The Copperbelt University

School of Mathematics And Natural Sciences

Department of Mathematics

MA 110 : (Mathematical Methods I) : Test Two

Friday - July 22, 2022

Instructions

(1). You must write your **Name, Computer number and Programme** of study on your answer sheet. Time allowed is **2 hours.**

(2). **Calculators** and use of **Cell phones** are **Not** allowed in this paper.

(3). There are **Four (4) questions** in this paper, **attempt all the questions** and show detailed working for full credit.

QUESTION ONE

(a) Find the center and radius of the circle whose equation is

$$x^2 + y^2 + 8x - 2y + 13 = 0.$$

(5 marks)

(b) Write down the constant term in the expansion of $\left(x - \dfrac{1}{2x^2}\right)^9$.

(5 marks)

(c) Prove that $\log_a\left(\dfrac{A}{B}\right) = \log_a(A) - \log_a(B)$.

(5 marks)

(d) Use **Crammer's method** to solve the linear system of the equation

$$\begin{aligned} 3x - 4y &= -11 \\ -5x + y &= 7. \end{aligned}$$

(5 marks)

QUESTION TWO

(a) Change the repeating decimal $3.\overline{7}$ to its reduced form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ using sum to infinity of a geometric series.

(5 marks)

(b) Use Mathematical induction to prove that $2^n \geqslant n + 1$ for all possible integer n.

(5 marks)

(c) Find the equation of the tangent at the point $(3, 1)$ on the circle

$$x^2 + y^2 - 4x + 10y - 8 = 0.$$

(5 marks)

(d) Graph the function $f(x) = 2^{(x-3)} + 2$ and obtain its inverse on the same axes.

(5 marks)

QUESTION THREE

(a) Solve $25^x - 5^x = 12$.

(5 marks)

(b) Express $\frac{2x^2 - 5x + 7}{(x-2)(x-1)^2}$ into a partial fraction.

(5 marks)

(c) What is the common ratio of the G.P. $(\sqrt{2} - 1) + (3 - 2\sqrt{2}) + \cdots$? Find the third term of progression.

(5 marks)

(d) How long will it take K2000 to double itself at 13% interest compounded continously?

(5 marks)

QUESTION FOUR

(a) Find the first term and the general expansion of $\frac{1}{(2-3x)^3}$ in ascending power of x. State the range of value of x for which this expansion is valid.

(5 marks)

(b) Show that the general term of an arithmetic sequence is given by

$$a_n = a_1 + (n-1)d.$$

(5 marks)

(c) Solve the equation $\log_2 x = \log_4(x + 6)$.

(5 marks)

(d) Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}.$$

(5 marks)

THE END OF TEST

Q 1

(a) find the center and radius of the circle with eqn

$$x^2+y^2+8x-2y+13=0$$

[5]

Pointers:

express the eqn in the form $(x-h)^2+(y-k)^2=r^2$ by completing the square.

- make sure the coefficients of x^2 & y^2 are 1.

$$x^2+y^2+8x-2y+13=0$$

$$\underbrace{x^2+8x+\left(\frac{+8}{2}\right)^2}-\left(\frac{+8}{2}\right)^2+\underbrace{y^2-2y+\left(-\frac{2}{2}\right)^2}-\left(\frac{-2}{2}\right)^2=-13$$

$$\left(x+\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+\left(y-\frac{2}{2}\right)^2-\left(\frac{-2}{2}\right)^2=-13$$

$$(x+4)^2-(4)^2+(y-1)^2-(-1)^2=-13$$

$$(x+4)^2+(y-1)^2-16-1=-13$$

$$(x+4)^2+(y-1)^2=-13+16+1$$

$$(x+4)^2+(y-1)^2=4$$

center is $(-4,+1)$, radius $=\sqrt{4}=2$

1b 0965944100 ③

Write down the constant in the expansion of $\left(x - \frac{1}{2x^2}\right)^9$

[5]

*Pointers:

