BRAINFUSE TUTORIALS

Test 1 – Sunday, 8th January 2023

Time allowed: 2:30hrs

Instructions – Attempt all four (4) questions. All questions carry equal marks.

- Indicate your name, TG and computer number on your answer sheet.
- Full credit will only be given when necessary work is shown.
- Calculators are not allowed for this test.

1. a) Let R be the universal set and A = (-7,3], B = (0,8), C = {x : x ≤ 10, x £R }

   i. Find the set C – B and display your answer on the number line.

   ii. Using the sets above, verify that A' n B' = (A u B)'

   b) Express [(P n Q) u (P – Q)]' in its simplest form.

   c) (i) Determine whether the function

   $$f(x) = \frac{x}{x^3 - 1}$$

   is even or odd or neither.

   ii. Let * be a binary operation on the set of integers Z defined by x * y = 1 + xy. Determine whether the operation is commutative and/or associative.

   [8, 10, 7]

2. a) (i) Rationalise the denominator of

   $$\frac{5}{2-x\sqrt{3}}\ ,\ x\ Ε\ Z$$

   (ii) Find the real and imaginary parts of

   $$\frac{2-i}{4i+(1+i)^4}$$

   (iii) Find the complex square root in the form a + b i where a and are real numbers

   3 + 4 i

   b) Given the following functions

   (i) $g(x) = -\frac{x+4}{x^2+3x-4}$

   Sketch the graph of each of the above functions, indicating the intercepts, vertical and horizontal asymptotes if they exist.

Indicate the domain and range in each case.

(c) (i) Given that f $(x) = \frac{3}{x^2-1}$, g $(x) = x + 1$, find the domain of f o g

(ii) Let f $(x) = \frac{x^2-1}{x^2+1}$, $x \geq 0$, find its inverse and domain. [7, 10, 8]

3. a) Complete the square of the quadratic function f $(x) = -2x^2 - 12x + 7$. Hence ;

(i) Find the turning point and the x and $y$intercepts

(ii) Sketch the graph of y = f (x)

b) (i) When divided by $(x + 2)$, the expression $5x^3 - 3x^2 + ax + 7$ leaves a remainder of r. When the expression $4x^3 + ax^2 + 7x - 4$ is divided by $(x + 2)$, there is a remainder of $2r$. Find the values of the constants a and r.

(ii) Use synthetic division to find the quotient and remainder when $2x^3 + x^2 - 32x - 16$ is divided by $2x + 1$.

c) (i) The quadratic polynomial I $x^2 - 2x - 3$ is a factor of the quartic function given as f $(x) = 3x^4 - x^3 - 21x^2 - 11x + 6$. Find all the zeros of the function f. hence sketch the graph of f (x).

(ii) Sketch f $(x) = |x - 5| + |x + 3|$

4. a) Solve the inequalities

(i) $3 + \sqrt{2x-5} \leq 6$ (ii) $\left|\frac{x+2}{2}\right| \leq 2$

b) (i) The roots of the quadratic equation $5x^2 - 3x - 1 = 0$ are α and β. Find a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$

(ii) Let $z = x + iy$. If $z + \frac{1}{z}$ is real, show that either z is real or $x^2 + y^2 = 1$.

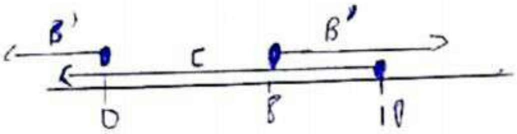
c) Solve the equation $\sqrt{1-x} - \sqrt{x-2} = 1$ for real values of x.

