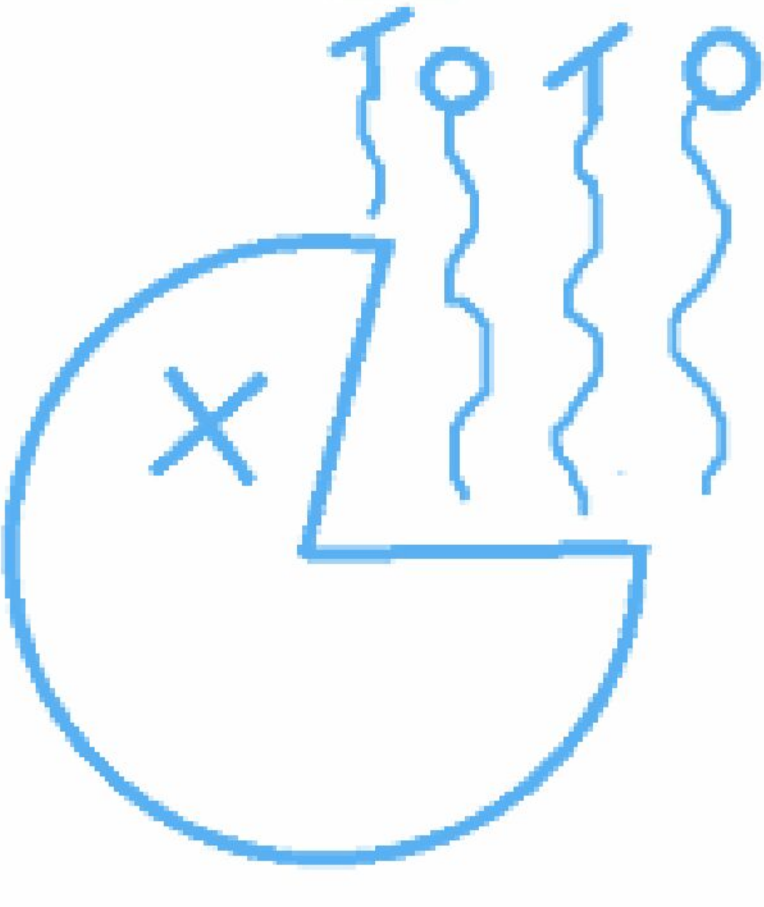
2023 MA 110 test 1 ☣☢



MA110 - MATHEMATICAL METHODS

Time allowed: Two hours (2:00 hours)

Instructions:

1. You must write your **Name**, your **Computer Number and programme of study** on your answer sheet.
2. Calculators are **not** allowed in this paper.
3. There are three (3) questions in this paper, Attempt **All** questions and show **detailed** working for full credit

---

QUESTION ONE

1) a) (i) If $C \subset D$, then simplify if possible
$C' \cup D'$ **(2.5 marks)**

2.5 (ii) Express 1.171717……… as a fraction $\frac{a}{b}$ in its simplest form where a and b are integers and $b \neq 0$. **(2.5 marks)**

b) Consider the binary operation $a * b = a + b - 2ab$, where $a \, and \, b$ are real numbers.

(i) Is * a binary operation on the set of real numbers? Give reason for your answer. **(1) Mark**

(ii) Is the operation * commutative? If not give a counter example. **(1)Mark**

(iii) Find the value of 1*(2*3) and (1*2)*3 and state whether * is associative **(3) Marks**

3) c) Given the rational function $f(x) = \frac{x+2}{x-2}$. Sketch its graph indicating its domain and range, all the asymptotes and intercepts. **(5 Marks)**

d) Prove that $\sqrt{2}$ is an irrational number **(5 Marks)**

e) Let $f(x) = \frac{x+1}{x-1}$ and $g(x) = \sqrt{x}$ .Find $(g \, o \, f)(x)$ and determine the domain **(5 Marks)**

(2, 2)

$-3+2-2($ (-1, 1)(-3, 2)

