THE COPPERBELT UNIVERSITY
SCHOOL OF MATHEMATICS AND NATURAL SCIENCES
DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MA 110 - MATHEMATICAL METHODS I
TEST 1

DATE: 8th December 2017
DURATION: 3 Hours
MARKS: 100

READ THE FOLLOWING INSTRUCTIONS

1. Write your **NAME, PROGRAM, COMPUTER NUMBER AND GROUP on the cover of your answer sheet.**
2. This is a **THREE Hours** test. Cell phones are **NOT** allowed.
3. Attempt **ALL** questions. Answers to questions should fully be explained. A correct but unclear answer will not get full marks.
4. No pencil work (except for graph sketching) or any work in red ink will be marked.
5. Use of correction fluid or **"Tip-Ex"** and **calculators** are **NOT** allowed.

Question 1

(a) Let $E = \{3,4,5,6\}$, $F = \{0,2,4,6,8\}$ and the universal set $X = \{0,1,2,\ldots,10\}$. Find $E \cap F'$. **[1 mark]**

(b) Let

$$f(x) = \frac{2}{x^2-2}, \text{ and } g(x) = \frac{1}{\sqrt{x+1}}$$

Find $(g \circ f)(-2)$. **[4 marks]**

(c) Use long division to divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$. **[3 marks]**

(d) If A and B are subsets of the universal set U such that $A \subset B$. Simplify

(i) $A' \cup B'$. **(ii)** $A - B$. **[2+2 marks]**

(e) Find the possible values of λ and k if the expression $3x^4 + \lambda x^3 + kx + 4$ is exactly divisible by $x - 1$ and leaves a remainder of 18 when divided by $x + 2$. **[5 marks]**

$x = 1$

$x = -2$

Question 2

(a) Prove the De-Morgan's law; $(A \cap B)' = A' \cup B'$. [5 marks]

(b) Is the function $f(x) = |x| + x^2$ even, odd or neither? Justify your answer. [3 marks]

(c) Solve the inequality below and write the solution set in interval notation

$$\frac{x+4}{x+1} \leq \frac{x-2}{x-4}.$$ [5 marks]

(d) Use Factor Theorem and synthetic division to factorise
$f(x) = 6x^4 - 19x^3 + 17x^2 - x - 3$ completely. [4 marks]

$\frac{b^2 - 4ac}{2a}$

Question 3

(a) Sketch the following piecewise defined function

$$f(x) = \begin{cases} |x+2| & \text{if } -4 \leq x < 0 \\ x^2 & \text{if } 0 \leq x \leq 2 \\ 3 & \text{if } 2 < x \leq 3 \end{cases}$$ [4 marks]

$K = \frac{81 - 4(1)(c)}{2}$

$\frac{76}{2}$ $\frac{38}{2}$ 76

$K = 38c$

(b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational. [4 marks]

(c) Solve the polynomial equation $2x^5 - 5x^4 + x^3 + x^2 - x + 6 = 0$. [5 marks]

(d) Solve the following inequality and write the solution in interval notation
$3|2x+1| + 1 \leq 7$. [4 marks]

Question 4

(a) Let the universal set be the set of real numbers, with $A = (3, 8]$, $B = (2, 7)$, $C = [1, 5]$ and $D = [6, \infty)$. Find (i) $(A \cup C)$ (ii) $(B \cap D)$ (iii) $(A \cup C) - (B \cap D)$ [1+1+2 marks]

(b) Is the binary operation $*$ defined by $a * b = a + b - ab$ both commutative and associative. Justify your answer. [3 marks]

(c) Sketch the graph of the polynomial $f(x) = -(x-3)(x-2)^3(x+1)^2$. [5 marks]

(d) Solve the equation $|2x + 11| = 7$. [2 marks]

(e) The roots of the equation $x^2 - 9x + K = 0$ are α and $\alpha + 1$. Find the value of K. [3 marks]

$-x + 3 = 0$ $-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$-x = -3$

$-81 - 7$

$-6 - 3$

$x^2 - 6x - 3x + K$

$x(x-6) - 3(x-K)$

$+1$ -2

-3

$-3 + 1$

-2

$2x^5 + 5x^4 + x^3 + x^2 - x + 6$

5 8 3 3 -3 9

$x + 2$

$x + 1 \mid 2x^5 + 5x^4 + x^3 +$

2 3

2 3 -2

2 5

$x + 3 = 0$

Question 5

(a) Express

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\sqrt{3}-1$$

in the form $a+b\sqrt{3}$ where a and b are rational numbers.

