THE COPPERBELT UNIVERSITY
SCHOOL OF MATHEMATICS AND NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS
2015/2016 Academic Year
MA110 - Mathematical Methods
Test I
5th September, 2015.

Time Allowed : Three hours (3:00 hrs)

Instructions:

1. You must write your **Name**, your **Computer Number and programme of study** on your answer sheet.
2. Calculates are **not** allowed in this paper.
3. There are five (5) questions in this paper, Attempt **All** questions and show **detailed** working for full credit.

Question 1

a) If C *and* D are disjoint, simplify if possible $(C \cup D)'$

b) Express $2.\overline{143}$ in the form of $\frac{a}{b}$ where a and b are integers, $b \neq 0$.

c) Rationalize the denominator and express the final answer in simplest radical form for $\frac{5\sqrt[3]{y^2}}{4\sqrt[4]{x}}$

d) Sketch and determine the domain and the range of the function $f(x) = \frac{2}{x^2+4}$

e) Prove that if $a + c = b + c$ *then* $a = b$ when $a, b, c \in R$.

Question 2

a) Let binary operation $*$ defined $a * b = a - b + ab$ where a and $b \in R$, solve $|x * 2| = 1$.

b) Rationalize the denominator of $\frac{1}{(\sqrt{2}+1)(\sqrt{3}-1)}$.

c) Determine whether $f(x) = x^2 + 1$ is even, odd or neither.

d) Solve the equation $\sqrt{-2x-7} + \sqrt{x+9} = \sqrt{8-x}$.

e) Solve $x^4 + 3x - 2 = 0$.

Question 3

a) Solve for x and y given that $\frac{x}{1+i} - \frac{y}{2-i} = \frac{1-5i}{3-2i}$.

b) Sketch $f(x) = 2 + 3\sqrt{-x+1}$ and determine its range and domain.

c) The roots of the equation $2x^2 + 6x - 15 = 0$ are α and β. Find the value of $(\alpha - \beta)$.

d) Prove that $\sqrt{3}$ is an irrational number.

e) Solve $\frac{3x+2}{x-1} > 0$ expressing the set of solution sets in interval notation.

Question 4

a) Verify that the two given functions are inverses of each other $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

b) Express $5 - x - 2x^2$ in the form $a - b(x+c)^2$ and hence or otherwise find its maximum value and the value of x where this occurs.

c) Using the associative and distributive properties of union and intersection of sets . Show that

$$A \cup B = (A \cap B) \cup (A \cap B') \cup (A' \cap B)$$

d) Solve for x given $|3x+1| < |4-2x|$.

e) What type of roots does the equation $5x^2 - 3x + 1 = 0$ have ?

Question 5

a) Determine whether $f(x) = x^2 - 2$ is one-to-one . If it is, find the inverse and graph both the function and its inverse.

b) Given that $z + \frac{1}{z} = k$, where k is a real number, prove that either z is real or $|z| = 1$

c) Given the set $X = \{0, 1, 2, 3\}$. Determine whether the operations $+$, $-$, $\times$ are binary operations on X

d) Determine whether or not $x + 3i$ is a factor of $f(x) = x^4 + 14x^2 + 45$.

e) Solve $\frac{4}{x-2} + \frac{x}{x+1} = \frac{x^2-2}{x^2-x-2}$.

f) Sketch $f(x) = |x^2 + 5x + 4| - 2$ and determine its domain and range.

END

THE COPPERBELT UNIVERSITY

SCHOOL OF MATHEMATICS AND NATURAL SCIENCES

NAME: [redacted]

SIN: 15005953

COURSE: MA 110

PROGRAMME: NON- QUOTA

LECTURER: [redacted]

DATE: OCTOBER 5th, 2015.

TEST 1.

