}}=0\right)$$

Substitute (2) & (3) values in (1)

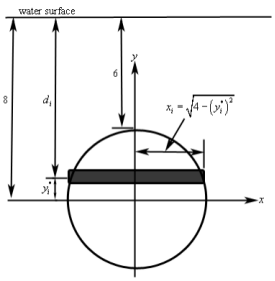
$$I = \lim_{t\to-\infty}\int_t^0 x^3e^{-x^4}dx + \lim_{t\to\infty}\int_0^t x^3e^{-x^4}dx$$

$$= \frac{-1}{4}+\frac{1}{4} = 0$$

Hence $\int_{-\infty}^{\infty} x^3e^{-x^4}dx$ is converges.

17 **Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.**

Solution: Given, the top of the circular plate is 6 meters. Assume that the origin of the axis system is at the center of the plate. Split the plate into n horizontal strips each of width Δy and choose a point y_i^* from each strip.



The depth below the water surface of each strip is $d_i = 8 - y_i^*$

and that in turn gives us the pressure on the strip,

$$P_i = \rho g d_i \qquad \text{where } \rho - \text{Density of water} = 1000\,kg/m^3$$

$$g - \text{Gravitaional Force} = 9.81\,N/kg$$

$$P_i = 9810\left(8-y_i^*\right)$$

The area of each strip is $A_i = 2\sqrt{4-\left(y_i^*\right)^2}\ \Delta y$ $\qquad \left[\because A_i\,(\text{Area}) = \text{length}\times\text{breadth}\right]$

The hydrostatic force on each strip is $F_i = P_iA_i = 9810\left(8-y_i^*\right)(2)\sqrt{4-\left(y_i^*\right)^2}\,\Delta y$

$$= 19620\left(8-y_i^*\right)\sqrt{4-\left(y_i^*\right)^2}\,\Delta y$$

The total force on the plate is,

$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 19620\left(8-y_i^*\right)\sqrt{4-\left(y_i^*\right)^2}\,\Delta y$$

$$= 19620\int_{-2}^{2}(8-y)\sqrt{4-y^2}\,dy$$

$$= 19620\int_{-2}^{2} 8\left(\sqrt{4-y^2}\right)dy - 19620\int_{-2}^{2}\left(y\sqrt{4-y^2}\right)dy$$

$$= 156960\int_{-2}^{2}\left(\sqrt{4-y^2}\right)dy - 19620\int_{-2}^{2}\left(y\sqrt{4-y^2}\right)dy$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(x) \text{ is an odd function } (i.e)\ f(-x) = -f(x)\right]$$

$$\left[\because y\sqrt{4-y^2} \text{ is an odd function}, \int_{-2}^{2} y\sqrt{4-y^2}\,dy = 0\right]$$

$$= 156960 \times 2\int_0^2 \left(\sqrt{4-y^2}\right) dy - 19620(0)$$

$$= 313920\int_0^2 \left(\sqrt{4-y^2}\right) dy \qquad \left[\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 313920\left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2$$

$$= 313920\left[\frac{2}{2}\sqrt{4-4} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right) - \frac{0}{2}\sqrt{4-0} - \frac{4}{2}\sin^{-1}\left(\frac{0}{2}\right)\right]$$

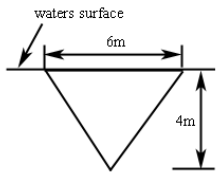
$$= 313920\left[0 + 2\sin^{-1}(1) - 0 - 2\sin^{-1}(0)\right]$$

$$= 313920\left[\cancel{2}\left(\frac{\pi}{\cancel{2}}\right) - 2(0)\right]$$

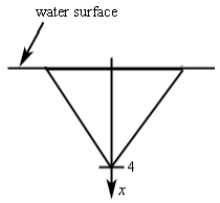
$$= 313920\pi$$

Hence, the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water is 313920π.

18 **Determine the hydrostatic force on the following triangular plate that is submerged in water as shown.**



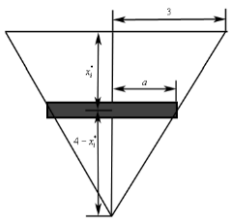
Solution: The sketch of the triangular plate with the axis system is



this system is orient the x-axis so that positive x is downward

$\therefore$ The water surface is $x = 0$ and the depth of the triangle is $x = 4$.

To find the hydrostatic force break the triangle into n horizontal strips each of equal width Δx and in each interval $[x_{i-1}, x_i]$ choose any point x_i^*. The height of this strip is Δx and the width is $2a$.



If two triangles are similar then their corresponding sides are equal in ratio.

$$\frac{3}{4}=\frac{a}{4-x_i^*}\Rightarrow a=3-\frac{3}{4}x_i^*$$

since Δx is small therefore the pressure on this strip is constant, the pressure is

$$P_i=\rho g d_i \quad \text{where } \rho-\text{Density of water}=1000\,kg/m^3$$

$$g-\text{Gravitaional Force}=9.81\,N/kg$$

$$P_i=\rho g d_i=1000(9.81)x_i^*=9810x_i^*$$

and the hydrostatic force on each strip is,

$$F_i=P_iA \quad \text{where A =area of the strip}=2a\Delta x$$

$$=P_i(2a\Delta x)$$

$$=9810x_i^*(2)\left(3-\frac{3}{4}x_i^*\right)\Delta x$$

$$=19620x_i^*\left(3-\frac{3}{4}x_i^*\right)\Delta x$$

The approximate hydrostatic force on the plate is the sum of the forces on all the strips.

$$F\approx\sum_{i=1}^{n}19620x_i^*\left(3-\frac{3}{4}x_i^*\right)\Delta x$$

Taking the limit will give the exact hydrostatic force

$$F=\lim_{n\to\infty}\sum_{i=1}^{n}19620x_i^*\left(3-\frac{3}{4}x_i^*\right)\Delta x$$

The hydrostatic force is

$$F=\int_0^4 19620\left(3x-\frac{3}{4}x^2\right)dx$$

$$=\left[19620\left(\frac{3}{2}x^2-\frac{1}{4}x^3\right)\right]_0^4$$

$$=19620\left[\left(\frac{3}{2}(4)^2-\frac{1}{4}(4)^3\right)-\left(\frac{3}{2}(0)^2-\frac{1}{4}(0)^3\right)\right]$$

$$=19620\left[\left(\frac{3}{2}(16)-\frac{1}{4}(64)\right)-(0-0)\right]$$

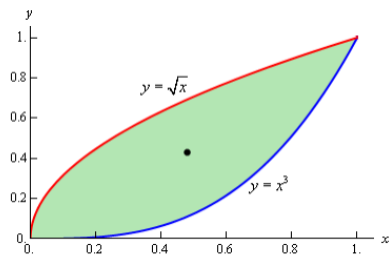
$$=19620(3(8)-(16))$$

$$=19620(24-16)=19620(8)$$

$$=156960$$

19 **Determine the center of mass for the region bounded by $y=x^3$ and $y=\sqrt{x}$**

Solution: The two curves intersect at $x=0$ and $x=1$ and here is a sketch of the region with the center of mass marked with a box.



Let $f(x)=\sqrt{x}, g(x)=x^3$

We'll first get the area of the region.

$$A=\int_0^1 \left(f(x)-g(x)\right)dx$$

$$=\int_0^1 \left(\sqrt{x}-x^3\right)dx$$

$$=\int_0^1 \left(\sqrt{x}\right)dx-\int_0^1 \left(x^3\right)dx$$

$$=\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1-\left[\frac{x^{3+1}}{3+1}\right]_0^1=\left[\frac{x^{\frac{1+2}{2}}}{\frac{1+2}{2}}\right]_0^1-\left[\frac{x^{3+1}}{3+1}\right]_0^1$$

$$=\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1-\left[\frac{x^4}{4}\right]_0^1$$

$$=\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1-\frac{1}{4}\left[x^4\right]_0^1$$

$$=\frac{2}{3}\left[(1)^{\frac{3}{2}}-(0)^{\frac{3}{2}}\right]-\frac{1}{4}\left[(1)^4-(0)^4\right]$$

$$=\frac{2}{3}[1-0]-\frac{1}{4}[1-0]=\frac{2}{3}-\frac{1}{4}=\frac{8-3}{12}=\frac{5}{12}$$

Now the moments, again without density, are

$M_x=\int_a^b \frac{1}{2}\left(f(x)^2-g(x)^2\right)dx$ $=\int_0^1 \frac{1}{2}\left(x-x^6\right)dx$ $=\frac{1}{2}\left[\int_0^1 (x)dx-\int_0^1 \left(x^6\right)dx\right]$ $=\frac{1}{2}\left[\left(\frac{x^2}{2}\right)_0^1-\left(\frac{x^7}{7}\right)_0^1\right]$ $=\frac{1}{2}\left[\left(\frac{(1)^2}{2}-\frac{(0)^2}{2}\right)-\left(\frac{(1)^7}{7}-\frac{(0)^7}{7}\right)\right]$ $=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{7}\right]$ $=\frac{1}{2}\left[\frac{7-2}{14}\right]$ $=\frac{5}{28}$	$M_y=\int_a^b x\left(f(x)-g(x)\right)dx$ $=\int_0^1 x\left(\sqrt{x}-x^3\right)dx$ $=\int_0^1 \left(x^{\frac{3}{2}}-x^4\right)dx$ $=\int_0^1 \left(x^{\frac{3}{2}}\right)dx-\int_0^1 \left(x^4\right)dx$ $=\left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1-\left[\frac{x^{4+1}}{4+1}\right]_0^1$ $=\left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1-\left[\frac{x^5}{5}\right]_0^1$

$$= \frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^1 - \frac{1}{5}\left[x^5\right]_0^1$$
$$= \frac{2}{5}\left[(1)^{\frac{5}{2}} - (0)^{\frac{5}{2}}\right] - \frac{1}{5}\left[(1)^5 - (0)^5\right]$$
$$= \frac{2}{5}[1-0] - \frac{1}{5}[1-0]$$
$$= \frac{2}{5} - \frac{1}{5} = \frac{2-1}{5} = \frac{1}{5}$$

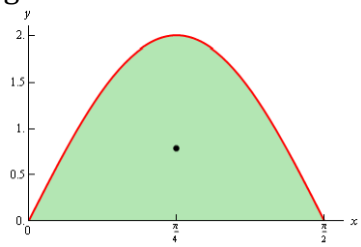
The coordinates of the center of mass is,

$$\bar{x} = \frac{M_y}{m} = \frac{1/5}{5/12} = \frac{12}{5} \times \frac{1}{5} = \frac{12}{25}$$
$$\bar{y} = \frac{M_x}{m} = \frac{5/28}{5/12} = \frac{12}{5} \times \frac{5}{28} = \frac{3}{7}$$

The coordinates of the center of mass are $\left(\frac{12}{25}, \frac{3}{7}\right)$.

20 **Determine the center of mass for the region bounded by $y = 2\sin(2x)$, $y = 0$ on the interval $\left[0, \frac{\pi}{2}\right]$**

Solution: Here is a sketch of the region with the center of mass denoted with a dot.



