MATHEMATICS

DEC160 – TECHNICAL

Time allowed: Two hours (2:00 hours) TEST 3 2023

QUESTION ONE

a) Find the period, amplitude and phase shift of $f(x) = 1 + \frac{1}{3} sin\left(2x + \frac{\pi}{4}\right)$ and sketch the curve.

b) Solve the trigonometric equation $sinx + cosx = \sqrt{2}$ if $0 \leq x \leq 2\pi$

c) Use De Moivre's theorem to find the indicated power of $(1 + i)^{20}$
Express results in $a + bi$

d) Find the cube roots of $2 + 2i$

e) Sketch the graph of the polar equation.

$$r^2 = sin2\theta$$

f) Solve for real values of x, given that $sinh^2 x - 3coshx = 3$

QUESTION TWO

a) Find the limit of the quotients as $h \to 0$ $\frac{\sqrt{x+h} - \sqrt{x}}{h}$

b) Determine the interval for which $f(x)$ is continuous.

$$f(x) = \sqrt{4 - x^2}$$

c) Differentiate the function $f(x) = sinx$ using the first principle

d) Use the **Second derivative Test** to find the local extrema of

$$f(x) = x^4 + 8x^2 + 10$$

e) Find the derivative $y'(x)$

(ii) $xe^y - 3ysinx = 1$ $\quad$ $y = \ln\left(\frac{x^3 + 2}{(x+3)^5}\right)$

f) Show that $\frac{d}{dx} tan^{-1} x = \frac{1}{1+x^2}$

g) Evaluate the following indefinite integrals

(i) $\int_0^{\pi^2} \frac{cos\sqrt{x}}{\sqrt{x}} dx$ (ii) $\int \ln x \, dx$ (iii) $\int \frac{2x^2 - 5x + 2}{x(x^2+1)} dx$

h) Let $f(x) = 1 + \sin^2 x + \sin x$ be a function defined on the interval $0 < x < 2\pi$.

(i) Find all the critical points of the function on $0 < x < 2\pi$.

(ii) Find the tangent to the graph of the function at a point where $x = \pi$.

QUESTION 1

(a) $F(x) = 1 + \frac{1}{3}\sin\left(2x + \frac{\pi}{4}\right)$

* Amplitude $= \left|\frac{1}{3}\right| = \frac{1}{3}$

* Period $= \frac{360}{2} = 180$ or $\frac{2\pi}{2} = \pi$

But since we are dealing with rads, π is the best answer.

