Tutorial Sheet 11

Rotational motional

1. A wheel rotates with an angular acceleration $\alpha = 6at - 2b$. At $t = 0$, the wheel has an angular speed ω_0 and angular position θ_0. Write down the equations for the angular speed ω and angular position θ as a function of time t.

2. A flywheel of mass 500 kg and one meter diameter makes 500 r.p.m. Assuming the mass to be concentrated at the rim, calculate the angular velocity, moment of inertia and energy of the flywheel.

3. A boy stands at the centre of a turn table with his two arms stretched. The turn table is set rotating with an angular speed of 40 r.p.m. How much is the angular speed of the boy if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume the turn table to rotate without friction.

4. A wheel rotates in such a way that its angular displacement θ as a function of time t is given by:

$$\theta = t^3 + 2t^2 - 2$$

Where θ is in radians and t is in seconds.

At $t = 2$ seconds, find its angular velocity ω and angular acceleration α

5. Consider a solid disc of radius r and mass m initially at rest on an inclined surface. Show that if the disc rolls without slipping from a height h, it reaches the bottom of the incline with speed

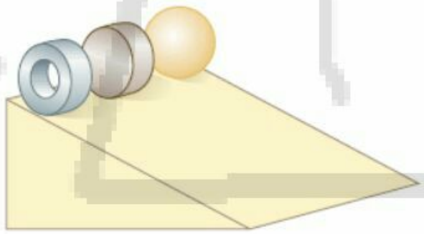
$$v = \sqrt{\frac{4gh}{3}}$$

6. Explain why the speed of rotation of an ice skater increases when she assumes a position with folded arms than a position with stretched arms.

7. A uniform solid sphere of radius r and mass m starts from rest at the top of an incline of height h and rolls down. How fast is the moving when it reaches the bottom? Assume that it rolls smoothly and that friction energy losses are negligible. Take $I = \frac{2}{5}mr^2$ for a uniform sphere.

8. A force of 7.0 N is applied to a string wound on the rim of a 20 cm diameter wheel. How much work is done by this force as it turns the wheel through 30^{o}?

9. Three objects of uniform density — a solid sphere, a solid cylinder, and a hollow cylinder — are placed at the top of an incline as shown in the Figure below. They are all released from rest at the same elevation and roll without slipping. Which object reaches the bottom first? Which reaches it last? Try this at home and note that the result is independent of the masses and the radii of the objects.



MR. AMON CHILESHE

TUTORIAL SHEET 11: ROTATIONAL MOTION

Question (1)

Data

Angular acceleration $\alpha = 6at - 2b$

$\omega = ?$

$\theta = ?$

- The instantaneous angular acceleration of a particle moving in a circular path or a rotating rigid object is

$$\alpha = \frac{d\omega}{dt}$$

$$d\omega = \alpha dt \text{ (integrating both sides)}$$

$$\int d\omega = \int \alpha dt$$

$$\omega - \omega_0 = \int_0^t \alpha dt$$

$$\omega - \omega_0 = \int_0^t (6at - 2b) dt$$

$$\omega - \omega_0 = \left[\frac{6}{2}at^2 - \frac{2}{1}bt\right]_0^t$$

$$\omega - \omega_0 = 3at^2 - 2bt$$

$$\therefore \omega = \omega_0 + 3at^2 - 2bt$$

- The instantaneous angular speed of a particle moving in a circular path or of a rigid body rotating about a fixed axis is

$$\omega = \frac{d\theta}{dt}$$

$$d\theta = \omega dt \text{ (integrating both sides)}$$

$$\int d\theta = \int_0^t \omega\, dt$$

$$\theta - \theta_0 = \int_0^t (\omega_0 + 3at^2 - 2bt)\, dt$$

$$\theta - \theta_0 = \left[\omega_0 t + \frac{\cancel{3}}{\cancel{3}} at^3 - \frac{\cancel{2}}{\cancel{2}} bt^2 \right]_0^t$$

$$\theta - \theta_0 = \omega_0 t + at^3 - bt^2$$

$$\therefore\ \underline{\theta = \theta_0 + \omega_0 t + at^3 - bt^2}$$

Additional answers

If $\omega_0 = 0$ and $\theta_0 = 0$ at $t = 0$, then

$$\underline{\omega = 3at^2 - 2bt \quad \text{and} \quad \theta = at^3 - bt^2}$$

Question (2)

