Week 8: Introduction to Risk Measures and Measuring Algorithms Performance – Part 2

AF3214 Python Programming for Accounting and Finance

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R508, 8:30 am – 11:20 am, Wednesdays, Semester 2, AY 2024-25

Agenda

- Motivation
- Measuring Risk and Reward
- Mean-Variance Analysis
- The Efficient Frontier
- The Tangency Portfolio

Motivation

What is a Portfolio?

- A portfolio is simply a specific combination of securities, usually defined by portfolio weights that sum to 1:

➢ *Such as gold, stocks, funds, derivatives, property, bonds, etc.*

$$\{w_1, w_2, \ldots, w_n\}, \quad w_1 + w_2 + \cdots + w_n = 1$$

$$w_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + \cdots + N_n P_n}$$

N_i, Real numbers: 0, 100, -200…

- Portfolio weights can sum to 0 (dollar-neutral portfolios), they can be positive (long positions) or negative (short positions).
- Assumption: Portfolio weights summarize all relevant information there is to know about your investment.

Motivation

Example 1. Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

No more worries about prices and shares anymore.
Only focus on portfolio weights and the returns of your securities multiplied by those weights

Motivation

Example 2. Your broker informs you that you only need to keep \$50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks on margin. You withdraw \$50,000 to use for other purposes, leaving \$50,000 in the account. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	\$50	\$10,000	20%
B	1,000	\$60	\$60,000	120%
C	750	\$40	\$30,000	60%
Riskless bond	-50,000	\$1	-\$50,000	-100%
Total			\$50,000	100%

Motivation

Example 3. You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80%. What are your portfolio weights for this investment?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	1	-$400,000	-$400,000	-400%
Total			$100,000	100%

Leverage is a two-edged sword. When things are working well, it gives you a boost. When things are not working well, it can hurt you on the downside as well.

Motivation

Example 4. You own 100 shares of stock A, and you have shorted 200 shares of stock B. Your portfolio is summarized by the following weights.

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	100	\$50	\$5,000	???
B	-200	\$25	-\$5,000	???

Zero net-investment portfolios do not have portfolio weights in percentages (because the denominator is 0)—we simply use dollar amounts instead of portfolio weights to represent long and short positions

Motivation

So *why* go through all this trouble?

Why Not Pick The Best Stock Instead of Forming a Portfolio?

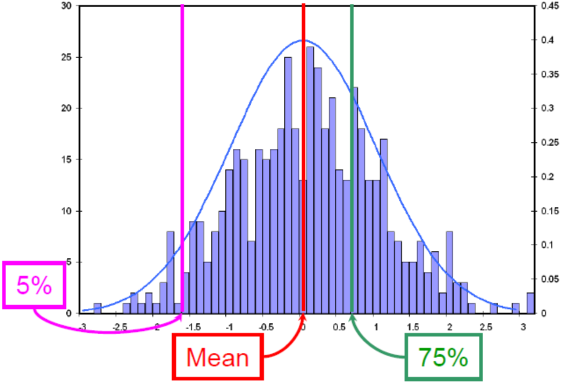
- We don't know which stock is best!
- Portfolios provide diversification, reducing unnecessary risks.
- Portfolios can enhance performance by focusing bets, e.g., AI.
- Portfolios can customize and manage risk/reward trade-offs.

How Do We Construct a "Good" Portfolio?

- What does "good" mean? - > High return(mean) and low risk(sd)
- What characteristics do we care about for a given portfolio?
    - Risk and reward
- Investors like higher expected returns
- Investors dislike risk

Measuring Risk and Reward

- Reward is typically measured by return
- Higher returns are better than lower returns.
- But what if returns are unknown?
- Assume returns are random, consider the distribution of returns.

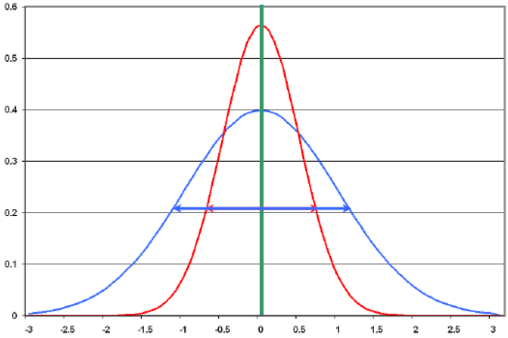


Several possible measures:

- Mean: central tendency
- 75%: upper quartile
- 5%: losses

Measuring Risk and Reward

- Reward is typically measured by return
- How about risk?
- Likelihood of loss (negative return).
- But loss can come from positive return (e.g., short position).
- A **symmetric** measure of dispersion is **variance** or **std. dev**.