END

Brainfuse Tutorials Marking Key

a (i) $A = [-7, 3]$, $B = (0, 8)$, $C = \{x : x \le 10, x \in R\}$

$C - B = C \cap B'$



[3]

$C \cap B' = (-\infty, 0] \cup [8, 10]$

3

(ii) $A' = (-\infty, -7] \cup (3, \infty)$

$B' = (-\infty, 0] \cup [8, \infty)$ [7]

$A' \cap B' = (-\infty, -7] \cup [8, \infty)$

Now $A \cup B = (-7, 8)$

$(A \cup B)' = (-\infty, -7] \cup [8, \infty)$

7

Thus since $A' \cap B' = (A \cup B)'$, thus verified.

b Let. $[(P \cap Q) \cup (P \cap Q')]' \ni x$

$x \notin (P \cap Q) \cup (P \cap Q')$ 5 [5]

$x \notin P \cap Q$ and $x \notin P \cap Q'$

$x \notin P$ or $x \notin Q$ and $x \notin P$ or $x \notin Q'$

$x \in P'$ or $x \in Q'$ and $x \in P'$ or $x \in Q$

$x \in (P' \cup P') \cap (Q' \cup Q)$

$x \in P' \cap U$, $x \in P'$

c(i) $f(x) = \frac{x}{x^3 - 1}$ [3]

$f(-x) = \frac{-x}{-x^3 - 1}$

<u>Neither</u>

3

(ii) $x * y = 1 + xy$

$y * x = 1 + yx$

Since $y * x = x * y$, thus commutative.

$x * (y * z) = (x * y) * z$

let $y * z = p$

$x * p = 1 + xp$

but $p = 1 + yz$

thus $x * (y * z) = 1 + x(1 + yz)$

$= \underline{1 + x + yzx}$

now $(x * y) * z$

let $x * y = \phi$

$\phi * z = 1 + \phi z$

But $\phi = 1 + y*$ [7]

thus $(x * y) * z = 1 + (1 + yx)z$

$= \underline{1 + z + yxz}$

now Since $(x * y) * z \neq x * (y * z)$ thus not associative.

7

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Question 2(a)

a(i) $\frac{5}{2-x\sqrt{3}} \times \frac{(2+x\sqrt{3})}{2+x\sqrt{3}}$ [2]

$= \frac{10+5x\sqrt{3}}{4-3x^2}$

8

$= \frac{10}{4-3x^2} + \frac{5x\sqrt{3}}{4-3x^2}$

b) $\frac{2-i}{4i+(1+i)^4}$ [3]

$= \frac{2-i}{4i+(1+i)^2(1+i)^2}$

$= \frac{2-i}{4i+(2i)(2i)}$

3

$= \frac{2-i}{4i-4}$

$= \frac{2-i}{-4+4i} \times \frac{(-4-4i)}{\cancel{-4} -4-4i}$

$= \frac{-8-8i+4i-4}{16+16}$

$= \frac{-12-4i}{32} = -\frac{3}{8} - \frac{i}{8}$

ii) $(\sqrt{3+4i})^2 = (a+bi)^2$ [5]

$3+4i = a^2-b^2+2abi$

$4 = 2ab \ldots \text{eq (i)}$

$3 = a^2-b^2 \ldots \text{eq (ii)}$

$a = \frac{2}{b}$

$3 = \left(\frac{2}{b}\right)^2 - b^2$

$\Rightarrow 3b^2 = 4 - b^4$

$\Rightarrow b^4 + 3b^2 - 4 = 0$ 8

let $b^2 = y$

$y^2+3y-4=0$ [factorise the quad]

$y=1$

$b^2 = 1$

$b = \sqrt{1}$, $b = \pm 1$

$a = \frac{2}{b}$, $a = \frac{2}{1}$, $a = 2$

~~that~~ thus $\sqrt{3+4i} = \pm(2+i)$

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b. $f(x) = \frac{x+4}{x^2+3x-4} = \frac{x+4}{(x+4)(x-1)}$

[5]