Substitute into $\binom{n}{r} a^{n-r} b^r$, then distribute the r for b. Then equate the power of the first x to the power of the power of the second x that is a denominator.

$$\binom{n}{r} a^{n-r} b^r \qquad \therefore\ a = x$$

$$b = \frac{-1}{2x^2}$$

$$n = 9$$

first substitution

$$\binom{9}{r} x^{9-r} \left(\frac{-1}{2x^2}\right)^r \Rightarrow \binom{9}{r} x^{9-r} \left(\frac{-1^r}{2^r x^{2r}}\right)$$

equate

$$9 - r = 2r$$

$$9 = 2r + r$$

$$9 = 3r$$

$$r = 3$$

Then substitute into the general formula and evaluate.

$$\binom{9}{3} x^{9-3} \left(\frac{-1}{2x^2}\right)^3 \Rightarrow \binom{9}{3} \cancel{x^6} \left(\frac{-1}{8\cancel{x^6}}\right)$$

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$$\binom{9}{3} \times \left(\frac{-1}{8}\right) \Rightarrow \frac{9!}{(9-3)!\,3!} \times -\frac{1}{8}$$

$$\Rightarrow \frac{\cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{6!}\ \cancel{3} \cdot 2 \cdot 1} \times \frac{-1}{\cancel{8}}$$

$$\Rightarrow \frac{3 \times 7 \times -1}{2} = -\frac{21}{2}$$

$\boxed{1c}$ Prove $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$

Let $\log_a A = P$, $\log_a B = Q$

convert to Their exponential forms

$$a^P = A \quad , \quad a^Q = B$$

$$A \div B = a^P \div a^Q$$

when dividing Things with The same base it is as good as subtracting their powers.

$$A \div B = a^{P-Q}$$

$$\frac{A}{B} = a^{P-Q}$$

* introduce $\log_a$ on both sides.

$$\log_a\left(\frac{A}{B}\right) = \log_a a^{(P-Q)}$$

logs can make powers become coefficients and $\log_a a = 1$.

$$\log_a\left(\frac{A}{B}\right) = (P-Q)\underbrace{\log_a a}$$

$$\log_a\left(\frac{A}{B}\right) = P-Q$$

replace P & Q

$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Hence Proved.

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1d Use crammers method to solve

$$3x - 4y = -11$$
$$-5x + y = 7$$

Pointers:

Convert the equations to their matrix form by arranging the coefficients

$$\begin{pmatrix} 3 & -4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ 7 \end{pmatrix}$$

find $d0$, dx and dy

$d0$: determinat of original matrix

dx: replace first column with answers then find the determinat

dy: replace second column with answers then find the determinat.

$d0$

$$\begin{vmatrix} 3 & -4 \\ -5 & 1 \end{vmatrix} \Rightarrow (3 \times 1) - (-4 \times -5)$$
$$3 - (20)$$
$$-17$$

dx

$$\begin{vmatrix} -11 & -4 \\ 7 & 1 \end{vmatrix} \Rightarrow (-11 \times 1) - (-4 \times 7)$$
$$-11 + 28$$
$$+17$$

dy $\begin{vmatrix} 3 & -11 \\ -5 & 7 \end{vmatrix} \Rightarrow (3\times7) - (-11\times-5)$

$21 - 55$

-34

$\therefore \; x = \frac{dx}{d0}, \quad y = \frac{dy}{d0}$

$x = \frac{17}{-17} \qquad y = \frac{-34}{-17}$

$x = -1 \qquad y = 2$

confirm by plugging into any one of the eqns.

$3x - 4y = -11$

$3(-1) - 4(2)$

$-3 - 8$

-11 ✓✓

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Q2

(a) Convert $3.\bar{7}$ to the form $\frac{a}{b}$ using G.P sum to infinity [5]

Pointers

- use $S_\infty = \frac{a_1}{1-r}$
- Divide $3.\bar{7}$ into a repeating part and a non repeating part.

$$3.\bar{7} \Rightarrow \underbrace{3}_{\text{non repeating}} + \underbrace{0.\bar{7}}_{\text{convert the repeating part to a series}}$$

e.g $0.\bar{a} = 0.a + 0.0a$

$0.\overline{ab} = 0.ab + 0.00ab$

$$0.\bar{7} = \underbrace{0.7 + 0.07 + 0.007}$$

then apply sum to infinity on the repeating part.

