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ORIGINAL ARTICLE

Why golden rectangle is used so often by architects: A mathematical approach

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Abstract It is often found in the literature that many researchers have studied or documented the use of golden rectangle or Fibonacci rectangle in architectural design. In this way, a lot of well-known architects in the history, knowingly or unknowingly, have employed either the golden rectangle or the Fibonacci rectangle in their works. Using some mathematical tools, this paper tried to approach one of the properties of the golden rectangle (or the Fibonacci rectangle) and its significance to architectural design, which could lead to state one hypothesis about why architects have used them so often.

This work begins with an algorithm which constructs a Fibonacci rectangle and a golden rectangle. Then adjacency among the squares (which are arranged inside them) is defined, by considering each square as a room or an architectural space. At the end, using some tools of the graph theory, it has been proved that they are one of the best arrangements of squares (or rectangles) inside a rectangle, from the point of view of connectivity.

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1. Introduction

1.1. The golden ratio

The *golden number* is, in a sense, the most natural real number, since it can be written as:

$$\varphi = [\overline{1}],$$

without reference to any numbering system. This is standard abbreviation for the continued fraction expansion

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

From this, it can be seen that $\varphi - 1 = \frac{1}{\varphi}$, so that φ is also equal to $\frac{1+\sqrt{5}}{2}$. The golden number is also the limit of the sequence of *convergents*

$$(p_n/q_n)_{n=0}^{\infty} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \dots\right),$$

which involves the successive Fibonacci numbers. This is not just one more occurrence of the Fibonacci numbers; in fact, a classical result (see Hardy & Wright [1], Chapter 10) is that the sequence of convergents is a sequence of *best approximations* to φ by rational numbers, in a very specific sense. So, instead of saying that $\varphi = 1.61803\dots$, which is meaningful

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only in the decimal system, it is better to consider φ as a sequence of convergents

$$\varphi = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \ldots\right).$$

1.2. Relation between the golden rectangle and the Fibonacci rectangle

The *golden rectangle* is a rectangle whose ratio of width over height is equal to φ with a geometric property as follows:

One can remove a square with side length one from a rectangle of sides $1 \times \varphi$ and obtain a new rectangle, with sides $\frac{1}{\varphi} \times 1$, which is similar to the original one. Hence, the construction can be repeated (see Walser [2], Chapter 3). The golden rectangle and logarithmic spiral are shown in Fig. 1. The *logarithmic spiral* is centered at the point O which is the intersection of the two diagonals BD and CE_1. Its radius r is reduced by a factor φ each time the angle θ is decreased by $\pi/2$.

The golden rectangle is considered as one of the shape for representing φ in two dimensions (refer [3]). Because of this, φ and golden rectangle have same properties as well as the most visually pleasing constructions.

In a *Fibonacci sequence*, each of its term is obtained from the sum of the two preceding terms i.e. $F_{n+1} = F_n + F_{n-1}$ for $n > 1$ where $F_0 = F_1 = 1$. Here each F_n is called Fibonacci number. A *Fibonacci rectangle* is a rectangle with side lengths x and y such that either x/y or y/x is equal to F_{n+1}/F_n for some non-negative integer n. Naturally, one can construct such a rectangle by successively introducing squares of side lengths $F_0, F_1, F_2, \ldots$ as shown in Fig. 5. It can be easily seen that the ratio of two successive Fibonacci numbers (F_{n+1}/F_n) approaches φ.

If only the arrangement of squares is considered, the two Figures, golden rectangle and Fibonacci rectangle, look similar, whereas, geometrically it is far more convenient to use the Fibonacci rectangle in comparison with the golden rectangle, because, it is comparatively easy to draw a rectangle with integer dimensions (say 5×8) than a rectangle having rational or irrational dimensions respectively.

1.3. Golden rectangle, Fibonacci rectangle and architecture

The φ or golden rectangle has been found in the natural world through human proportions and through growth patterns of many living plants, animals, and insects. Basically, it has been always considered that φ is the most pleasing proportion to human eyes [4,5]. The presence of φ in the design of the Pyramids represents that the Egyptians were aware of the number. A Greek sculptor and mathematician, Phidias (490–430 BC), was first to study and apply Phi, to the design of sculptures for the Parthenon (example of Doric architecture, the main temple of the goddess Athena) [5,6].

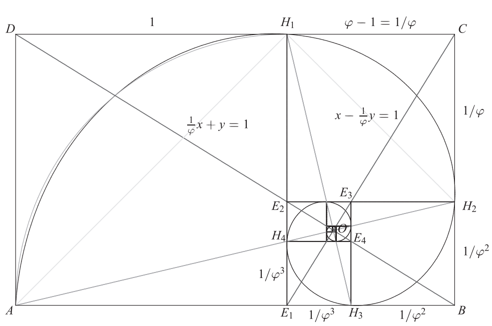

Figure 1 The golden rectangle and logarithmic spiral.