DATA

- $M = 500 Kg$
- $r = 0.5m$
- $f = 500$ r.p.m (500 revolutions per minute)

$f = \frac{500 rev}{min} \times \frac{1 min}{60 s}$ (f is the frequency)

$f = 8.33 rev/s$

By assuming the mass to be concentrated at the rim

(i) Angular velocity

$\omega = 2\pi f$

$\omega = 2\pi \times (8.33)$

$\therefore \omega = 52.34 \ rad/s$

(ii) Moment of Inertia.

The moment of inertial is a measure of the resistance of an object to changes in it's rotational motion. The moment of inertial of a rigid object is

$I = \int r^2 dm$

$I = Mr^2$

$I = (500) \times (0.5)^2$

$\therefore I = 125 \ Kg.m^2$

(iii) If a rigid object rotates about a fixed axis with angular speed ω, its rotational kinetic energy can be written

$K_R = \frac{1}{2} I \omega^2$

$= \frac{1}{2} \times (125) \times (52.34)^2$

$\therefore K_R = 1.71 \times 10^5 J$

Question (3)

Angular momentum is a vector quantity that has magnitude $I\omega$ and is directed along the axis of rotation. If the net torque on a body is zero, its angular momentum will remain unchanged in both magnitude and direction. This is the law of angular momentum. i.e

$$I_i \omega_i = I_f \omega_f$$

Data

Let ω_1 = Initial angular velocity = 40 r.p.m

ω_2 = final angular velocity = ?

I_1 = the moment of inertia of the boy with stretched hands or the initial value of moment of inertia.

I_2 = the moment of inertia of the boy with folded hands i.e $I_2 = \frac{2}{5} I_1$

- Since no external force acts on the body, the angular momentum L is constant

$$L_i = L_f$$

$$L_1 = L_2$$

$$I_1 \omega_1 = I_2 \omega_2$$

$$\cancel{I_1} \times (40\ r.p.m) = \left(\frac{2}{5} \cancel{I_1}\right) \omega_2$$

$$\omega_2 = \frac{5 \times 40\ r.p.m}{2}$$

$$\therefore \omega_2 = 100\ r.p.m$$

Question ④

Data

$$\theta = t^3 + 2t^2 - 2$$

- At t = 2 seconds its angular velocity ω is defined as the limiting value of the ratio $\Delta\theta/\Delta t$ when Δt approaches zero. thus

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\omega = \left.\frac{d\theta}{dt}\right|_{t=2}$$

$$\omega = \left.\frac{d}{dt}\left(t^3 + 2t^2 - 2\right)\right|_{t=2}$$

$$\omega = \left.2t^2 + 4t\right|_{t=2}$$

$$\therefore \ \omega = 16 \text{ rad/s}$$

- At t = 2 seconds, its angular acceleration α is defined as the limiting value of the ratio $\Delta\omega/\Delta t$ when Δt approaches zero.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \left.\frac{d\omega}{dt}\right|_{t=2}$$

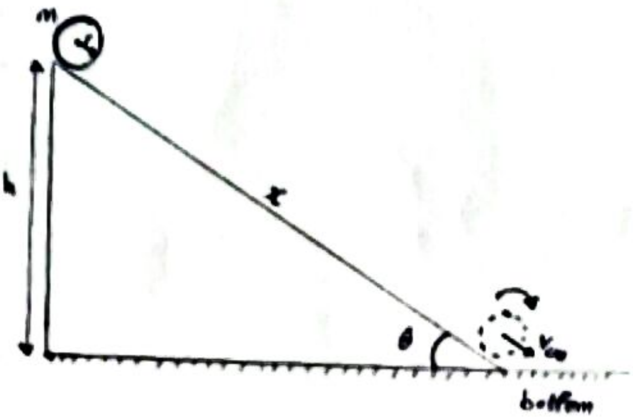
$$\alpha = \left.\frac{d}{dt}\left(2t^2 + 4t\right)\right|_{t=2}$$