* Phase shift $= 2x + \frac{\pi}{4} = 0$

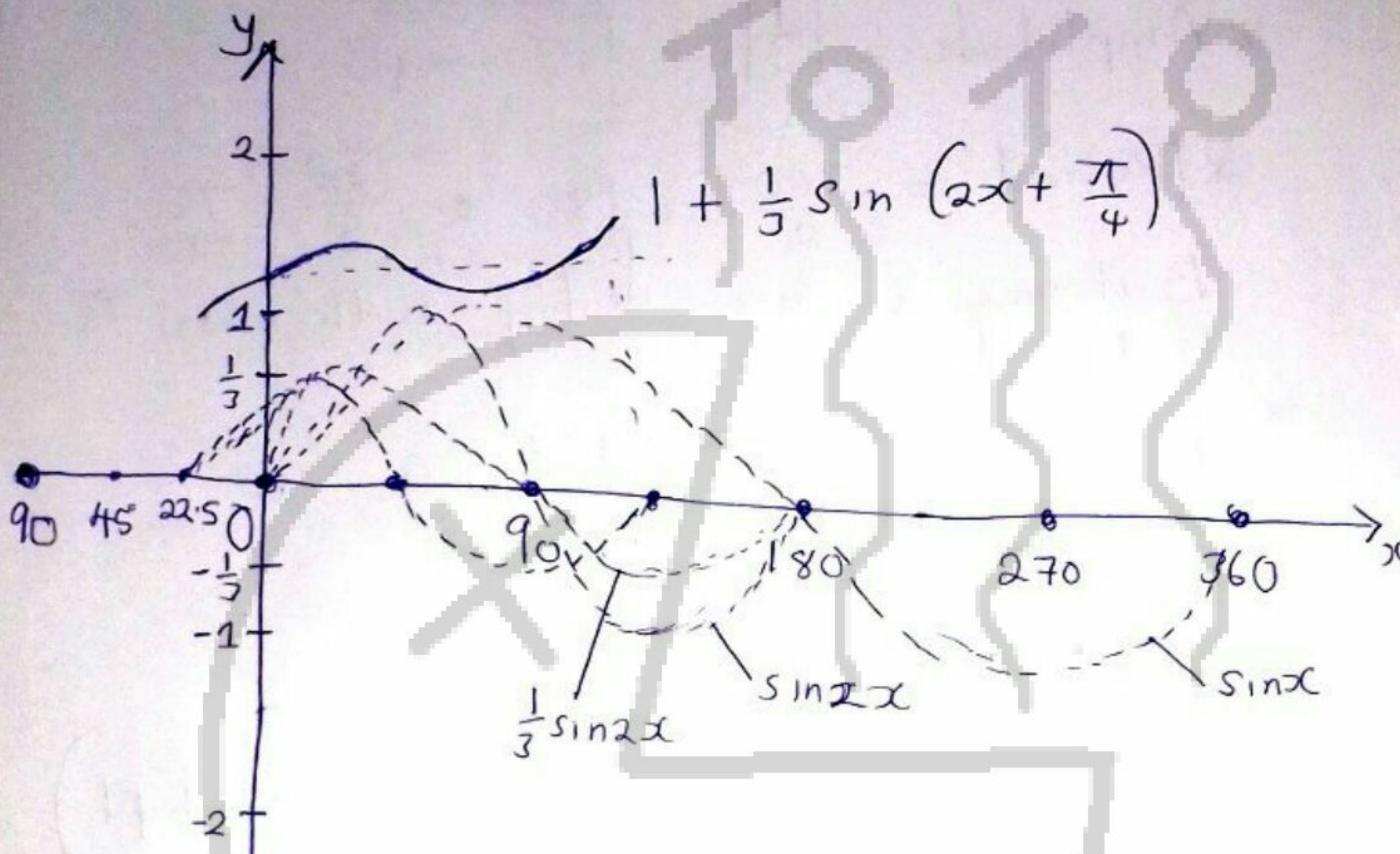
$$2x = \left|-\frac{\pi}{4}\right|$$

$$\frac{8x}{8} = \frac{\pi}{8}$$

$$x = \frac{\pi}{8} \text{ (phase shift)}$$

Let's try to sketch

$1 + \frac{1}{3}\sin\left(2x + \frac{\pi}{4}\right)$



* Let's divide 0°, 180° and 360° by 2 to sketch sin2x

$\frac{180°}{2} = 90°$ $\quad \frac{360°}{2} = 180°$ $\quad \frac{0}{2} = 0°$

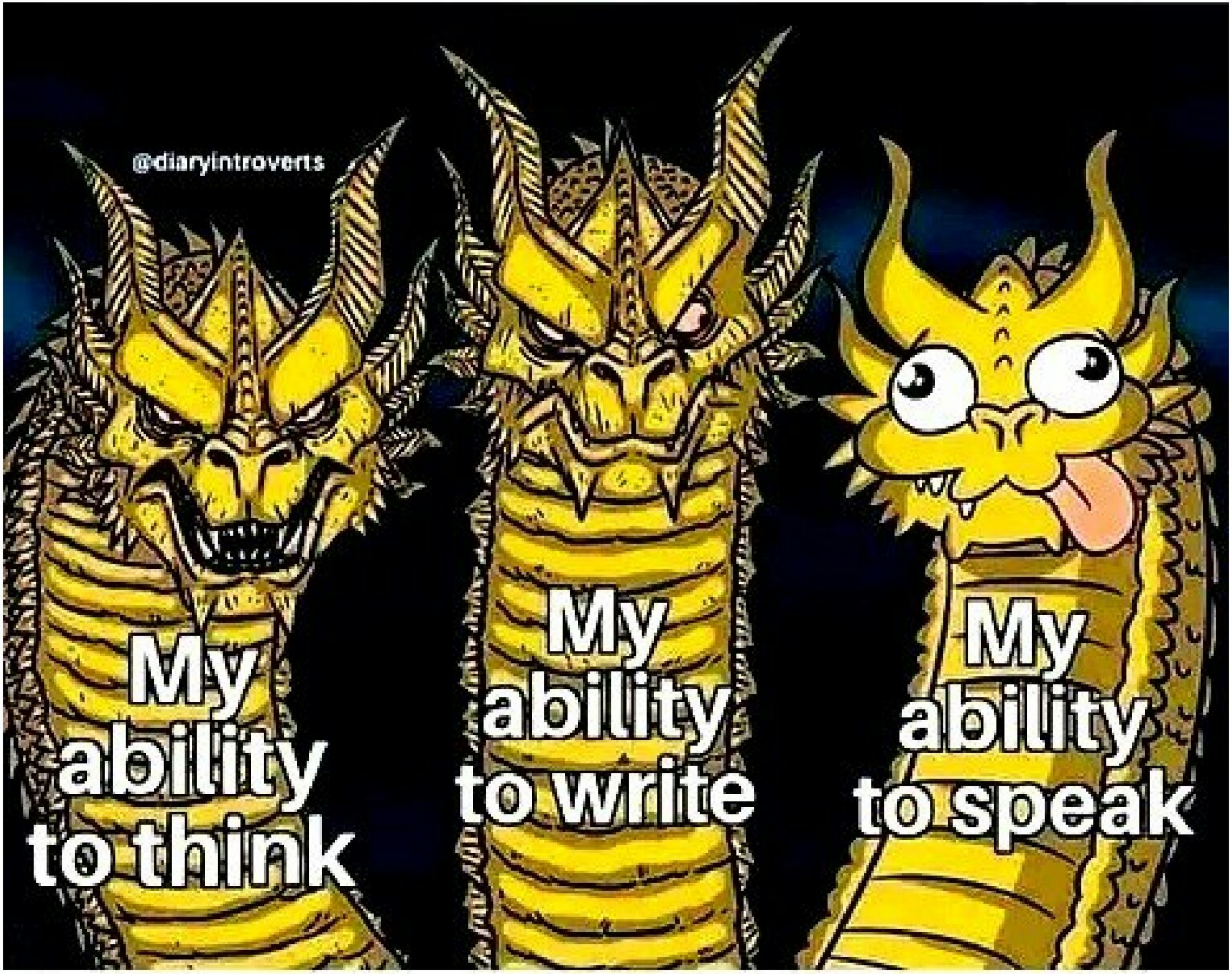
* Let's move the graph to the required amplitude $\left(\frac{1}{3}\right)$
* Next let's do a phase shift ... $\left(x + \frac{\pi}{8}\right)$