Around 1200 AD, Leonardo Fibonacci (1170–1250 AD), an Italian born mathematician found φ in a numerical series (known as Fibonacci series) and named it *divine proportion*, due to which, Fibonacci series can be used to construct the golden rectangle [3]. The design of Notre Dame in Paris, which was built in between 1163 and 1250, appears to have golden rectangle in number of its key proportions. Likewise, the Renaissance artists used the golden rectangle in their various paintings and sculptures to achieve balance and esthetic beauty [7].

Also, φ is been favorite to many key architects in history, such as, Palladio, Le Corbusier, Pacioli, and Leonardo Da Vinci. Palladio's Villa La Rotonda is designed using φ, see Figs. 12–14 (for details of Palladio's work, refer [8], *I Quattro Libri dell'Architettura*). Le Corbusier himself wrote *Modulor I and II* (refer [9]) where for instance he displayed the composition and drawing of his 'Modulor' figure by using the golden ratio. As a sculptor and applied mathematician, George Hart [10] delves the work of Pacioli and Leonardo Da Vinci given in the book *De Divina Proportione*.

In addition, many recent publications discussed the golden ratio and architectural designs that exist at the different moments of history. In 1986, Burckhardt [11] studied the presence of golden rectangle in a house of 1871 in Basel. In 2000, Mark Reynolds [12] presented the use of the golden section in generating the geometry of Pazzi Chapel of Santa Croce in Florence. In 2013, Fernández-Llebrez and Fran [13] discussed the presence of the golden section and the Fibonacci sequence in the compositional scheme of the Roman Catholic Church Pastoor Van Ars, built by Aldo van Eyck in The Hague in 1968.

It is hard to find many publications showing the presence of Fibonacci sequence in the geometrical composition of architectural design; two of them are mentioned as follows:

Bartoli [14] discussed the case of Palazzo della Signoria, where the Fibonacci rectangle has been found (see Figs. 15 and 16). In the same way, Park and Lee [15] published the underlying design of the Braxton-Shore house by Rudolph Schindler, which is based on the Fibonacci sequence.

From the work of many authors and researchers, it can be easily seen that many buildings and architectural designs (from ancient to contemporary time) have been developed using the golden rectangle or the Fibonacci rectangle, but it is difficult to find a mathematical or logical reason behind it. That is why, this paper tried to provide a possible mathematical explanation to this question in terms of adjacency among the rooms of an architectural design.

2. Rectangular arrangements and connectivity

2.1. Rectangular arrangements

A *rectangular arrangement* is defined as an arrangement of sub-rectangles inside a bigger rectangle where:

1. All the sub-rectangles are *normalized*, i.e. they have only horizontal and vertical sides. For example, Fig. 2(A) is prohibited.
2. There is no *overlapping* among the regions of sub-rectangles, therefore, Fig. 2(B) is not allowed.
3. For obtaining a rectangular arrangement, each sub-rectangle is arranged one by one such that:
   3.1 A new sub-rectangle must be *adjacent* to at least one of the existing sub-rectangles. For example, Fig. 2(C) and (D) are not permitted. Adjacency among the sub-rectangles has been defined in next section.
   3.2 After drawing a sub-rectangle, the composition of the sub-rectangles must be rectangular. It can be clearly seen in Fig. 2(E) that after drawing 2nd and 3rd sub-rectangles, the overall composition is rectangular but in Fig. 2(F), after drawing 3rd sub-rectangle, composition is not rectangular, hence, it is not a rectangular arrangement.
4. There is no *extra space* or *empty spaces* inside the arrangement.

A *rectangular arrangement* is denoted by R^A and R^A made up of n sub-rectangles i.e. of order n, is denoted by $R^A(n)$. Clearly, Fibonacci rectangle or golden rectangle satisfies all the conditions of a R^A. For example, a Fibonacci rectangle shown in Fig. 5(G) is a rectangular arrangement of order seven.

2.2. Adjacency and connectivity

Every rectangle has four sides which are termed *walls* of the rectangle. A part of a wall of a rectangle is its *sub-wall*. Mathematically, if W is a wall of side length j, a *sub-wall* of W is a proper connected part of W. Hence, it is a closed interval of length k strictly included in W, i.e. $0 < k < j$.

The two distinct rectangles are *adjacent* if they share a wall or a sub-wall. For better understanding of concept of adjacency, refer to Fig. 3. In Fig. 3(a), rectangles A and B are adjacent because they share a wall; in Fig. 3(b), rectangles A and B share a sub-wall, hence they are adjacent; in Fig. 3(c), rectangle A shares a sub-wall with rectangle B while rectangle B shares its wall with rectangle A, hence they are adjacent. But in the last case i.e. Fig. 3(d), rectangles A and B are not adjacent.