GROUP: D

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...tion 1

$(A \cup B)'$

$(E)'$

$= \underline{\underline{\varnothing}}$

b. $2.\overline{143}$

let $x = 2.\overline{143}$ $\cdots$(i)

$100x = 2143.\overline{143}$ $\cdots$(ii)

(ii) − (i)

$(100x - x) = 2143.\overline{143} - 2.\overline{143}$

$\frac{99x}{99} = \frac{2141}{99}$

$x = \frac{2141}{99}$

$\therefore 2.\overline{143} = \underline{\underline{\frac{2141}{99}}}$

2

c. $\frac{5\sqrt[3]{y^2}}{4\sqrt[4]{x}}$

$= \frac{5\sqrt[3]{y^2}}{4\sqrt[4]{x}} \cdot \frac{(\sqrt[4]{x})^3}{(\sqrt[4]{x})^3}$

$= \frac{5(\sqrt[3]{y^2})(\sqrt[4]{x})^3}{4x^{1/4+3/4}} = \frac{5(\sqrt[3]{y^2})(\sqrt[4]{x^3})}{4x} = \underline{\underline{\frac{5\sqrt[12]{x^9y^8}}{4x}}}$

3

=

d. $f(x) = 2/x^2+4$

$D_{f(x)}$: $x^2+4 \neq 0$

$= \{x \mid x^2+4 \neq 0\}$.

$R_{f(x)} = \{f(x) \mid f(x)$

$f(x) = 2x^0 / x^2+4$

Horizontal Asymptote: since $x^0 < x^2$ then, $y = 0$.

Vertical Asymptote: Set, $x^2+4=0$

$x^2 = -4$

No vertical Asymptote.

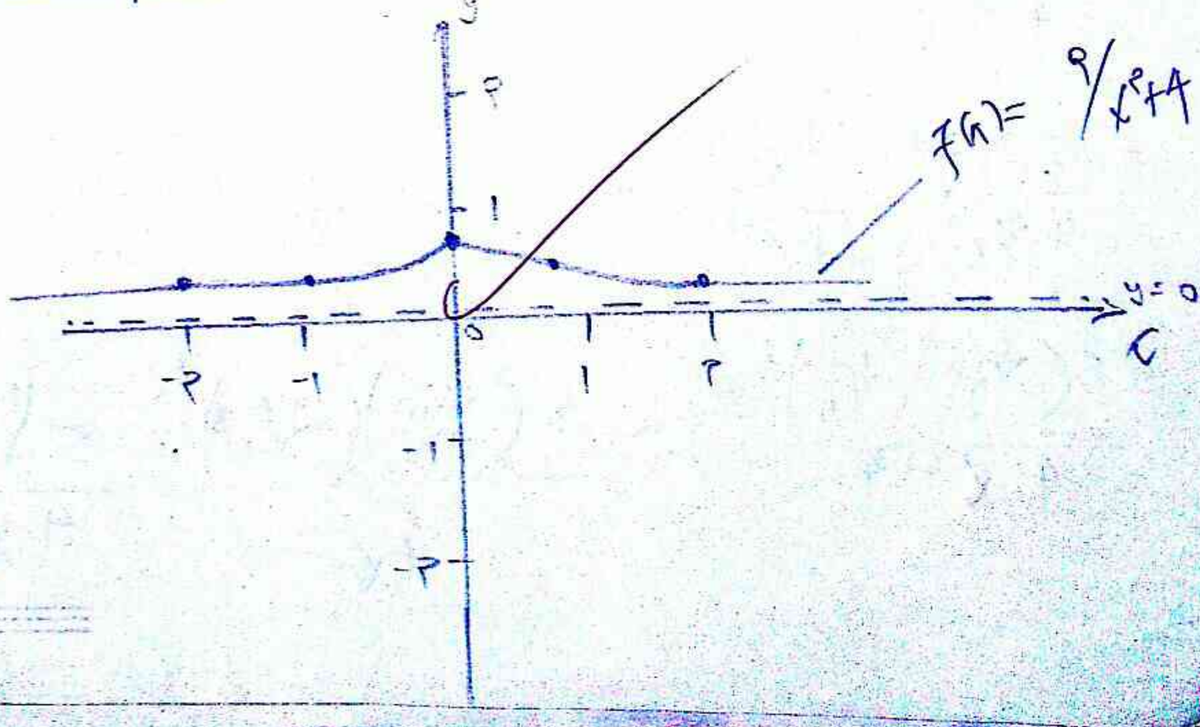
x-axis intercept ($y=0$): $0 = 2/x^2+4$, no x-intercept.

y-axis intercept ($x=0$), $y = 2/(0)^2+4$

$= \frac{2}{4} = 0.5$.