[4 marks]

(b) Verify that

$$f(x) = 4x-5 \text{ and } g(x) = \frac{x+5}{4}$$

are inverse functions of each other.

[4 marks]

(c) Express $f(x) = 1-6x-x^2$ in the form $f(x) = a(x+h)^2+k$ where a, h and k are rational numbers. Hence, write down the coordinates of the turning point of the graph $f(x) = 1-6x-x^2$.

[5 marks]

(d) Solve the equation $2x-11\sqrt{x}+12 = 0$.

[4 marks]

Question 6

(a) Solve for x and y where x and y are real numbers

$$(x+yi)-i = i(x+yi)+5.$$

[4 Marks]

(b) The equation $Kx^2-2Kx+2K = 1$ where K is a constant has two real solutions.

(i) Show that K satisfies the inequality

$$K^2-K \leq 0.$$

[2 Marks]

(ii) Hence, find the set of all possible values of K.

[3 Marks]

(c) For the following rational function

$$f(x) = \frac{2x^2-2}{x^2-4},$$

(i) Determine the x-intercepts and the y-intercept.

[3 Marks]

(ii) Find the horizontal and the vertical asymptotes.

[2 Marks]

(iii) Sketch the graph of f.

[3 Marks]

2017 Test 1 by Toto!

#Question 1

(a) $X = \{0, 1, 2, \ldots 10\}$

$E = \{3, 4, 5, 6\}$

$F' = \{1, 3, 5, 7, 9, 10\}$

$E \cap F' = \{3, 5\}$

b. $f(x) = \dfrac{2}{x^2 - 2}$ $\qquad g(x) = \dfrac{1}{\sqrt{x+1}}$

$$g \circ f(x) = \frac{1}{\sqrt{\frac{2}{x^2-2} + 1}}$$

$$g \circ f(2) = \frac{1}{\sqrt{\frac{2}{(-2)^2 - 2} + 1}}$$

$$= \frac{1}{\sqrt{\frac{2}{2} + 1}}$$

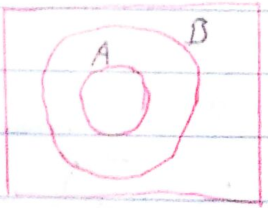
$$g \circ f(-2) = \frac{1}{\sqrt{1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

c $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$

$$\begin{array}{r} x^2 - 3x + 1 \\ 4x + 5 \,\big)\, 4x^3 - 7x^2 - 11x + 5 \\ -(4x^3 + 5x^2) \qquad\qquad\quad \\ -12x^2 - 11x \qquad \\ -(-12x^2 - 15x) \qquad \\ 4x + 5 \\ -(4x + 5) \\ 0 \end{array}$$

$$\therefore \frac{4x^3 - 7x^2 - 11x + 5}{4x + 5} = x^2 - 3x + 1$$

d # If $A \subset B$ then ;



(i) $A' \cup B' = A'$

~~cannot be simplified~~

(ii) $A - B$

$A \cap B'$

$\underline{\underline{\emptyset \text{ or } \{\ \}}}$

e $3x^4 + \lambda x^3 + kx + 4$

$x - 1 = 0$ $\qquad\qquad$ $x + 2 = 0$

$x = 1$ $\qquad\qquad$ $x = -2$

when ~~you~~ they is exactly divisible meaning remainder is zero. So we shall form two equations

Equ.. 1;

$$3(1)^4 + \lambda(1)^3 + k(1) + 4 = 0$$
$$3 + \lambda + k + 4 = 0$$
$$\lambda + k = -7 \dots (i).$$