$$\alpha = \left.4t + 4\right|_{t=2}$$

$$\therefore \ \alpha = 12 \text{ rad/s}^2$$

Question ⑤

Diagram



The diagram shows a solid disc rolling down an incline. Mechanical energy of the solid disc is conserved if no slipping occurs. If the frictional force between the surface and the solid disc is neglected, all the potential energy at the top (h) is converted to the total kinetic energy when it begins to roll down the incline. In this case, the total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass. i.e

$$PE_{top} = KE_{bottom}$$

$$mgh = KE_{tot}$$

$KE_R = \frac{1}{2} I\omega^2 \longrightarrow$ rotational kinetic energy

$KE_T = \frac{1}{2} mv^2 \longrightarrow$ translational kinetic energy

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2 \quad \left(\text{where } v = r\omega \text{ and } \omega = \frac{v}{r}\right)$$

$$mgh = \frac{1}{2} mv^2 + \frac{1}{2} I\left(\frac{v}{r}\right)^2$$

$$mgh = \frac{1}{2} mv^2 + \frac{Iv^2}{2r^2}$$

we need to find the moment of inertia of a solid disc using

$$I = \int r^2 dm \text{ ——— } ①$$

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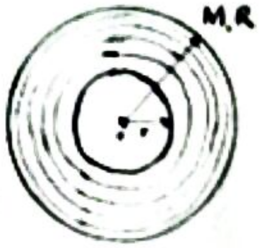
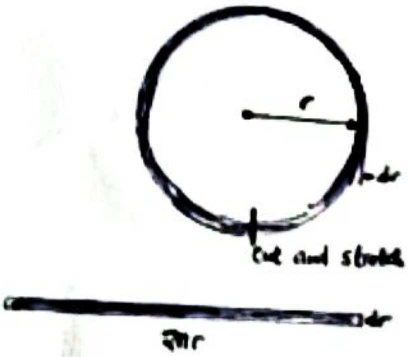


R ≡ is the total radius of the solid disc
r ≡ is the radius of each piece of ring

If we take the inner piece ring and stick it



dr ≡ thickness of the ring
dm ≡ mass of each ring

$$dm = \sigma dA$$

where

$$\sigma = \frac{\text{Total mass}}{\text{Total Area}} = \frac{M}{\pi R^2}$$

and

dA ≡ area of each ring

$$dA = 2\pi r dr$$

- using equation ①

$$I = \int r^2 dm$$

$$I = \int r^2 \sigma dA$$

$$I = \int r^2 \left(\frac{M}{\cancel{\pi} R^2}\right) \times \left(2\cancel{\pi} r dr\right)$$

$$I = \frac{2M}{R^2}\int_0^R r^3 dr$$

$$I = \frac{\cancel{2}M}{R^2}\left[\frac{r^4}{\cancel{4}}\right]_0^R$$

$$I = \frac{M}{\cancel{R^2}}\left[\frac{R^{\cancel{4}}}{2}\right]$$

$$\therefore \quad I = \frac{1}{2} MR^2$$

In this question, the total radius is r

$$\therefore I = \frac{1}{2} Mr^2$$

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$$mgh = \frac{1}{2}mv^2 + \frac{\left(\frac{1}{2}mr^2\right)v^2}{2r^2}$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2 + \frac{1}{2}\cancel{m}v^2\cancel{r^2} \times \frac{1}{2\cancel{r^2}}$$

$$\left(gh = \frac{v^2}{2} + \frac{v^2}{4}\right) \times 4$$

$$4gh = 2v^2 + v^2$$

$$4gh = 3v^2$$

$$v^2 = \frac{4gh}{3}$$

$$\therefore \underline{v = \sqrt{\frac{4gh}{3}}}$$ Hence Shown

Question 6

Diagrams

Before folding the arms (stretched arms)

Data

ω_1 = Initial angular velocity

I_1 = moment of Inertia

L_1 = Initial angular momentum

$$L_1 = I_1 \omega_1 \quad \text{—— (1)}$$

where

$$\omega_1 = \frac{L_1}{I_1} \quad \text{—— (2)}$$

After folding the arms (unstreched arms)