$2-3-2(-3)(2)$; $2-3-2$ $2+2-2(2)(2)$ $4-8$

CamScanner

QUESTION TWO

4 a) Using the associative and distributive properties of unio and intersection of sets .Show that

$$A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$$ (5 Marks)

5 b) Let α and β be the roots of the quadratic equation $3x^2 + 2x + 5 = 0$. Find a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ without calculating α and β (5 Marks)

3 c) Solve the given radical function inequality $\sqrt{2} - \sqrt{x+6} \leq -\sqrt{x}$ (5 Marks)

5 d) Solve for x and y given that:

$$\frac{x}{1+i} - \frac{y}{2-i} = \frac{1-5i}{3-2i}$$ (5 Marks)

5 e) Show that the function f defined by $f(x) = \frac{2x}{x-1}$ $x \in R$, is a bijection on R on to $\{y \in R : y \neq 2\}$ (5 Marks)

QUESTION THREE

2 a) Use the Rational root theorem to solve $x^3 - 4x^2 + 8 = 0$ (5 Marks)

5 b) Rationalize the denominator $\frac{1}{(\sqrt{2}+1)(\sqrt{3}-1)}$ (5 Marks)

2.5 c) (i) Determine whether the function $f(x) = x^4 + x^2 + 1$ even, odd or neither. (2.5 marks)

2.5 (ii) Let $A = \{x \in \mathbb{R} : -4 \leq x < 2\}$ and B= $\{x \in \mathbb{R} : x \geq -1\}$.
Find a) $A \cap B$ b) A' (2.5 marks)

3 d) What are the dimensions of the largest rectangular field which can be enclosed by 1200 m of fencing? (5 Marks)

3 e) Sketch the graph of $f(x) = |2x + 1|$. On the same diagram sketch also the graph of $g(x) = \sqrt{1-2x}$ and , hence, find the values such that $\sqrt{1-2x} > |2x + 1|$ (5 Marks)

$\frac{x}{1+i}$, $\frac{1-i}{1-i}$

$A = L \times B$

$= A \times m$

$A = [m][m]$

$A = [m]^{1+1} = m^2$

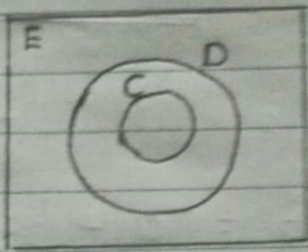
CamScanner

MA 110 - 2023 - Test 1

Qustion One

(a, i) If $C \subset D \rightarrow$ C is a subset of D

$C' \cup D'$



Answer = C'

Method 2

let $x \in C' \cup D'$

$x \in C'$ or D'

$x \notin C$ or D

$x \notin C \cap D$

$x \in (C \cap D)'$

now that can be simplify to just C'

(ii) $1.171717\ldots \rightarrow 1.\overline{17}$

$x = 1.\overline{17}$ ..... (i)

$10x = 117.\overline{17}$ .... (ii)

equ (i) from equ (ii)

$100x - x = 117.\overline{17} - 1.\overline{17}$

$99x = 116$

$$\frac{99x}{99} = \frac{116}{99}$$

$$x = \frac{116}{99}$$

b, i $a * b = a + b - 2ab$

let $a = 1$, $b = 2$

$1 * 2 = 1 + 2 - 2(1)(2)$
$= 3 - 4$
$= -1$

Reason for the answer !!!

$\therefore$ The above is a binary operation on all real numbers.

ii # For it to be commutative $\boxed{a * b = b * a}$

Example :

$1 * 2 = 2 * 1$

$1 + 2 - 2(1)(2) = 2 + 1 - 2(2)(1)$

$3 - 4 = 3 - 4$

$-1 = -1$

$\therefore$ It is commutative !!