$f(x) = 2\sin 2x, g(x) = 0$

Let's first get the area of the region.

$$A = \int_a^b \left(f(x) - g(x)\right)dx$$
$$= \int_0^{\pi/2} \left(2\sin(2x) - 0\right)dx$$
$$= \int_0^{\pi/2} 2\sin(2x)\,dx$$
$$= \cancel{2}\left[\frac{-\cos 2x}{\cancel{2}}\right]_0^{\pi/2}$$
$$= -\left(\cos 2\left(\frac{\pi}{2}\right) - \cos 2(0)\right)$$
$$= -(\cos\pi - \cos 0)$$
$$= -(-1-1) = -(-2) = 2$$

Now, the moments (without density since it will just drop out) are

$$M_x = \int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2 \right) dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(\left(2\sin(2x)\right)^2 - (0)^2 \right) dx$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(4\sin^2(2x) - 0\right) dx$$

$$= \int_0^{\pi/2} 2\sin^2(2x)\, dx$$

$$= \int_0^{\pi/2} \left(1 - \cos(4x)\right) dx$$

$$= \int_0^{\pi/2} (1)\, dx - \int_0^{\pi/2} \cos(4x)\, dx$$

$$= [x]_0^{\pi/2} - \left[\frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - 0\right] - \frac{1}{4}\left[\sin 4\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

$$= \frac{\pi}{2} - \frac{1}{4}\left[\sin 2\pi - \sin(0)\right]$$

$$= \frac{\pi}{2} - \frac{1}{4}[0 - 0]$$

$$= \frac{\pi}{2}$$

$$M_y = \int_a^b x\left(f(x) - g(x) \right) dx$$

$$= \int_0^{\pi/2} x\left(2\sin(2x) - 0\right) dx$$

$$= \int_0^{\pi/2} 2x\sin(2x)\, dx$$

(integrating by parts)

$u = 2x$	$dv = \sin(2x)\, dx$
$\frac{du}{dx} = 2$	$\int dv = \int \sin(2x)\, dx$
$du = 2dx$	$v = \frac{-\cos 2x}{2}$

$$= \left[\cancel{2}x\left(\frac{-\cos 2x}{\cancel{2}}\right)\right]_0^{\pi/2} - \int_0^{\pi/2}\left(\frac{-\cos 2x}{\cancel{2}}\right)\cancel{2}\, dx$$

$$= \left[-x\cos 2x\right]_0^{\pi/2} + \int_0^{\pi/2} (\cos 2x)\, dx$$

$$= \left[\frac{-\pi}{2}\cos 2\left(\frac{\pi}{2}\right) + (0)\cos 2(0)\right] + \left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \left[\frac{-\pi}{2}\cos\pi + 0\right] + \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 2(0)\right]$$

$$= \frac{-\pi}{2}(-1) + \frac{1}{2}\left[\sin\pi - \sin 0\right]$$

$$= \frac{\pi}{2} + \frac{1}{2}[0 - 0]$$

$$= \frac{\pi}{2}$$

The coordinates of the center of mass are

$$\bar{x} = \frac{M_y}{A} = \frac{\pi/2}{2} = \frac{\pi}{4} \qquad \bar{y} = \frac{M_x}{A} = \frac{\pi/2}{2} = \frac{\pi}{4}$$

So, the center of mass for this region is $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

I-YEAR B.E./B.TECH. (COMMON TO ALL BRANCHES)
MATRICES AND CALCULUS(MA3151) QUESTION BANK

UNIT V – MULTIPLE INTEGRALS

PART- A

1 **Evaluate** $\int_0^2\int_0^1 y\,dydx$

Solution:

Let $I=\int_0^2\int_0^1 y\,dydx$

$$=\int_0^2\left[\int_0^1 y\,dy\right]dx=\int_0^2\left[\frac{y^2}{2}\right]_0^1 dx=\frac{1}{2}\int_0^2(1^2-0)dx$$

$$=\frac{1}{2}\int_0^2 dx=\frac{1}{2}[x]_0^2=\frac{1}{2}(2-0)=\frac{1}{2}(2)=1 \quad \therefore \int_0^2\int_0^1 y\,dydx=1$$

2 **Evaluate** $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$ **(APRIL/MAY 2017)**

Solution:

Let $I=\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta=\int_0^{\pi}\left[\int_0^{\cos\theta} r\,dr\right]d\theta$

$$=\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta}d\theta=\frac{1}{2}\int_0^{\pi}\left(\cos^2\theta-0\right)d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\cos^2\theta d\theta=\frac{1}{2}\int_0^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \qquad \left[\text{WKT}\cos^2\theta=\frac{1+\cos 2\theta}{2}\right]$$

$$=\frac{1}{4}\int_0^{\pi}(1+\cos 2\theta)\,d\theta=\frac{1}{4}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$=\frac{1}{4}\left[\left(\pi+\frac{\sin(2\pi)}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right] \qquad [\sin 2\pi=0;\ \sin 0=0]$$

$$=\frac{1}{4}\left[(\pi+0)-\left(0+\frac{0}{2}\right)\right]=\frac{1}{4}(\pi)$$

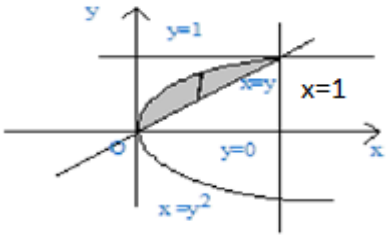
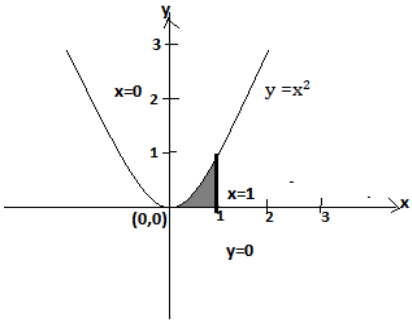
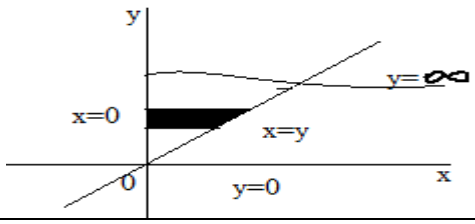
$$\therefore \int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta=\frac{\pi}{4}$$

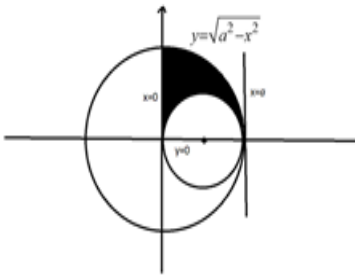
3 **Evaluate** $\int_1^2\int_0^{x^2} x\,dydx$ **(NOV/DEC 2019)**

Solution:

Given $I=\int_1^2\int_0^{x^2} x\,dydx$

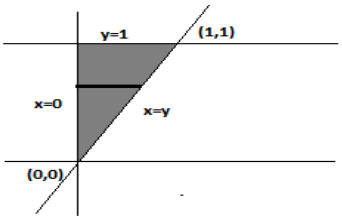
$$=\int_1^2 x\left(\int_0^{x^2} dy\right)dx$$

	$=\int_1^2 x\left(y\right)_0^{x^2} dx=\int_1^2 x\left(x^2\right)dx$ $=\int_1^2 x^3dx=\left(\frac{x^4}{4}\right)_1^2=\left(\frac{2^4}{4}-\frac{1^4}{4}\right)$ $=\left(\frac{16}{4}-\frac{1}{4}\right)$ $I=\frac{15}{4}$
4	**Change the order of integration in $\int_0^1\int_{y^2}^{y} f\left(x,y\right)dxdy$** **(NOV/DEC 2018)**
	Solution: Given $I=\int_0^1\int_{y^2}^{y} f\left(x,y\right)dxdy$ Limits of $x:y^2$ to y and Limits of $y:0$ to 1 By changing the order of integration New limits is $0\le x\le 1 \& x\le y\le\sqrt{x}$ $I=\int_{x=0}^{x=1}\int_{y=x}^{y=\sqrt{x}} f\left(x,y\right)dydx$ 
5	**Check the correct form of $\int_0^1\int_0^{x^2}(1+x)dxdy$**
	Solution: Given $\int_0^1\int_0^{x^2}(1+x)dxdy$ The limits for the inner integral are functions of x, then the first integral is with respect to y $\therefore \int_0^1\int_0^{x^2}(1+x)dxdy=\int_0^1\int_0^{x^2}(1+x)dydx$ [correct form] 
6	**Transform the integration $\int_0^\infty\int_0^y dx\,dy$ into polar co-ordinates.**
	Solution: Let $x=r\cos\theta$ and $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$ Limits of r: 0 to ∞ and Limits of θ: $\frac{\pi}{4}$ to $\frac{\pi}{2}$ 

	$\int_0^{\infty}\int_0^{y} dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=\infty} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left(\int_0^{\infty} r\,dr\right)d\theta$
7	**Shade the region of integration in** $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$
	Solution: The given order of the integration is not in correct form. $\int_{x=0}^{x=a}\int_{y=\sqrt{ax-x^2}}^{y=\sqrt{a^2-x^2}} dydx$ (correct form)  Limits of x: 0 to a and Limits of y: $\sqrt{ax-x^2}$ to $\sqrt{a^2-x^2}$ $y=\sqrt{ax-x^2}$ (Upper limit of y) $\Rightarrow y^2=\left(\sqrt{ax-x^2}\right)^2$ (Squaring on both sides) $\Rightarrow y^2=ax-x^2$ $x^2+y^2-ax=0$ $x^2+y^2-2x\left(\frac{a}{2}\right)=0$ $x^2-ax+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+y^2=0$ $\left(\text{add and sub by } \left(\frac{a}{2}\right)^2\right)$ $\left(x-\frac{a}{2}\right)^2+(y-0)^2=\left(\frac{a}{2}\right)^2 \quad \ldots(1)$ which is a circle with center at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$. $y=\sqrt{a^2-x^2}$ (Lower limit of y) $y^2=\left(\sqrt{a^2-x^2}\right)^2$ (Squaring on both sides) $y^2=a^2-x^2$ $x^2+y^2=a^2$ $(x-0)^2+(y-0)^2=a^2 \quad \ldots(2)$ which is a circle with center at (0,0) and radius *a.* Draw the circles (1) and (2), and shade the region bounded by the circles, $x=0$ & $x=a$
8	**Find the area bounded by the lines $x=0, y=1, y=x$ using double integration.**
	Solution: $\text{Area}=\iint_R dxdy=\int_0^1\int_0^y dxdy=\int_0^1[x]_0^y dy=\int_0^1(y-0)dy$

$$= \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left(1^2 - 0\right) = \frac{1}{2}$$



9 **Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$.**

Solution:

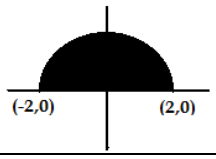
$$I = \int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 e^x e^y e^z\,dz\,dy\,dx$$

$$= \left[\int_0^1 e^x dx\right]\left[\int_0^1 e^y dx\right]\left[\int_0^1 e^z dx\right] = \left[e^x\right]_0^1\left[e^x\right]_0^1\left[e^x\right]_0^1$$

$$= \left(e^1 - e^0\right)\left(e^1 - e^0\right)\left(e^1 - e^0\right) \qquad [\text{WKT}, e^0 = 1]$$

$$= \left(e^1 - 1\right)\left(e^1 - 1\right)\left(e^1 - 1\right) = \left(e^1 - 1\right)^3$$

10 **Evaluate $\iint_R dxdy$, where R is the shaded region in the figure.**



Solution:

$$\iint_R dxdy = \text{Area of the shaded region} = 2 \times \textit{First Quadrant} = 2\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} dy\,dx$$

$$= 2\int_{x=0}^{2} [y]_0^{\sqrt{4-x^2}}\,dx = 2\int_{x=0}^{2}\left[\sqrt{4-x^2} - 0\right]dx = 2\int_{x=0}^{a}\sqrt{4-x^2}\,dx$$

$$\left[WKT, \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 2\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= 2\left[\left(\frac{2}{2}\sqrt{4-2^2} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right)\right) - \left(\frac{0}{2}\sqrt{4-0} + \frac{4}{2}\sin^{-1}\left(\frac{0}{2}\right)\right)\right]$$

$$= 2\left[\left(0 + \frac{4}{2}\sin^{-1}(1)\right) - (0+0)\right] = 2\left[\left(2\times\frac{\pi}{2}\right) - (0+0)\right] = 2\pi \quad \text{Square Units}$$

11 **Evaluate $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dx\,dy\,dz$**

Solution:

$$I = \int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dx\,dy\,dz = \int_0^4\int_0^x\left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx \qquad \left(\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right)$$

$$=\frac{1}{2}\int_0^4\int_0^x\left[\left(\sqrt{x+y}\right)^2-0\right]dy\,dx=\frac{1}{2}\int_0^4\int_0^x(x+y)dy\,dx$$

$$=\frac{1}{2}\int_0^4\left[xy+\frac{y^2}{2}\right]_0^x dx=\frac{1}{2}\int_0^4\left[\left(x^2+\frac{x^2}{2}\right)-0\right]dx$$

$$=\frac{1}{2}\int_0^4\left(x^2+\frac{x^2}{2}\right)dx=\frac{1}{2}\int_0^4\left(\frac{2x^2+x^2}{2}\right)dx$$

$$=\frac{3}{4}\int_0^4 x^2dx=\frac{3}{4}\left[\frac{x^3}{3}\right]_0^4=\frac{3}{4}\left[\frac{4^3}{3}-0\right]=\frac{\cancel{3}}{\cancel{4}}\left(\frac{4^{\cancel{3}\,2}}{\cancel{3}}\right)=4^2=16$$

12 **Express the region $x\ge 0, y\ge 0, z\ge 0, x^2+y^2+z^2\le 1$ by triple integration.(NOV/DEC 2019)**

Solution:

Given region is the positive octant of the sphere.