Variance Measures Spread:

- **Blue** distribution is "riskier".
- Extreme outcomes more likely.
- This measure is symmetric.

Measuring Risk and Reward

Assumption

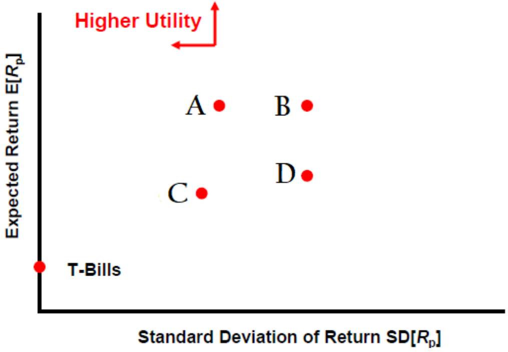
- Investors like high expected returns but dislike high volatility
- Investors care only about the expected return and volatility of their overall portfolio
  - Not individual stocks in the portfolio
  - Investors are generally assumed to be well-diversified

Questions: How much does a stock contribute to the risk and return of a portfolio, and how can we choose portfolio weights to optimize the risk/reward characteristics of the overall portfolio?

Mean-Variance Analysis

Objective

- Assume investors focus only on the expected return and variance (or standard deviation) of their portfolios: higher expected return is good, higher variance is bad.
- Develop a method for constructing optimal portfolios



Portfolio Characteristics

A portfolio's characteristics are determined by the returns of its assets and its weights in them.

Mean (expected) returns:

Asset	1	2	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\cdots$	$\bar{r}_n$

Variances and co-variances:

	r_1	r_2	$\cdots$	r_n
r_1	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
r_2	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
r_n	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

Covariance of an asset with itself is its variance.

Portfolio returns: Two assets

The portfolio return is a weighted average of the individual returns:

$$r_p = \sum_{i=1}^{n} w_i \times r_i \longrightarrow r_p = w_1 \times r_1 + w_2 \times r_2$$

Example. Suppose you invest \$600 in IBM and \$400 in Merck for a month. If the realized return is 2.5% on IBM and 1.5% on Merck over the month, what is the return on your total portfolio?

The portfolio weights are

$$w_{\text{IBM}} = 600/1000 = 60\% \qquad w_{\text{Merck}} = 400/1000 = 40\%$$

$$\begin{aligned} r_p &= \frac{(600)(0.025) + (400)(0.015)}{1000} \\ &= (0.6)(0.025) + (0.4)(0.015) \\ &= 2.1\% \end{aligned}$$

Portfolio returns: Two assets – Cont'd

Expected return on a portfolio with two assets、

Expected portfolio return: $\bar{r}_p = w_1 \times \bar{r}_1 + w_2 \times \bar{r}_2$

Unexpected portfolio return: $r_p - \bar{r}_p = w_1(r_1 - \bar{r}_1) + w_2(r_2 - \bar{r}_2)$

Variance of return on a portfolio with two assets

The variance of the portfolio return: $\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$

which is also the sum of all entries of the following table

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$w_1^2\sigma_1^2$	$w_1w_2\sigma_{12}$
$w_2\tilde{r}_2$	$w_1w_2\sigma_{12}$	$w_2^2\sigma_2^2$

Portfolio returns: Two assets – Cont'd

Example. Consider again investing in IBM and Merck stocks.