A *graph* $G = (V, E)$ is a mathematical structure consisting of two finite sets V and E. The elements of V are called *vertices* and the elements of E are called *edges*. Each edge has a set of one or two vertices associated with it, which are called its *end-points*. A *simple graph* has neither self-loops nor multi-edges. Two vertices u and v are called *adjacent* if an edge exists between them. In an undirected graph G, two vertices u and v are *connected* if G contains a path (or walk) from u to v.

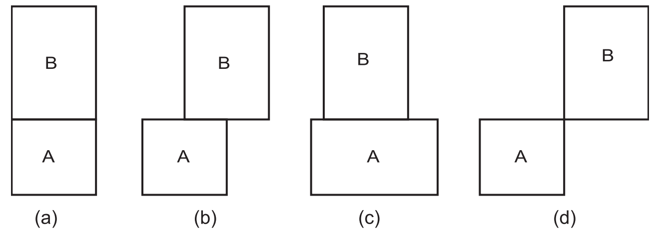

Figure 3 Describing adjacency among the sub-rectangles of a rectangular arrangement.

A graph is *connected* if for every pair of vertices u and v, there is a walk from u to v. For details about these definitions refer to Gross and Yellen [16], Chapter 1.

The *adjacency graph of a* R^A, is a simple undirected graph, obtained by representing each of its sub-rectangle as a *vertex* and then drawing an *edge* between any two vertices; if the corresponding sub-rectangles are *adjacent*. The corresponding adjacency graph and its number of edges are denoted by G^R and E^R respectively.

The *degree of connectivity of a* R^A is given in the terms of connectivity of the corresponding adjacency graph. Therefore, comparison of connectivity of the different rectangular arrangements is done by comparing connectivity of the corresponding adjacency graphs. In the literature, there exist various measures to compare the connectivity of two adjacency graphs having the same number of vertices. For example, they can be compared on the basis of number of edges, diameter, average distance, number of cycles etc. In this work, the *number of edges* in the adjacency graph is regarded as a measure of connectivity. For further details about these measures, see Rodrigue et al. ([17], Chapter 2, *measures and indices of graph theory*).

If the two connected adjacency graphs have same number of vertices then the adjacency graph having more edges is *more connected*. For better understanding, refer to Fig. 4, where R_1^A and R_2^A are rectangular arrangements with their adjacency graphs G_1^R and G_2^R respectively. It can be clearly seen that G_1^R and G_2^R have equal number of vertices, i.e. 5 and different number of edges, i.e., 7 and 8 edges respectively. Therefore, R_2^A is more connected than R_1^A.

2.3. Why connectivity

In architectural terms, it can be easily observed that the overall connectivity of the floor plan and the adjacency among the spaces are essential. For example, while constructing a house, logically, it is preferred to have dining room close to kitchen; at the same time there is no point in having playing room close to the library. During this research, no work has been found

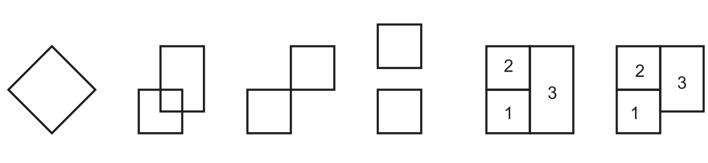

Figure 2 Conditions for constructing a rectangular arrangement.

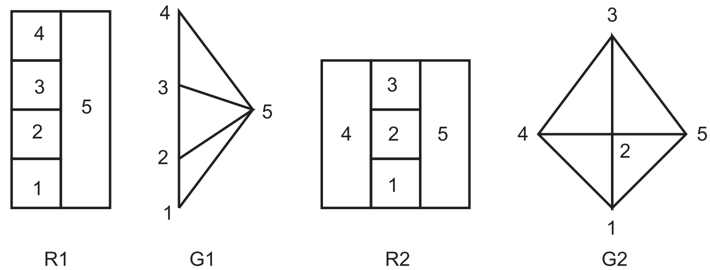

Figure 4 Comparing the connectivity of two different rectangular arrangements.

where connectivity has been considered for the design of a floor plan. But, in the literature, it can be seen that without mentioning connectivity, architects have used this concept in some form or other. For example, Roth [18] has considered adjacency among the cells for constructing a rectangular floor plan. In the coming sections, it would be interesting to see that this concept of adjacency and connectivity would help in providing a mathematical answer, to the question of using the Fibonacci rectangle and golden rectangle so often by architects.

3. Construction of a Fibonacci rectangle

Lemmas 1 and 2 are well known results that already exist in the literature in same or other form.

Lemma 1. *The process of cutting off a square can be done by starting from any Fibonacci rectangle. The ratios of width over height for the successive residual rectangles run through all quotients of the form* $F_n/F_{n-1}, F_{n-1}/F_{n-2}$, *etc., until the last square has been reached, which is of size 1. Then the residual rectangle is a square of side length* $F_1 = F_0 = 1$.