4

x	-2	-1	0	1	2
y	0.25	0.4	0.5	0.4	0.25



c. $a + c = b + c$, $a = b$, $a, b, c \in \mathbb{R}$.

$a + c - c = b + c - c$ (using additive inverse of c).

$a + 0 = b + 0$

$a = b$

2

Question 2.

a. $a * b = a - b + ab$.

$x = a$

$b = 2$.

4

$= |x * 2| = 1$

$= |x - 2 + 2x| = 1$

$= |3x - 2| = 1$

$= 3x - 2 = 1$ or $-(3x - 2) = 1$

$= 3x - 2 + 2 = 1 + 2$

$= 3x = 3$

$= x = 3$

$3x - 2 = -1$

$3x - 2 + 2 = -1 + 2$

$3x = 1$

$x = 1/3$

b. $\dfrac{1}{(\sqrt{2} + 1)(\sqrt{3} - 1)}$

$= \dfrac{1}{(\sqrt{2} + 1)(\sqrt{3} - 1)} \cdot \dfrac{(\sqrt{2} - 1)(\sqrt{3} + 1)}{(\sqrt{2} - 1)(\sqrt{3} + 1)}$

$= \dfrac{(\sqrt{2} - 1)(\sqrt{3} + 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1) + (\sqrt{2} + 1)(\sqrt{3} + 1) + (\sqrt{3} - 1)(\sqrt{2} - 1) + (\sqrt{3} - 1)(\sqrt{3} + 1)}$

$= \dfrac{(\sqrt{2} - 1)(\sqrt{3} + 1)}{(2 - 1) + (\sqrt{6} + \sqrt{2} + \sqrt{3} + 1) + (\sqrt{6} - \sqrt{3} - \sqrt{2} + 1) + (3 - 1)}$

b. $\dfrac{\sqrt6+\sqrt2-\sqrt3-1}{(\cancel{1+1+1+2}) + (\sqrt6+\sqrt6) + (\sqrt2-\sqrt2+\sqrt3-\sqrt3)}$

$= \dfrac{(\sqrt2-1)(\sqrt3+1)}{5+2\sqrt6}$

$= \dfrac{(\sqrt2-1)(\sqrt3+1)\cdot(\cancel{5}-2\sqrt6)}{(5+2\sqrt6)\cdot(5-2\sqrt6)}$

$= \dfrac{(\sqrt2-1)(\sqrt3+1)(5-2\sqrt6)}{25+4(6)}$

$= \dfrac{(\sqrt2-1)(\sqrt3+1)(5-2\sqrt6)}{25+24} = \dfrac{(\sqrt2-1)(\sqrt3+1)(5-2\sqrt6)}{49}$

C. $= f(x) = x^2+1$.

$= f(-x) = f(x)$, : for even function

$= f(-x) = -f(x)$, : for odd function.

$\Rightarrow f(-x) = (-x)^2+1$

$= x^2+1$

= Hence $f(x) = x^2+1$ is an even function.

d. $\sqrt{-2x-7} + \sqrt{x+9} = \sqrt{8-x}$

$\left(\sqrt{-2x-7} + \sqrt{x+9}\right)^2 = \left(\sqrt{8-x}\right)^2$

$$1. = \left(\sqrt{-2x+7} + \sqrt{x+9}\right)^2 = \left(\sqrt{8-x}\right)^2$$

$$= -2x+7+2\left(\left(-2x+7\right)\left(x+9\right)\right) + x+9 = 8-x$$

$$= -\cancel{x}+16+2\left(-2x^2-18x+7x+63\right) = 8-\cancel{x}$$

$$= 16-8+2\left(-2x^2-11x+63\right)=0$$

$$= 8+2\left(-2x^2-11x+63\right)=0$$

$$= 8-4x^2-22x+126=0$$

$$= -4x^2-22x+134=0 \quad (\div 2)$$

$$= 2x^2+11x-67=0$$

$$= 2x^2+11x = 67$$

$$= x^2+\tfrac{11}{2}x = \tfrac{67}{2}.$$

$$= x^2+\tfrac{11}{2}x+\left(\tfrac{11}{4}\right)^2 = \tfrac{67}{2}+\left(\tfrac{11}{4}\right)^2$$