$= x = \left|\frac{\pi}{8}\right| = \frac{\pi}{8}$

$= 22.5°$

* we will subtract 180, 90 and 0°

we will subtract 22.5° from 180°, 90° and 0° to get a phase shift.


@diaryintroverts
My ability to think
My ability to write
My ability to speak

b) Solve the trigonometric equation

$\sin x + \cos x = \sqrt{2}$ if $0 \leq x \leq 2\pi$

Solution

Let's Square both sides

$(\sin x + \cos x)^2 = (\sqrt{2})^2$

$(\sin x + \cos x)(\sin x + \cos x) = 2$

$\sin^2 x + \sin x \cos x + \cos^2 x = 2$

$\sin^2 x + \cos^2 x + 2 \sin x \cos x = 2$

Recall $\sin^2 x + \cos^2 x = 1$

therefore we replace It with 1.

$1 + 2 \sin x \cos x = 2$

$2 \sin x \cos x = 2 - 1$

$2 \sin x \cos x = 1$

recall again $\sin 2A = 2 \sin A \cos A$

therefore $\sin 2x = 2 \sin x \cos x$

$\sin 2x = 1$

$2x = \sin^{-1}(1)$

$\frac{2x}{2} = \frac{90}{2}$

$x = 45° = \frac{\pi}{4}$

Let's try to prove by plugging in $\frac{\pi}{4}$ Into the equation and see IF it will give us $\sqrt{2}$.

$$\sin x + \cos x = \sqrt{2}$$

$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2}$$

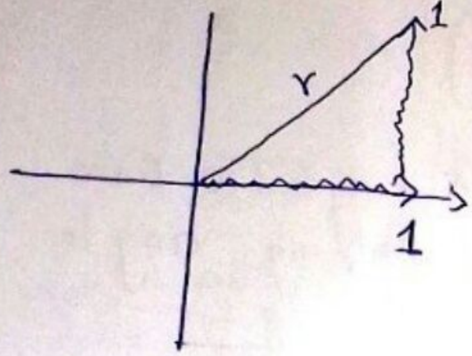
$$\frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$$

Therefore $x = \frac{\pi}{4}$

(c) De moivre's theorem to Solve $(1+i)^{20}$ in Form $a+bi$.

Solution

Let's sketch



$r = \sqrt{1^2+1^2}$

$r = \sqrt{2}$

then $\theta = \tan^{-1}(1) = 45°$

Since $Z = r(\cos\theta + i\sin\theta)$

then $a+bi = \sqrt{2}\,(\cos 45 + i\sin 45)$

Let's not Forget to power It to 20!