Proof. Removing a square of side length F_n from a rectangle with sides $F_{n+1} \times F_n$ yields a rectangle with sides $F_n \times F_{n-1}$,since $F_{n+1} - F_n = F_{n-1}$. And this process can be repeated until a rectangle with sides $F_2 \times F_1 = 2 \times 1$ is obtained. The next square to be cut off has side length $F_1 = 1$ and the residual rectangle is a square of side length 1. For further details, refer to Fig. 5 from (G) to (A) and see Walser [1], page 39. □

Lemma 2. *One can construct a sequence of Fibonacci rectangles by reversing the process described in Lemma 1. Starting from two unit squares one above another, one first adjoins a square of side length 2 to their right, so as to obtain a Fibonacci rectangle with sides* $F_3 \times F_2$. *Then one adjoins a square of side length* F_3 *below, so as to obtain a Fibonacci rectangle with sides* $F_5 \times F_3$, *etc. following a clockwise movement.*

Proof. The possibility of this construction of a sequence of Fibonacci rectangles follows from the relation $F_{n+1} = F_n + F_{n-1}$. For details refer to Fig. 5 from (A) to (G), where a sequence of Fibonacci rectangles is displayed. □

A Fibonacci rectangle having n squares is called a Fibonacci rectangle of order n and it is denoted by $F^R(n)$. The golden rectangle of order n is denoted by Go^R. The logarithmic spiral can be replaced by its natural approximation by quarter circles in the successive squares appearing in this construction. It is termed *Fibonacci spiral*. For example, the Fibonacci spiral is shown in Fig. 5(G).

4. Maximum number of edges in the adjacency graph of a rectangular arrangement

4.1. Adjacent sides and adjacent numbers

This section provides an important result, which will be further used to prove that both the golden rectangle and the Fibonacci rectangle are best connected.

It is obvious to see that a R^A consists of sub-rectangles but, it is also a rectangle having four sides. Each of its side is termed

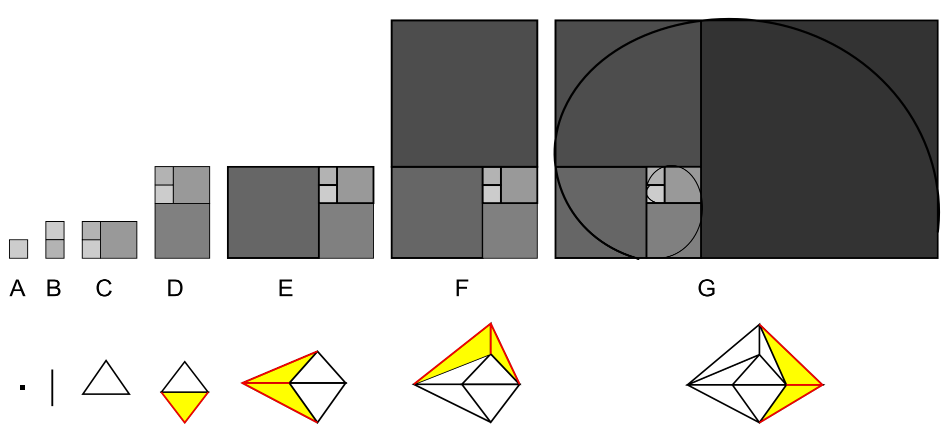

Figure 5 A sequence of Fibonacci rectangles, corresponding adjacency graphs and a Fibonacci spiral.

adjacent side. For example, in Fig. 6, the drawn R^A has four adjacent sides which are illustrated in bold and are called *left*, *upper*, *right* and *lower* adjacent sides. Also, each adjacent side is made up of different number of walls of the sub-rectangles. These numbers are termed adjacent numbers. The adjacent numbers, associated with left, upper, right and lower adjacent sides are denoted by k_1, k_2, k_3 and k_4 respectively. The concept of adjacent numbers has been demonstrated in Fig. 6.

The total number of sides of the sub-rectangles of a R^A facing the outer world is collectively called the *adjacent sum*, i.e., $S^R = k_1 + k_2 + k_3 + k_4$.

Theorem 1. *For any rectangular arrangement of order n, the total sum of number of edges in the corresponding adjacency graph and adjacent sum is equal to* $3n + 1$ *i.e.*

$$E^R(n) + S^R(n) = 3n + 1.$$

Proof. This result is proved by induction. First, it is required to check that it holds for $n = 1$, then it is assumed to be true for $n = k$ and at the end it would be verified for $n = k + 1$.

For $n = 1$, it can be directly seen that $E^R(1) = 0$ and $S^R(1) = 4 = 3n + 1$. Hence, the result is obvious for $n = 1$.

Now, suppose that, the result holds for $n = k$ i.e. $E^R(k) + S^R(k) = 3k + 1$.