$$= \left(x+\tfrac{11}{4}\right)^2 = \tfrac{67}{2}+\tfrac{121}{16}$$

$$= \left(x+\tfrac{11}{4}\right)^2 = \frac{536+121}{16}$$

$$= \sqrt{\left(x+\tfrac{11}{4}\right)^2} = \sqrt{\frac{657}{16}}$$

$$= x+\tfrac{11}{4} = \pm\frac{\sqrt{657}}{4}$$

$$= x = -\frac{11}{4} \pm \frac{\sqrt{657}}{4}$$

$$= x = -\frac{11}{4}+\frac{\sqrt{657}}{4} \quad \text{or} \quad x = \frac{-11}{4}-\frac{\sqrt{657}}{4}$$

$$= \frac{-11+\sqrt{657}}{4} \qquad = \frac{-11-\sqrt{657}}{4}$$

2

$x^4 + 8x - 2 = 0$

$c = -2$; $\pm 1, \pm 2$

$d = 1$; ± 1

$c/d = \pm 1, \pm 2.$

$x^4 + 8x - 2 = 0$ cannot be solved as c/d cannot fit into the given expression to make it equal 0

Question 3.

a. $\frac{x}{1+i} - \frac{y}{2-i} = \frac{1-5i}{3-2i}$

$= \frac{x(2-i) - y(1+i)}{(1+i)(2-i)} = \frac{1-5i}{3-2i}$

$= \frac{2x - ix - y - iy}{2 - i + 2i - i^2} = \frac{1-5i}{3-2i}$

$= \frac{(2x-y) + (-x-y)i}{2+1+i} = \frac{1-5i}{3-2i}$

$= \frac{(2x-y) + (-x-y)i}{3+i} = \left(\frac{1-5i}{3-2i}\right) \cdot \frac{(3+2i)}{(3+2i)}$