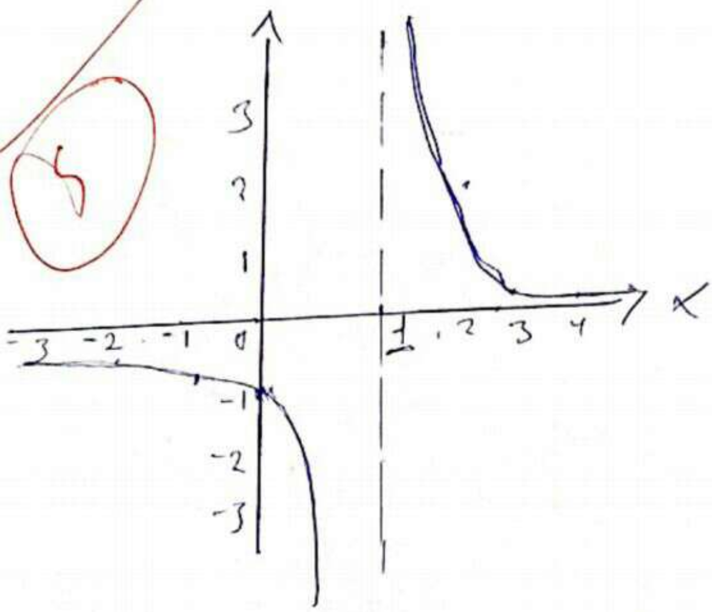
$= \frac{1}{x-1}$

VA: $x-1=0$

$x=1$

HA $= 0$

~~when y = 0~~



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x	0	-1	-2
y	-1	$-\frac{1}{2}$	$-\frac{2}{3}$

x	2		
y	1		

c $f(x) = \frac{3}{x^2-1}$, $g(x) = x+1$

(4)

$f \circ g = \frac{3}{(x+1)^2 - 1}$

$= \frac{3}{x^2+2x}$

2

Domain: $\frac{3}{x(x+2)}$

0

Let $x(x+2) = 0$, $x=0$, $x=-2$

Domain $= \{x \mid x \in \mathbb{R}$, but $x \neq 0$ and $x \neq -2y\}$

[3]

(i) $f(x) = \dfrac{x^2-1}{x^2+1}$

$y = \dfrac{x^2-1}{x^2+1}$

~~xy~~ $yx^2 + y = x^2 - 1$

$x^2y - x^2 = -y - 1$

$x^2(y-1) = -y-1$

$x = \sqrt{\dfrac{-y-1}{y-1}}$

[7]

$f(x)^{-1} = \sqrt{\dfrac{x+1}{1-x}}$

x

~~Domain~~ Domain $\dfrac{x+1}{1-x} \geq 0$

Critical points $x+1=0$ $1-x=0$

$x=-1$, $x=1$

$x \leq -1$	$-1 \leq x < 1$	$x > 1$
$-$	$+$	$-$

let $x = -2$; $\dfrac{-2+1}{1-(-2)} = \dfrac{(-)}{+} = -$

Ans; $-1 \leq x < 1$

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Question 3

$-2\left(x^2+6x-\frac{7}{2}\right]$, taking 6; $6\times\frac{1}{2} = (3)^2$

$-2\left((x+3)^2-9-\frac{7}{2}\right]$

$-2\left((x+3)^2-\frac{25}{2}\right]$

$f(x) = -2(x+3)^2+25$

$TP = \left[-3, +25\right]$

5

y - Intercept; $x = 0$

$y = 7$, $[0, 7]$

3

x - Intercept $y = 0$

$0 = -2x^2 - 12x + 7$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{12 \pm \sqrt{12^2 - 4(-2)(7)}}{-4}$