z varies from 0 to $\sqrt{1-x^2-y^2}$

y varies from 0 to $\sqrt{1-x^2}$

x varies from 0 to 1

$$\therefore \text{I}=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

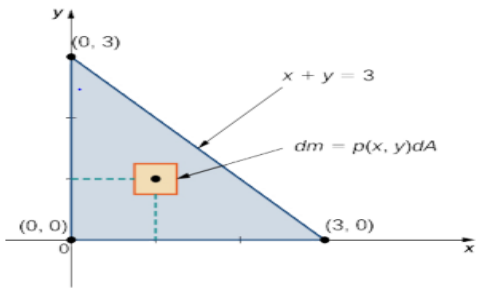
13 **Consider a triangular lamina R with vertices (0,0),(0,3),(3,0) and with density $\rho(x,y)=xy\,kg/m^2$. Find the moments M_x and M_y**

Solution:

Given $\rho(x,y)=xy$

The equation of line joining (0,3) and (3,0)

WKT the equation of line is



$$y-y_1=\frac{(y_2-y_1)}{(x_2-x_1)}(x-x_1)$$

$$\therefore y-3=-1(x-0)$$

$$\Rightarrow y-3=-x$$

$$\Rightarrow x+y=3$$

The moment $M_x=\iint_R y\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} y(xy)\,dydx$

$$=\int_0^3\int_0^{3-x} xy^2dydx=\int_0^3\left(\int_0^{3-x} xy^2dy\right)dx=\int_0^3 x\left(\frac{y^3}{3}\right)_0^{3-x}dx$$

$$=\frac{1}{3}\int_0^3 x\left[(3-x)^3-0\right]dx=\frac{1}{3}\int_0^3 x\left(3^3-3(3^2)x+3(3)x^2-x^3\right)dx=\int_0^3\left(\frac{\overset{9}{\cancel{27}}\,x}{\cancel{3}}-\frac{\overset{9}{\cancel{27}}\,x^2}{\cancel{3}}+\frac{\overset{3}{\cancel{9}}\,x^3}{\cancel{3}}-\frac{x^4}{3}\right)dx$$

$$=\left(\frac{9x^2}{2}-\frac{9x^3}{3}+\frac{3x^4}{4}-\frac{x^5}{15}\right)_0^3=\left(\frac{9(3)^2}{2}-\frac{\cancel{9}^{3}(3)^3}{\cancel{3}}+\frac{3(3)^4}{4}-\frac{(3)^5}{15}\right)=\left(\frac{81}{2}-81+\frac{243}{4}-\frac{\cancel{243}^{81}}{\cancel{15}_{5}}\right)$$

$$=\left(\frac{810-1620+1215-324}{20}\right)=\frac{81}{20}$$

The moment $M_y=\iint_R x\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x}x(xy)\,dydx=\int_0^3\int_0^{3-x}x^2ydydx$

$$=\int_0^3\left(\int_0^{3-x}x^2ydy\right)dx=\int_0^3x^2\left(\frac{y^2}{2}\right)_0^{3-x}dx=\frac{1}{2}\int_0^3x^2\left[(3-x)^2-0\right]dx$$

$$=\frac{1}{2}\int_0^3x^2\left[\left(3^2-2(3)x+x^2\right)\right]dx=\frac{1}{2}\int_0^3\left(3^2x^2-6x^3+x^4\right)dx=\frac{1}{2}\left(\frac{\cancel{9}^{3}x^3}{\cancel{3}}-\frac{\cancel{6}^{3}x^4}{\cancel{4}_{2}}+\frac{x^5}{5}\right)_0^3$$

$$=\left(3(3)^3-\frac{3(3)^4}{2}+\frac{3^5}{5}\right)=\frac{1}{2}\left(81-\frac{243}{2}+\frac{243}{5}\right)=\frac{810-1215+486}{20}=\frac{81}{20}$$

14 **Find the mass of the lamina of density $\rho(x,y)=x+y$ occupying the region R under the curve $y=x^2$ in the interval $0\le x\le 2$**

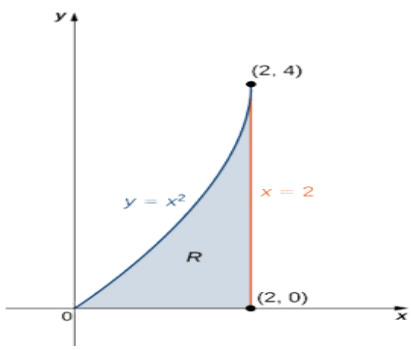
To compute the mass m

$$m=\iint_R\rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x^2}(x+y)dydx$$

$$=\int_0^2\left[\int_0^{x^2}(x+y)dy\right]dx=\int_0^2\left[xy+\frac{y^2}{2}\right]_0^{x^2}dx$$

$$=\int_0^2\left[x\left(x^2\right)+\frac{\left(x^2\right)^2}{2}\right]dx=\int_0^2\left[x^3+\frac{x^4}{2}\right]dx=\left[\frac{x^4}{4}+\frac{x^5}{10}\right]_0^2$$

$$=\frac{2^4}{4}+\frac{2^5}{10}=\frac{\cancel{16}^{4}}{\cancel{4}}+\frac{\cancel{32}^{16}}{\cancel{10}_{5}}=\frac{20+16}{5}=\frac{36}{5}$$



15 **Find the moment of inertia of a lamina covering the inside of the unit circle, with density function $\rho(x,y)=1-x^2-y^2$**

Solution: The moment of inertia of the entire lamina is

$$I=\iint_D(x^2+y^2)\rho(x,y)dA$$

$$=\iint_D(x^2+y^2)\left(1-x^2-y^2\right)dA$$

By polar co-ordinates, $r^2=x^2+y^2$

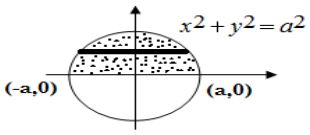
$$I=\iint_D r^2\rho(x,y)dA=\int_0^{2\pi}\int_0^1r^2\left(1-r^2\right)rdrd\theta=\int_0^{2\pi}\int_0^1(r^3-r^5)drd\theta$$

$$=\int_0^{2\pi}\left(\int_0^1(r^3-r^5)dr\right)d\theta=\int_0^{2\pi}\left(\frac{r^4}{4}-\frac{r^6}{6}\right)_0^1d\theta=\int_0^{2\pi}\left[\left(\frac{1^4}{4}-\frac{1^6}{6}\right)-0\right]d\theta$$

$$= \int_0^{2\pi}\left(\frac{6-4}{24}\right)d\theta = \int_0^{2\pi}\left(\frac{\cancel{2}^{1}}{\cancel{24}_{12}}\right)d\theta = \int_0^{2\pi}\frac{d\theta}{12} = \frac{1}{12}[\theta]_0^{2\pi} = \frac{1}{12}(2\pi - 0) = \frac{\cancel{2}^{1}\pi}{\cancel{12}_{6}} = \frac{\pi}{6}$$

16 **Change the order of integration in** $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left(x^2+y^2\right)dx\,dy$

Solution:



Given $I = \int_{x=-a}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}}\left(x^2+y^2\right)dydx$ (correct form)

Limits of x : $-a$ to a and limits of y : 0 to $\sqrt{a^2-x^2}$

$y=0$ and $y=\sqrt{a^2-x^2} \Rightarrow y^2 = a^2 - x^2 \Rightarrow x^2+y^2 = a^2$

$x=-a$ and $x=a$

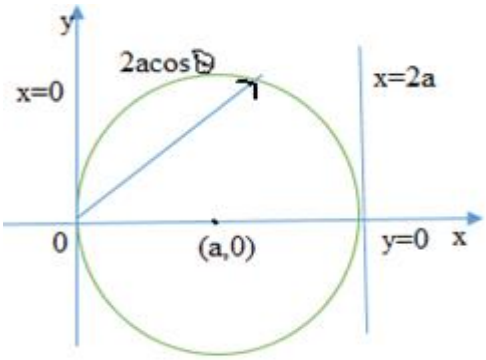
By changing the order of Integration

Limits of x : $-\sqrt{a^2-y^2}$ to $\sqrt{a^2-y^2}$ and limits of y : 0 to a

$$I = \int_{y=0}^{y=a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\left(x^2+y^2\right)dx\,dy$$

17 **Transform the integration** $\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$ **to polar co-ordinates.**

Solution:



Given $\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$

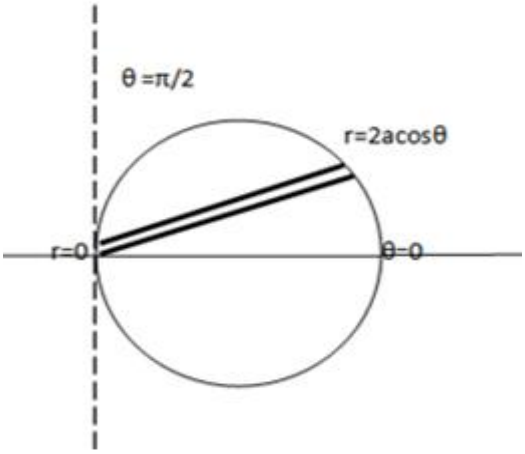
Where limits of x is from 0 to 2a (i.e) $x=0$ and $x=2a$ and

Limits of y is from 0 to $\sqrt{2ax-x^2}$ (i.e) $y=0$ and $y=\sqrt{2ax-x^2}$

Let $x = r\cos\theta$ and $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\sin^2\theta+\cos^2\theta\right) = r^2$

$\left[\because \sin^2\theta+\cos^2\theta = 1\right]$



Here $y=\sqrt{2ax-x^2}$ (Squaring on both sides)

$y^2 = 2ax - x^2$

$x^2+y^2-2ax = 0$

$r^2 - 2ar\cos\theta = 0$

$r(r-2a\cos\theta) = 0$

$r=0;\ (r-2a\cos\theta)=0$

$r=0;\ r=2a\cos\theta$

$\therefore$ Limits of r : 0 to $2a\cos\theta$

$x=0 \Rightarrow r\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \cos^{-1}(0) \Rightarrow \theta = \frac{\pi}{2}$

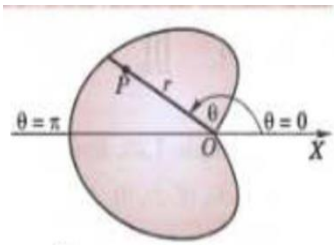
$y=0 \Rightarrow r\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = c\sin^{-1}(0) \Rightarrow \theta = 0$

$\therefore$ Limits of θ: 0 to $\frac{\pi}{2}$

$$\therefore \int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}} \left(x^2+y^2\right)dy\,dx = \int_{0}^{\pi/2}\int_{0}^{2a\cos\theta} r^3 dr\,d\theta$$

18 **Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about its axis.**

Solution:



In polar co-ordinates, the volume of the solid formed by the revolution is $= \iint_A 2\pi r \sin\theta\, r dr\, d\theta$

$$\text{Required Volume} = \int_0^{\pi}\int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi}\left[\left(a(1-\cos\theta)^3 - 0\right)\right]\sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1-\cos\theta)^3 \sin\theta\, d\theta$$

Let $t = 1-\cos\theta$
$dt = \sin\theta\, d\theta$

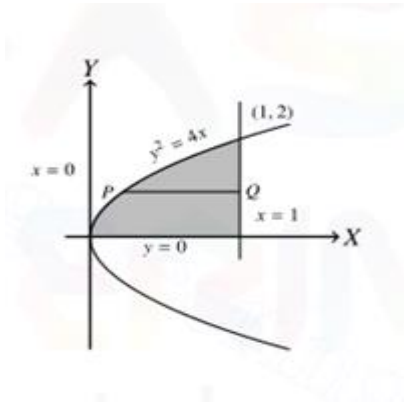
θ	0	π
t	0	2

$$= \frac{2\pi a^3}{3}\int_0^2 (t)^3 dt$$

$$= \frac{2\pi a^3}{3}\left(\frac{t^4}{4}\right)_0^2 = \frac{2\pi a^3}{3}\frac{1}{4}\left(2^4-0\right) = \frac{2\pi a^3}{3}\frac{1}{\cancel{4}}\left(\cancel{4}\times 4\right) = \frac{8\pi a^3}{3}$$

19 **Change the order of integration in $\int_0^1\int_0^{2\sqrt{x}} f(x,y)\,dx\,dy$.**

Solution:



Given $I = \int_0^1\int_0^{2\sqrt{x}} f(x,y)\,dydx$

Limits of x: 0 to 1 and Limits of y: 0 to $2\sqrt{x}$

By changing the order of integration

Limits of y: 0 to 2 and Limits of x: $\frac{y^2}{4}$ to 1

$$I = \int_0^1\int_0^{2\sqrt{x}} f(x,y)\,dydx = \int_{y=0}^{y=2}\int_{x=\frac{y^2}{4}}^{x=1} f(x,y)\,dxdy$$