For $n = k$, there are k sub-rectangles. If $(k+1)^{th}$ sub-rectangle, say R_{k+1}, is added to any one of the adjacent side of $R^A(k)$ (say *left*), then R_{k+1} would be adjacent to all the sub-rectangles which are a part of the left side. Therefore, when R_{k+1} is added to $R^A(k)$, the following changes occur in the set of adjacent numbers $\{k_1, k_2, k_3, k_4\}$ corresponding to the $R^A(k)$ (see Fig. 7):

1. k_1 becomes one.
   In this case, $E^R(k)$ gets increased by k_1 while $S^R(k)$ gets reduced by $(k_1 - 1)$.
2. k_2 and k_4 get increased by one.
   Because of this, $S^R(k)$ gets increased by 2.
3. k_3 remains unchanged.From above three points, it can be easily seen that $E^R(k+1) + S^R(k+1) = \{E^R(k) + k_1\} + \{S^R(k) + 2 - (k_1 - 1)\} = E^R(k) + S^R(k) + 3 = 3k + 1 + 3 = 3(k+1) + 1$, as required. □

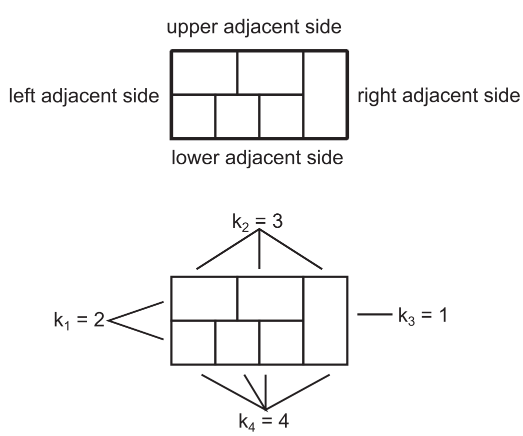

Figure 6 The adjacent sides and adjacent numbers associated with a rectangular arrangement.

4.2. Maximum number of edges

A *planar graph* is a graph that can be embedded in the plane in such a way that no two edges cross each other.

Lemma 3. *If G is a (connected) simple, finite planar graph with n vertices* ($n \geqslant 3$)*, then G has at most* $3n - 6$ *edges.*

Proof. For the proof, refer to Diestel [19], Corollary 4.2.10. □

Theorem 2. *Any rectangular arrangement of order n has at most* $3n - 7$ *edges in its adjacency graph provided that* $n > 3$.

Proof. In a R^A, there is no overlapping among the sub-rectangles (refer condition 2, Section 2.1) therefore corresponding G^R is always planar. Also, G^R is simple because of symmetric property of adjacency (i.e., if any sub-rectangle A is adjacent to a sub-rectangle B then the sub-rectangle B would be adjacent to the sub-rectangle A). And G^R is connected because of condition 3.1, Section 2.1.

It should be noted that, in this Theorem, wherever the word planar graph is used, it means a simple, connected, planar graph and the word edge corresponds to the edge in the corresponding G^R.

Since $G^R(n)$ is planar, it follows from Lemma 3 that $E^R(n) \leqslant 3n - 6$. Therefore, to prove our result it is enough to show that $E^R(n) \neq 3n - 6$.

The result is proved by induction, starting from the case $n = 4$. It can be easily verified that the only graph having $3n - 6 = 6$ edges is the complete graph K_4 which is illustrated in Fig. 8. Therefore, to prove the result for $n = 4$, it is only required to show that there does not exist any $G^R(4)$ which is isomorphic to K_4.

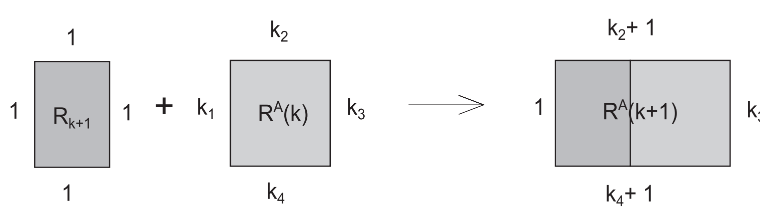

Figure 7 Showing the changes in the adjacent numbers when a sub-rectangle is added to the left of a $R^A(k)$.

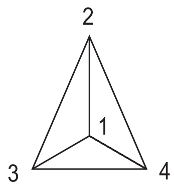

Figure 8 Graph of order 4 having 6 edges.