Equ... 2

$$3(-2)^4 + \lambda(-2)^3 + k(-2) + 4 = 18$$
$$48 - 8\lambda - 2k + 4 = 18$$
$$-8\lambda - 2k = 18 - 52$$
$$-8\lambda - 2k = -34 \;] \div -2$$

$$4\lambda + k = 17 \dots (i)$$

Solve them manje!

$$-\left|\begin{array}{l} \lambda + k = -7 \\ 4\lambda + k = 17 \end{array}\right.$$

$$-3\lambda = -24$$
$$\lambda = 8$$

next up, k

$\lambda + k = -7$

$8 + k = -7$

$k = -15$

$\therefore \lambda = 8$ & $k = -15$

Question 2

(i) $(A \cap B)' = A' \cup B'$

yes the statements are important!!!

To prove $(A \cap B)' = A' \cup B'$ we need to show that $(A \cap B)' \subset A' \cup B'$ ...(i) and that $A' \cup B' \subset (A \cap B)'$

To prove (i) let $x \in (A \cap B)'$

$x \notin A \cap B$

$x \notin A$ or $x \notin B$

$x \in A'$ or $x \in B'$

$x \in A'$ or B'

$x \in A' \cup B'$

Thus, $(A \cap B)' \subset A' \cup B'$

To prove (ii) let $x \in A' \cup B'$

$x \in A'$ or B'

$x \in A'$ or $x \in B'$

$x \notin A$ or $x \notin B$

$x \notin A \cap B$

$x \in (A \cap B)'$

Thus $A' \cup B' \subset (A \cap B)'$

$\therefore$ Since both the statements (i) & (ii) have been proven true it means $(A \cap B)' = A' \cup B'$.

(b) $f(x) = |x| + x^2$

\# if even $f(x) = f(-x)$
if odd $f(-x) = -f(x)$

$f(-x) = |-x| + (-x)^2$
$f(-x) = x + x^2$

$\therefore$ since $f(x) = f(-x)$ the function is even

$-f(x) = -[|x| + x^2]$
$-f(x) = -|x| - x^2$

$\therefore$ $f(-x) \neq -f(x)$ hence not odd

C

$$\frac{x+4}{x+1} < \frac{x-2}{x-4}$$

$$\frac{x+4}{x+1} - \frac{x-2}{x-4} < 0$$

$$\frac{(x+4)(x-4) - (x-2)(x+1)}{(x+1)(x-4)} < 0$$

$$\frac{x^2 - 16 - (x^2 - x - 2)}{(x+1)(x-4)} < 0$$

$$\frac{x^2 - 16 - x^2 + x + 2}{} \le 0$$

$$\frac{x - 14}{(x+1)(x-4)} \le 0$$

Critical points

$x - 14 = 0$ $\quad$ $x + 1 = 0$ $\quad$ $x - 4 = 0$

$x = 14$ $\quad$ $x = -1$ $\quad$ $x = 4$

$x < -1$ $\quad$ $-1 < x < 4$ $\quad$ $4 < x < 14$ $\quad$ $x > 14$

-1 $\quad$ 4 $\quad$ 14

Range	$x < -1$	$-1 < x < 4$	$4 < x < 14$	$x > 14$
Test value	-2	0	5	15
$x - 14$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$	$+$
$x - 4$	$-$	$-$	$+$	$+$
Quotent	$-$ ✓	$+$	$-$ ✓	$+$