$\left(\sqrt{2}\,(\cos 45 + i\sin 45)\right)^{20}$

* The power will be the power of the resultant $(\sqrt{2})$ and be the coefficient of the angles given!

$$(\sqrt{2})^{20}\left(\cos 45 + i\sin 45\right)^{20}$$

$$(2)^{\frac{1}{2}\times 20}\left(\cos(20\times 45) + i\sin(20\times 45)\right)$$

$$2^{10}\left(\cos 900 + i\sin 900\right)$$

$$1024\left(\cos 900 + i\sin 900\right)$$

let's keep on subtracting 360 from 900 to get a good helping angle

$$900 - 360 = 540$$

$$540 - 360 = \underline{180} *$$

$$1024\left(\cos 180 + i\sin 180\right)$$

$$1024\left(-1 + 0\right)$$

$$\underline{\underline{-1024}}$$

(d) Find the cube roots of $2+2i$

let's express it first in $r(\cos\theta + i\sin\theta)$

Since $x=2$ $y=2i$

$$r = \sqrt{2^2+2^2} = \underline{\underline{\sqrt{8}}}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = 45^\circ$$

We apply the formula!

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta+360k}{n}\right) + i\sin\left(\frac{\theta+360k}{n}\right)\right]$$

$n = 3$

therefore $k = 0, 1, 2$

$$\sqrt[3]{\sqrt{8}}\left[\cos\left(\frac{45}{3}\right) + i\sin\left(\frac{45}{3}\right)\right] \qquad k = \underline{\underline{0}}!$$

$$\sqrt[3]{8^{\frac{1}{2}}}\left[\cos 15 + i\sin 15\right]$$

$$8^{\frac{1}{2}\times\frac{1}{3}}\left[\cos 15 + i\sin 15\right]$$

$$8^{\frac{1}{6}}\left[\cos 15 + i\sin 15\right]$$

$$\sqrt[6]{8}\left[\cos 15 + i\sin 15\right]$$

When $k = 1$

$$\sqrt[6]{8}\left[\cos\left(\frac{\theta + 360k}{n}\right) + i\sin\left(\frac{\theta + 360k}{n}\right)\right]$$

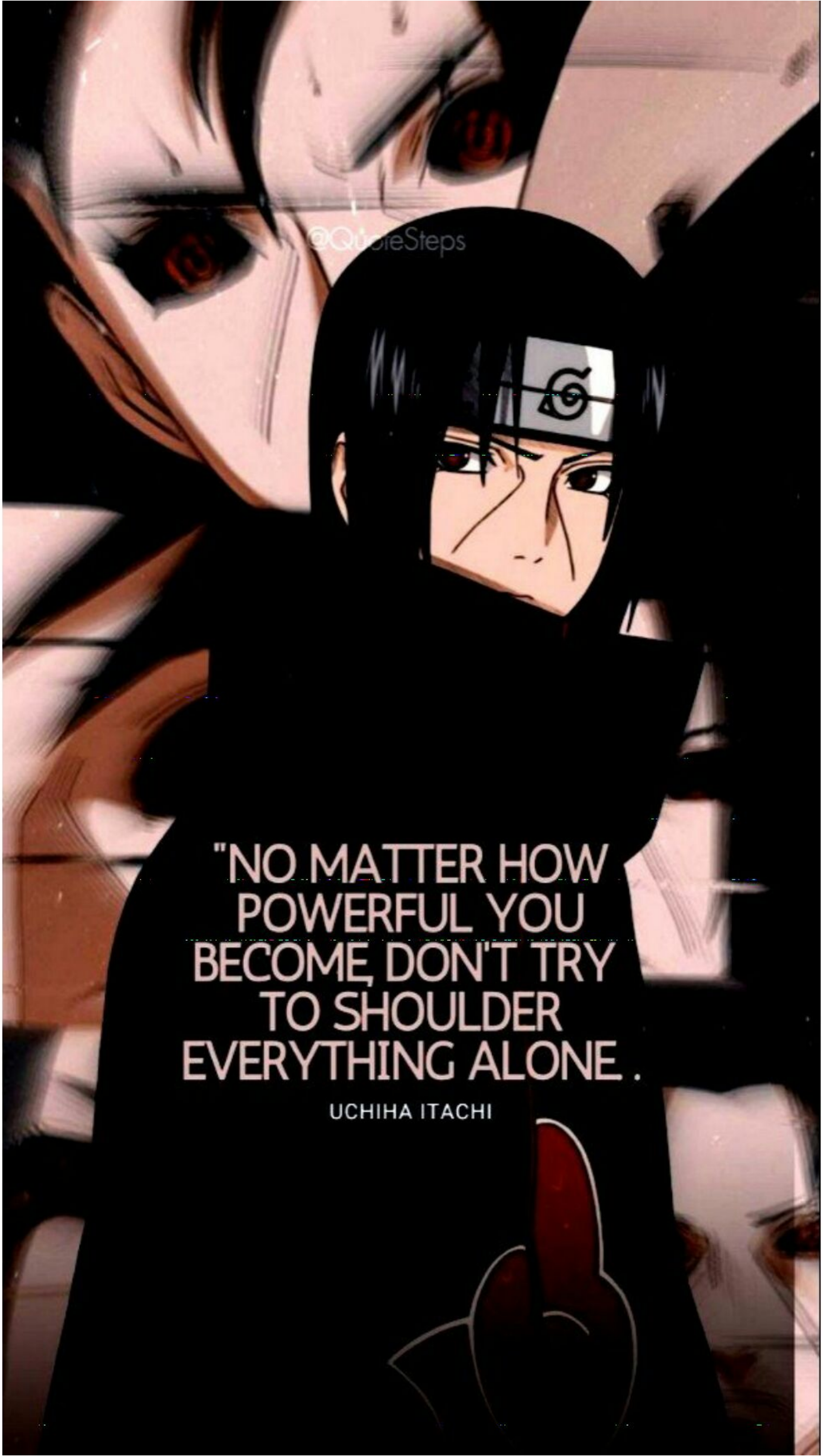
$$\sqrt[6]{8}\left[\cos\left(\frac{45 + 360}{3}\right) + i\sin\left(\frac{45 + 360}{3}\right)\right]$$