$= \frac{(2x-y) + (-x-y)i}{3+i} = \frac{3+2i-15i-10i^2}{(3)^2-(2i)^2}$

$= \frac{(2x-y) + (-x-y)i}{3+i} = \frac{3+10-13i}{9+4}$

$= \frac{(2x-y) + (-x-y)i}{3+i} = \frac{13-13i}{13}$

$= \frac{(2x-y) + (-x-y)i}{3+i} = 1-i$

$= (2x-y) + (-x-y)i = (1-i)(3+i)$

$= (2x-y) + (-x-y)i = 3+i-3i-i^2$

$= (2x-y) + (-x-y)i = 3+1-2i$

$= (2x-y) + (-x-y)i = 4-2i$

$= 2x - y = 4 \ldots (i)$

$-x - y = -2 \ldots (ii)$

$2x - y = 4$
$-x - y = -2$
$3x = 6$
$x = 2$

$-x - y = -2$
$-(2) - y = -2$
$-2 - y = -2$
$-y = -2 + 2$
$-y = 0$
$y = 0$

D

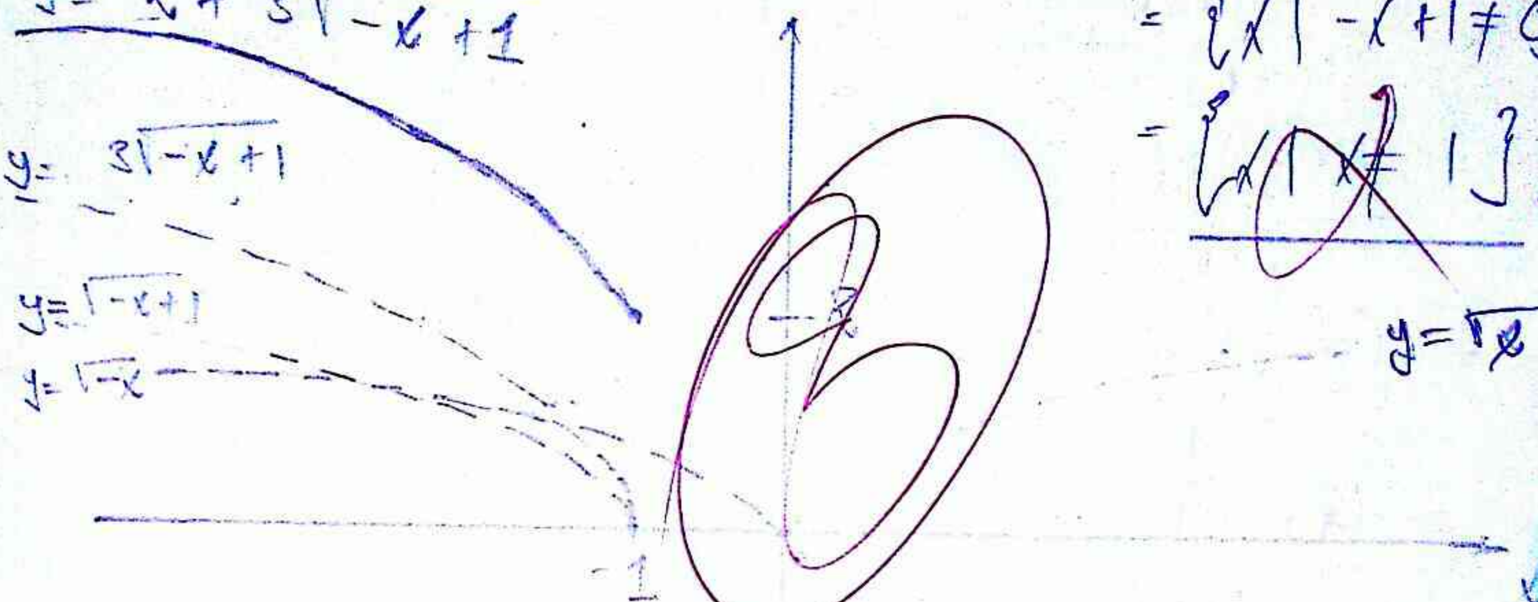
b. $f(x) = 2 + 3\sqrt{-x+1}$.

$y = 2 + 3\sqrt{-x+1}$

$y = 3\sqrt{-x+1}$

$y = \sqrt{-x+1}$

$y = \sqrt{-x}$



B

$D_f: -x+1 \neq 0$
$= \{x \mid -x+1 \neq 0\}$
$= \{x \mid x \neq 1\}$.

$y = \sqrt{x}$

$R_f = \{f(x) \mid f(x) > 0\}$

c. $2x^2 + 6x - 15 = 0$.

$(\alpha - \beta) = ?$

$\alpha + \beta = -b/a$
$= -6/2$
$= -3$.

$\alpha\beta = c/a$
$= -15/2$

$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$
$= (\alpha^2 + \beta^2) - 2\alpha\beta$
$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$

$\therefore (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$(\alpha-\beta) = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \pm\sqrt{(-3)^2 - 4(-15/2)}$

$= \pm\sqrt{9+80}$

$= \pm\sqrt{89}$

$\therefore (\alpha-\beta) = \sqrt{39}$ or $(\alpha-\beta) = -\sqrt{39}.$

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d. $\sqrt{3}$,

We prove by contradiction i.e. $\sqrt{3} = a/b$

and $\frac{a}{b}$ have no common factor.

$\sqrt{3} = a/b$

$(\sqrt{3})^2 = (a/b)^2$

$3 = a^2/b^2$

$a^2 = 3b^2.$

$a^2 = 3b^2$

$a =$ odd.

$a = 3k.$

$a^2 = 9k^2.$

$a^2 = 3b^2$

$9k^2 = 3b^2$

$b^2 = 3k^2.$

$b =$ odd.

let $\Big\rbrack$ $a =$ odd, $b =$ odd $\Big\rbrack$ contradiction.

Since a and b are odd and have a common factor, i.e. $3k$, $\sqrt{3}$ is not rational.

$\therefore \sqrt{3}$ is irrational.