$a_1 = 0.7$ or $\frac{7}{10}$

$$r = \frac{0.07 \times 100}{0.7 \times 100} = \frac{7}{70} = \frac{1}{10}$$

$$\therefore a_1 = \frac{7}{10}, \quad r = \frac{1}{10}$$

$$S_\infty = \frac{a_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} \Rightarrow \frac{\frac{7}{10}}{\frac{10}{10}-\frac{1}{10}} \Rightarrow \frac{\frac{7}{10}}{\frac{9}{10}}$$

$$\Rightarrow \frac{7}{10} \div \frac{9}{10} \Rightarrow \frac{7}{\cancel{10}} \times \frac{\cancel{10}}{9} = \frac{7}{9}$$

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Now add the non repeating part to the fraction that the sum to infinity formular gave you.

$$\frac{3}{1} + \frac{7}{9}$$

$$\frac{27 + 7}{9} = \frac{34}{9}$$

$$\therefore \; 3.\bar{7} = \frac{34}{9}$$

Q2b use mathematical induction to prove that $2^n \geq n + 1$ for all possible integers n. [5]

PROOF BY INDUCTION — Don't forget the title.

Step 1 Let $n = 1$

$2^1 \geq 1 + 1$

$2 \geq 2$ true statement.

Step 2 let $n = k, k \in \mathbb{Z}^+$

$2^k \geq k + 1$ --- (i)

true statement.

Step 3 let $n = k+1, k \in \mathbb{Z}^+$

$2^{k+1} \geq k + 1 + 1$

$2^{k+1} \geq k + 2$ --- (ii) true statement

Step 4 use eqn(i) to derive eqn(ii)

$2(2^k \geq k + 1)$

$2^{k+1} \geq 2k + 2$

$2^{k+1} \geq 2k + 2 \geq k + 2$

using the transitive rule

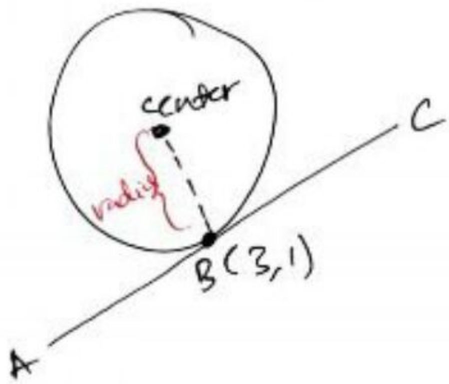
$2^{k+1} \geq k + 2$

Hence Proved.

Q2c find the equation of the tangent to the point $(3,1)$ on the circle $x^2+y^2-4x+10y-8=0$

Pointers:

Sketch (Your sketch may not be accurate)



by expressing in the form $(x-a)^2+(y-b)^2=r^2$

* find the center of the circle Δ then find the gradient of the line connecting the center to B(3,1).
* The radius is the normal to the tangent so their gradients are related by the eqn $M_{rad} \times M_{tangent} = -1$
* After finding the gradient of the tangent, use the point (3,1) and the eqn $y-y_1=m(x-x_1)$.