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

Consider the equally weighted portfolio: $w_1 = w_2 = 0.5$

Mean of portfolio return: $\bar{r}_p = (0.5)(0.0149) + (0.5)(0.0100) = 1.25\%$

Variance of portfolio return:

$$\begin{aligned} \sigma_p^2 &= (0.5)^2(0.007770) + (0.5)^2(0.003587) + (2)(0.5)^2(0.002095) \\ &= 0.003888 \\ \sigma_p &= 6.23\% \end{aligned}$$

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$(0.5)^2(0.007770)$	$(0.5)^2(0.002095)$
$w_2\tilde{r}_2$	$(0.5)^2(0.002095)$	$(0.5)^2(0.003587)$

Portfolio returns: Three assets

- Expected portfolio return:

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2 + w_3\bar{r}_3$$

- The variance of the portfolio return:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23}$$

Example: IBM, Merck and Intel returns have covariance matrix:

	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Merck}}$	$\tilde{r}_{\text{Intel}}$
$\tilde{r}_{\text{IBM}}$	0.007770	0.002095	0.001189
$\tilde{r}_{\text{Merck}}$	0.002095	0.003587	0.000229
$\tilde{r}_{\text{Intel}}$	0.001189	0.000229	0.009790

What is the risk (StD) of the equally weighted portfolio?

$$\begin{aligned} \sigma_p^2 &= \left(\tfrac{1}{3}\right)^2 \times (\text{Sum of all entries of covariance matrix}) = 0.003130 \\ \sigma_p &= 5.59\% \end{aligned}$$

Portfolio returns: Multiple assets

We now consider a portfolio of n assets:

$$\{w_1, w_2, \ldots, w_n\}, \quad \sum_i w_i = 1$$

1. The return on the portfolio is:

$$r_p = \sum_{i=1}^{n} w_i \times r_i$$

2. The expected return on the portfolio is:

$$\bar{r}_p = \mathrm{E}[r_p] = w_1\bar{r}_1 + w_2\bar{r}_2 + \cdots + w_n\bar{r}_n$$

3. The variance of portfolio return is:

$$\sigma_p^2 = \mathrm{Var}[\tilde{r}_p] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}, \quad \sigma_{ii} = \sigma_i^2$$

4. The volatility (Std Dev) of portfolio return is:

$$\sigma_p = \sqrt{\mathrm{V}[\tilde{r}_p]} = \sqrt{\sigma_p^2}$$

Portfolio returns: Multiple assets – Cont'd

The variance of portfolio return can be computed by summing up all the entries to the following table:

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n2}$	$\cdots$	$w_n^2 \sigma_n^2$

- The variance of a sum is not just the sum of variances! We also need to account for the covariances.
- *Question*: In order to calculate return variance of a portfolio, we need:

  a) portfolio weights
  b) individual variances
  c) all covariances

Modern Portfolio Theory – Cont'd

For us to maximize returns and minimize risk, we will need to properly allocate the assets of our choosing into proper weights.

We will be using the modern portfolio theory, introduced by Harry Markowitz.

In March 1952, Harry Markowitz, a 25 year old graduate student from the University of Chicago, published "Portfolio Selection" in the Journal of Finance. The paper opens with:

A Little History

The Journal of FINANCE
The Journal of THE AMERICAN FINANCE ASSOCIATION

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PORTFOLIO SELECTION*

Harry Markowitz

First published: March 1952 | https://doi.org/10.1111/j.1540-6261.1952.tb01525.x | Citations: 3,893

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the

Thirty eight years later, this paper would earn him a Nobel Prize in economic sciences.

Mean-Variance Analysis

Example: Stock A has an average monthly return of 1.75% and a std dev of 9.73%. Stock B has an average monthly return of 1.08% and a std dev of 6.23%. Their correlation is 0.37 or 37%. How would a portfolio of the two stocks perform? → Depends

$$E[R_p] = w_A 1.75 + w_B 1.08$$

You change the weights, you change the performance

$$Var[R_p] = w_A^2 9.73^2 + w_B^2 6.23^2 + 2 w_A w_B (0.37 \times 9.73 \times 6.23)$$

w_A	w_B	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	1	1.08	38.8	6.23
0.25	0.75	1.25	36.2	6.01
0.50	0.50	1.42	44.6	6.68
0.75	0.25	1.58	64.1	8.00
1	0	1.75	94.6	9.73
1.25	-0.25	1.92	136.3	11.67

negative 25% of your wealth into B short selling B, using the proceeds to take an extra large position in A

risk not 50-50, risks don't aggregate in a linear way because of covariances and correlations

Mean-Variance Analysis

How do you annualize the monthly return to get an annual return?