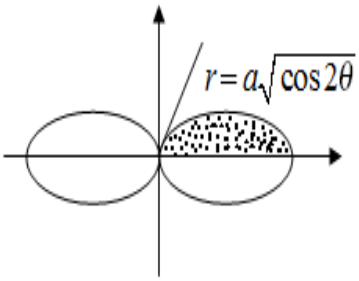
20 **Compute the entire area bounded by $r^2 = a^2\cos 2\theta$.**

Solution:

Given $r^2 = a^2\cos 2\theta \Rightarrow r = a\sqrt{\cos 2\theta}$

Area $= 4 \times$ Area in first quadrant

$$A = \iint_R r\,dr\,d\theta = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$



$$= 4\int_{\theta=0}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{\cancel{4}^{2}}{\cancel{2}}\int_{\theta=0}^{\pi/4}\left(a^2\cos 2\theta\right)d\theta$$

$$= 2a^2\int_{\theta=0}^{\pi/4}\cos 2\theta\, d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= 2a^2\left[\frac{\sin 2(\pi/4)}{2} - 0\right]$$

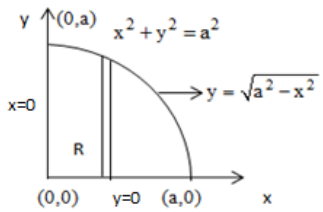
$$= \frac{\cancel{2}a^2}{\cancel{2}}\sin 2\left(\frac{\pi}{4}\right) = a^2\sin\frac{\pi}{2} = a^2$$

PART- B

1 **Evaluate $\iint xydxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. (NOV/DEC 2019)**

Solution:

Let $I = \iint xy\ dxdy$



From the circle $x^2 + y^2 = a^2$, we have

$y^2 = a^2 - x^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$

Given region is the positive quadrant of the circle.

Integrating over a vertical strip (i.e.) w.r.t. y from $y = 0$ to $y = +\sqrt{a^2 - x^2}$ on the circle then w.r.t. x from $x = 0$ to $x = a$

$$\therefore I = \iint xy\ dxdy = \int_{x=0}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_0^a\left[\int_0^{\sqrt{a^2-x^2}} y\,dy\right]x\,dx$$

$$= \int_0^a\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\,dx$$

$$= \frac{1}{2}\int_0^a\left[\left(\sqrt{a^2-x^2}\right)^2 - 0\right]x\,dx$$

$$= \frac{1}{2}\int_0^a\left(a^2 - x^2\right)x\,dx$$

$$= \frac{1}{2}\int_0^a \left(a^2 x - x^3\right) dx$$

$$= \frac{1}{2}\left[a^2 \frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(a^2 \frac{a^2}{2} - \frac{a^4}{4}\right) - \left(a^2 \frac{0^2}{2} - \frac{0^4}{4}\right)\right]$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0)$$

$$= \frac{1}{2}\left(\frac{2a^4 - a^4}{4}\right) = \frac{1}{2}\left(\frac{a^4}{4}\right)$$

$$\therefore \iint xy \; dxdy = \frac{a^4}{8}$$

2 **Evaluate $\iint xydxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**

Solution:

Let $I = \iint xy \; dxdy$

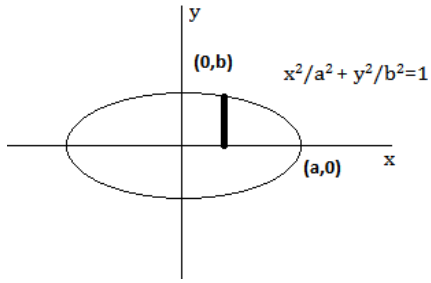
From the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have, $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(\frac{a^2 - x^2}{a^2}\right) \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$

Given region is the positive quadrant of the ellipse.

Integrating over a vertical strip

(i.e.) w.r.t. y from $y = 0$ to $y = +\frac{b}{a}\sqrt{a^2 - x^2}$ on the ellipse

then w.r.t. x from $x = 0$ to $x = a$



$$\therefore I = \iint xy\,dx\,dy = \int_{x=0}^{x=a} \int_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \int_{x=0}^{a} \left(\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y dy\right) x\,dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\left[\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 - 0\right] dx$$

$$= \frac{1}{2}\int_0^a x\left[\frac{b^2(a^2-x^2)}{a^2}\right]dx$$

$$= \frac{1}{2}\frac{b^2}{a^2}\int_0^a x(a^2-x^2)dx$$

$$= \frac{b^2}{2a^2}\int_0^a \left(a^2x - x^3\right)dx$$

$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\left(a^2\frac{a^2}{2} - \frac{a^4}{4}\right) - \left(a^2\frac{0^2}{2} - \frac{0^4}{4}\right)\right]$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0)$$

$$= \frac{b^2}{2a^2}\left(\frac{2a^4 - a^4}{4}\right)$$

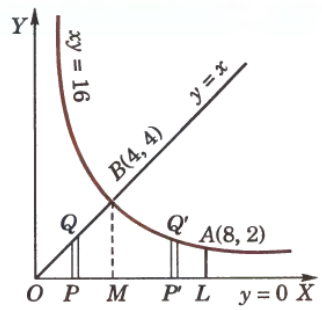
$$= \frac{b^2}{2a^2}\left(\frac{a^4}{4}\right)$$

$$\therefore \iint xy\,dx\,dy = \frac{a^2b^2}{8}$$

3 **Evaluate $\iint_R x^2dxdy$ where R is the region in the first quadrant bounded by the lines $x = y, y = 0, x = 8$ and the curve $xy = 16$.**

Solution:

Solution:



Given $\iint_R x^2dxdy$

The line $x = 8$ intersects the hyperbola $xy = 16$ at $(8, 2)$ and the line $y=x$ intersects this hyperbolaat (4,4). From the diagram the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

$$\iint_R x^2dxdy = \iint_{OMB} x^2dxdy + \iint_{MLAB} x^2dxdy$$

$$= I_1 + I_2$$

To evaluate $I_1 = \iint_{OMB} x^2dxdy$:

Integrating over a vertical strip (i.e.) w.r.t. y from $y = 0$ to $y = x$ then w.r.t. x from $x = 0$ to $x = 4$

$$I_1 = \iint_{OMB} x^2 dxdy = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2 dxdy$$

$$= \int_0^4 \left(\int_0^x dy \right) x^2 dx$$

$$= \int_0^4 x^2 \left[y \right]_0^x dx$$

$$= \int_0^4 x^2 \left(x - 0 \right) dx$$

$$= \int_0^4 x^3 dx = \left[\frac{x^4}{4} \right]_0^4$$

$$= \left[\frac{1}{4}\left(4^4 - 0 \right) \right] = 4^3$$

$$I_1 = 64$$

To evaluate $I_2 = \iint_{MLAB} x^2 dxdy$:

Integrating over a vertical strip (i.e.) w.r.t. y from $y = 0$ to $y = \frac{16}{x}$ then w.r.t. x from $x = 4$ to $x = 8$

$$I_2 = \iint_{MLAB} x^2 dxdy = \int_{x=4}^{x=8}\int_{y=0'}^{y=\frac{16}{x}} x^2 dxdy$$

$$= \int_4^8 \left(\int_{0'}^{\frac{16}{x}} dy \right) x^2 dx$$

$$= \int_4^8 x^2 \left[y \right]_0^{\frac{16}{x}} dx$$

$$= \int_4^8 x^2 \left(\frac{16}{x} - 0 \right) dx$$

$$= 16\int_4^8 \frac{x^{\cancel{2}}}{\cancel{x}} dx = 16\left[\frac{x^2}{2} \right]_4^8$$

$$= \frac{\overset{8}{\cancel{16}}}{\cancel{2}}\left(8^2 - 4^2 \right) = 8(64 - 16)$$

$$I_2 = 384$$

	$\therefore \iint_R x^2 dxdy = I_1 + I_2 = 64 + 384$ $\therefore \iint_R x^2 dxdy = 448$
4	**Evaluate** $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dxdy$. **(NOV/DEC 2016)**
	Solution: **Given** $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dxdy$ $= \int_{x=0}^{x=a} \left(\int_{y=0}^{y=\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2 - y^2}\, dy \right) dx$ $\left(\because \int \sqrt{a^2-x^2}\, dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} \right] \right)$ $= \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2} sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$ $= \int_0^a \left[\left(\frac{y}{2}\sqrt{a^2-x^2-\left(\sqrt{a^2-x^2}\right)^2} + \frac{a^2-x^2}{2} sin^{-1}\frac{\cancel{\sqrt{a^2-x^2}}}{\cancel{\sqrt{a^2-x^2}}} \right) - 0 \right] dx$ $= \int_0^a \left[\left(\frac{y}{2}\sqrt{\cancel{a^2} - \cancel{x^2} - \cancel{a^2} + \cancel{x^2}} + \frac{a^2-x^2}{2} sin^{-1}(1) \right) - 0 \right] dx$ $= \int_0^a \left[0 + \frac{a^2-x^2}{2} sin^{-1}(1) \right] dx$ $= \int_0^a \frac{a^2-x^2}{2}\left(\frac{\pi}{2}\right) dx$ $\left[sin^{-1}(1) = \frac{\pi}{2} \right]$ $= \frac{\pi}{4}\int_0^a \left(a^2 - x^2\right) dx = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} - 0 \right]$ $= \frac{\pi}{4}\left[\frac{3a^3 - a^3}{3} \right] = \frac{\pi}{4}\left(\frac{2a^3}{3} \right)$ $I = \frac{a^3\pi}{6}$
5	**Find the area bounded by $y^2 = 4x$ and $x^2 = 4y$ by using double integral. (NOV/DEC 2019)**
	Solution: Area $= \iint dxdy$ The region is $x^2 = 4y$ and $y^2 = 4x$ $x^2 = 4y$.....(1)

$y^2 = 4x$.....(2)

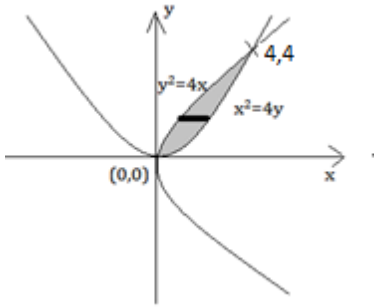
To find the point of intersection:

From eqn (2) $x = \frac{y^2}{4}$, substitute in (1) $\left(\frac{y^2}{4}\right)^2 = 4y$

$\Rightarrow \frac{y^4}{16} = 4y \Rightarrow y^4 = 64y \Rightarrow y\left(y^3 - 64\right) = 0$

$\Rightarrow y = 0$ and $y = 4$

when $y = 0 \Rightarrow x = 0$ and $y = 4 \Rightarrow x = \frac{16}{4} = 4$



The point of intersection is $(0,0)$ and $(4,4)$

$\therefore$ Limits

$y: 0$ to 4

$x: \frac{y^2}{4}$ to $2\sqrt{y}$

The required area $= \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\, dy$

$$= \int_0^4 [x]_{\frac{y^2}{4}}^{2\sqrt{y}} dy$$

$$= \int_0^4 \left(2y^{\frac{1}{2}} - \frac{y^2}{4}\right) dy$$

$$= \left(\frac{2y^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{y^3}{4\times 3}\right)_0^4 = \left(\frac{2y^{3/2}}{3/2} - \frac{y^3}{12}\right)_0^4$$

$$= \left(\frac{2(4)^{3/2}}{3/2} - \frac{(4)^3}{12} - 0\right)$$

$$= \left(\frac{4}{3}4\sqrt{4} - \frac{4^3}{12}\right)$$

$$= \frac{32}{3} - \frac{16}{3}$$

Area $= \frac{16}{3}$ sq. units.

6 **Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ (NOV/DEC 2018)**

Solution:

Given $y^2 = 4 - x$ and $y^2 = x$

$y^2 = 4 - x \Rightarrow y^2 = -(x-4)$

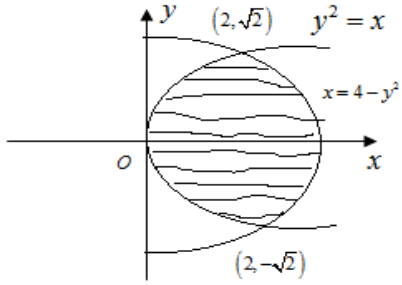
it is a parabola with vertex $(4,0)$

$y^2 = x$ is a parabola with vertex $(0,0)$.