Remember $\frac{x-14}{(x+1)(x-4)}$ ≤ 0 meaning $(-)$

$x < -1$ or $4 < x \le 14$

$$(-\infty, -1) \cup (4, 14]$$

d $f(x) = 6x^4 - 19x^3 + 17x^2 - x - 3$

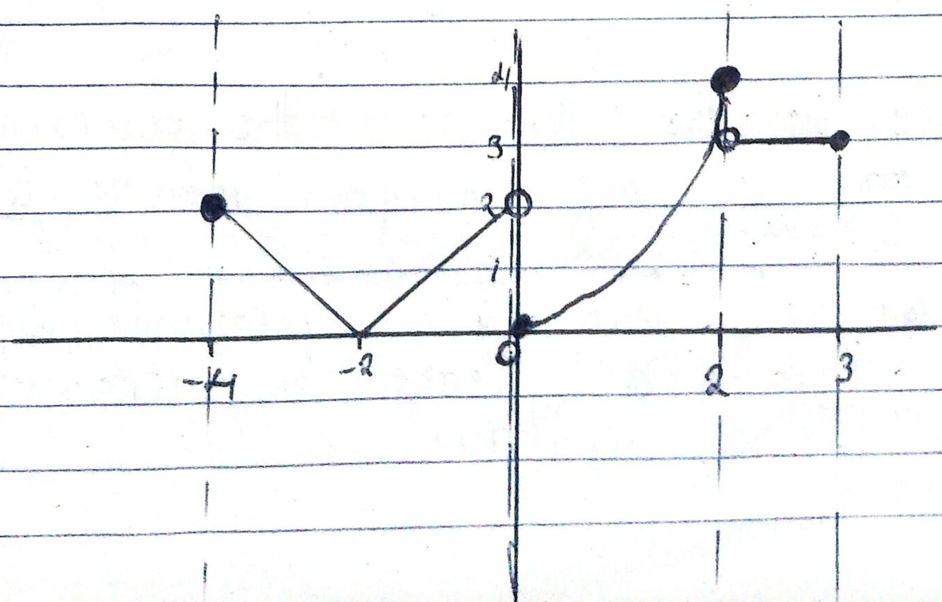
1	6	-19	17	-1	-3
		6	-13	4	3
1	6	-13	4	3	0
		6	-7	-3	
	6	-7	-3	0	

$$6x^2 - 7x - 3 = 0$$
$$6x^2 + 2x - 9x - 3$$
$$2x(3x+1) - 3(3x+1) = 0$$
$$(3x+1)(2x-3) = 0$$

$\therefore$ by factor theorem factors of $f(x)$ are;
$\underline{(x-1)(3x+1)(2x-3)}$

Question 3

(a) $f(x) = \begin{cases} |x+2| & \text{if } -4 \leq x < 0 \\ x^2 & 0 \leq x \leq 2 \\ 3 & 2 < x \leq 3 \end{cases}$



b $\sqrt{2} + \sqrt{3}$

~~To prove that this is irrational we need to~~

Proof by contradiction

Assume $\sqrt{2} + \sqrt{3}$ is rational such that it can be expressed in the form a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$

$$\left(\sqrt{2} + \sqrt{3}\right)^2 = \left(\frac{a}{b}\right)^2$$

$$2 + 2.\sqrt{2}\sqrt{3} + 3 = \frac{a^2}{b^2}$$

$$2\sqrt{6} + 5 = \frac{a^2}{b^2}$$

$$2\sqrt{6} = \frac{a^2}{b^2} - 5$$

$$\left. 2\sqrt{6} = \frac{a^2 - 5b^2}{b^2} \right] \times \frac{1}{2}$$

$$\sqrt{6} = \frac{a^2 - 5b^2}{2b^2}$$

$\therefore$ since on the R.H.S of the expression a, b, 2 and 5 are integers meaning $\frac{a^2 - 5b^2}{2b^2}$ is rational. But $\sqrt{6}$ is irrational which is a contradiction. Hence by contradiction method $\sqrt{2} + \sqrt{3}$ is irrational.