5

2. $\dfrac{3x+2}{x-1} > 0$

Set $3x+2=0$

$3x=-2$

$x=-2/3$.

Set $x-1=0$

$x=1$

$-\infty < x < -2/3$, $-2/3 < x < 1$, $1 < x < \infty$.

	$x=-1$	0	2
		$-2/3$	1
$(3x+2)$	$-$	$+$	$+$
$(x-1)$	$-$	$-$	$+$
Quotient $f(x)$	$+$	$-$	$+$

So $-\infty < x < -2/3$ and $1 < x < \infty$ are the solution sets

or $(-\infty, -2/3) \cup (1, \infty)$.

5

Question 4.

a. $f(x) = x^3+1$, $g(x) = \sqrt[3]{x-1}$,

for inverse function: $(f \circ g)(x) = x. = (g \circ f)(x$

$= (f \circ g)(x) = f[g(x)]$

$= (\sqrt[3]{x-1})^3 + 1$

$= ((x-1)^{\frac{1}{3}})^3 + 1$

$= x - 1 + 1$

$= x.$

2

Show that $(f \circ g)(x) = (g \circ f)(x)$
$= x$

$\therefore$ since $(f \circ g)(x) = x$, it follows that $f(x) = x^3+1$ and $g(x) = \sqrt[3]{x-1}$ are inverses of each other.

b. $5 - x - 2x^2$, $a - b(x+c)^2$.

$= -2x^2 - x + 5$

$= -2(x^2 + \frac{1}{2}x - \frac{5}{2})$

$= -2(x^2 + \frac{1}{2}x + (\frac{1}{4})^2 - (\frac{1}{4})^2 - \frac{5}{2})$

$= -2((x+\frac{1}{4})^2 - \frac{1}{16} - \frac{5}{2})$

4

$= -2((x+\frac{1}{4})^2 - \frac{41}{16})$

$= -2(x+\frac{1}{4})^2 + \frac{41}{8}$

b. $-2\left(x + \frac{1}{4}\right) + \frac{41}{8}$

$\frac{41}{8} - 2\left(x + \frac{1}{4}\right)($

$\therefore\ 5 - x - 2x^2 = \frac{41}{8} - 2((x + \frac{1}{4})$

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c. $A \cup B = (A \cap B) \cup (A \cap B') \ (A' \cap B)$

R.H.S

$= (A \cap B) \cup (A \cap B') \cup (A' \cap B)$

$= A \cap (B \cup B') \cup (A' \cap B)$

$= (A \cap \mathcal{E}) \cup (A' \cap B)$

$= A \cup (A' \cap B)$

$= (A \cup A') \cap (A \cup B)$

$= \mathcal{E} \cap (A \cup B)$

$= A \cup B$

$\therefore$ R.H.S = L.H.S

$\therefore\ A \cup B = A \cup B$ hence shown.

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d. $|3x + 1| < |4 - 2x|$

$= (3x + 1)^2 < (4 - 2x)^2$

$= 9x^2 + 6x + 1 < 16 - 16x + 4x^2$

$= 9x^2 - 4x^2 + 6x + 16x + 1 - 16 < 0$

$= 5x^2 + 22x - 15 < 0$.

1. $5x^2 + 22x - 15 < 0$.

$= (5x - 3)(x + 5) < 0$

$= 5x - 3 = 0$ | $x + 5 = 0$

$= x = 3/5$ | $x = -5$

$-\infty < x < -5$, $-5 < x < 3/5$, $3/5 < x < \infty$

good!

	$x = -6$	$x = 0$	$x = 1$
$(5x-3)$	$-$	$-$	$+$
$(x+5)$	$-$	$+$	$+$
product	$+$	$-$	$+$

7

$\therefore$ Solution set

$= -5 < x < 3/5$

$= (-5, 3/5)$

$5x \quad -3$

$x \quad 5$

$25x - 3x = 22x$

2. $5x^2 - 3x + 1 = 0$

D = discriminate.

$a = 5$

$b = -3$

$c = 1$

$D = b^2 - 4ac$

$= (-3)^2 - 4(5)(1)$

$= 9 - 20$

$= -11$.