(iii) $1 * (2 * 3)$;

$(2 * 3) = 2 + 3 - 2(2)(3)$
$= 5 - 12$
$(2 * 3) = -7$

$1 * -7$;

$1 * -7 = 1 + (-7) - 2(1)(-7)$
$= -6 + 14$
$\underline{\underline{1 * -7 = 8}}$

$(1 * 2) * 3$;

$1 * 2 = 1 + 2 - 2(1)(2)$
$= 3 - 4$
$1 * 2 = -1$

$-1 * 3$

$-1 * 3 = -1 + 3 - 2(-1)(3)$
$= +2 + 6$
$\underline{\underline{-1 * 3 = 8}}$

$\therefore$ since $1 * (2 * 3) = (1 * 2) * 3$ then the operation is associative !!

C $f(x) = \dfrac{x + 2}{x - 2}$

Vertical Asymptote ;

$x - 2 = 0$
$\underline{\underline{\# \; x = 2}}$

Horizontal Asymptote ;

$f(x) = \dfrac{x + 2}{x - 2}$

$= \dfrac{1}{1}$

$\underline{\underline{\# \; H.A = 1}}$

Domain ; $f(x) = \frac{x+2}{x-2}$

$$x - 2 = 0$$
$$x = 2$$

$\therefore$ Domain is all real numbers but 2

Range ; $f(x) = \frac{x+2}{x-2}$, find it inverse :

$$y = \frac{x+2}{x-2}$$

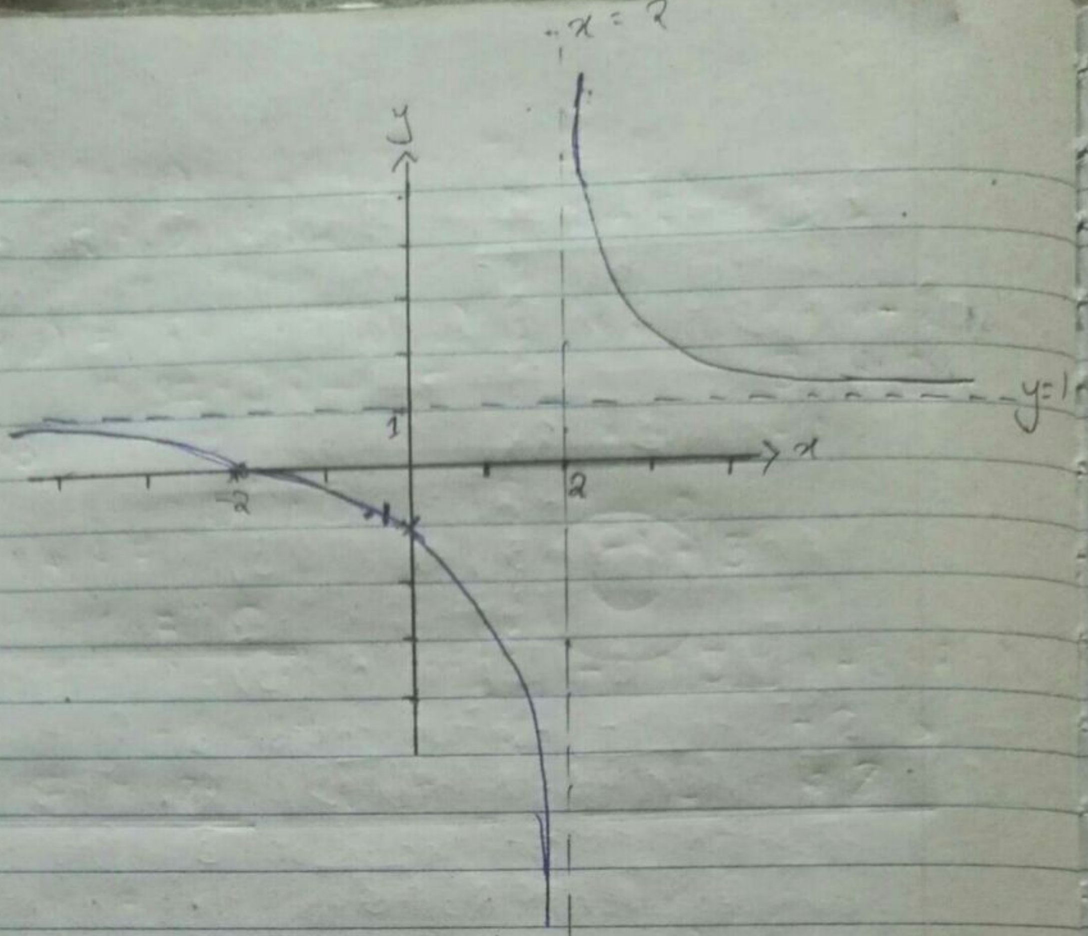
$$xy - 2y = x + 2$$
$$xy - x = 2 + 2y$$
$$\frac{x\cancel{(y-1)}}{\cancel{y-1}} = \frac{2+2y}{y-1}$$