`Forget compounding: 12 * monthly return`

`w/ compounding: (1 + monthly return)^12-1`

What about risk?

two random variables `a,b`, what's the variance of `a + b?`

`var(a + b) = var(a) + var(b) +2cov(a+b)`

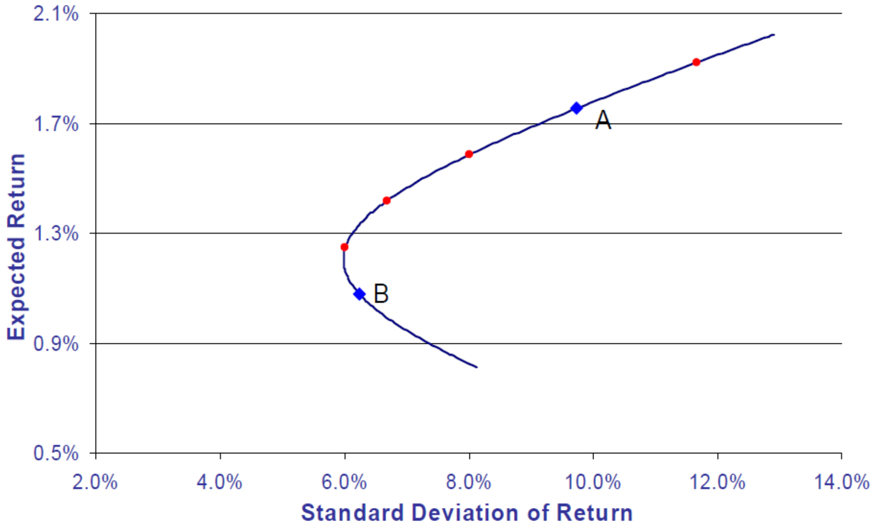
Now let's assume there are no cycles, there are no predictabilities, there's no regularities, there's no correlation, every month is independent of every other month. What is the annualized risk?

`12 * monthly variance`

`Sqrt(12) * monthly std.dev`

Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of Stock A and B



Mean-Variance Analysis

Assume correlations are changing over time

Example (cont'd): Suppose the correlation between A and B changes. What if it equals −1.0? 0.0? 1.0?

correlation

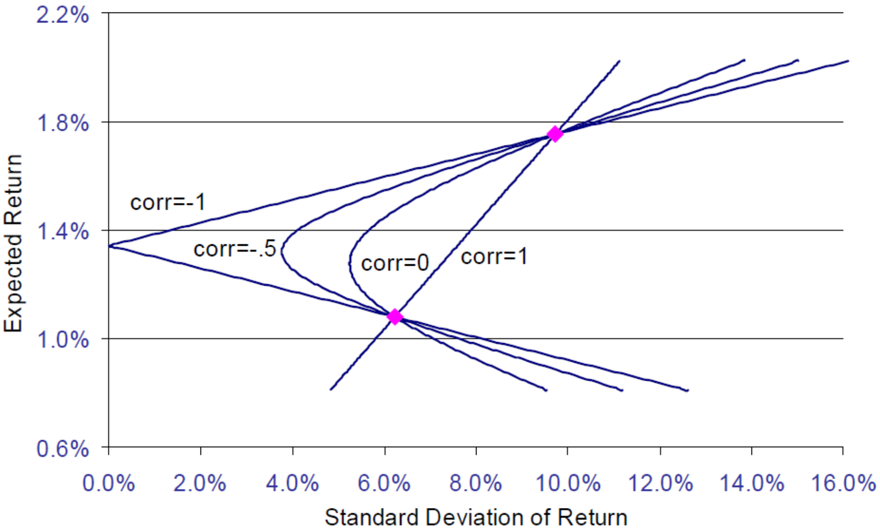
$$E[R_p] = w_A 1.75 + w_B 1.08$$

$$Var[R_p] = w_A^2 9.73^2 + w_B^2 6.23^2 + 2 w_A w_B (\rho_{AB} \times 9.73 \times 6.23)$$

			Std dev of portfolio		
w_A	w_B	$E[R_P]$	corr = -1	corr = 0	corr = 1
0	1	1.08%	6.23%	6.23%	6.23%
0.25	0.75	1.25	2.24	5.27	7.10
0.50	0.50	1.42	1.75	5.78	7.98
0.75	0.25	1.58	5.74	7.46	8.85
1	0	1.75	9.73	9.73	9.73

Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of Stock A and B



Mean-Variance Analysis

Example: Let's think about your retirement plans by investing between Treasury bills (T-Bill) and the stock market (S&P 500). The T-Bill rate is 0.12% monthly. You expect the stock market to have a monthly return of 0.75% with a standard deviation of 4.25%.

$$E[R_P] = \omega_{Tbill} 0.12 + \omega_{Stk} 0.75$$

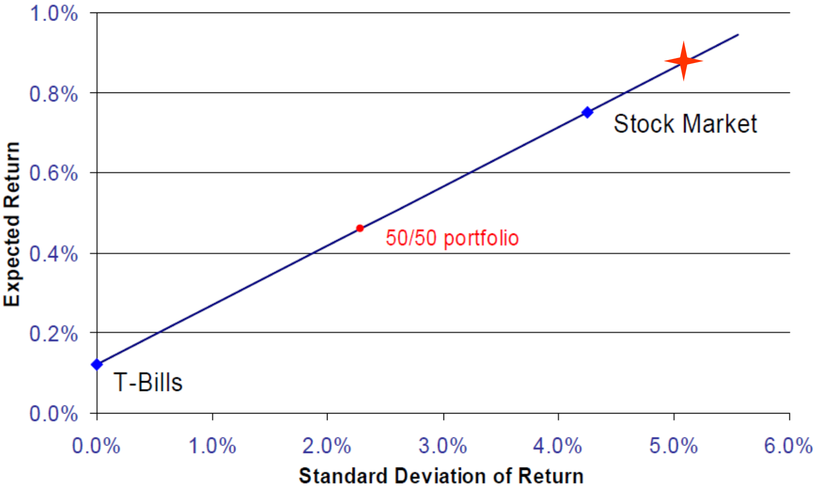
$$Var[R_P] = \omega_{Tbill}^2 0.0^2 + \omega_{Stk}^2 4.25 + 2\omega_{Tbill}\omega_{Stk}(0.00 \times 0.00 \times 4.25)$$

$$\sigma_P = \sqrt{Var[R_P]} = \omega_{Stk} 4.25$$

w_{Stk}	w_{Tbill}	$E[R_P]$	$var(R_P)$	$stdev(R_P)$
0	1	0.12	0.00	0.00
0.33	0.67	0.33	1.97	1.40
0.67	0.33	0.54	8.11	2.85
1	0	0.75	18.06	4.25

Mean-Variance Analysis

Mean/SD Trade-Off for Portfolios of Stock T-Bill and S&P 500



Mean-Variance Analysis

$$E[R_p] = w_A\mu_A + w_B\mu_B$$

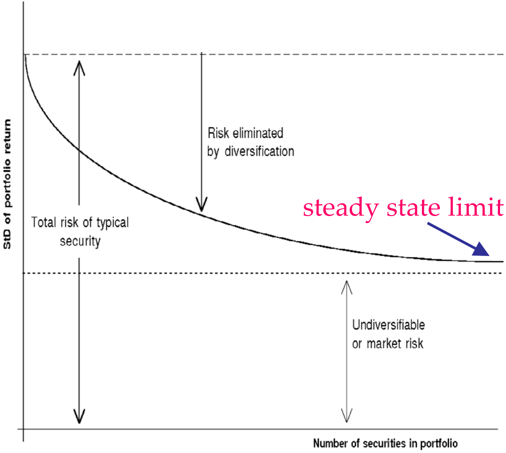
$$Var[R_p] = w_A^2{\sigma_A}^2 + w_B^2{\sigma_B}^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{AB}$$

Short Summary

- $E[R_P]$ is a weighted average of stocks' expected returns
- StdDev(R_P) is smaller if stocks' correlation is lower. It is **less than** a weighted average of the stocks' standard deviations (unless perfect correlation)
- The graph of portfolio mean/StdDev is nonlinear
- If we combine T-Bills with any risky stock, portfolios plot along a straight line

Impact of Diversification on Portfolio Risk

- Remaining risk known as systematic or market risk
- Due to common factors that cannot be diversified
- Example: S&P 500
- Other sources of systematic risk may exist:
  - Credit; Liquidity; Volatility; Business Cycle; Value/Growth
- Provides motivation for linear factor models