To find the points of intersection:

$y^2 = x.....(1)$

$y^2 = 4 - x.....(2)$



Substitute (1) in (2)

$x = 4 - x \Rightarrow 2x = 4 \Rightarrow x = 2$

When $x = 2$

From (1), $y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

The points of intersection are $(2,\sqrt{2})$ & $(2,-\sqrt{2})$

$\therefore$ Limits :

$y : -\sqrt{2}$ to $\sqrt{2}$

$x : y^2$ to $4 - y^2$

The required area $= \int_{y=-\sqrt{2}}^{y=\sqrt{2}} \int_{x=y^2}^{x=4-y^2} dxdy$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} [x]_{y^2}^{4-y^2} \, dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (4 - y^2 - y^2) dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2y^2) dy$$

$\because$ for even function $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$$= 2\int_0^{\sqrt{2}} (4 - 2y^2) dy$$

$$= 2\left[4y - 2\frac{y^3}{3}\right]_0^{\sqrt{2}}$$

$$= 2\left[\left(4\sqrt{2} - 2\frac{(\sqrt{2})^3}{3}\right) - (0)\right] \qquad \left[\because (\sqrt{2})^3 = 2\sqrt{2}\right]$$

$$= 2\left[4\sqrt{2} - 2\frac{(2\sqrt{2})}{3}\right] = 2\left[4\sqrt{2} - \frac{4}{3}\sqrt{2}\right] = 2 \times 4\sqrt{2}\left[1 - \frac{1}{3}\right]$$

$$= 8\sqrt{2}\left(\frac{2}{3}\right) = \frac{16\sqrt{2}}{3} \text{ sq. units.}$$

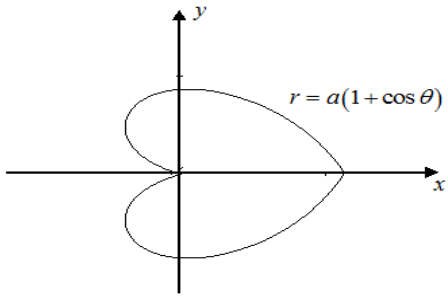
7 **Find the area of the cardioid $r = a(1 + \cos\theta)$**

Solution:

The curve is symmetrical about the initial line.

The required area $= 2 \times$ Upper half of the Cardioid

$$= 2\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1+\cos\theta)} r \, dr \, d\theta$$



$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi}\left[a^2(1+\cos\theta)^2 - 0\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\cos^2\theta + 2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\left(\frac{1+\cos 2\theta}{2}\right)+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\frac{1}{2}+\frac{\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= a^2\int_0^{\pi}\left[\frac{3}{2}+\frac{\cos 2\theta}{2}+2\cos\theta\right]d\theta$$

$$= a^2\left[\frac{3}{2}\theta+\frac{\sin 2\theta}{2\times 2}+2\sin\theta\right]_0^{\pi}$$

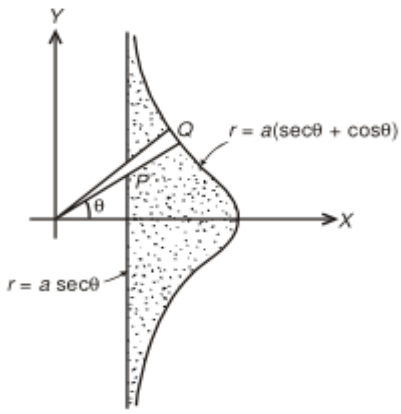
$$= a^2\left[\left(\frac{3}{2}\pi+\frac{\sin 2\pi}{4}+2\sin\pi\right)-\left(0+\frac{\sin 0}{4}+2\sin 0\right)\right]$$

$$= a^2\left[\left(\frac{3}{2}\pi+0+0\right)-(0+0+0)\right]$$

$$= \frac{3}{2}\pi a^2 \text{ Sq.units..}$$

8 **Find the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote**

Solution:



Given $r = a(\sec\theta + \cos\theta)$

i.e., $r = a\left(\frac{1}{\cos\theta} + \cos\theta\right)$ contains cosine terms only and hence it is symmetrical about the initial axis.

Further for $\theta = 0, r = 2a$ and $\boldsymbol{r}$ goes on decreasing above and below the initial axis as $\boldsymbol{\theta}$ approaches to $\frac{\pi}{2}$ and at $\theta = \frac{\pi}{2}, r = \infty$

Clearly, the required area is the doted region in which r varies along the radial strip from $r = a\sec\theta$ to $r = a(\sec\theta + \cos\theta)$ and finally strip slides between $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

$\therefore$ Area $= 2 \times$ Upper part of the region

$$= 2\int_0^{\pi/2} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/2} \left[\left(a(\sec\theta+\cos\theta)\right)^2 - \left(a\sec\theta\right)^2\right] d\theta \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$= a^2\int_0^{\pi/2} \left[\left(\frac{1+\cos^2\theta}{\cos\theta}\right)^2 - \left(\frac{1}{\cos\theta}\right)^2\right] d\theta$$

$$= a^2\int_0^{\pi/2} \left[\left(\frac{1+2\cos^2\theta+\cos^4\theta}{\cos^2\theta}\right) - \left(\frac{1}{\cos^2\theta}\right)\right] d\theta$$

$$= a^2\int_0^{\pi/2} \left(\frac{\cancel{1}+2\cos^2\theta+\cos^4\theta\cancel{-1}}{\cos^2\theta}\right) d\theta$$

$$= a^2\int_0^{\pi/2} \left(\frac{2\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} + \frac{\overset{\cos^2\theta}{\cancel{\cos^4\theta}}}{\cancel{\cos^2\theta}}\right) d\theta$$

$$= a^2\int_0^{\pi/2} \left(\cos^2\theta+2\right) d\theta \qquad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= a^2\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}+2\right) d\theta$$

$$= a^2\int_0^{\pi/2} \left(\frac{1+\cos 2\theta+4}{2}\right) d\theta$$

$$= a^2\int_0^{\pi/2} \frac{\left(5+\cos 2\theta\right)}{2} d\theta$$

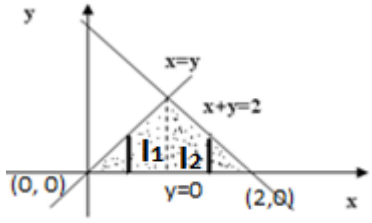
$$= \frac{a^2}{2}\left[5\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(5\left(\pi/2\right) + \frac{\sin 2\left(\pi/2\right)}{2}\right) - 0\right]$$

$$= \frac{a^2}{2}\left[5\left(\pi/2\right) + 0\right] = \frac{5\pi a^2}{4}$$

9 **Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ and evaluate it (NOV/DEC 2020)**

Solution:



Given $I = \int_{y=0}^{y=1} \int_{x=y}^{x=2-y} xy\ dx\,dy$

Limits of x: y to $2-y$ and limits of y: 0 to 1

i.e., $x = y$ and $x = 2-y \Rightarrow x+y=2$

By changing the order of integration

$I_1 \Rightarrow$ Limits of x: 0 to 1 and

Limits of y: 0 to x

$I_2 \Rightarrow$ Limits of x: 1 to 2 and

Limits of y: 0 to $2-x$

$$\boldsymbol{I = I_1 + I_2}$$

$$I = \int_0^1 \int_0^x xy\,dydx + \int_1^2 \int_0^{2-x} xy\,dydx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \int_0^1 x\left(\frac{x^2}{2} - 0\right)dx + \int_1^2 x\left(\frac{(2-x)^2}{2} - 0\right)dx$$

$$= \int_0^1 \frac{x^3}{2}dx + \int_1^2 \left(\frac{x(2^2 + x^2 - 4x)}{2}\right)dx$$

$$=\frac{1}{2}\int_0^1 x^3dx+\frac{1}{2}\int_1^2\left(4x+x^3-4x^2\right)dx$$

$$=\frac{1}{2}\left(\frac{x^4}{4}\right)_0^1+\frac{1}{2}\left(\frac{\cancel{4}^{2}x^2}{\cancel{2}}+\frac{x^4}{4}-\frac{4x^3}{3}\right)_1^2$$

$$=\left[\frac{1}{8}\left(1^4-0\right)\right]+\frac{1}{2}\left[\left(2(2)^2+\frac{(2)^4}{4}-\frac{4(2)^3}{3}\right)-\left(2(1)^2+\frac{(1)^4}{4}-\frac{4(1)^3}{3}\right)\right]$$

$$=\frac{1}{8}+\frac{1}{2}\left[\left(8+\frac{\cancel{16}^{4}}{\cancel{4}}-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$=\frac{1}{8}+\frac{1}{2}\left(10-\frac{28}{3}-\frac{1}{4}\right)$$

$$=\frac{1}{8}+\frac{1}{2}\left(\frac{120-112-3}{12}\right)$$

$$=\frac{1}{8}+\frac{5}{24}=\frac{3+5}{24}=\frac{8}{24}=\frac{1}{3}$$

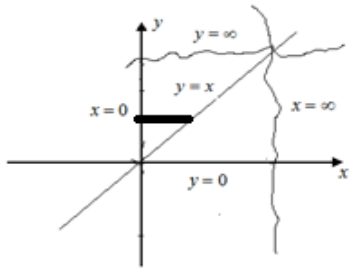
10 **Change the order of integration in $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dy\,dx$ and hence evaluate it. (APR/MAY 2019)**

Solution:

Given $\int_{x=0}^{x=\infty}\int_{y=x}^{y=\infty}\frac{e^{-y}}{y}dy\,dx$

x limit: $x=0$ to $x=\infty$

y limit: $y=x$ to $y=\infty$



By changing the order of integration, consider the horizontal strips.

x limit: $x=0$ to $x=y$

y limi: $y=0$ to $y=\infty$

$$\int_{x=0}^{x=\infty}\int_{y=x}^{y=\infty}\frac{e^{-y}}{y}dy\,dx=\int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=y}\left(\frac{e^{-y}}{y}\right)dx\right]dy$$

$$=\int_0^\infty\left(\frac{e^{-y}}{y}\right)[x]_0^y\,dy$$

$$=\int_0^\infty\left(\frac{e^{-y}}{y}\right)(y-0)\,dy$$

$$=\int_0^\infty \cancel{y}\left(\frac{e^{-y}}{\cancel{y}}\right)dy$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^\infty$$

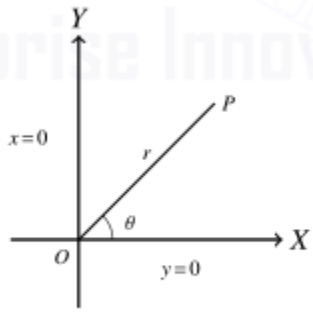
$$= -\left[e^{-\infty} - e^0 \right]$$

$$= -(0-1) = 1$$

11 **Transform the integral into polar co-ordinates and hence evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ and hence find the value of $\int_0^\infty e^{-x^2} dx$.**

Solution:

Let us transform this integral into polar co-ordinates by taking $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$



$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$$\mathrm{I} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

let $r^2 = t \Rightarrow 2rdr = dt \Rightarrow rdr = \frac{dt}{2}$

$$= \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2}\, d\theta$$

when $r = 0, t = 0$ and when $r = \infty, t = \infty$

$$\mathrm{I} = \frac{1}{2}\int_0^{\pi/2} -\left[e^{-\infty} - e^0 \right] d\theta \qquad \left[\because e^{-\infty} = 0; e^0 = 1 \right]$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}[\theta]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{2} - 0 \right]$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4} \text{.....(1)}$$

To find $\int_0^\infty e^{-x^2} dx$

Deduction:

From (1), $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\, dx\,dy = \frac{\pi}{4}$$

$$\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \frac{\pi}{4} \text{.....(2)}$$