$$x = \frac{2+2y}{y-1}$$

$$y = \frac{2+2x}{x-1}$$

$$x - 1 = 0$$
$$x = 1$$

$\therefore$ Range is all real numbers but 1



(d) $\sqrt{2}$

Proof by Contradiction

Suppose $\sqrt{2}$ is rational. Then it can be written in the form a/b where $a, b \in Z$ and $b \neq 0$ with no comon factor.

$$\sqrt{2} = \frac{a}{b} \quad \ldots (i)$$

$$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2 \quad \ldots (ii)$$

Since a^2 has a comon factor of 2 meaning a also has a common factor of 2. this can be written as $a = 2k$ wher k is an integer

$$a^2 = 2b^2$$
$$(2k)^2 = 2b^2$$
$$4k^2 = 2b^2$$

$$b^2 = 2k^2 \quad \ldots (iii)$$

b^2 has a common factor of 2 therefore b also has a common factor of 2. Thus a and b have a common factor of 3. This contradicts our earlier assumption that a and b have no common factor. Therefore by method of contradiction we have proved that $\sqrt{2}$ is an irrational number.

(e) $f(x) = \dfrac{x+1}{x-1}$, $g(x) = \sqrt{x}$

$$(g \circ f)(x) = \sqrt{\frac{x+1}{x-1}}$$

# Domain of $f(x)$;	Domain of $g(x)$
$x - 1 = 0$,	$\sqrt{x}$
$x = 1$	$x \geq 0$
∴ Domain is all real numbers except 1	∴ Domain is all real numbers ≥ 0

$\therefore$ The Domain of $(g \circ f)(x)$ is the intersection of the domains of the two functions

$\therefore$ Domain of $(g \circ f)(x)$ is all real numbers greater than 0 but except 1

$x = x$

(e) $f(x) = \dfrac{x+1}{x-1}$ $\qquad g(x) = \sqrt{x}$

$$(g \circ f)(x) = \sqrt{\frac{x+1}{x-1}}$$

$$\frac{x+1}{x-1} \geq 0$$

$x+1 = 0$ $\qquad$ or, $x - 1 = 0$

$x = -1$ $\qquad$ $x = 1$

These are the critical points

	$x < -1$	$-1 < x < 1$	$x > 1$
	$x = -2$	$x = 0$	$x = 2$
$x+1$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
	$+$	$-$	$+$

$x \leq -1$ or $x > 1$

$(-\infty, -1] \cup (1, \infty)$

Question two

(a.) $A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$

R.H.S

$(A \cap B) \cup (A \cap B') \cup (A' \cap B)$

$[A \cap (B \cup B')] \cup (A' \cap B)$

$A \cap U \cup (A' \cap B)$

$A \cup (A' \cap B)$

$(A \cup A') \cap (A \cup B)$

$U \cap (A \cup B)$

$A \cup B = L.H.S$

b. $3x^2 + 2x + 5 = 0$

$a = 3, \ b = 2, \ c = 5$

$\alpha + \beta = -b/a$ $\qquad \alpha\beta = c/a$

$\alpha + \beta = -2/3$ $\qquad \alpha\beta = \frac{5}{3}$

Sum of the roots; | Product or roots

$\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ | $\frac{1}{\alpha^2} \times \frac{1}{\beta^2}$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$= \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$$

$$\# \left\{ \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} \right\}$$

$$\# \left\{ = \frac{\left(-\frac{2}{3}\right)^2 - 2\left(\frac{5}{3}\right)}{\left(\frac{5}{3}\right)^2} \right\}$$