$$x^2+y^2-4x+10y-8=0$$

$$x^2-4x+(-2)^2-(-2)^2+y^2+10y+(+5)^2-(+5)^2=+8$$

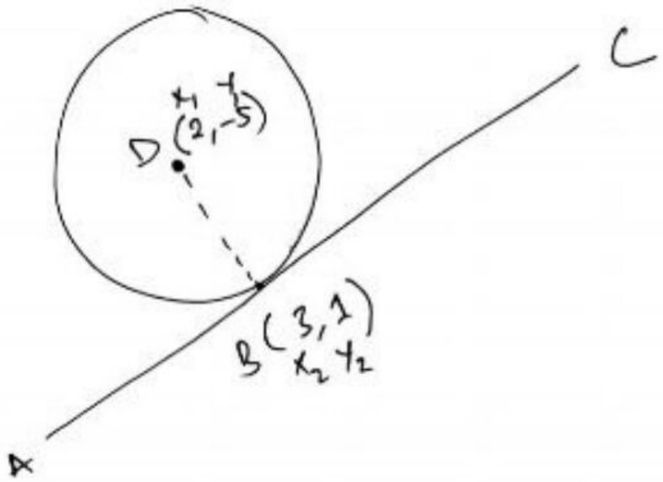
$$(x-2)^2-4+(y+5)^2-25=-8$$

$$(x-2)^2+(y+5)^2=+8+4+25$$

$$(x-2)^2+(y+5)^2=37$$

$$(x-2)^2 + (y+5)^2 = 37$$

center is $(2,-5)$



* This is just a rough sketch, it doesn't have to be accurate.

$$\text{Gradient}_{\substack{BD \\ (\text{radius})}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - -5}{3 - 2} = \frac{6}{1} = 6$$

$$m_1 m_2 = -1$$

$$m_{tan} = \frac{-1}{m_{radius}}$$

$$m_{tan} = -\frac{1}{6}$$

Then plug into $y - y_1 = m(x - x_1)$

$$\underset{x_1\ \ y_1}{(3, 1)}$$

$$y - 1 = -\frac{1}{6}(x - 3)$$

$$y = -\frac{1}{6}x + \frac{3}{6} + \frac{6}{6}$$

$$y = -\frac{1}{6}x + \frac{9}{6} \;//$$

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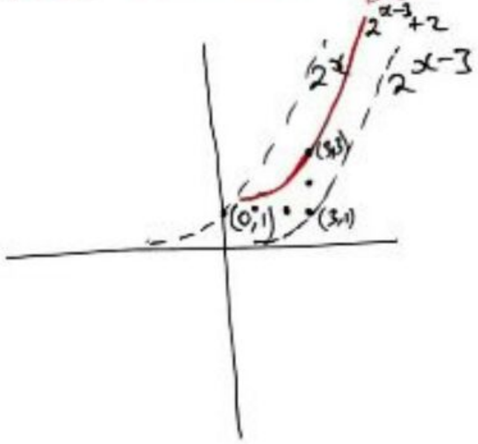
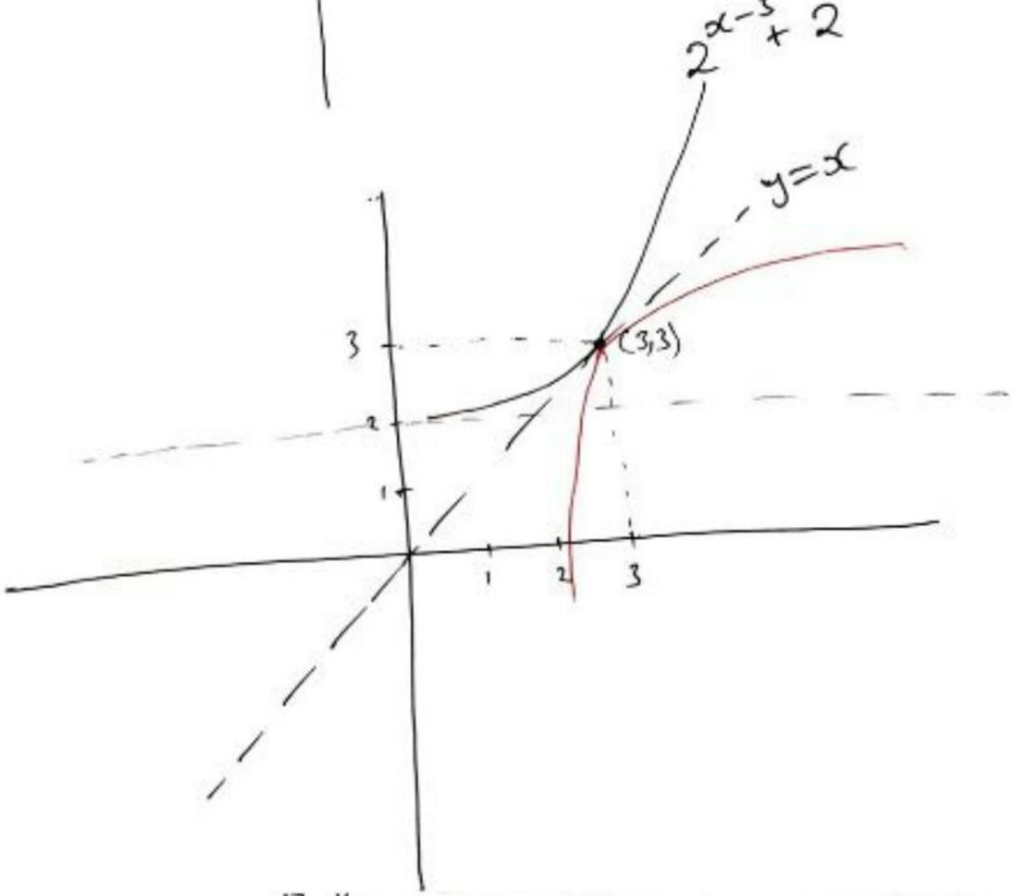
Q 2d