	$Put\ y = x \Rightarrow dy = dx$ $(2) \Rightarrow \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-x^2}dx = \frac{\pi}{4}$ $\Rightarrow \left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}$ $\Rightarrow \int_0^\infty e^{-x^2}dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$
12	**Evaluate $\iint_D xy\sqrt{1-x-y}\,dxdy$ where D is the region bounded by x = 0,y = 0, x + y = 1 using the transformation x + y = u,y = uv** **(NOV/DEC 2020)**
	Solution: Given $x + y = u, y = uv$ $\Rightarrow x = u - uv = u(1-v)$ *and* $y = uv$ $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ $= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$ $= (1-v)u - (-uv)$ $= u - uv + uv = u$ Since x = 0 $\Rightarrow u(1-v) = 0 \Rightarrow u = 0,\ v = 1$ y = 0 $\Rightarrow uv = 0 \Rightarrow u = 0,\ v = 0$ $x + y = 1 \Rightarrow u = 1$ $\sqrt{1-x-y} = \sqrt{1-(x+y)} = \sqrt{1-u} = (1-u)^{\frac{1}{2}}$ $\iint f(x,y)dxdy = \iint f(u,v)\frac{\partial(x,y)}{\partial(u,v)}dudv$ $I = \iint_D xy\sqrt{1-x-y}\,dxdy = \int_0^1\int_0^1 [u(1-v)][uv]\left[(1-u)^{\frac{1}{2}}\right]ududv$ $= \int_0^1\int_0^1 u^3(1-u)^{\frac{1}{2}}v(1-v)dudv$ $= \int_0^1 u^3(1-u)^{\frac{1}{2}}du \int_0^1 v(1-v)dv$ put $t = 1-u \Rightarrow dt = -du$ u: 0, 1 t: 1, 0 $I = \int_0^1 (1-t)^3 t^{1/2}(-dt)\int_0^1 v - v^2 dv \qquad \because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ $= -\int_0^1 \left(1^3 - 3(1^2)t + 3(1)t^2 - t^3\right)t^{1/2}dt \int_0^1 v - v^2 dv$

$$=-\int_0^1\left(1-t^3+3t^2-3t\right)t^{1/2}\,dt\int_0^1 v-v^2dv$$

$$=-\int_0^1\left(t^{1/2}-t^3t^{1/2}+3t^2t^{1/2}-3t^1t^{1/2}\right)dt\int_0^1 v-v^2dv$$

$$=-\int_0^1\left(t^{1/2}-t^{7/2}+3t^{5/2}-3t^{3/2}\right)dt\left[\frac{v^2}{2}-\frac{v^3}{3}\right]_0^1$$

$$=-\left(\frac{t^{3/2}}{3/2}-\frac{t^{9/2}}{9/2}+\frac{3t^{7/2}}{7/2}-\frac{3t^{5/2}}{5/2}\right)_1^0\left[\left(\frac{1^2}{2}-\frac{1^3}{3}\right)-0\right]$$

$$=-\left[0-\left(\frac{1}{3/2}-\frac{1}{9/2}+\frac{3}{7/2}-\frac{3}{5/2}\right)\right]\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\left[\frac{2}{3}-\frac{2}{9}+\frac{6}{7}-\frac{6}{5}\right]\left[\left(\frac{1}{6}\right)-0\right]$$

$$=\left[\frac{2}{3}-\frac{2}{9}+\frac{6}{7}-\frac{6}{5}\right]\left(\frac{1}{6}\right)$$

$$=\left(\frac{210-70+270-378}{315}\right)\left(\frac{1}{6}\right)$$

$$=\left(\frac{\cancel{32}^{16}}{315}\right)\left(\frac{1}{\cancel{6}_{3}}\right)$$

$$=\frac{16}{945}$$

13 **Evaluate by changing into polar coordinates** $\int_0^a\int_y^a \frac{x^2dxdy}{\sqrt{x^2+y^2}}$ **(APR/MAY 2019)**

Solution:

Given $\int_0^a\int_y^a \frac{x^2dxdy}{\sqrt{x^2+y^2}}$ Cartesian form:

Limits of y : from 0 to a (i.e) $y=0$ and $y=a$ and

Limits of x : from y to a (i.e) $x=0$ and $x=a$

Let us transform this integral in polar co-ordinates by taking $x=r\cos\theta$, $y=r\sin\theta$ and $dx\,dy=r\,dr\,d\theta$

$x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2$

$\Rightarrow\sqrt{x^2+y^2}=\sqrt{r^2}=r$

$x=0\Rightarrow r\cos\theta=0\Rightarrow r=0$

$x=a\Rightarrow r\cos\theta=a\Rightarrow r=\frac{a}{\cos\theta}\Rightarrow r=a\sec\theta$

$\therefore$ Limits of r : 0 to $a\sec\theta$

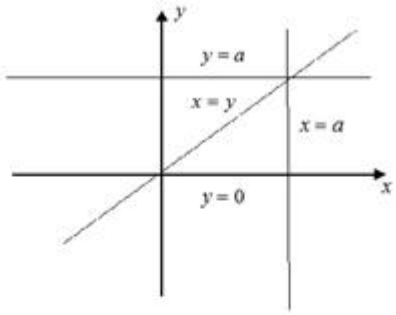
$x=y\Rightarrow r\cos\theta=r\sin\theta\Rightarrow\cos\theta=\sin\theta\Rightarrow 1$

$$= \frac{\sin\theta}{\cos\theta} \Rightarrow 1 = \tan\theta \Rightarrow \tan^{-1}(1) = \theta \Rightarrow \frac{\pi}{4} = \theta$$

$$y = 0 \Rightarrow r\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = \sin^{-1}(0) \Rightarrow \theta = 0$$

$\therefore$ Limits of θ: 0 to $\frac{\pi}{4}$

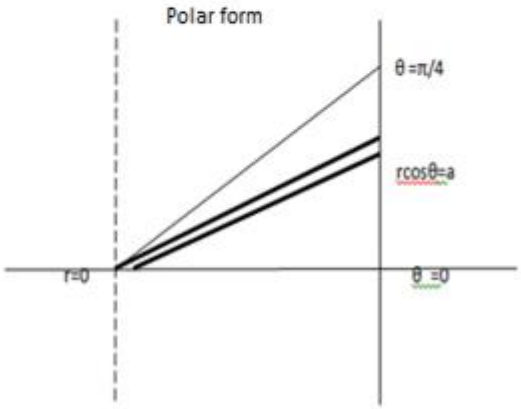
$$\int_0^a \int_y^a \frac{x^2 dxdy}{\sqrt{x^2 + y^2}} = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\cancel{r}} \cancel{r}\, dr d\theta$$



$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} r^2\cos^2\theta dr d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3}\right]_0^{a\sec\theta} \cos^2\theta d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} a^3\sec^3\theta\cos^2\theta d\theta$$



$$= \frac{a^3}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta \frac{1}{\sec^2\theta} d\theta$$

$$= \frac{a^3}{3}\int_0^{\frac{\pi}{4}} \sec\theta d\theta$$

$$= \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3}\left[\log(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}) - \log(\sec 0 + \tan 0)\right] \qquad \left[\because \sec\frac{\pi}{4} = \frac{1}{\sqrt{2}}; \tan\frac{\pi}{4} = 1\right]$$

$$= \frac{a^3}{3}\left[\log(\sqrt{2}+1) - \log(1+0)\right] \qquad \left[\because \sec 0 = 1; \tan 0 = 0; \log 1 = 0\right]$$

$$= \frac{a^3\log(\sqrt{2}+1)}{3}$$

14 **Evaluate $\iint_R r^2 \sin\theta\, dr d\theta$ where R is the semi circle $r = 2a\cos\theta$ above the initial line**

Solution:

The region R of integration is the semi circle $r = 2a\cos\theta$ above the initial line

Where,limits of r : from 0 to $2a\cos\theta$ and limits of θ : from 0 to $\frac{\pi}{2}$

Hence the desired area $= \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta dr d\theta$

$$= \int_0^{\pi/2} \left(\int_0^{2a\cos\theta} r^2 dr\right) \sin\theta d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} \sin\theta d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (2a)^3 \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta$$

Reduction formulae,

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \begin{cases} \dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5)...}{(m+n)(m+n-2)(m+n-4)(m+n-6)...}\dfrac{\pi}{2} & \text{if m \& n are even} \\ \dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5)...}{(m+n)(m+n-2)(m+n-4)(m+n-6)...}.1 & \text{otherwise} \end{cases}$$

$$= \frac{8a^3}{3}\left(\frac{(3-1)}{(3+1)(3+1-2)}\right) = \frac{8a^3}{3}\left(\frac{2}{4(2)}\right) = \frac{8a^3}{3}\left(\frac{2}{8}\right) = \frac{2a^3}{3}$$

15 **Evaluate $\iiint_V xyz\,dxdydz$ where V is the volume of the positive octant of the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates** **(NOV/DEC 2020)**

Solution:

Changing into polar spherical co ordinates by putting

$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$dxdydz = r^2 \sin\theta \; drd\theta d\phi$

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which

Limits of r: 0 to 1

Limits of θ: 0 to $\pi/2$ and

Limits of ϕ: 0 to $\pi/2$

Therefore, volume of the sphere,

$$= 8 \times \text{Volume of the first octant (in Polar form)}$$

$$= \int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r\sin\theta\cos\phi \, r\sin\theta\sin\phi \, r\cos\theta r^2 \sin\theta \, drd\theta d\phi$$

$$= \left[\int_0^1 r^5 dr \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta \, d\theta \int_0^{\frac{\pi}{2}} \cos\phi\sin\phi \, d\phi\right]$$

$$= \left[\frac{r^6}{6}\right]_0^1 \int_0^{\frac{\pi}{2}} \sin^3\theta\cos\theta \, d\theta \int_0^{\frac{\pi}{2}} \cos\phi\sin\phi \, d\phi$$

$$=\frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\int_0^{\frac{\pi}{2}}\cos\phi\sin\phi\,d\phi \qquad [\because 2\cos\phi\sin\phi=\sin 2\phi]$$

$$=\frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\int_0^{\frac{\pi}{2}}\left[\frac{\sin 2\phi}{2}\right]d\phi$$

$$=\frac{1}{6}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\left[\frac{-\cos 2\phi}{4}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{-1}{24}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta\left[\cos 2\frac{\pi}{2}-\cos 0\right]$$

$$=\frac{-1}{24}(-1-1)\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta$$

$$=\frac{2}{24}\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\,d\theta$$

Reduction Formulae,

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta=\begin{cases}\dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5)...}{(m+n)(m+n-2)(m+n-4)(m+n-6)...}\dfrac{\pi}{2} & \text{if m \& n are even}\\ \dfrac{(m-1)(m-3)(m-5)....(n-1)(n-3)(n-5)...}{(m+n)(m+n-2)(m+n-4)(m+n-6)...}.1 & \text{otherwise}\end{cases}$$

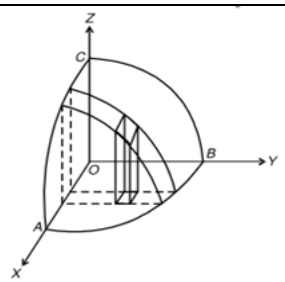
$$=\frac{1}{12}\left(\frac{(3-1)}{(4)(4-2)}\right)=\frac{1}{12}\left(\frac{(2)}{(4)(2)}\right)=\frac{1}{48}$$

Hence, Volume of the sphere is 1/48.

16 **Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$**

Solution:

The required volume $V=8\times$ Volume in the first octant



Here z varies from z=0 to z=c$\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

y varies from y=0 to y=b$\sqrt{1-\frac{x^2}{a^2}}$

x varies from x=0 to x=a

$$\text{Hence, } V=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dzdydx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}(z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dydx$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dydx$$

$$=\frac{8c}{b}\int_0^a\left(\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\,dy\right)dx$$

$$=\frac{8c}{b}\int_0^a\left(\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{k^2-y^2}\,dy\right)dx \qquad \text{where } k^2=b^2\left(1-\frac{x^2}{a^2}\right)\Rightarrow k=b\sqrt{1-\frac{x^2}{a^2}}$$

$$=\frac{8c}{b}\int_0^a\left(\int_0^k\sqrt{k^2-y^2}\,dy\right)dx$$

$$=\frac{8c}{b}\int_0^a\left[\frac{y}{2}\sqrt{k^2-y^2}+\frac{k^2}{2}\sin^{-1}\frac{y}{k}\right]_0^k dx$$