It is proved by contradiction. Let's say that there exist a $G^R(4)$ with 6 edges. In this case (see Fig. 8), each sub-rectangle would be adjacent to 3 other sub-rectangles. Now, if it is assumed that the first sub-rectangle is adjacent to the remaining three sub-rectangles, then there exist the following 3 possibilities regarding adjacency among the sub-rectangles:

1. *Any 3 walls of the first sub-rectangle are shared by other 3 sub-rectangles.*Clearly two of the other sub-rectangles must be situated on opposite sides of the first sub-rectangle. Hence, these sub-rectangles cannot be adjacent (see Fig. 9).
2. *Only two walls of the first sub-rectangle are shared by other 3 sub-rectangles.*There are only 3 possibilities, as shown in Fig. 10. And in all these cases, R_2 cannot be adjacent to R_3.
3. *Only one wall of the first sub-rectangle is shared by other 3 sub-rectangles.*There is only one possibility that is shown in Fig. 11. Clearly, R_2 cannot be adjacent to R_4.

Above, it is proved that $E^R(4) \neq 6$ for any $G^R(4)$ or $R^A(4)$. Now assume that $E^R(k) \neq 3k-6$ i.e. $E^R(k) \leqslant 3k-7$ (where $k \geqslant 4$). At this stage, it is required to prove that, $E^R(k+1) \neq 3(k+1)-6$.

Let's consider that $E^R(k) = 3k-7$. Then to have $E^R(k+1) = 3(k+1)-6 = 3k-3 = 3k-7+4 = E^R(k)+4$, four edges need to be added to $G^R(k)$.

Now Theorem 1 provides the result $E^R(k) + S^R(k) = 3k+1$ which implies that $S^R(k) = (3k+1) - (3k-7) = 8$ i.e. $k_1 + k_2 + k_3 + k_4 = 8$. This means that to obtain

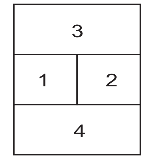

Figure 9 $R^A(4)$ with two sub-rectangles on opposite sides of the first sub-rectangle.

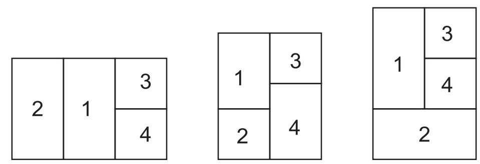

Figure 10 $R^A(4)$ where only two walls of the first sub-rectangle are being shared.

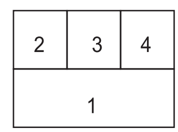

Figure 11 $R^A(4)$ where only one wall of the first sub-rectangle is being shared.

$G^R(k+1)$, if 4 edges are added to $G^R(k)$, the adjacent number associated with at least one adjacent side should be 4. Let's consider that, $k_1 = 4$. Here $S^R(k) = 8$ and $k_1 = 4$ which implies that $k_2 + k_3 + k_4 = 4$, i.e., only possible set is $\{k_2, k_3, k_4\} = \{2, 1, 1\}$. The set $\{k_2, k_3, k_4\} = \{2, 1, 1\}$ is only possible when $n = 2$ and both the sub-rectangles are adjacent, i.e., the case like $F^R(2)$ (see Fig. 5(B)). But in our induction

Figure 12 Villa La Rotonda in Vicenza.

hypothesis, it has been assumed that $n \geqslant 4$. Hence, it is not possible to add 4 edges to $G^R(k)$ to obtain $G^R(k+1)$, when $G^R(k)$ has $3k-7$ edges.

If $E^R(k)$ is even smaller, i.e., say $E^R(k) = 3k-7-j$, then the value of k_1 becomes $j+4$ while, the set $\{2,1,1\}$ remains unchanged, i.e., $\{k_2,k_3,k_4\} = \{2,1,1\}$ as before. This means that, it is not possible to add 4 or more edges to $G^R(k)$ to acquire $G^R(k+1)$. Hence, $E^R(k+1) \neq 3(k+1)-6$, as asserted. □

Corollary 1. *For the Fibonacci rectangle and golden rectangle of order n when $n > 4$, the new n^{th} sub-rectangle R_n is always adjacent to existing sub-rectangles R_{n-1}, R_{n-3} and R_{n-4}.*

Proof. For a $F^R(n)$ and $Go^R(n)$ when $n > 4$, clearly R_{n-1}, R_{n-3} and R_{n-4} always share a wall with R_n. For example, in Fig. 5(E), R_5 is adjacent to R_1, R_2 and R_4; in Fig. 5(F), R_6 is adjacent to R_2, R_3 and R_5; in Fig. 5(G), R_7 is adjacent to R_3, R_4 and R_6. By induction, it can be seen that, every new sub-rectangle R_n is adjacent to the three previously drawn sub-rectangles R_{n-1}, R_{n-3} and R_{n-4}. □

Theorem 3. *The number of edges in the adjacency graph of the Fibonacci rectangle and golden rectangle of order n is equal to $3n-7$ when $n > 3$.*

Proof. For all the Fig. 5(D)–(G), it can be easily computed and verified that $E^R(n) = 3n-7$. If n is increased by one, then from Corollary 1, a new sub-rectangle would be adjacent to 3 existing sub-rectangles, and hence $E^R(n+1) = E^R(n) + 3 = 3n-7+3 = 3(n+1)-7$ (see Fig. 5(E)–(G) where newly added edges are shown in red color). By induction, $E^R(n) = 3n-7$ when $n > 3$. □