$$\sqrt[6]{8}\left[\cos 135 + i\sin 135\right]$$

When $k = 2$

$$\sqrt[6]{8}\left[\cos\left(\frac{45 + (360 \times 2)}{3}\right) + i\sin\left(\frac{45 + (360 \times 2)}{3}\right)\right]$$

$$\sqrt[6]{8}\left[\cos 255 + i\sin 255\right]$$


@QuoteSteps
"NO MATTER HOW POWERFUL YOU BECOME, DON'T TRY TO SHOULDER EVERYTHING ALONE.
UCHIHA ITACHI

(f) Solve $\sinh^2 x - 3\cosh x = 3$

$\sinh^2 x - 3\cosh x - 3 = 0$

let's apply the Identity $\sinh^2 x = \cosh^2 x - 1$

$\cosh^2 x - 1 - 3\cosh x - 3 = 0$

$\cosh^2 x - 3\cosh x - 4 = 0$

let $\cosh x = N$

$N^2 - 3N - 4 = 0$

Sum = -3
product = -4
Factors = $-4, 1$

$N^2 - 4N + N - 4 = 0$

$N(N-4) 1(N-4) = 0$

$N+1 = 0 \qquad N = 4$

$N = -1 \qquad \cosh x = 4$

$\underbrace{\cosh x = -1}_{\text{invalid}}$

$\frac{e^x + e^{-x}}{2} = 4$

$e^x + e^{-x} = 8$

$(e^x + e^{-x} - 8 = 0)e^x$

$e^{2x} + 1 - 8e^x = 0$

$e^{2x} - 8e^x + 1 = 0$

Solve quadratic Equation (Nzelu za pala ine!)

QUESTION 2

(a) $h \to 0 \quad \dfrac{\sqrt{x+h} - \sqrt{x}}{h}$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\left(\sqrt{x+h}\right) + \left(\sqrt{x}\right)\right)}$$

$$\lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$\lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (\text{conjugate})$$

$$= \frac{1}{2\sqrt{x}} \frac{\left(\sqrt{x}\right)}{\left(\sqrt{x}\right)}$$

$$= \frac{\sqrt{x}}{2x}$$

$$\#$$

(b) Determine the Interval for which $F(x)$ is continuous.

$$F(x) = \sqrt{4 - x^2}$$

$$4 - x^2 \geqslant 0$$

$$(2 - x)(2 + x) \geqslant 0$$

$2 - x = 0 \qquad 2 + x = 0$

$x = \underline{\underline{2}} \qquad x = \underline{\underline{-2}}$

We draw a sign graph

	(-3) -2	(0) 2	(3)
$2 - x$	+	+	−
$2 + x$	−	+	+
overall	−	+	−

We are looking for the Positive region because of $\geqslant$ sign!

there the interval is $[-2, 2]$

The SPIRAL

Your INTUITION calls you to make gradual changes in your life that lead towards ongoing evolution, development, and transcendence. Pure consciousness can see past the repeating cycles of the mind and urges you to move beyond what you think you know.

When God is about to take you to a Greater level, everything will go crazy. People will disappoint you, friends will betray you. You will be left all alone, in that very moment God will be your only source and He alone will take the GLORY!!