$x = \frac{12 \pm \sqrt{200}}{-4}$

$x = \frac{12-\sqrt{200}}{-4}$ or $\frac{12+\sqrt{200}}{-4}$

$x = \frac{6-5\sqrt{2}}{-2}$ or $\frac{6+5\sqrt{2}}{-2}$

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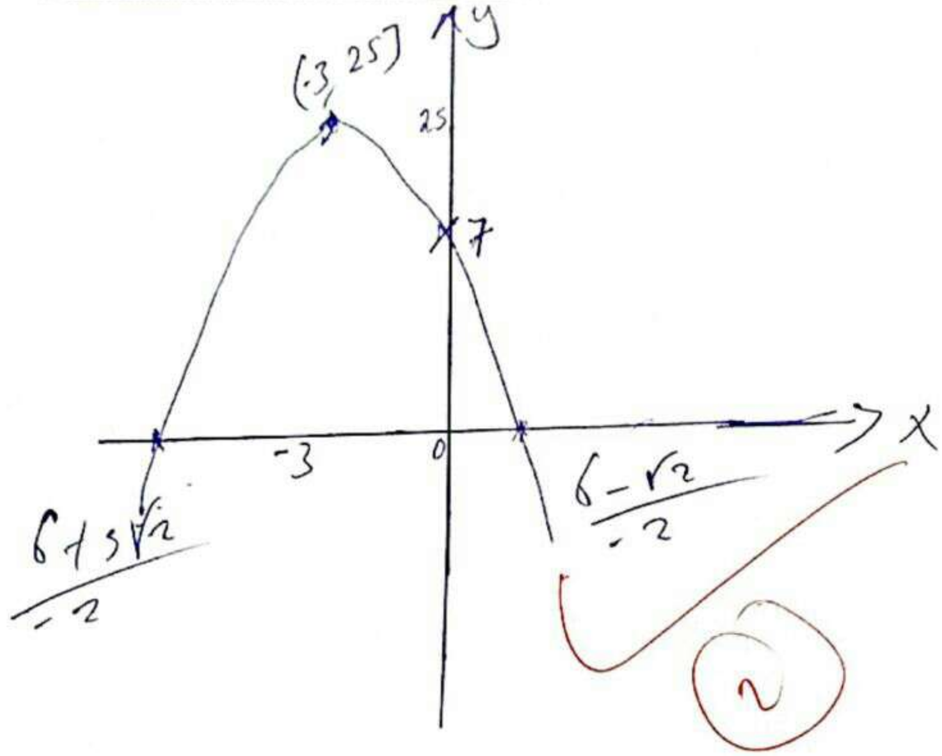

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b) $f(x) = 5x^3 - 3x^2 + ax + 7$

let $x = -2$

$f(-2) = 5(-2)^3 - 3(-2)^2 + a(-2) + 7$

$f(-2) = -40 - 12 - 2a + 7$

$f(-2) = -45 - 2a$

but $f(-2) = r$

$r = -45 - 2a \quad \ldots \quad$ eq (1)

5

$g(x) = 4x^3 + ax^2 + 7x - 4$

$g(-2) = 4(-2)^3 + a(-2)^2 + 7(-2) - 4$

$= -32 + 4a - 14 - 4$

$g(-2) = -50 + 4a$

but $g(-2) = 2r$

$2r = -50 + 4a \quad \ldots \quad$ eq (11)

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Solving eq(1) and eq(2) simultaneously -

$r = -45 + 2a$

$2(-45 + 2a) = -50 - 4a$

$-90 + 4a = -50 - 4a$

$-90 + 50 = -8a$

$-40 = -8a$

$\underline{a = 5}$

4

$r = -45 + 2(5)$

$\underline{r = -35}$

5(c) let $2x + 1 = 0$

$x = -\frac{1}{2}$

$$
\begin{array}{r|rrrr}
-\frac{1}{2} & 2 & 1 & -32 & -16 \\
 & \downarrow & -1 & 0 & 16 \\
\hline
 & 2 & 0 & -32 & 0
\end{array}
$$

$2x^2 - 32 = 0$

$2(x^2 - 16)$

$2(x-4)(x+4) = 0$

8

Quotient $2x^2 - 32$, Remainder is Zero

C(i) x^2-2x-3

~~factors~~

$P=-3$ $S=-2$

factors -3 and 1

$(x-3)(x+1)$

let $x=3$, $x=-1$

$$\begin{array}{r|rrrrr} 3 & 3 & -1 & -21 & -11 & 6 \\ & \downarrow & 9 & 24 & 9 & -6 \\ \hline & 3 & 8 & 3 & -2 & 0 \end{array}$$