From Theorem 2, the adjacency graph of any rectangular arrangement of order n can have at most $3n-7$ edges. It means that, if connectivity of two arrangements having same order is compared, by comparing their number of edges, then a rectangular arrangement of order n having $3n-7$ edges is *best connected*. In Theorem 3, it is shown that, in the adjacency graph of Fibonacci rectangle and golden rectangle of order n, the number of edges is $3n-7$. Hence, *Fibonacci rectangle and golden rectangle are one of the best connected rectangular arrangements.*

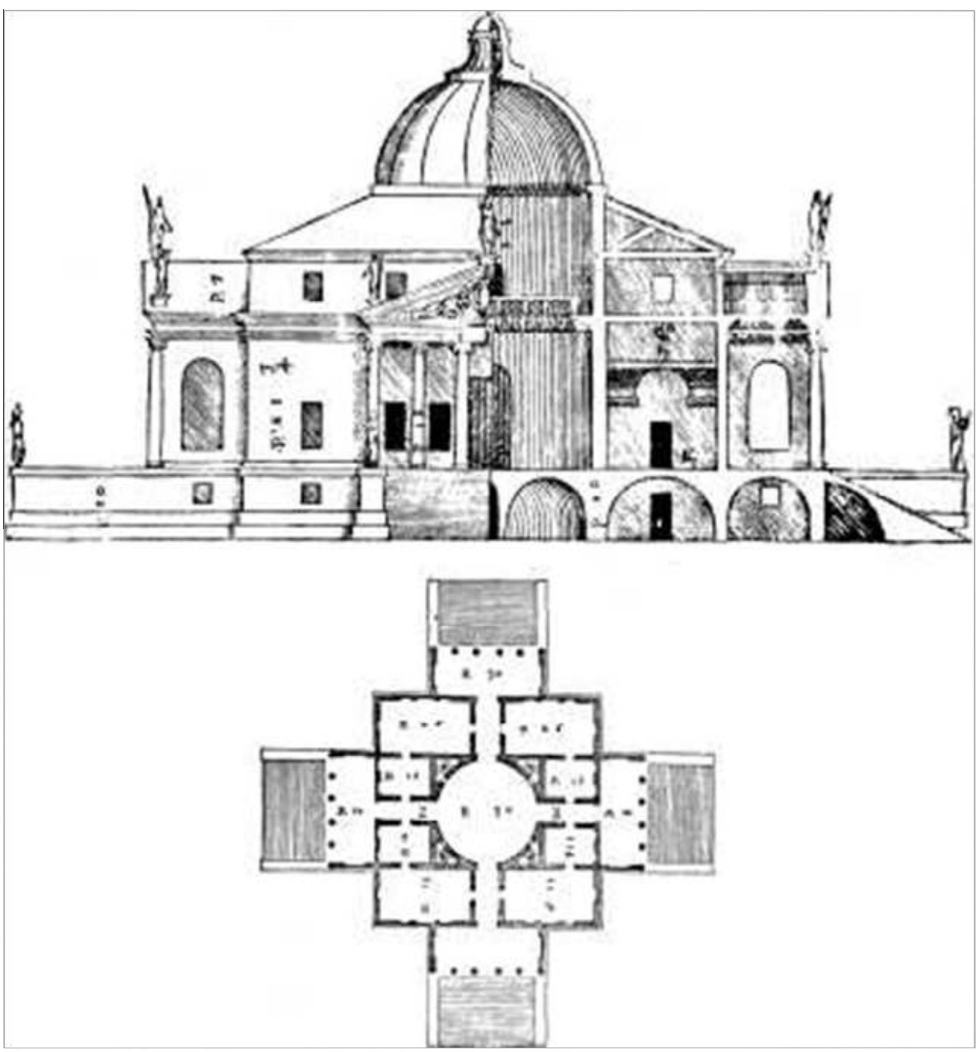

Figure 13 A floor Plan diagram of the Villa La Rotonda.

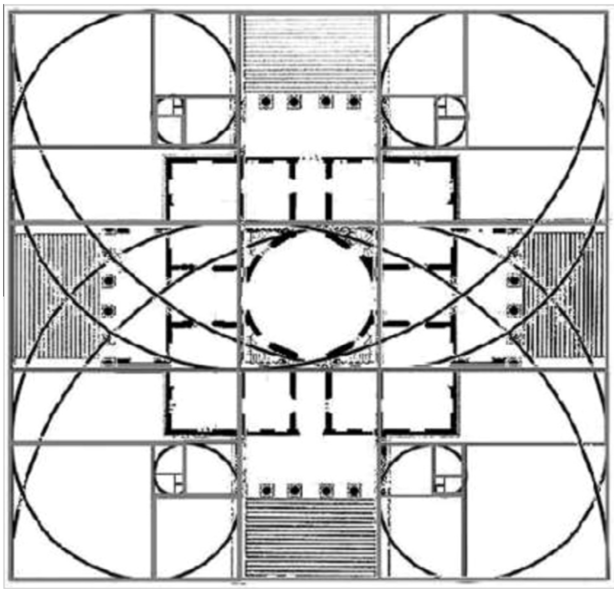

Figure 14 A golden section diagram of Villa La Rotonda.