$$\frac{4}{9} - \frac{10}{3}$$

$$-\frac{26}{9} \div \frac{25}{9}$$

$$-\frac{26}{9} \times \frac{9}{25}$$

$$-\frac{26}{25}$$

$$\frac{1}{\alpha^2} \times \frac{1}{\beta^2} = \frac{1}{\alpha^2\beta^2}$$

$$\frac{1}{(\alpha\beta)^2} = \frac{1}{\left(\frac{5}{3}\right)^2}$$

$$= \frac{1}{\frac{25}{9}}$$

product $= \frac{9}{25}$

\# $x^2 -$ (sum of roots) x + (product of roots)

$$x^2 - \left(\frac{-26}{25}\right)x + \left(\frac{9}{25}\right)$$

$$\left\{x^2 + \frac{26}{25}x + \frac{9}{25}\right\} \times 25$$

$$\underline{25x^2 + 26x + 9}$$

(c.) $\sqrt{2} - \sqrt{x+6} \leq -\sqrt{x}$

$$(\sqrt{2} + \sqrt{x})^2 \leq (\sqrt{x+6})^2$$

$$2 + 2\sqrt{2}\sqrt{x} + x \leq x + 6$$

$$2 + 2\sqrt{2x} + x \leq x + 6$$

$$2\sqrt{2x} \leq x - x + 6 - 2$$

$$2\sqrt{2x} \leq 4$$

$$(\sqrt{2x})^2 \leq (2)^2$$

$$\frac{2x}{2} \leq \frac{4}{2}$$

$$x \leq 2$$

But from the question itself getting the two radicals $\sqrt{x+6}$ & $-\sqrt{x}$

$\sqrt{x+6}$ $\sqrt{x}$

$x+6 \geq 0$ $x \geq 0$

$x \geq -6$

The intersection of these two is $x \geq 0$

Finally intersection of $x \geq 0$ and $x \leq 2$ is $0 \leq x \leq 2$

$[0, 2]$

(d.) $\frac{x}{1+i} - \frac{y}{2-i} = \frac{1-5i}{3-2i}$

$\frac{x(2-i) - y(1+i)}{(1+i)(2-i)} = \frac{1-5i}{3-2i} \cdot \left(\frac{3+2i}{3+2i}\right)$

$\frac{2x - xi - y - yi}{2 - i + 2i + 1} = \frac{3 + 2i - 15i + 10}{9 + 4}$

$\frac{2x - y - xi - yi}{3 + i} = \frac{13 - 13i}{13}$

$\frac{2x - y - xi - yi}{3+i} = 1 - i$

$$2x - y - xi - yi = 3 + i - 3i + 1$$

$$2x - y - xi - yi = 4 - 2i$$

$$2x - y = 4 \quad \ldots\ldots (i)$$
$$-xi - yi = -2i \quad \ldots\ldots (ii)$$

$$-\left|\begin{array}{l} 2x - y = 4 \\ -x - y = -2 \end{array}\right.$$

$$\frac{\cancel{3}x}{\cancel{3}} = \frac{6}{3}$$

$$\underline{\underline{x = 2}}$$

$$2x - y = 4$$

$$2(2) - y = 4$$

$$4 - 4 = y$$

$$\underline{\underline{y = 0}}$$

~~For Injective~~;

For bijection we need to prove that it is Injective and ~~Super~~ Surjective;

For Injective; $f(a) = f(b) = a = b$.

$$f(a) = \frac{2a}{a-1}, \quad f(b) = \frac{2b}{b-1}$$

$$\therefore \frac{2a}{a-1} = \frac{2b}{b-1}$$

$$2a(b-1) = 2b(a-1)$$

$$\cancel{2ab} - 2a = \cancel{2ba} - 2b$$

$$\frac{-2a}{-2} = \frac{-2b}{-2}$$

$$a = b$$

$\therefore$ the function is Injective;