😇🙏❤️

(c) Differentiate $f(x) = \sin x$ using First principle.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

recall $\sin(x+h) = \sin x \cos h + \sin h \cos x$

$$\lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$\lim_{h \to 0} \frac{\sin x \cos h - \sin x + \sin h \cos x}{h}$$

$$\lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h}{h} \cos x$$

apply $\frac{\cos h - 1}{h} = 0$ and $\frac{\sin h}{h} = 1$

$$\therefore \sin x \lim_{h \to} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$\sin x \times 0 + \cos x \times 1$$

$$0 + \cos x$$

$$= \underline{\underline{\cos x}}$$

(d) Use Second derivative test to find the local extrema of

$f(x) = x^4 + 8x^2 + 10$

let's find $f'(x)$ first

$f'(x) = \underline{4x^3 + 16x}$

$4x^3 + 16x = 0$

$4x(x^2 + 4] = 0$

$4x = 0 \quad x^2 + 4 = 0$

* awe they should have made a mistake here It was Supposed to be $-8x^2$

$F(x) = x^4 - 8x^2 + 10$

$= 4x^3 - 16x$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x = 0, \quad x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$\frac{4x}{4} = \frac{0}{4} \qquad x = 2, \quad x = -2$

$x = 0$ (these are critical points)

let's differentiate $4x^3 - 16x$

$4x^3 - 16x$

$f''(x) = 12x^2 - 16$

our critical points are $0, -2, 2$

lets start plugging them in the second derivative

$F''(0) = 12(0)^2 - 16$

$= -16$

We have a negative there $x = 0$ is a maxima.

$F''(-2) = 12(-2)^2 - 16$

$= 12(4) - 16$

$= +32$

+ positive $x = -2$ is a minima.

and also $x = 2$ is minima.

(e)(i) $xe^y - 3y\sin x = 1$

We apply implicit differentiation

let's deal with one item at a time

sime all items on LHS are products

① xe^y

$$\frac{dy}{dx} = V\frac{du}{dx} + u\frac{dv}{dx} \qquad x = U,\ e^y = V$$

$$= e^y(1) + x(e^y)\frac{dy}{dx}$$

$$= e^y + xe^y\left(\frac{dy}{dx}\right)$$

(ii) $-3y\sin x$

let $U = y \qquad V = \sin x$

$$\frac{dy}{dx} = V\frac{du}{dx} + U\frac{dv}{dx}$$

$$= \sin x(1)\left(\frac{dy}{dx}\right) + y(\cos x)$$

$$= \sin x\left(\frac{dy}{dx}\right) + y\cos x$$

$$-3\sin x\left(\frac{dy}{dx}\right) - 3y\cos x$$

let's combine them now

$$e^y + xe^y\left(\frac{dy}{dx}\right) - 3\sin x\left(\frac{dy}{dx}\right) - 3y\cos x = 0$$

$$xe^y\left(\frac{dy}{dx}\right) - 3\sin x\left(\frac{dy}{dx}\right) = 3y\cos x - e^y$$

$$\frac{dy}{dx} = \frac{3y\cos x - e^y}{xe^y - 3\sin x}$$

(ii) $y = \ln\left(\frac{x^3+2}{(x+3)^5}\right)$

$y = \ln\left((x^3+2)(x+3)^{-5}\right)$

let $U = x^3+2\,(x+3)^{-5}$ (using chain rule)

then $y = \ln U$

$\frac{dy}{du} = \frac{1}{U}$ (i)

let's now differentiate $U = x^3+2(x+3)^{-5}$

We are going to use the Power rule and Chain rule again on this Lot!

$$\frac{du}{dx} = \underbrace{3x^2}_{\text{power rule}} + \underbrace{2(x+3)^{-5}}_{\text{we should use chain rule.}}$$

$U = x+3$ $\qquad y = U^{-5}$

$\frac{du}{dx} = 1$ $\qquad \frac{dy}{du} = -5U^{-6}$

$\frac{du}{dx} \times \frac{dy}{du} = 1 \times -5U^{-6} = -5U^{-6}$

Not Forgetting a 2

$2 \times (-5U^{-6}) = -10U^{-6}$

$-10(x+3)^{-6}$

So $\dfrac{du}{dx} = 3x^2 - 10(x+3)^{-6}$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= \left(3x^2 - 10(x+3)^{-6}\right) \times \frac{1}{U}$$

$$= \frac{3x^2 - 10(x+3)^{-6}}{U}$$

$$= \frac{3x^2 - 10(x+3)^{-6}}{x^3 + 2(x+3)^{-5}}$$

(f) show that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

recall that $\tan y = x$

then $\tan^{-1}(x) = y$

let's differentiate $\tan y = x$ implicitly

$\frac{dy}{dx}$ of $\tan x = \sec^2 x$

then $\frac{dy}{dx}$ of $\tan y = \sec^2 y \left(\frac{dy}{dx}\right)$

$$\therefore \frac{\cancel{\sec^2 y}\left(\frac{dy}{dx}\right)}{\cancel{\sec^2 y}} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

recall that $1 + \tan^2 y = \sec^2 y$

then $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

but $\tan y = x$

so $\tan^2 y = x^2$

therefore $\frac{dy}{dx} = \frac{1}{1+x^2}$

15 years old

50 years old


d/dx
e^x
e^x
e^x
e^x
e^x
d/dx

(g) (i) Evaluate

$$\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

let's Use U substitution here.