Graph $f(x) = 2^{x-3} + 2$ and obtain its inverse on the same axes.

Pointers: we get the inverse of a log or exponential graph by reflecting the graph we have about the mirror line $y = x$.

* find what $2^{x-3} + 2$ looks like.





$y = 2^{x-3} + 2$ * lets try and find the inverse. (13)

$y - 2 = 2^{x-3}$

$\log_2(y-2) = \log_2 2^{(x-3)}$

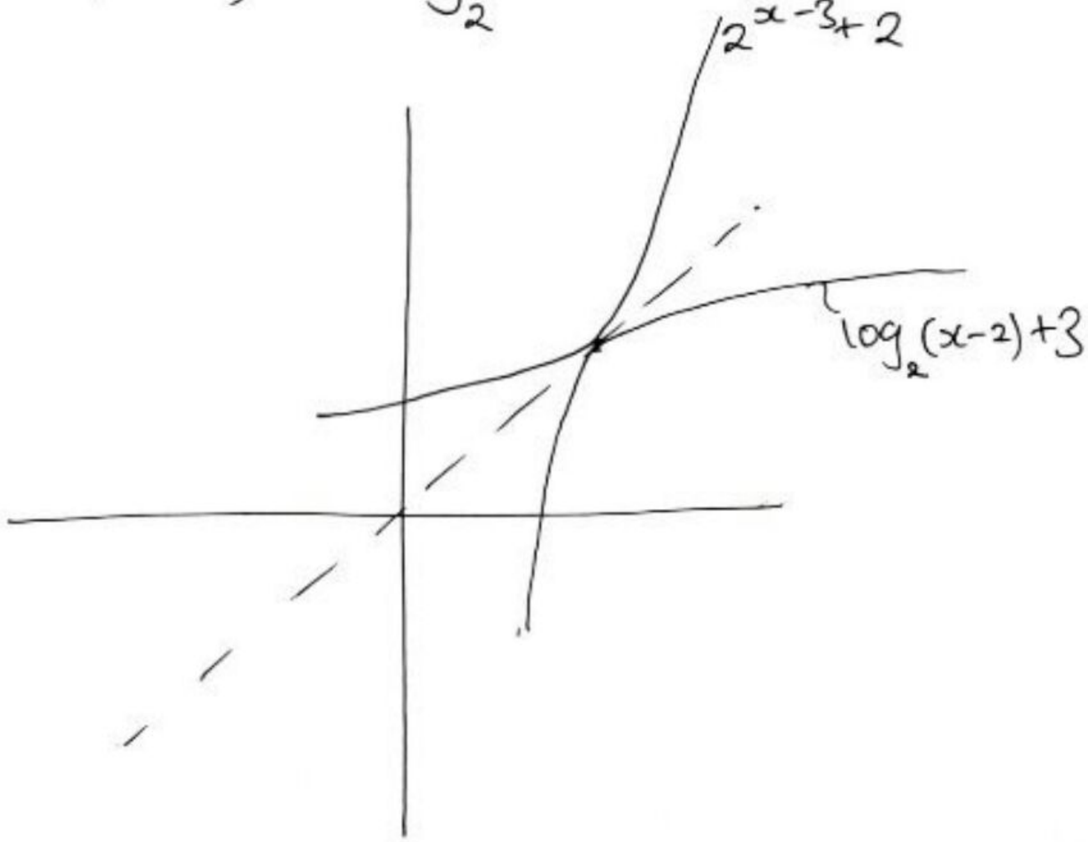
$\log_2(y-2) = x - 3$

$\log_2(y-2) + 3 = x$

$\therefore x = \log_2(y-2) + 3$

$\therefore$ replace
x with $f^{-1}(x)$
and y with x

$\therefore f^{-1}(x) = \log_2(x-2) + 3$



Q3 0965944100

(a) Solve $25^x - 5^x = 12$ [5]

Pointers:

write 25^x in terms of 5^x

$$25^x \Rightarrow (5^2)^x \Rightarrow (5^x)^2$$

$$(5^x)^2 - 5^x = 12$$

replace 5^x with P

$$P^2 - P - 12 = 0$$

S: -1 P: -12 F: $+3$ -4

$$P^2 + 3P - 4P - 12 = 0$$

$$P(P+3) - 4(P+3) = 0$$

$$(P-4)(P+3) = 0$$

$$P = 4 \quad , \quad P = -3$$

$$5^x = 4 \quad , \quad \cancel{5^x = -3}$$

invalid solution

←introduce any log you want

$\log_5 5^x = \log_5 4$

$x = \log_5 x$,

or $\ln 5^x = \ln 4$

$$x = \frac{\ln 4}{\ln 5}$$

or $\log 5^x = \log 4$

$$x = \frac{\log 4}{\log 5}$$

all are correct