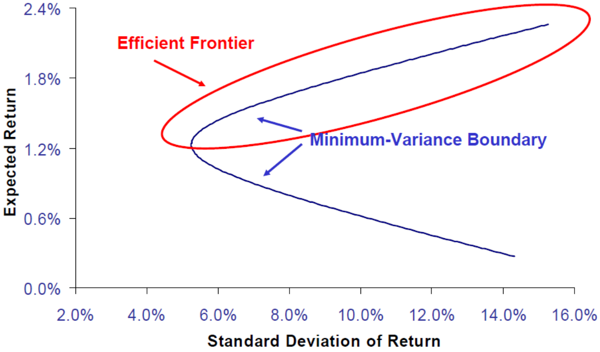


This graph basically shows that: as you add more and more securities, even though the correlations are bouncing around, there ends up being some kind of a steady state limit to what the variance of your overall portfolio is.

The Efficient Frontier

Given portfolio expected returns & variances, how should we pick the best weights?

- All feasible portfolios lie inside a bullet-shaped region, called the minimum-variance boundary or frontier.
- The efficient frontier is the top half of the minimum-variance boundary.
- Rational investors should select portfolios from the efficient frontier.
- The way we got this bullet-shaped curve is from taking different weighted averages of the securities that we have access to as investments.



The Efficient Frontier

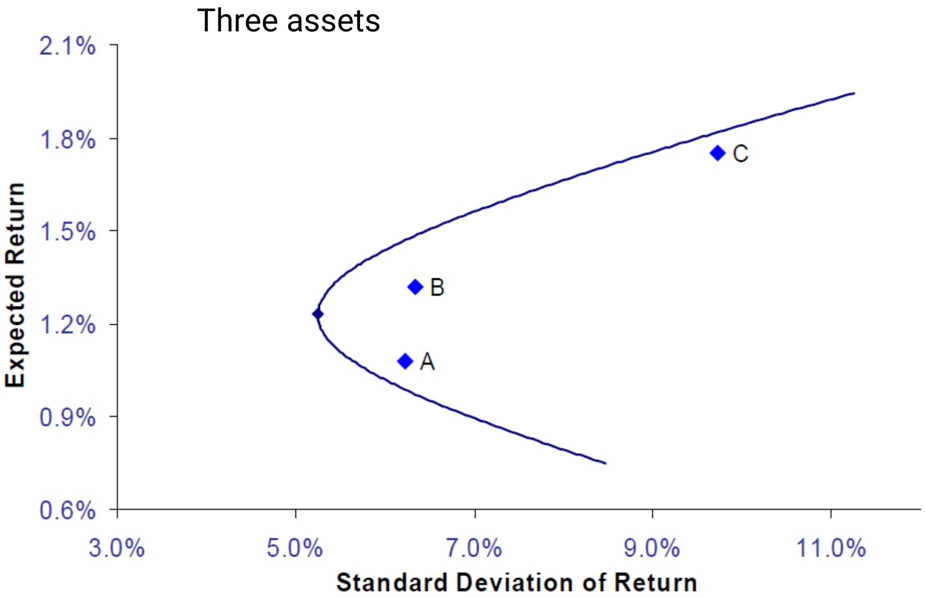
Example: You can invest in any combination of stock A, B, and C. What portfolio would you choose?

			Variance / covariance		
Stock	Mean	Std dev	A	B	C
A	1.08	6.23	38.80	16.13	22.43
B	1.32	6.34	16.13	40.21	23.99
C	1.75	9.73	22.43	23.99	94.63

$$\mathbf{E[R_P]} = (w_A \times 1.08) + (w_B \times 1.32) + (w_C \times 1.75)$$

$$\begin{aligned}\mathbf{var(R_P)} = &(w_A^2 \times 38.80) + (w_B^2 \times 40.21) + (w_C^2 \times 94.63) + \\ &(2 \times w_A \times w_B \times 16.13) + (2 \times w_A \times w_C \times 22.43) \\ + \quad &(2 \times w_B \times w_C \times 23.99)\end{aligned}$$