For Surjective, $y = \dfrac{2x}{x-1}$

$$xy - y = 2x$$

$$xy - 2x = y$$

$$x = \frac{y}{y-2}, \quad f(x)^{-1} = \frac{x}{x-2}$$

$\therefore$ the function is Surjective;

Since the function is both ~~Injective~~ injective and surjective $\therefore$ the function is a bijective on R

Question 7

a) $x^3 - 4x^2 + 8$;

possible roots $p/q = [\pm 1, \pm 2, \pm 4, \pm 8]$

let $x = 2$;

$f(2) = (2)^3 - 4(2)^2 + 8$

$= 8 - 16 + 8$

$= 0$

using synthetic division.

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 0 & 8 \\ & \downarrow & 2 & -4 & -8 \\ \hline & 1 & -2 & -4 & 0 \end{array}$$

$x^2 - 2x - 4 = 0$

$P = -4$, $S = -2$, find

~~factors~~ $= 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4 - 4(1)(-4)}}{2}$$

$$x = \frac{2 \pm \sqrt{20}}{2}$$

$$x = \frac{2 \pm 2\sqrt{5}}{2}$$

$x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$

$\therefore x = 2, \; x = 1 + \sqrt{5}, \; x = 1 - \sqrt{5}$

$$\frac{1}{(\sqrt{2}+1)(\sqrt{3}-1)} \times \frac{(1-\sqrt{2})(-1-\sqrt{3})}{(1-\sqrt{2})(-1-\sqrt{3})}$$

$$= \frac{\cancel{\quad}}{\cancel{[(\sqrt{2})^2 - 1][\sqrt{\quad}}}$$

$$\frac{(1-\sqrt{2})(-1-\sqrt{3})}{[(1)^2-(\sqrt{2})^2][(1)^2-(\sqrt{3})^2]}$$

$$\frac{(1-\sqrt{2})(-1-\sqrt{3})}{(1-2)(1-3)}$$

$$\frac{(1-\sqrt{2})(-1-\sqrt{3})}{(-1)(-2)} = \frac{(1-\sqrt{2})(-1-\sqrt{3})}{2}$$

(c)(i) $f(x) = x^3 + x^2 + 1$

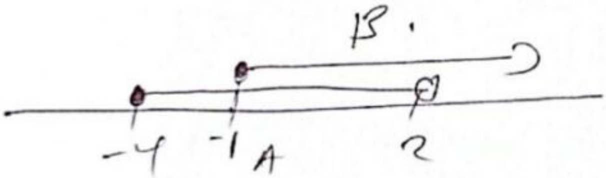
$f(-x) = (-x)^3 + (-x)^2 + 1$

$= -x^3 + x^2 + 1$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

$\therefore$ the function is Neither.

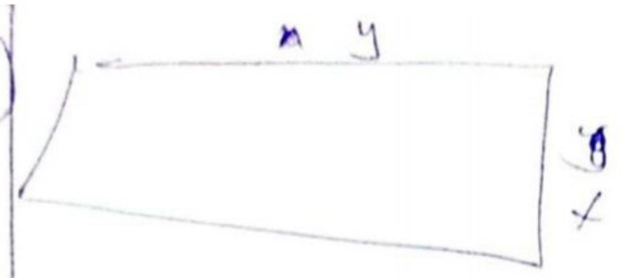
c ii) $A = -4 \leq x < 2$, $B = x \geq -1$



$A \cap B$: $[-1, 2)$

$A' = (-\infty, 4) \cup [2, \infty)$

d)



Perimeter, $2x + 2y = 1200$

$x + y = 600$

$y = 600 - x$

Area, $= xy$

$= x(600 - x)$

$600x - x^2$

$-1(x^2 - 600x + 0] \Rightarrow$ Complete the square.