3b) Express $\dfrac{2x^2-5x+7}{(x-2)(x-1)^2}$ into a partial fraction. (15)

[5]

$$\left[\frac{2x^2-5x+7}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right](x-2)(x-1)^2$$

$$2x^2-5x+7 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$$

Let $x = 1$

$$2-5+7 = 0 + 0 + C(1-2)$$

$$4 = -C$$

$$C = -4$$

Let $x = 2$

$$2(2)^2 - 5(2) + 7 = A(2-1)^2 + 0 + 0$$

$$8 - 10 + 7 = A$$

$$A = 5$$

Let $x = 0$

$$0-0+7 = \overset{5}{A}(0-1)^2 + \overset{?}{B}(0-2)(0-1) + \overset{-4}{C}(0-2)$$

$$7 = 5(1) + B(-2\times-1) + (-4\times-2)$$

$$7 = 5 + 2B + 8$$

$$7 = 13 + 2B$$

$$7 - 13 = 2B$$

$$-6 = 2B$$

$$B = -3$$

$A = 5$
$B = -3$
$C = -4$

$$\therefore \frac{5}{x-2} + \frac{-3}{x-1} + \frac{-4}{(x-1)^2}$$

$\boxed{3c}$ what is the common ratio of the G.P $(\sqrt{2}-1)+(3-2\sqrt{2})+\dots$?
find the third term of the G.P

[5]

$$r = \frac{a_{k+1}}{a_k} = \frac{3-2\sqrt{2}}{\sqrt{2}-1}$$

* rationalize the denominator.

$$r = \frac{3-2\sqrt{2}}{\sqrt{2}-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{\sqrt{2}(3-2\sqrt{2})+1(3-2\sqrt{2})}{(\sqrt{2})^2-(1)^2}$$

$$r = \frac{3\sqrt{2}-2(2)+3-2\sqrt{2}}{2-1} \Rightarrow \frac{\sqrt{2}-1}{1} = \sqrt{2}-1$$