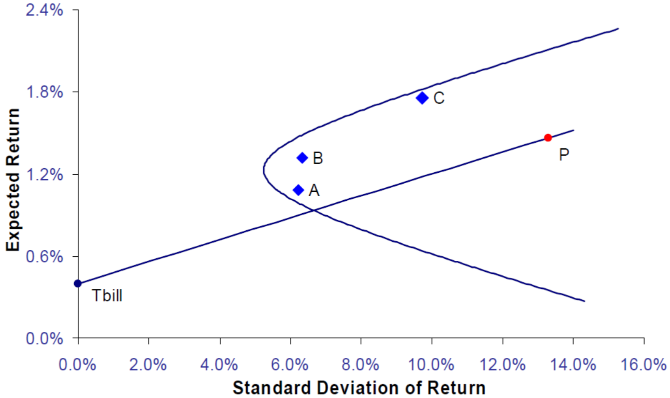
The Efficient Frontier



The Efficient Frontier

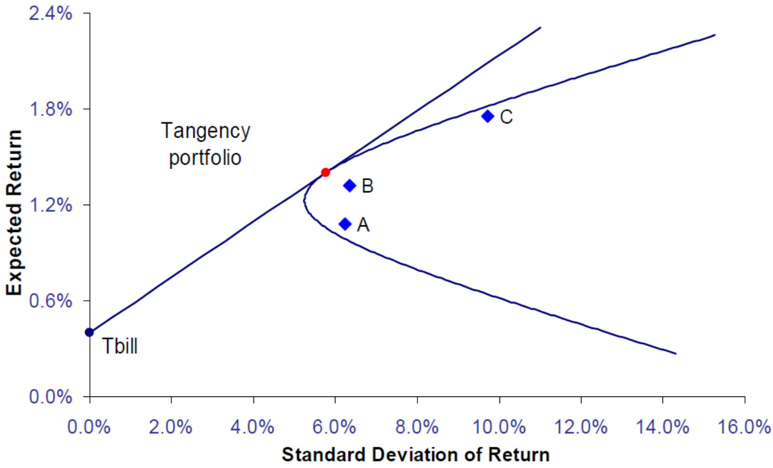
The Tangency Portfolio

- If there is also a riskless asset (T-Bills), all investors should hold exactly the same stock portfolio!
- All efficient portfolios are combinations of the riskless asset and a unique portfolio of stocks, called the tangency portfolio.
  - ➢ In this case, efficient frontier becomes straight line



The Efficient Frontier

Question: If I were to give you the choice of mixing T-Bills with only one portfolio, just one, which would it be? Which would you prefer?



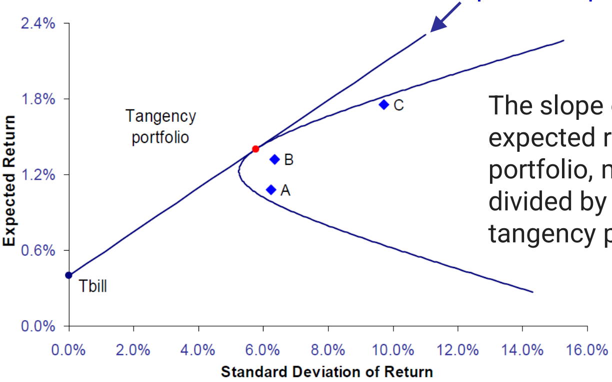
Sharpe Ratio

A measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\text{Sharpe Ratio} \equiv \frac{\mathrm{E}[R_p] - r_f}{\sigma_p} \quad \text{(higher is better!)}$$

The tangency portfolio has the highest possible Sharpe ratio of any portfolio.

Slope = Sharpe Ratio



The slope of that line is equal to the expected return of that tangency portfolio, minus the T-Bill rate, divided by the volatility of that tangency portfolio.

Two Kinds of Risks

Risk comes in two types:

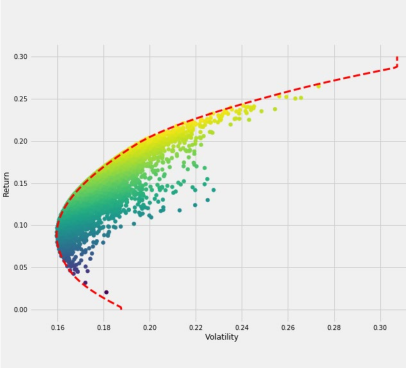
- **Diversifiable risk**: Results from uncontrollable or random events that are **firm-specific**; can be eliminated through diversification.
- **Non-diversifiable risk**: market risk or systematic risk due to macro (business cycle, inflation, etc.) / market conditions (liquidity) / political events.