let $U = \sqrt{x}$ (from Cos)

$$\frac{du}{dx} = \frac{1}{2} \times x^{-\frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{x}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$dx = 2\sqrt{x}\, du$$

$$\int_0^{\pi^2} \frac{\cos U}{\sqrt{x}}\, 2\sqrt{x}\, du$$

$$\int_0^{\pi^2} 2\cos u\, du$$

$$2\int_0^{\pi^2} \cos u\, du$$

$$2\left[\sin U\right]_0^{\pi^2}$$

$$2\left[\sin\sqrt{x}\right]^{\pi^2}$$

$$2\left[\sin\sqrt{\pi^2}\right] = 2\left[\sin\pi\right]$$

$$= \underline{\underline{0}}$$

(ii) $\int \ln x \, dx$

Here we will Use Integration by parts Formula.

$$\int u\,dv = uv - \int v\,du$$

We will use the mnemonic LIATE to choose our U

L — Logs, I — Inverse trig, A — Algebra, T — Trig, E — Exponents

We have $1 \quad \ln x$

$$\begin{array}{cc} 1 & \ln x \\ \downarrow & \downarrow \\ v & U \end{array}$$

$U = \ln x$

$du = \frac{1}{x}\,dx$

$dv = 1\,dx$

$V = x$ (integral of V)

$$\int \ln x \, dx = \ln x \times x - \int x \frac{1}{x}\,dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

(iii) $\int \frac{2x^2-5x+2}{x(x^2+1)}\,dx$

Here we will first resolve the function into its partial fractions.

$$\frac{2x^2-5x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$2x^2-5x+2 = A(x^2+1) + (Bx+C)(x)$$

$$2x^2-5x+2 = A(x^2+1) + (Bx+C)(x)$$

let $x = 0$

$$2 = A(0+1)$$

$$A = 2$$

$$2x^2-5x+2 = Ax^2+A+Bx^2+xC$$

$$Ax^2+Bx^2 = 2x^2 \qquad Cx = -5x$$

$$A+B = 2 \qquad \therefore C = -5$$

$$2+B = 2$$

$$B = 0$$

$$\int\left(\frac{2}{x}\right)dx + \int\left(\frac{-5}{x^2+1}\right)dx$$

$$\int \frac{2}{x}\,dx + \int \frac{-5}{x^2+1}\,dx$$

$$2\int \frac{1}{x}\,dx - 5\int \frac{1}{x^2+1}\,dx$$

$$= \underline{\underline{2\ln|x| - 5\ln|x^2+1| + C}}$$

(h) (i) $1 + \sin^2 x + \sin x$

let's differentiate then equate expression to 0.

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \underbrace{\sin x \sin x}_{\text{Apply product rule}} + \sin x + 1$$