* to find the third term, derive the general formula then replace n with 3.

$$a_n = \underbrace{a_1}_{(\sqrt{2}-1)} \times \underbrace{r}_{\sqrt{2}-1}{}^{n-1}$$

$$a_n = (\sqrt{2}-1)^1(\sqrt{2}-1)^{n-1}$$

* add the powers since the bases are the same.

$$a_n = (\sqrt{2}-1)^{n-1+1} = (\sqrt{2}-1)^n$$

Third ~~mean~~ term in $n=3$

$$a_n = (\sqrt{2}-1)^n$$

$$a_3 = (\sqrt{2}-1)^3$$

* we can quickly use (pascals) binomial $(a+b)^3$ then replace $a=\sqrt{2}$, $b=-1$

1

1 1

1 2 1

1 3 3 1 ✓

$$(a+b)^3 = 1a^3b^0 \quad 3a^2b^1 \quad 3a^1b^2 \quad 1a^0b^3$$

$$= a^3+3a^2b+3ab^2+b^3$$

$a=\sqrt{2}$
$b=-1$

$$\Rightarrow (\sqrt{2})^3+3(\sqrt{2})^2(-1)+3(\sqrt{2})(-1)^2+(-1)^3$$

$$2\sqrt{2}+3(2)(-1)+3\sqrt{2}-1$$

$$5\sqrt{2}+-6-1$$

$5\sqrt{2}-7$ is the third term.

* or you could multiply $(\sqrt{2}-1)(\sqrt{2}-1)(\sqrt{2}-1)$ to confirm

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Quantom
Inspired.

Q3d How long does it take K2000 to double itself at 13% interest compounded continously. [5]

Pointers: Pay attention to words like doubled (x2), tripled (x3) Quadruple (x4) - - .

use $A = Pe^{rt}$

time

$\frac{\%}{100}$

Amount you want to end up with.

Amount you're putting in.

$A = 2000 \times 2 = 4000$

$$\frac{4000}{2000} = \frac{2000}{2000} e^{0.13t}$$

$2 = e^{0.13t}$ * introduce $\log_e$ or Ln on both sides

$\ln 2 = \ln e^{0.13t}$

$\ln 2 = 0.13t(\log_e e)$

$\therefore \ln e = 1$

$\log_e e = 1$.

$t = \frac{\ln 2}{0.13} \approx \frac{0.693}{0.13}$ //

you can leave it here

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Quantom.

$\boxed{Q4}$

(a) find the first term and general expansion of $\frac{1}{(2-3x)^3}$ in ascending powers of x and state the values for which the expansion is valid.

$$\frac{1}{(2-3x)^3} \Rightarrow (2-3x)^{-3} \Rightarrow \left[2\left(1-\frac{3x}{2}\right)\right]^{-3}$$

$$\Rightarrow 2^{-3}\underbrace{\left(1-\frac{3}{2}x\right)^{-3}}_{\text{expand}}$$

$$2^{-3}\left[a^n + na^{n-1}b^1 + \frac{n(n-1)a^{n-2}b^2}{2!}\right]$$

$$2^{-3}\left[1 + (-3)\left(-\frac{3x}{2}\right) + (-3)(-4)\left(-\frac{3x}{2}\right)^2\left(\frac{1}{2}\right)\right]$$

$$\frac{1}{8}\left[1 + \frac{9x}{4} + \cancel{12}^{3} \times \frac{9x^2}{\cancel{4}_{1}} \times \frac{1}{2}\right]$$

$$\frac{1}{8}\left[1 + \frac{9x}{4} + \frac{27x^2}{2}\right]$$

$$\Rightarrow \frac{1}{8} + \frac{9x}{32} + \frac{27x^2}{16} + \cdots$$

* first term is $\frac{1}{8}$ //

(b) Show that the general term of an AP is $a_n = a_1 + (n-1)d$. (20)

* Let a_1 be our first term.
* to get a_2 we add d to a_1

$$a_1, \quad a_2, \quad a_3, \ldots a_n$$

$$a_1 \quad a_1 + 1d \quad a_1 + 2d$$

$\therefore$ notice that the coefficient of d is always less than the position by 1.