c $f(x) = 2x^5 - 5x^4 + x^3 + x^2 - x + 6 = 0$

-1	2	-5	1	1	-1	6
		-2	7	-8	7	-6
2	2	-7	8	-7	6	0
		4	-6	4	-6	
$\frac{3}{2}$	2	-3	2	-3	0	
		3	0	3		
	2	0	2	0		

$$2x^2 + 0x + 2$$
$$2x^2 + 2 = 0$$
$$x^2 + 1 = 0$$

$$\therefore x = -1 \quad , \quad x = 2 \quad , \quad x = 3/2$$

(d) $3|2x+1| + 1 \leq 7$

$$3|2x+1| + 1 - 1 \leq 7 - 1$$

$$3|2x+1| \leq 6$$

$$|2x+1| \leq 2$$

$2x + 1 \leq 2$ or $2x + 1 \geq -2$

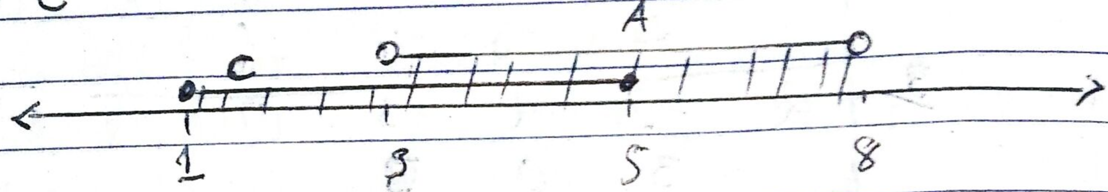
$2x \leq 1$ $\qquad$ $2x \geq -3$

$x \leq \frac{1}{2}$ $\qquad$ $x \geq -\frac{3}{2}$

Question 4

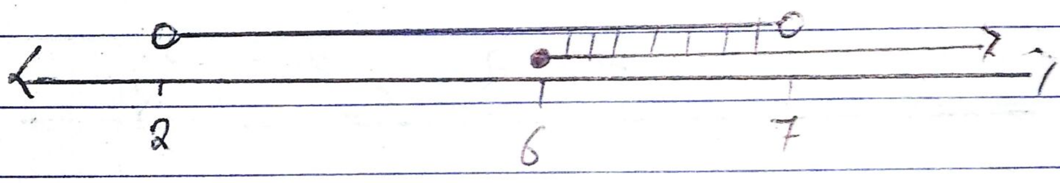
(a) $A = (3, 8]$ $B = (2, 7)$ $C = [1, 5]$ $D = [6, \infty)$

(i) $A \cup C$



$A \cup C = [1, 8)$

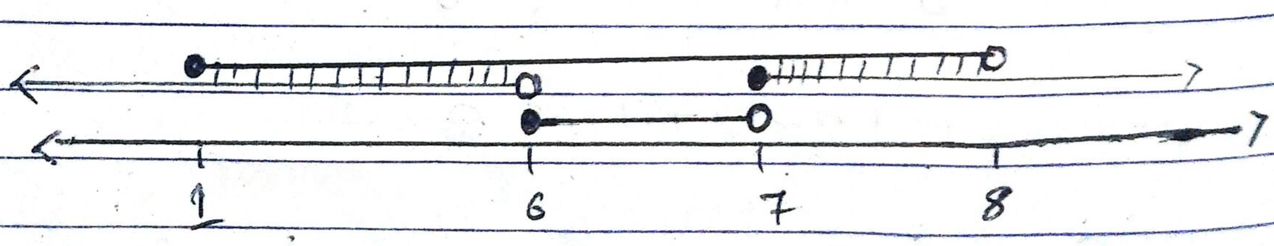
(ii) $B \cap D$



$B \cap D = [6, 7)$

(ii) $(A \cup C) - (B \cap D)$

$(A \cup C) \cap (B \cap D)'$



$(A \cup C) - (B \cap D) = [1, 6) \cup [7, 8)$

b $a * b = a + b - ab$

for commutative

$$a * b = b * a$$
$$2 * 1 = 1 * 2$$
$$2 + 1 - (2)(1) = 1 + (2) - (1)(2)$$
$$3 - 2 = 3 - 2$$
$$1 = 1$$

$\therefore$ since $a * b = b * a$ the operation is commutative

for associative

$$(1 * 2) * 3 = 1 * (2 * 3)$$

$1 * 2 = 1 + 2 - (2)(1)$
$= 3 - 2$
$1 * 2 = 1$

$2 * 3 = 2 + 3 - (2)(3)$
$= 5 - 6$
$2 * 3 = -1$

$1 * 3 = 1 + 3 - (1)(3)$
$= 4 - 3$
$1 * 3 = 1$

$1 * -1 = 1 + (-1) - (1)(-1)$
$= 0 + 1$
$1 * -1 = 1$

$(1 * 2) * 3 = 1$ $1 * (2 * 3) = 1$

- Hence it is associative too !!