(Using the formula) $\left(\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$

$$=\frac{8c}{b}\int_0^a\left[\left(\frac{k}{2}\sqrt{k^2-k^2}+\frac{k^2}{2}sin^{-1}\frac{k}{k}\right)-0\right]dx$$

$$=\frac{8c}{b}\int_0^a\left[0+\frac{k^2}{2}sin^{-1}(1)\right]dx \qquad \left[\because sin^{-1}1=\frac{\pi}{2}\right]$$

$$=\frac{8c}{b}\int_0^a\frac{k^2}{2}sin^{-1}(1)\,dx$$

$$=\frac{8c}{b}\int_0^a\frac{k^2}{2}.\frac{\pi}{2}dx$$

$$=\frac{\overset{2}{\cancel{8}}\pi c}{\cancel{4}b}\int_0^a k^2dx$$

$$=\frac{2\pi c}{b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx$$

$$=\frac{2\pi cb^{\cancel{2}}}{\cancel{b}}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$$

$$=2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$=2\pi bc\left[\left(a-\frac{a^{\cancel{3}}}{3\cancel{a^2}}\right)-0\right]$$

$$=2\pi bc\left(\frac{3a-a}{3}\right)$$

	$= 2\pi bc\left(\frac{2a}{3}\right)$ $= \frac{4}{3}\pi abc$ cubic units
17	**Evaluate** $\int_0^{2a}\int_0^{x}\int_y^{x}(xyz)\,dz\,dy\,dx$. **(NOV/DEC 2019)**
	Solution: Given $\int_0^{2a}\int_0^{x}\int_y^{x}(xyz)\,dz\,dy\,dx$ $\int_{x=0}^{x=2a}\int_{y=0}^{y=x}\int_{z=y}^{z=x}(xyz)\,dz\,dy\,dx = \int_0^{2a}\int_0^{x} xy\left(\frac{z^2}{2}\right)_y^x dydx$ $\left(\because \int x^n dx = \frac{x^{n+1}}{n+1}+c\right)$ $= \frac{1}{2}\int_0^{2a}\int_0^{x} xy\left(x^2-y^2\right)dydx$ $= \frac{1}{2}\int_0^{2a}\int_0^{x} x^3y - xy^3 dydx$ $= \frac{1}{2}\int_0^{2a}\left(\frac{x^3y^2}{2}-\frac{xy^4}{4}\right)_0^x dx$ $= \frac{1}{2}\int_0^{2a}\left(\frac{x^5}{2}-\frac{x^5}{4}\right)_0^x dx$ $= \frac{1}{8}\int_0^{2a} x^5 dx = \frac{1}{8}\left(\frac{x^6}{6}\right)_0^{2a}$ $= \frac{1}{8}\left(\frac{(2a)^6}{6}\right) = \frac{\cancel{64}^{4}\,a^6}{\cancel{48}_{3}}$ $= \frac{4a^6}{3}$
18	**Evaluate** $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dxdydz$
	Solution: $I = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dzdydx$ $= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dydx$

$$=\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\left(\sqrt{1-x^2-y^2}\right)^2-0\right]dydx$$

$$=\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy\left(1-x^2-y^2\right)dydx$$

$$=\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} x\left(y-x^2y-y^3\right)dydx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dydx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{\left(1-x^2\right)y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{2}\int_0^1 x\left[\left(\frac{\left(1-x^2\right)\left(\sqrt{1-x^2}\right)^2}{2}-\frac{\left(\sqrt{1-x^2}\right)^4}{4}\right)-\left(\frac{\left(1-x^2\right)(0)}{2}-\frac{0}{4}\right)\right]dx$$

$$=\frac{1}{2}\int_0^1 x\left[\left(\frac{\left(1-x^2\right)\left(\sqrt{1-x^2}\right)^2}{2}-\frac{\left(\sqrt{1-x^2}\right)^4}{4}\right)-0\right]dx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{\left(1-x^2\right)\left(1-x^2\right)}{2}-\frac{\left(1-x^2\right)^2}{4}\right]dx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{2\left(1-x^2\right)^2}{4}-\frac{\left(1-x^2\right)^2}{4}\right]dx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{\left(1-x^2\right)^2}{4}\right]dx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{\left(1-2x^2+x^4\right)}{4}\right]dx$$

$$=\frac{1}{8}\int_0^1 (x-2x^3+x^5)dx$$

	$= \frac{1}{8}\left[\frac{x^2}{2} - \frac{\cancel{2}x^4}{\cancel{4}_{2}} + \frac{x^6}{6}\right]_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$
19	**Find the volume of the cylinder bounded by $x^2 + y^2 = 4$ and then planes $y + z = 4$ and $z = 0$ using triple integral (NOV/DEC 2020)**
	Solution: Here z varies from $z = 0$ to $z = 4 - y$ y varies from $y = -\sqrt{4-x^2}$ to $y = \sqrt{4-x^2}$ x varies from $x = -2$ to $x = 2$ $V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-y} dzdydx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4]_0^{4-y} dydx$ $= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4-y]dydx = \int_{-2}^{2}\left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$ $= \int_{-2}^{2}\left[\left(4\sqrt{4-x^2} - \frac{\left(\sqrt{4-x^2}\right)^2}{2}\right) - \left(-4\sqrt{4-x^2} - \frac{\left(\sqrt{4-x^2}\right)^2}{2}\right)\right]dx$ $= \int_{-2}^{2}\left[\left(4\sqrt{4-x^2} - \frac{4-x^2}{2}\right) - \left(-4\sqrt{4-x^2} - \frac{4-x^2}{2}\right)\right]dx$ $= \int_{-2}^{2}\left(4\sqrt{4-x^2} - \frac{4-x^2}{2} + 4\sqrt{4-x^2} + \frac{4-x^2}{2}\right)dx$ $= \int_{-2}^{2}\left(4\sqrt{4-x^2} + 4\sqrt{4-x^2}\right)dx$ $= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx = 2\int_0^2 8\sqrt{4-x^2}\,dx \qquad \left(\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right)$ $= 16\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$ $\left[\because \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right]$ $= 16\left\{\left[\frac{x}{2}\sqrt{4-(2)^2} + \frac{4}{2}sin^{-1}\left(\frac{2}{2}\right)\right] - 0\right\}$ $= 16\left(0 + \cancel{2}\frac{\pi}{\cancel{2}}\right)$ $V = 16\pi$
20	**Evaluate $\iiint_V dx\,dy\,dz$ where V is the finite region of space (tetrahedron) bounded by the planes $x = 0, y = 0, z = 0$ & $2x + 3y + 4z = 12$. (NOV/DEC 2018)**
	Solution: Here x varies from 0 to 6

y varies from 0 to $\frac{12-2x}{3}$

z varies from 0 to $\frac{12-2x-3y}{4}$

$$\iiint_V dx\,dy\,dz = \int_0^6 \int_0^{\frac{12-2x}{3}} \int_0^{\frac{12-2x-3y}{4}} dz\,dy\,dx$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} [z]_0^{\frac{12-2x-3y}{4}} dy\,dx$$

$$= \int_0^6 \int_0^{\frac{12-2x}{3}} \left[\left(\frac{12-2x-3y}{4}\right)-0\right] dy\,dx$$

$$= \frac{1}{4}\int_0^6 \int_0^{\frac{12-2x}{3}} (12-2x-3y)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6 \left(12y-2xy-\frac{3y^2}{2}\right)_0^{\frac{12-2x}{3}} dx$$

$$= \frac{1}{4}\int_0^6 12\left(\frac{12-2x}{3}\right)-2x\left(\frac{12-2x}{3}\right)-\frac{3}{2}\left(\frac{12-2x}{3}\right)^2 dx$$

$$= \frac{1}{4}\int_0^6 \left[\frac{144-24x}{3}-\frac{24x}{3}+\frac{4x^2}{3}-\frac{3}{18}\left(144-48x+4x^2\right)\right] dx$$

$$= \frac{1}{4}\int_0^6 \left[\frac{144-24x}{3}-\frac{24x}{3}+\frac{4x^2}{3}-\frac{\cancel{3}}{\underset{6}{\cancel{18}}}\left(144-48x+4x^2\right)\right] dx$$

$$= \frac{1}{4}\int_0^6 \left[\frac{\overset{48}{\cancel{144}}}{\cancel{3}}-\frac{\overset{16}{\cancel{48}}\,x}{\cancel{3}}+\frac{4x^2}{3}-24+8x-\frac{2x^2}{3}\right] dx$$

$$= \frac{1}{4}\int_0^6 \left[48-16x+\frac{4x^2}{3}-24+8x-\frac{2x^2}{3}\right] dx$$

$$= \frac{1}{4}\int_0^6 \left[24-8x+\frac{2x^2}{3}\right] dx$$

$$= \frac{1}{4}\left[24x-8\left(\frac{x^2}{2}\right)+\frac{2}{3}\left(\frac{x^3}{3}\right)\right]_0^6$$

$$= \frac{1}{4}\left[24x-\frac{\overset{4}{\cancel{8}}\,x^2}{\cancel{2}}+\frac{2x^3}{9}\right]_0^6$$

$$=\frac{1}{4}\left[24(6)-4\left(6^2\right)+\frac{2}{9}\left(6^3\right)\right]$$

$$=\frac{1}{4}\left[144-144+\frac{\overset{48}{\cancel{432}}}{\cancel{9}}\right]$$

$$=\frac{1}{4}(0+48)=\frac{\overset{12}{\cancel{48}}}{\cancel{4}}=12$$

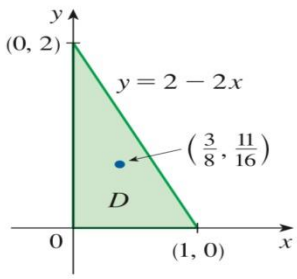
21 **Find the mass and center of mass of a triangular Lamina with vertices (0,0), (1,0) and (0,2) if the density function is $\rho(x,y)=1+3x+y$**

Solution:

Given the vertices (0,0), (1,0) and (0,2)

The equation of the passing through (1,0) and (0,2)

WKT the equation of line is



$$y-y_1=\frac{(y_2-y_1)}{(x_2-x_1)}(x-x_1)$$

$$\therefore y-0=-2(x-1)$$

$$\Rightarrow y=-2x+2$$

$$\Rightarrow y=2-2x$$

The mass of the lamina is

$$m=\iint_D \rho(x,y)\,dA$$

$[\because$ The equation of the upper boundary is $y=2-2x]$

$$=\int_0^1\int_0^{2-2x}(1+3x+y)\,dydx$$

$$=\int_0^1\left[y+3xy+\frac{y^2}{2}\right]_{y=0}^{y=2-2x}dx$$

$$=\int_0^1\left[\left((2-2x)+3x(2-2x)+\frac{(2-2x)^2}{2}\right)-0\right]dx$$

$$=\int_0^1\left(2-2x+6x-6x^2+\frac{4-8x+4x^2}{2}\right)dx$$

$$=\int_0^1\left(2+4x-6x^2+\frac{\cancel{2}\left(2-4x+2x^2\right)}{\cancel{2}}\right)dx$$

$$=\int_0^1\left(4-4x^2\right)dx$$

$$=4\int_0^1\left(1-x^2\right)dx=4\left[x-\frac{x^3}{3}\right]_0^1=\frac{4}{3}\left[3x-x^3\right]_0^1=\frac{4}{3}\left[(3-1)-0\right]=\frac{8}{3}$$

The coordinates $(\bar{x}\ \bar{y})$ of the center of the mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{My}{m} = \frac{1}{m}\iint_D x\rho(x,y)dA$$

$$\bar{y} = \frac{Mx}{m} = \frac{1}{m}\iint_D y\rho(x,y)dA$$

By using the above formulas,

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)dA = \frac{3}{8}\int_0^1\int_0^{2-2x}\left(x+3x^2+xy\right)dydx$$

$$= \frac{3}{8}\int_0^1\left[xy+3x^2y+x\frac{y^2}{2}\right]_{y=0}^{y=2-2x}dx$$

$$= \frac{3}{8}\int_0^1\left[x(2-2x)+3x^2(2-2x)+x\frac{(2-2x)^2}{2}\right]dx$$

$$= \frac{3}{8}\int_0^1\left[2x-2x^2+6x^2-2x^3+\frac{x(4-8x+4x^2)}{2}\right]dx$$

$$= \frac{3}{8}\int_0^1\left[2x-2x^2+6x^2-2x^3+\frac{\cancel{2}x\left(2-4x+2x^2\right)}{\cancel{2}}\right]dx$$

$$= \frac{3}{8}\int_0^1\left[4x-4x^3\right]dx = \frac{3\times\cancel{4}}{\underset{2}{\cancel{8}}}\int_0^1\left[x-x^3\right]dx$$

$$= \frac{3}{2}\int_0^1(x-x^3)dx = \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left[\frac{2-1}{4}\right] = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)dA = \frac{3}{8}\int_0^1\int_0^{2-2x}\left(y+3xy+y^2\right)dydx$$

$$= \frac{3}{8}\int_0^1\left[\frac{y^2}{2}+3x\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=0}^{y=2-2x}dx$$

$$= \frac{3}{8}\int_0^1\left[\left(\frac{(2-2x)^2}{2}+\frac{3x(2-2x)^2}{2}+\frac{(2-2x)^3}{3}\right)-0\right]dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{\left(4-8x+4x^2\right)}{2}+\frac{3x\left(4-8x+4x^2\right)}{2}+\frac{\left(8-24x+24x^2-8x^3\right)}{3}\right)dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{4-8x+4x^2+12x-24x^2+12x^3}{2}+\frac{8-24x+24x^2-8x^3}{3}\right)dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{12x^3-20x^2+4x+4}{2}+\frac{8-24x+24x^2-8x^3}{3}\right)dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{\cancel{2}\left(6x^3-10x^2+2x+2\right)}{\cancel{2}}+\frac{8-24x+24x^2-8x^3}{3}\right)dx$$