$3x^3+8x^2+3x-2$

$$\begin{array}{r|rrrr} -1 & 3 & 8 & 3 & -2 \\ & \downarrow & -3 & -5 & 2 \\ \hline & 3 & 5 & -2 & 0 \end{array}$$

$3x^2+5x-2$

$P=-6$, $S=5$, factors $+6$ and -1

$3x^2+6x-x-2$

$3x(x+2)-1(x+2)$

$(3x-1)(x+2)=0$

Thus factors are $(3x-1)(x+2)(x-3)(x+1)$

Thus roots are $x=\frac{1}{3}$, $x=-2$, $x=3$, $x=-1$

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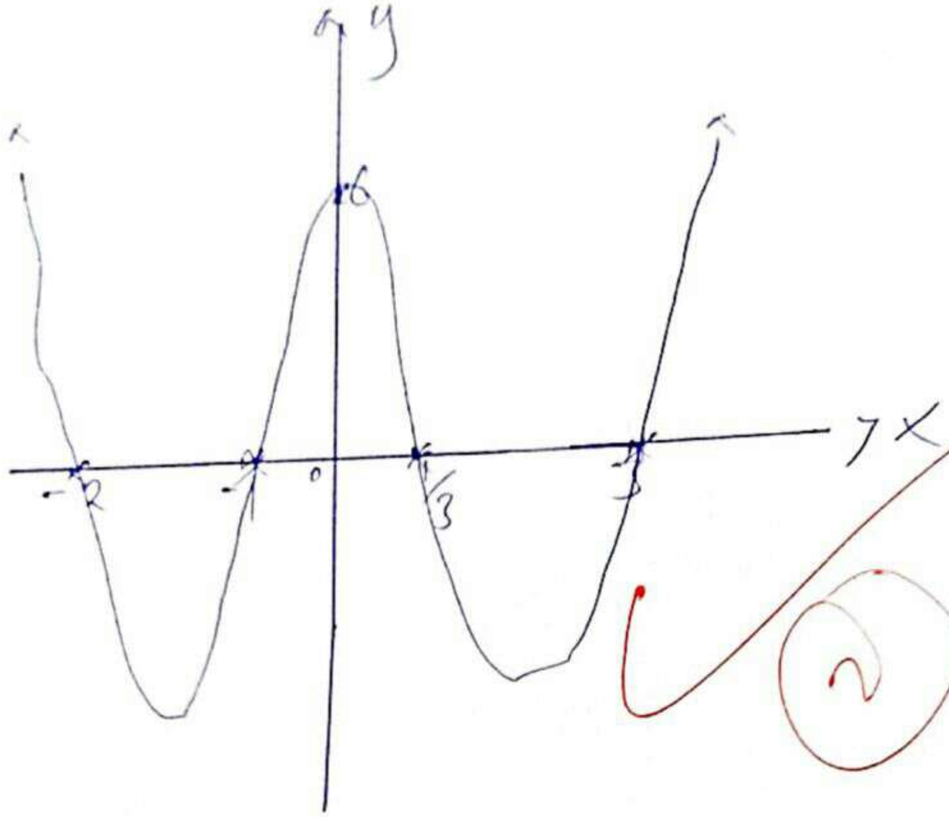

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C(ii) $f(x) = |x-5| + |x+3|$

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$$|x-5| = \begin{cases} x-5 & \text{if } x-5 \geq 0 \\ -(x-5) & \text{if } x-5 < 0 \end{cases}$$

$$= \begin{cases} x-5 & \text{if } x \geq 5 \\ -x+5 & \text{if } x < 5 \end{cases}$$

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases}$$

$$= \begin{cases} x+3 & \text{if } x \geq -3 \\ -(x+3) & \text{if } x < -3 \end{cases}$$

4

	$x<-3$	$-3 \leq x < 5$	$x \geq 5$
$\lvert x-5 \rvert$	$-x+5$	$-x+5$	$x-5$
$\lvert x+3 \rvert$	$-x-3$	$x+3$	$x+3$