c $f(x) =$

$$f(x) = -(x-3)(x-2)^3(x+1)^2$$

C O B E → Even

C → Cross, O → Odd, B → Bounce

$x - 3 = 0$ $x - 2 = 0$ $x + 1 = 0$

$x = 3$ $x = 2$ $x = -1$

~~Bounce~~   ~~Bounce~~   Bounce

Cross   Cross

Extra point (when $x = -2$) → ?

$$y = (x-3)(x-2)^3(x+1)^2$$

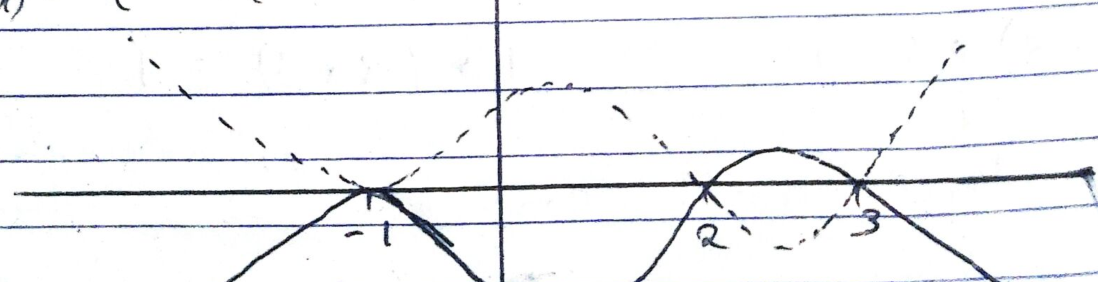
$$y = (-2-3)(-2-2)^3(-2+1)^2$$

$$y = (-)(-)^3(-)^2$$

$$y = (-)(-)(+)$$

$$y = (+)$$

$$f(x) = (x-3)(x-2)^3(x+1)^2$$



$$f(x) = -(x-3)(x-2)^3(x+1)^2$$

d $|2x+11| = 7$

$2x + 11 = 7$ or $2x + 11 = -7$

$2x = 7 - 11$ $2x = -7 - 11$

$2x = -4$ $2x = -18$

$\underline{x = -2}$ $\underline{x = -9}$

Question 5

(a) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1} + \sqrt{3} - 1$

get the fraction and rationalise it the add your result to $\sqrt{3} - 1$

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$\frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2-(1)^2}$$

$$\frac{3+2\sqrt{3}+1}{3-1}$$

$$\frac{4+2\sqrt{3}}{2}$$

$$\frac{4}{2}+\frac{2\sqrt{3}}{2}$$

$$2+\sqrt{3}$$

good then add to $\sqrt{3} - 1$

$$2+\sqrt{3}+\sqrt{3}-1$$

$$\underline{\underline{1+2\sqrt{3}}}$$

(b) $f(x) = 4x - 5$ $\qquad g(x) = \frac{x+5}{4}$

$$fog(x) = 4\left(\frac{x+5}{4}\right) - 5$$

$$= x + 5 - 5$$

$$\underline{\underline{fog(x) = x}}$$

$$gof(x) = \frac{(4x-5)+5}{4}$$

$$= \frac{4x-5+5}{4}$$

$$= \frac{4x}{4}$$

$$\underline{\underline{gof(x) = x}}$$

$\therefore$ since $fog(x) = gof(x) = x$ hence $g(x)$ and $f(x)$ are inverses of each other

c $\quad 1 - 6x - x^2$

$$-x^2 - 6x + 1$$

$$-1[x + 6x - 1]$$

$$-1[x^2 + 6x + (3)^2 - (3)^2 - 1]$$

$$-1[(x+3)^2 - 9 - 1]$$

$$-1\left[(x+3)^2 - 10\right]$$

$$\underline{\underline{-(x+3)^2 + 10}}$$

For turning point ;