$-600 \times \frac{1}{2}$

$= (300)^2$

$= -1[x^2 - 600x + (300)^2 - (300)^2]$

$= -1[(x - 300)^2 - 90{,}000]$

$= -1(x - 300)^2 + 90{,}000$

$\therefore$ let $x - 300 = 0$, $y = 90{,}000$

$x = 300$

$\therefore$ the width is 300m and ~~her~~ length $= 90000m$

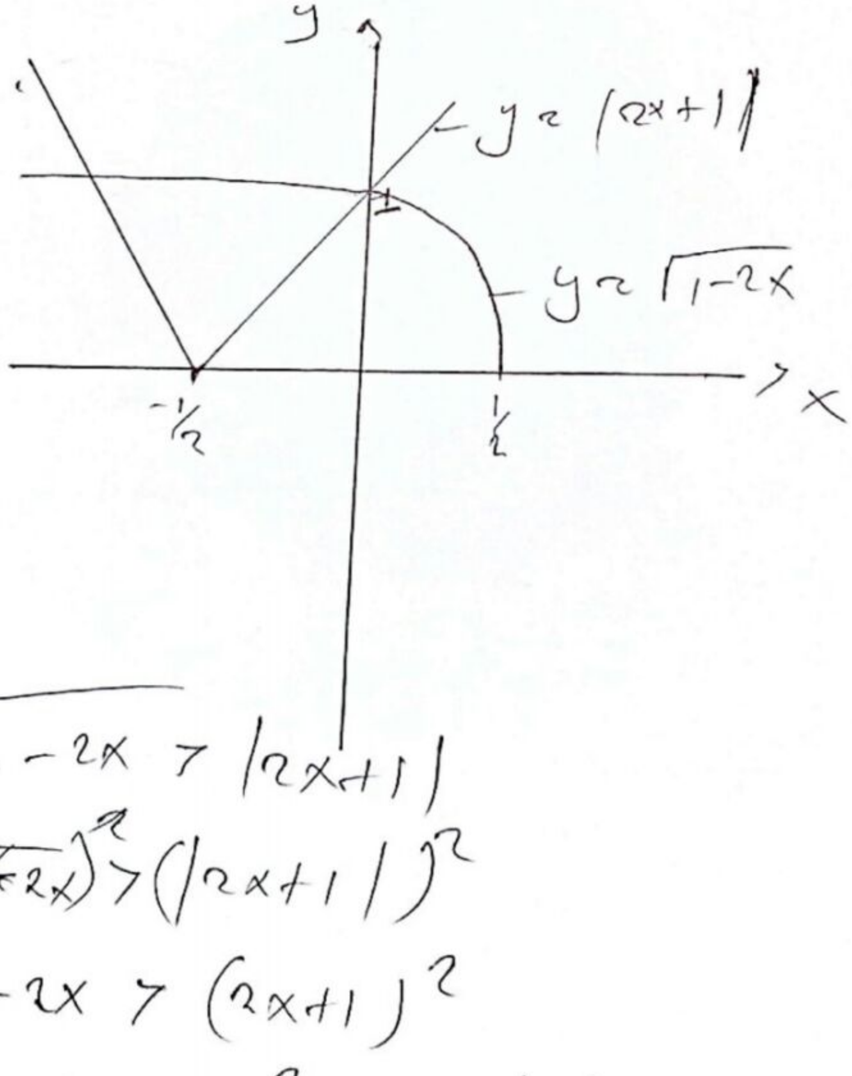
e) $f(x) = |2x+1|$

let $2x+1 = 0$

$x = -\frac{1}{2}$

and $1-2x = 0$ $[\sqrt{1-2x}]$

$x = \frac{1}{2}$



$\therefore \sqrt{1-2x} > |2x+1|$

$(\sqrt{1-2x})^2 > (|2x+1|)^2$

$1-2x > (2x+1)^2$

$\cancel{1} - 2x > 4x^2 + 4x + \cancel{1}$

$0 > 4x^2 + 6x$

$0 > 4x^2 + 6x$

$0 > 2x(2x + 3)$

Critical points;

$2x = 0 \quad 2x + 3 = 0$

$x = 0, \quad x = -\frac{3}{2}$

$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x > 0$
+	−	+

when $x = -4$

$2(-4)(2(-4) + 3)$

$(-) \times (-)$

$= +$

Ans; $-\frac{3}{2} < x < 0$