ω_2 = final angular velocity

I_2 = moment of Inertia

L_2 = final angular momentum

$$L_2 = I_2 \omega_2 \quad \text{—— (3)}$$

where

$$\omega_2 = \frac{L_2}{I_2} \quad \text{—— (4)}$$

LAW OF ANGULAR MOMENTUM

If there are no external fores acting on the system (ice skater) or if the net torque on the body is zero, then the angular momentum is Conserved or is Costant. meaning angular momentum before folding the arms is equal to angular momentum after folding the arms;

$$L_1 = L_2$$

$$I_1 \omega_1 = I_2 \omega_2 \quad \text{—— (5)}$$

Now we define moment of Inertia as follows;

$$I = mr^2 \quad \text{—— (6)}$$

when the ice skater draws the arms Inward, we have the following;

(i) The distance from the rotational axis decreases/reduces (r becomes small)

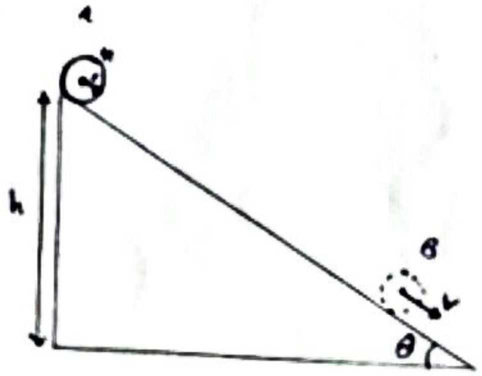
(ii) Some of the mass reduces/reduces (m = becomes smaller think about center of mass)

(iii) Therefore, if the mass reduces, then moment of inertial Inertia reduces.

$$I_1 > I_2 \quad \text{therefore} \quad \omega_1 < \omega_2 \quad \text{or} \quad \omega_2 > \omega_1$$

Question ②

Diagram



using the law of conservation of mechanical energy

$$PE_A = KE_B$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \quad \left(\text{where } v = r\omega \text{ and } \omega = \frac{v}{r}\right)$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{5}m\cancel{r^2}\right) \times \left(\frac{v}{\cancel{r}}\right)^2$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2 + \frac{1}{5}\cancel{m}v^2$$

$$\cancel{g = h = v}$$

$$gh = \frac{v^2}{2} + \frac{v^2}{5}$$

$$gh = \frac{7v^2}{10}$$

$$10gh = 7v^2$$

$$v^2 = \frac{10gh}{7}$$

$$\therefore v = \sqrt{\frac{10gh}{7}}$$

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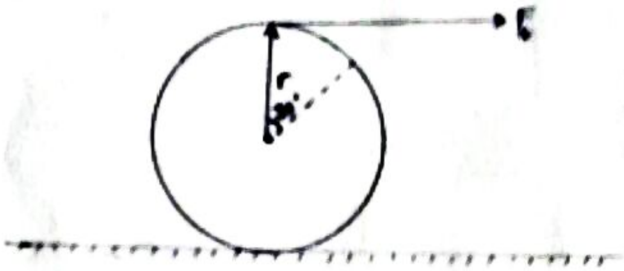
Question ⑧

Data

- Tangential force $F_t = 70N$
- Radius $r = 10cm = 0.1m$
- Angular displacement $\theta = 30°$

Diagram



Definition

The work (W) done on a rotating body during an angular displacement θ by a costant torque τ is given by

$$W = \tau\theta \text{ ——— } ①$$

where W is in joules and θ must be in radians.

And

The magnitude of the torque τ associated with a force F acting on an object is

$$\tau = Fd \text{ ——— } ②$$

where d is the moment arm of the force, which is the perpendicular distance from the rotation axis to the line of action of the force.

equation ② into ①

$$W = Fr\theta \quad (\theta \text{ should be in radians})$$

$$W = (7)\times(0.1)\times(0.52)$$

$$\therefore W = 0.364 J$$

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Question (9) : Answer

The Sphere will reach the bottom first, the hoop will reach the bottom last. If each Object has the ~~mass and~~ Same mass and the Same radius, they all have the Same torque due to gravity acting on them. The one with the Smallest moment of Inertia will have the largest angular acceleration and reach the bottom of the plane first.

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