$\cancel{f(x)+}$ $f(x) = |x-5| + |x+3| = \begin{cases} -2x+2 & \text{if } x < -3 \\ 8 & \text{if } -3 \leq x < 5 \\ 2x-2 & \text{if } x \geq 5 \end{cases}$

$y = -2x + 2 \quad ; \quad x < -3$

x	-3	-4	-5
y	8	10	12

$y = 8$

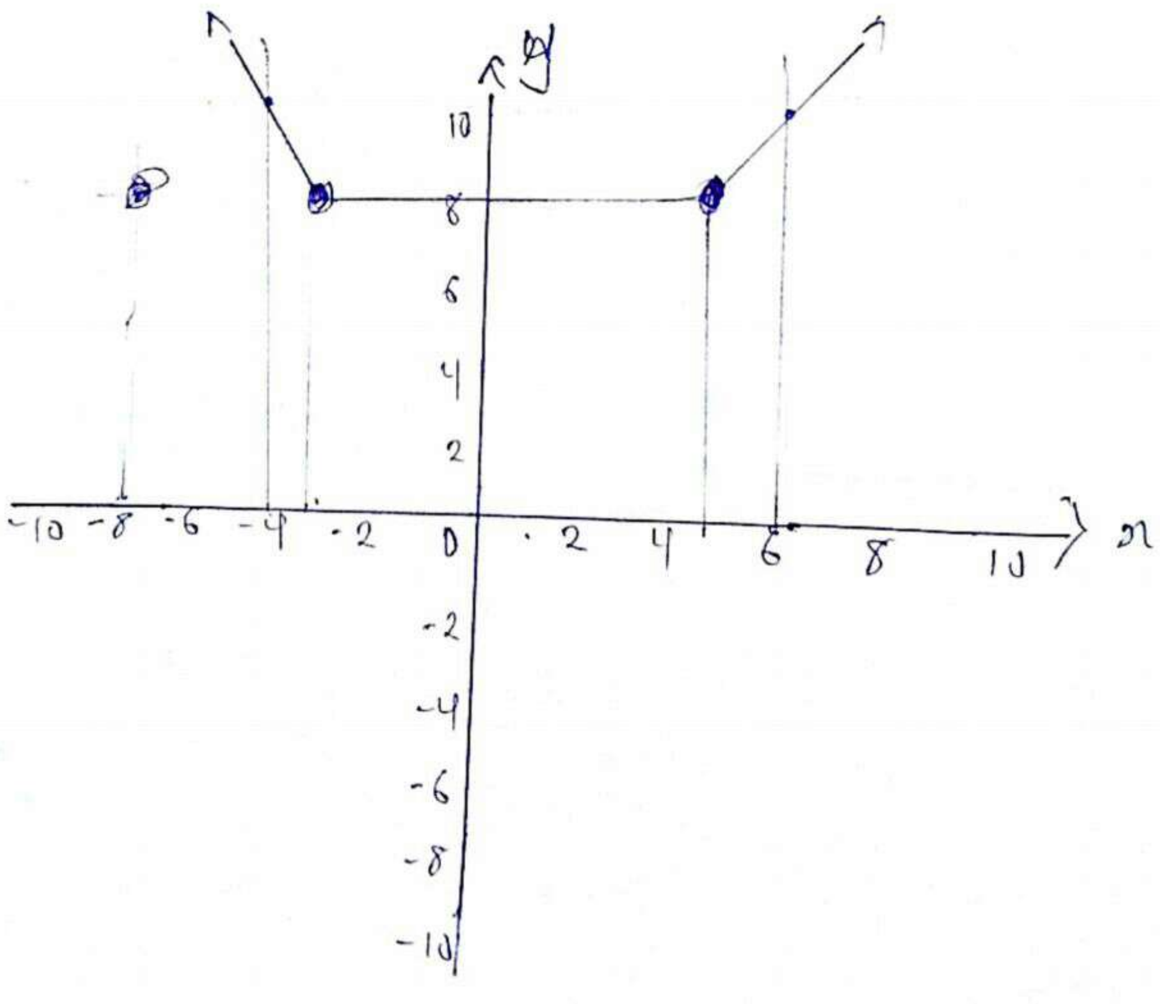
$-3 \leq x \leq 5$

x	-3	-2	-1	0	1	2	3	4	5
y	8	8	8	8	8	8	8	8	8

$y = 2x - 2 \;;$

x	5	6	7	8
y	8	10	12	

8



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4 (a) $3 + \sqrt{2x-5} \leq 6$

$\sqrt{2x-5} \leq 6-3$

$(\sqrt{2x-5})^2 \leq 3^2$

$2x - 5 \leq 9$

$2x \leq 14$

$x \leq 7$

3

3

(11) $\left|\frac{x+2}{2}\right| \leq 2$

$\cancel{-2 \leq}$

$-2 \leq \frac{x+2}{2} \leq 2$

$-2 \leq \frac{x+2}{2} \;/\; \frac{x+2}{2} \leq 2$

$0 \leq \frac{x+2}{2} + 2 \;/\; x \leq 4-2$

$x \leq 2$

$0 \leq \frac{x+6}{2}$

$-6 \leq x$

5

5

b(i) $5x^2 - 3x - 1 = 0$

$x^2 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)x - \frac{1}{\beta^2} \times \frac{1}{\alpha^2} = 0$

Sum: $\frac{1}{\alpha^2} + \frac{1}{\alpha^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$

where $\alpha + \beta = -\frac{b}{a}$

$= \frac{3}{5}$

5

and $\alpha\beta = \frac{c}{a}$

$\alpha\beta = -\frac{1}{5}$

$\therefore \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \left[\left(\frac{3}{5}\right)^2 - 2\left(-\frac{1}{5}\right)\right] \times \frac{1}{\left(\frac{1}{5}\right)^2}$