$y = 10$ $\quad\quad x + 3 = 0$

$x = -3$

$\underline{\underline{T.p(-3,10)}}$

(d) $2x - 11\sqrt{x} + 12 = 0$

$2(x^{\frac{1}{2}})^2 - 11x^{\frac{1}{2}} + 12 = 0$

let $a = x^{\frac{1}{2}}$

$2a^2 - 11a + 12 = 0$

$2a^2 - 8a - 3a + 12 = 0$

$2a(a-4) - 3(a-4) = 0$

$(a-4)(2a-3) = 0$

$a - 4 = 0$ or $2a - 3 = 0$

$a = 4$ $\quad\quad 2a = 3$

$a = \frac{3}{2}$

but $a = x^{\frac{1}{2}}$

$x^{\frac{1}{2}} = 4$ $\quad\quad x^{\frac{1}{2}} = \frac{3}{2}$

$(x^{\frac{1}{2}})^2 = (4)^2$ $\quad\quad (x^{\frac{1}{2}})^2 = \left(\frac{3}{2}\right)^2$

$x = 16$ $x = \frac{9}{4}$

Question 6

(a) $(x + yi) - i = i(x + yi) + 5$

$x + yi - i = xi - y + 5$

Equat real to real & imaginery to imaginery

$x = -y + 5$ $yi - i = xi$

$y - 1 = x,$

joining both equations together we get

$-y + 5 = y - 1$

$-y - y = -5 - 1$

$-2y = -6$

$y = 3$

$y - 1 = x$

$3 - 1 = x$

$2 = x$

$\therefore x = 2, y = 3$

b i) $kx^2 - 2kx + 2k = 1$

$$kx^2 - 2kx + 2k - 1 = 0$$

$a = k \qquad b = -2k \qquad c = 2k - 1$

$$D \geq 0$$
$$b^2 - 4ac \geq 0$$
$$(-2k)^2 - 4(k)(2k-1) \geq 0$$
$$4k^2 - 4(2k^2 - k) \geq 0$$
$$4k^2 - 8k^2 + 4k \geq 0$$
$$-4k^2 + 4k \geq 0 \quad \Big] \div -4$$

$$- \quad \underline{\underline{k^2 - k \leq 0}}$$

Hence shown !!

(ii) $k^2 - k \leq 0$

$$k(k-1) \leq 0$$

~~[illegible]~~ ~~[illegible]~~

\# Critical points:

$k = 0 \qquad k - 1 = 0$

$\qquad\qquad k = 1$

Range	$k < 0$	$0 < k < 1$	$k > 1$
Test value	-1	$\frac{1}{2}$	2
k	$-$	$+$	$+$
$k-1$	$-$	$-$	$+$
	$+$	$-$ ✓	$+$

$\underline{\underline{0 \leq k \leq 1}}$ or $\underline{\underline{[0, 1]}}$

e. $$f(x) = \frac{2x^2 - 2}{x^2 - 4}$$

(i) x-interecept [$y = 0$]

$$y = \frac{2x^2 - 2}{x^2 - 4}$$

$$0 = \frac{2x^2 - 2}{x^2 - 4}$$

$$2x^2 - 2 = 0$$

$$x^2 - 1 = 0$$

$$\sqrt{x^2} = \sqrt{1}$$

$$x = \pm 1$$

$x = 1$ or $x = -1$

$\underline{\underline{(1, 0) \text{ or } (-1, 0)}}$

y-intercept [$x = 0$]

$$y = \frac{2(0)^2 - 2}{(0)^2 - 4}$$

$$y = \frac{-2}{-4}$$

$$y = \frac{1}{2}$$

$\underline{\left(0, \frac{1}{2}\right)}$

(ii) Horizotal Asymptote

$$y = \frac{2}{1} = 2$$

$$\underline{y = 2}$$

Vertical Asymptote

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

$$\underline{x = 2} \qquad \underline{x = -2}$$

iii