$$= U\frac{dv}{dx} + V\frac{du}{dx} + \cos x$$

$$= \sin x \cos x + \sin x \cos x + \cos x$$

$$= 2\sin x \cos x + \cos x$$

$$2\sin x \cos x + \cos x = 0$$

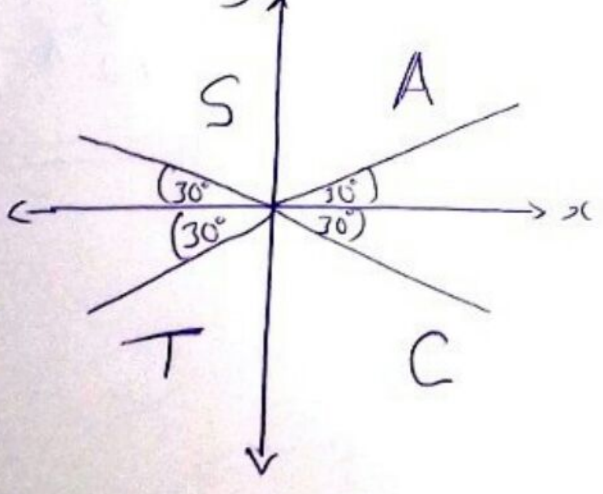
$$\cos x \,(2\sin x + 1) = 0$$

$$\cos x = 0 \qquad 2\sin x = -1$$

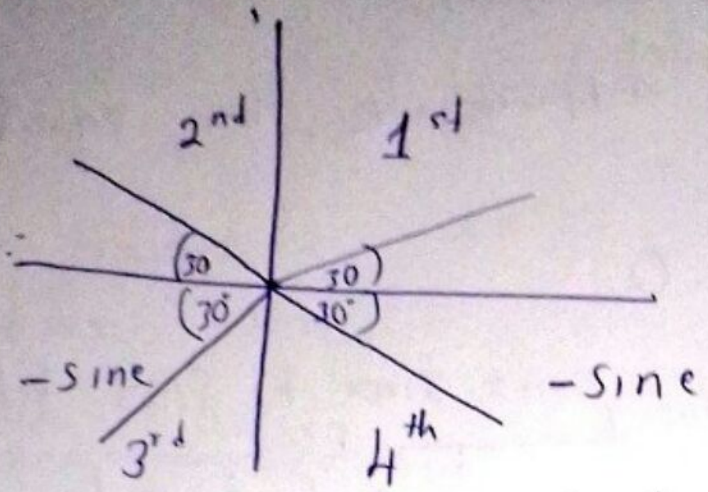
$$\cos x = 0 \qquad \sin x = \frac{-1}{2}$$

For cos $x = 90°$. and $270°$

For sine let's sketch



let's say our Helping angle is 30°.
Sine is negative in 3rd and 4th quadrant



therefore in 3^{rd} quadrant sine will be

$180° + 30° = 210° = \frac{7\pi}{6}$

In 4^{th} quadrant $360° - 30° = 330° = \frac{11\pi}{6}$

therefore our critical values are

$90° = \frac{3}{6}\pi = \frac{\pi}{2}$

$270° = \frac{3}{2}\pi$

Critical Values are $\frac{\pi}{2}, \frac{7\pi}{2}, \frac{3}{12}\pi, \frac{11\pi}{6}$

(11) To find the tangent, we must find the gradient first and the value of y since we have x.

let's substitute $x = \pi$ into our derivative first.

Solution:

$\frac{dy}{dx}$ was $2 \sin x \cos x + \cos x$

$$m = 2 \sin\pi \cos\pi + \cos\pi$$
$$= \underbrace{2 \sin 180}_{0} \cos 180 + \underbrace{\cos 180}_{}$$

makes it zero — we remain with!

$$m = \cos 180 = -1$$

let's find our y now by going back to our previous equation, that first 1. $(f(x) = 1 + \sin^2 x + \sin x)$

we will substitute $180°$ into it.

$$y = 1 + (\sin 180)^2 + \sin 180$$
$$y = 1$$

$$y - y_1 = m(x - x_1)$$

$y_1 = 1$ $m = -1$ $x = \pi$

$$y - 1 = -1(x - \pi)$$

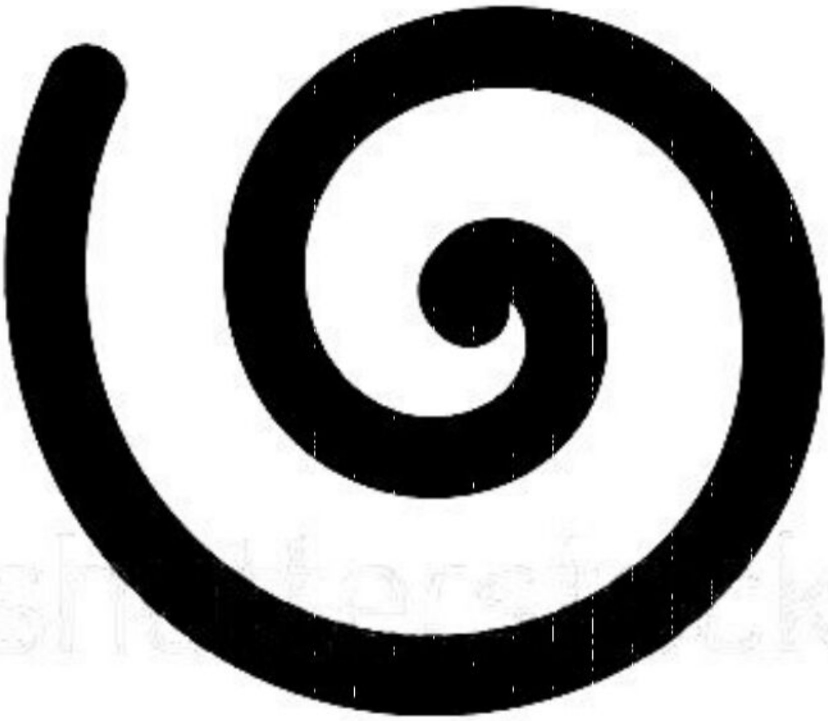
$$y - 1 = -x + \pi$$

$$y = -x + \pi + 1$$

$$y = -x + 180 + 1$$

$$y = -x + 181$$

$$y = -x + \frac{181\pi}{180}$$



Spiral