$= \left[\frac{9}{25} + \frac{2}{5}\right] \times 25$

$\frac{9 + 10}{25} \times 25$

4

$\underline{\underline{19}}$

product $= \frac{1}{(\alpha\beta)^2} = \underline{\underline{25}}$

thus $x^2 - 19x + 25 = 0$

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(ii) $z = x + yi$

$z + \frac{1}{z} \Rightarrow (x + yi) + \frac{1}{x+yi}$

$= x + yi + \left(\frac{1}{x+yi} \times \frac{x-yi}{x-yi}\right)$

$= x + yi + \left(\frac{x-yi}{x^2+y^2}\right)$

$= x + \frac{x}{x^2+y^2} + yi - \frac{yi}{x^2+y^2}$

now $yi - \frac{yi}{x^2+y^2} = 0$

$y\left(1 - \frac{1}{x^2+y^2}\right) = 0$

$1 - \frac{1}{x^2+y^2} = 0$

$\underline{\underline{x^2+y^2 = 1}}$. Hence shown

5

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$$\sqrt{1-x} - \sqrt{x-2} = 1$$

$$\left(\sqrt{1-x} - \sqrt{x-2}\right)^2 = 1^2$$

$$1-x - 2\sqrt{(1-x)(x-2)} + 1(x-2) = 1$$

$$-1 - 2\sqrt{x-2-x^2+2x} = 1$$

$$\left(-2\sqrt{-x^2+3x-2}\right)^2 = (2)^2$$

$$4(-x^2+3x-2) = 4$$

$$x^2 - 3x + 2 = \cancel{0}\ 1$$

7

7

$P = 2 \mid S = -3$

factors $\rightarrow$ -1 and -2

$x = 1$ and $x = 2$

$$x^2 - 3x + 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{3 \pm \sqrt{3^2 - 4(1)(3)}}{2}$$

$$x = \frac{3 + i\sqrt{3}}{2} \text{ or } \frac{3 - \sqrt{3}i}{2}$$