	$=\frac{3}{8}\int_0^1\left(\frac{18x^3-30x^2+6x+6+8-24x+24x^2-8x^3}{3}\right)dx$ $=\frac{\cancel{3}}{\underset{8}{\cancel{24}}}\int_0^1 10x^3-6x^2-18x+14dx$ $=\frac{\cancel{2}}{\underset{4}{\cancel{8}}}\int_0^1\left(5x^3-3x^2-9x+7\right)dx$ $=\frac{1}{4}\int_0^1\left(5x^3-3x^2-9x+7\right)dx$ $=\frac{1}{4}\left[5\frac{x^4}{4}-x^3-9\frac{x^2}{2}+7x\right]_0^1$ $=\frac{1}{4}\left[\left(\frac{5}{4}-1-\frac{9}{2}+7\right)-0\right]=\frac{1}{4}\left(\frac{5}{4}-\frac{9}{2}+6\right)$ $=\frac{1}{4}\left(\frac{5-18+24}{4}\right)=\frac{11}{16}$ The center of mass $(\bar{x}\ \bar{y})$ at the point $\left(\frac{3}{8},\frac{11}{16}\right)$
22	**Find the mass and center of mass of a lamina with density function $\rho(x,y)=6x$ covering the triangle D bounded by the x-axis, the line $y=x$ and the line $y=2-x$**
	Solution: The mass of a lamina, $m=\iint_D \rho(x,y)dA=\int_0^1\int_y^{2-y} 6xdxdy$ $=\int_0^1 6\left[\frac{x^2}{2}\right]_y^{2-y}dy$ $=\int_0^1 3\left[(2-y)^2-y^2\right]dy$ $=3\left[-\frac{(2-y)^3}{3}-\frac{y^3}{3}\right]_0^1$ $=-\left[(2-y)^3+y^3\right]_0^1$ $=-\left[\left((2-1)^3+1^3\right)-\left((2-0)^3+0\right)\right]$ $=-(2-8)$ $=6$ The centre mass of a lamina, $\bar{x}=\frac{My}{m}=\frac{1}{m}\iint_D x\rho(x,y)dA$ $\bar{y}=\frac{Mx}{m}=\frac{1}{m}\iint_D y\rho(x,y)dA$

By using the above formulas,

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{1}{6}\int_0^1\int_y^{2-y} 6x^2\,dxdy$$

$$= \frac{\cancel{6}}{\cancel{6}}\int_0^1\int_y^{2-y} x^2\,dxdy$$

$$= \frac{1}{3}\int_0^1 \left[x^3\right]_y^{2-y} dy$$

$$= \frac{1}{3}\int_0^1 (2-y)^3 - y^3 dy$$

$$= \frac{1}{3}\left[\frac{(2-y)^4}{-4} - \frac{y^4}{4}\right]_0^1$$

$$= -\frac{1}{3}\left[\frac{(2-y)^4}{4} + \frac{y^4}{4}\right]_0^1$$

$$= -\frac{1}{12}\left[\left((2-1)^4 + 1^4\right) - \left((2-0)^4 + 0\right)\right]$$

$$= -\frac{1}{12}(2-16) = \frac{\overset{7}{\cancel{14}}}{\underset{6}{\cancel{12}}}$$

$$= \frac{7}{6}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{6}\int_0^1\int_y^{2-y} 6xy\,dxdy$$

$$= \frac{\cancel{6}}{\cancel{6}}\int_0^1\int_y^{2-y} xy\,dxdy$$

$$= \frac{1}{2}\int_0^1 y\left[x^2\right]_y^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y\left[(2-y)^2 - y^2\right] dy$$

$$= \frac{1}{2}\int_0^1 4y - 4y^2 + y^3 - y^3 dy$$

$$= \frac{\overset{2}{\cancel{4}}}{\cancel{2}}\int_0^1 \left(y - y^2\right) dy$$

$$= 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

	$= 2\left[\left(\frac{1}{2}-\frac{1}{3}\right)-0\right]$ $= 2\left(\frac{1}{6}\right)$ $= \frac{1}{3}$ The center of mass is at the point $\left(\frac{7}{6},\frac{1}{3}\right)$
23	**Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x,y)=\rho$, center the origin, and radius a.**
	Solution: The boundary of D is the circle $x^2+y^2=a^2$ and in polar co-ordinates D is described by $0 \le \theta \le 2\pi,\quad 0 \le r \le a$ and let $y = r\sin\theta$ The moment of inertia of the lamina about the x-axis is $I_x = \iint_D y^2 \rho dA = \rho\int_0^{2\pi}\int_0^a (r\sin\theta)^2 r dr d\theta$ $= \rho\int_0^{2\pi}\left[\int_0^a r^3 \sin^2\theta dr\right]d\theta$ $= \rho\int_0^{2\pi}\sin^2\theta\left[\int_0^a r^3 dr\right]d\theta$ $= \rho\int_0^{2\pi}\sin^2\theta\left[\frac{r^4}{4}\right]_0^a d\theta$ $= \frac{\rho}{4}\int_0^{2\pi}\sin^2\theta\left(a^4-0\right)d\theta$ $= \frac{a^4\rho}{4}\int_0^{2\pi}\sin^2\theta\, d\theta$ $= \frac{a^4\rho}{4}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$ $= \frac{a^4\rho}{8}\int_0^{2\pi}\left(1-\cos 2\theta\right)d\theta$ $= \frac{a^4\rho}{8}\int_0^{2\pi}d\theta - \int_0^{2\pi}\cos 2\theta\, d\theta$ $= \frac{a^4\rho}{8}\left[(\theta)_0^{2\pi} - \left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}\right]$

$$= \frac{a^4\rho}{8}\left[(2\pi - 0) - \frac{1}{2}(\sin 4\pi - \sin 0)\right]$$

$$= \frac{a^4\rho}{\cancel{8}_{4}}(\cancel{2}\pi)$$

$$= \frac{\pi\rho a^4}{4}$$

The moment of inertia of the lamina about the y-axis is

$$I_y = \iint_D x^2\rho dA = \rho\int_0^{2\pi}\int_0^a (r\cos\theta)^2 r dr d\theta$$

$$= \rho\int_0^{2\pi}\int_0^a \left(r^2\cos^2\theta\right) r dr d\theta$$

$$= \rho\int_0^{2\pi}\int_0^a r^3\cos^2\theta\, dr d\theta$$

$$= \frac{\rho}{4}\int_0^{2\pi}\cos^2\theta\left(r^4\right)_0^a d\theta$$

$$= \frac{\rho}{4}\int_0^{2\pi}\cos^2\theta\left(a^4 - 0\right)d\theta$$

$$= \frac{a^4\rho}{4}\int_0^{2\pi}\cos^2\theta d\theta$$

$$= \frac{a^4\rho}{4}\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^4\rho}{8}\left[\int_0^{2\pi} d\theta + \int_0^{2\pi}\cos 2\theta\, d\theta\right]$$

$$= \frac{a^4\rho}{8}\left[(\theta)_0^{2\pi} + \left(\frac{\sin 2\theta}{2}\right)_0^{2\pi}\right]$$

$$= \frac{a^4\rho}{8}\left[(2\pi - 0) + \frac{1}{2}(\sin 4\pi - \sin 0)\right]$$

$$= \frac{a^4\rho}{\cancel{8}_{4}}(\cancel{2}\pi)$$

$$= \frac{\pi\rho a^4}{4}$$

$$I_0 = I_x + I_y = \frac{\pi\rho a^4}{4} + \frac{\pi\rho a^4}{4} = \frac{\cancel{2}\pi\rho a^4}{\cancel{4}_{2}} = \frac{\pi\rho a^4}{2}$$

The mass of the disk is $m = density \times area = \rho(\pi a^2)$

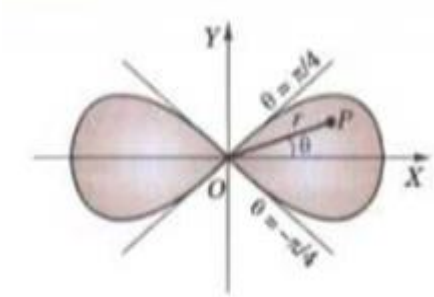
So the moment of inertia of the disk about the origin is written as

$$I_0 = \frac{\pi\rho a^4}{2} = \frac{1}{2}(\rho\pi a^2)a^2 = \frac{1}{2}ma^2$$

24 **Find the moment of inertia of the area bounded by the curve $r^2 = a^2\cos 2\theta$ about its axis**

Solution: Given curve is symmetrical about the pole and for half of the loop in the first

quadrant θ to $\frac{\pi}{4}$



Elementary area at $P(r,\theta) = r\,d\theta dr$

If ρ be the surface density, then elementary mass

$$= \rho r d\theta dr \qquad ....(i)$$

$\therefore$ Its total mass $M = 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho\, r dr d\theta$

$$= \cancel{4}^{2}\rho\int_0^{\pi/4}\left[\frac{r^2}{\cancel{2}}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= 2\rho\int_0^{\pi/4}\left[\left(a\sqrt{(\cos 2\theta)}\right)^2 - 0\right]d\theta$$

$$= 2a^2\rho\int_0^{\pi/4}\cos 2\theta\, d\theta \qquad .....(ii)$$

$$= 2a^2\rho\left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= \frac{\cancel{2}a^2\rho}{\cancel{2}}\left(\sin 2\theta\right)_0^{\pi/4}$$

$$= a^2\rho\left[\sin \cancel{2}\left(\frac{\pi}{\cancel{4}_{2}}\right) - \left(\sin 0\right)\right]$$

$$= a^2\rho\left[\sin\left(\pi/2\right)\right] = a^2\rho(1) = a^2\rho$$

Now the moment of inertia of the elementary mass (i) about the $x-$axis and let $y = r\sin\theta$

$$= \rho r\, d\theta dr\,.\,y^2 = \rho r\, d\theta dr(r\sin\theta)^2$$

$$= \rho r^3 \sin^2\theta\, dr d\theta$$

Hence the moment of inertia of the whole area

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, dr\, d\theta$$

$$= 4\rho\int_0^{\pi/4}\sin^2\theta\left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \frac{\cancel{4}\rho}{\cancel{4}}\int_0^{\pi/4}\sin^2\theta\left[r^4\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho\int_0^{\pi/4}\sin^2\theta\left[\left(a\sqrt{(\cos 2\theta)}\right)^4 - (0)\right]d\theta$$

$$= a^4\rho\int_0^{\pi/4}\sin^2\theta\cos^2 2\theta d\theta$$

$$= a^4\rho\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta d\theta$$

put $2\theta = \varphi, d\theta = \frac{d\varphi}{2}$

Limit of $\varphi = 0$ to $\pi/2$

$$= \rho a^4 \int_0^{\pi/2} \cos^2\varphi . \sin^2\frac{\varphi}{2}\frac{d\varphi}{2}$$

$$= \rho a^4 \int_0^{\pi/2} \cos^2\varphi . \left(\frac{1-\cos\varphi}{2}\right)\frac{d\varphi}{2} \qquad \left[\because 2\sin^2\frac{\varphi}{2} = 1-\cos\varphi\right]$$

$$= \frac{\rho a^4}{4}\int_0^{\pi/2} \cos^2\varphi . (1-\cos\varphi)\, d\varphi$$

$$= \frac{\rho a^2}{4}\int_0^{\pi/2} \left(\cos^2\varphi - \cos^3\varphi\right) d\varphi$$

$$= \frac{\rho a^2}{4}\left[\int_0^{\pi/2} \cos^2\varphi d\varphi - \int_0^{\pi/2} \cos^3\varphi d\varphi\right]$$

$$= \frac{\rho a^2}{4}\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - \left(\frac{2}{3}\right)\right] \qquad [\text{By reduction formula}]$$

$$= \frac{\rho a^2}{4}\left(\frac{\pi}{4} - \frac{2}{3}\right) = \frac{\rho a^2}{4}\left(\frac{3\pi - 8}{12}\right)$$

$$= \frac{\rho a^2}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8) \qquad [By(ii)]$$