* take the position to be n.

$$\therefore a_1, \quad a_1 + 1d, \quad a_1 + 2d, \ldots a_n$$

$\therefore a_1 = a_1 + (\text{Position} - 1)d, \quad a_2 = a_1 + (\text{Position} - 1)d \ldots$

Let position be n

$$\therefore a_n = a_1 + (\text{Position} - 1)d$$

$$a_n = a_1 + (n-1)d \;//.$$

4c Solve.

$$\log_2 x = \log_4 (x+6)$$

∴ Convert to $\log_2$

$$\log_2 x = \frac{\log_2 (x+6)}{\log_2 4}$$

write as 2^2

$$\log_2 x = \frac{\log_2 (x+6)}{\log_2 2^2}$$

∴ $\log_b b = 1$

$$\frac{\log_2 x}{1} = \frac{\log_2 (x+6)}{2}$$

$$2\log_2 x = \log_2 (x+6)$$

$$\log_2 x^2 = \log_2 (x+6)$$

$$\log_2 (x^2) - \log_2 (x+6) = 0$$

use $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

$$\log_2 \left(\frac{x^2}{x+6}\right) = 0$$

convert to its exponential form.

$$2^{\circ} = \frac{x^2}{x+6}$$

$$\frac{1}{1} \cancel{=} \frac{x^2}{x+6}$$

$$x^2 = x+6$$

$$x^2 - x - 6 = 0$$

S: -1 P: -6 F: $+2$ & -3

$$x^2 + 2x - 3x - 6 = 0$$

$$x(x+2) - 3(x+2) = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, \quad x = -2$$

Sobstitute each answer into The expression to see which would be invalid.

$\log_2 (3)$ ✓ | $\log_2 (-2)$ ✗

$\therefore$ $x = 3$ $x = -2$ (invalid)

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4d if $A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$

find the inverse of A [5]

$$\text{Inverse} = \frac{1}{\det} \times \text{Adjoint}.$$

<u>det</u>.

$$\begin{array}{ccc|cc} & & 4 & 6 & 1 \quad \Sigma_{top} = 11 \\ 3 & -1 & 2 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & -1 & 2 & 2 \\ & & -3 & -2 & 4 \quad \Sigma_{bot} = -1 \end{array}$$

$$\begin{aligned} \det &= \Sigma_{bot} - \Sigma_{top} \\ &= -1 - 11 \\ &= -12 \end{aligned}$$

<u>cofactors</u>.

$$\begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{pmatrix}$$

$\underline{a}: +(-1-2) = -3$ $\underline{b}: -(-1-2) = +3$, $\underline{c}: +(2-2) = 0$,

$\underline{d}: -(1-4) = +3$, $\underline{e}: +(-3-4) = -7$ $\underline{f}: -(6--2) = -8$

$\underline{g}: +(-1-2) = -3$ $\underline{h}: -(3-2) = -1$ $\underline{i}: +(3--1) = +4$

matrix of cofactors

$$\begin{pmatrix} -3 & 3 & 0 \\ 3 & -7 & -8 \\ -3 & -1 & 4 \end{pmatrix}$$

transpose of the matrix of co-factors.

$$\begin{pmatrix} -3 & 3 & -3 \\ 3 & -7 & -1 \\ 0 & -8 & 4 \end{pmatrix}$$

$$\text{Inverse} = \frac{1}{\det} \times \text{Adjoint}$$

$$= \frac{1}{-12} \begin{pmatrix} -3 & 3 & -3 \\ 3 & -7 & -1 \\ 0 & -8 & 4 \end{pmatrix} //$$

Stay Inspired.

Reach us on 0965944100 or 0974666569.