5. Discussion and future work

To reach at the conclusion, consider a very well known work of Palladio i.e. *Villa La Rotonda* shown in Fig. 12. Fig. 13 illustrates the Palladio's plan of Villa La Rotonda and Fig. 14 gives the golden section diagram of the same.

It is clear from Fig. 14 that, the Villa Rotonda is constructed using the following 4 golden rectangles:

For the golden rectangle at upper right side, the second square is situated above the first one and the other squares are arranged in clockwise direction, i.e., it has same arrangement of squares as in Fig. 5.

In the golden rectangle at lower left side, the position of first and second squares is swapped and then the squares are arranged in clockwise direction.

In the golden rectangle at upper left side, the second square is situated above the first one and anti-clockwise movement is considered.

For the golden rectangle at lower right side, the second square is situated below the first one and the other squares are arranged in anti-clockwise direction.

The adjacency graph of all these golden rectangles is same as in Fig. 5 with $3n - 7$ edges when $n > 3$. This shows that

Figure 15 The Palazzo della Signoria.

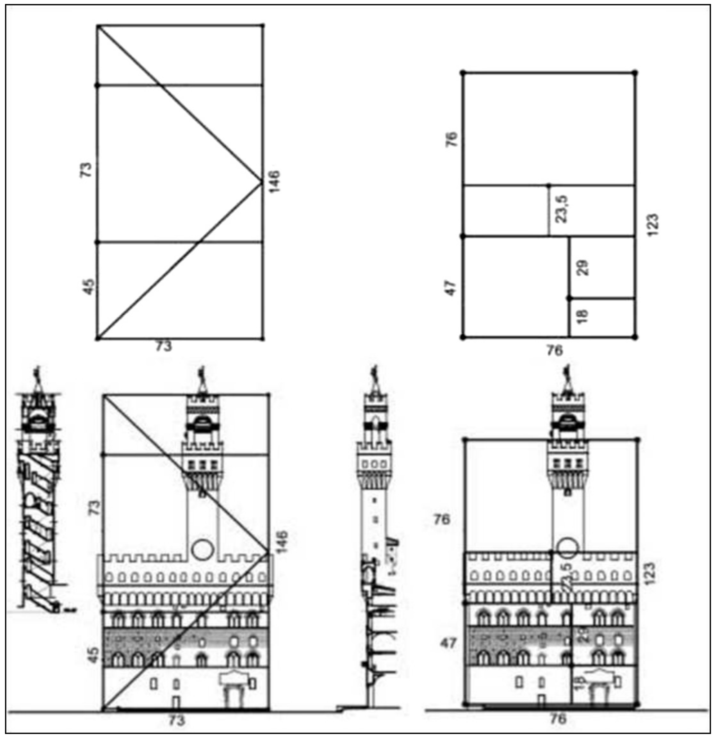

Figure 16 Front and partial sections of Palazzo Vecchio. For details about the Figure, refer to [14].

all the 4 rectangular arrangements considered in Fig. 14 are best connected. For further details about Villa Rotonda, refer to Palladio [8].

As Fibonacci rectangle is one of the best connected rectangular arrangements, to conclude its presence in architectural designs, consider one of the most important buildings of Gothic architecture: the *Palazzo della Signoria* in Florence, later widened and transformed into Palazzo Vecchio (see Fig. 15). It can be clearly seen from Fig. 16 that, for the designing of the plan the following Fibonacci sequence has been used:

$18, 29, 47, 76, 123, \ldots$

In the same fashion, many other architects have included the concept of golden rectangle or Fibonacci rectangle in their work, however, there is rarely any evidence (apart from cultural and esthetic reasons) which would explain why they have used this concept so often. In this paper, it has been illustrated that, connectivity is quite important from the point of view of architectural designs. However, in the existing architectural designs for the rectangular floor plans, it is very rare to find a solution which is best connected. For example, refer [18] (Figs. 5(a), 16 and 17) where rectangular floor plans are displayed for $n = 7$. For all these cases, number of edges are 12 which is not equal to $3n - 7 = 3 \times 7 - 7 = 14$. It means that the solutions are not best connected. But this paper proved that, both the golden rectangle and Fibonacci rectangle are best connected rectangular arrangements. This might provide one of the mathematical reasons for using them so often by architects.

As a future work, other rectangular arrangements can be searched that are best from the point of view of connectivity.

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