Certain risks cannot be diversified away.

The Efficient Frontier

The efficient frontier graphically represents portfolio that maximize returns for the risk assumed.

Returns are dependent on the investment combinations that make up the portfolio.

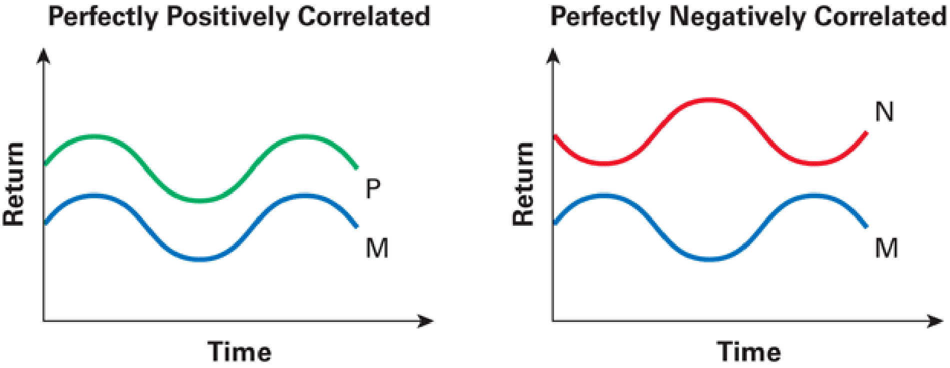


- Efficient frontier contains investment portfolios that offer the highest expected return for a specific level of risk.
- We should place a portfolio along the efficient frontier line.
- Optimal portfolios that are on the efficient frontier tend to have a higher degree of diversification.

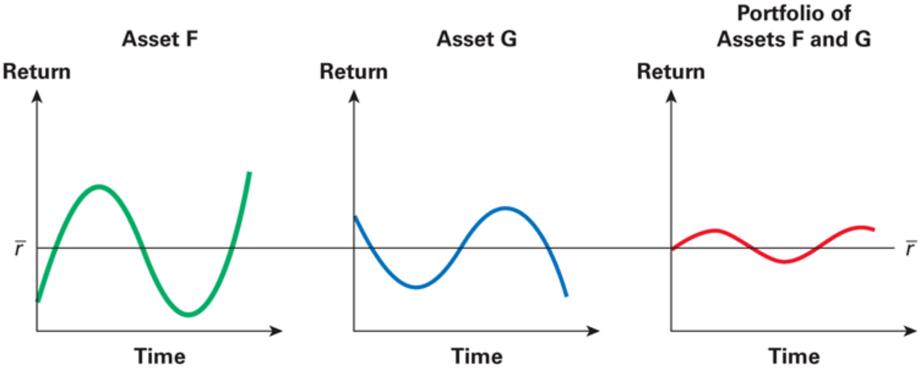
Why Diversification Works?

- Assets that are less than perfectly positively correlated tend to offset each other's movements, thus reducing the overall risk in a portfolio.
- The lower the correlation, the more the overall risk in a portfolio is reduced.
  - Assets with +1 correlation eliminate no risk.
  - Assets with less than +1 correlation eliminate some risk.
  - Assets with less than 0 correlation eliminate more risk.
  - Assets with -1 correlation eliminate all risk.

The Correlation Between Return Series



Combining Negatively Correlated Assets to Diversify Risk



Treynor Ratio

- Treynor Ratio is known as the reward to volatility ratio.
- It is a performance metric for determining how much excess return was generated for each unit of risk taken on by a portfolio.
- Risk in the Treynor ratio refers to systematic risk as measured by a portfolio's beta.

The Formula for the Treynor Ratio is:

$$\text{Treynor Ratio} = \frac{r_p - r_f}{\beta_p}$$

where:

r_p = Portfolio return

r_f = Risk-free rate

β_p = Beta of the portfolio

Modern Portfolio Theory – Cont'd

The End

Jupyter Notebook