2

Since $D < 0$,

then equation, $5x^2 - 3x + 1$ has two distinct and complex roots.

5. (a) $f(x) = x^2 - 2$,

$= f(a) = f(b)$

$= (a)^2 - 2 = (b)^2 - 2.$

$= a^2 - 2 = b^2 - 2$

$= a^2 = b^2 - 2 + 2$

$= a^2 = b^2$

$= \sqrt{a^2} = \sqrt{b^2}$

$= a = \pm b.$

$= a = b$ or $a = -b.$

$=$ Since $a = b$ or $a = -b$, this is not a one-to-one function.

for one-to-one,

$f(a) = f(b)$, a, b any number which is valid

Since $a = b$ or $a = -b$,

i.e. $a \to +b$, $a \to -b$

$f(x) = x^2 - 2$ should be a one-to-many function.

Q5. b. $z + 1/z = k$, either z is real or $|z| = 1$.

let $z = x + iy$

$= z + 1/z$

$$= \frac{z^2+1}{z} = \frac{(x+iy)^2+1}{(x+iy)} = k$$

$$= \frac{x^2 + 2ixy + i^2y^2 + 1}{x+iy} = k$$

$$= \frac{(x^2 - y^2) + 2ixy + 1}{(x+iy)} = k$$

$$= \frac{(x^2 - y^2 + 1) + 2ixy}{(x+iy)} = k$$

$= (x^2 - y^2 + 1) + 2ixy = k(x+iy)$

$= (x^2 - y^2 + 1) + 2ixy = kx + kiy$

$= 2ixy = kiy$ | $= x^2 - y^2 + 1 = kx$.

$= 2xy = ky$ | $= x^2 - y^2 + 1 = kx$

$= k = 2x$. | $= x^2 - y^2 + 1 = (2x)x$

$= x^2 - y^2 + 1 = 2x^2$

$= x^2 + y^2 = 1$

since $|z| = \sqrt{x^2+y^2} = \sqrt{1}$

$= 1$ Shown.

Since $\bar{z}+1/\bar{z} = k$, k being real, it follows that z also has to be real.

c. $X = \{0, 1, 2, 3\}$.

<u>Addition (+)</u>

$2+3=5$

$5 \notin X$

$\therefore$ <u>Addition not a binary operation on X</u>

<u>Subtraction (−)</u>

$2-3=-1$

$-1 \notin X$.

$\therefore$ <u>Subtraction not a binary operation on X.</u>

<u>Multiplication.</u>

$2 \times 3 = 6$

$6 \notin X$

$\therefore$ <u>Subtraction not a binary operation on X.</u>

d. $x+3i$, $f(x) = x^4+14x^2+45$,

for it to be a factor,

set $x+3i=0$

$x=-3i$

$f(x) = f(-3i) = 0$

$f(-3i) = (-3i)^4 + 14(-3i)^2 + 45$ $\qquad i^2 = -1$

$i^2 \times i^2 = 1$

$= 81i^4 + 14(9)i^2 + 45$

$= 81 + 126(-1) + 45$

$= 81 - 126 + 45$

$= 81 - 81$

$= 0.$

8

$\therefore x+3i$ is a factor of $f(x) = x^4 + 14x^2 + 45$ since $f(-3i) = 0.$

e. $\dfrac{4}{x-2} + \dfrac{x}{x+1} = \dfrac{x^2-2}{x^2-x-2}.$

$= \dfrac{4(x+1) + x(x-2)}{(x-2)(x+1)} = \dfrac{x^2-2}{x^2+x-2x-2}$

$= \dfrac{4x+4+x^2-2x}{(x-2)(x+1)} = \dfrac{x^2-2}{x(x+1)-2(x+1)}$

$= \dfrac{2x+x^2+4}{(x-2)(x+1)} = \dfrac{x^2-2}{(x+1)(x-2)}.$ since same denominator it follows:

Can't believe it!!

$= x^2+2x+4 = x^2 - 2$

$= x^2 - x^2 + 2x + x + 4 = 0$

$= 3x + 4 = 0$

$= 3x = -4$

$= x = -4/3.$

e

5. (f). $f(x) = |x^2 + 5x + 4| - 2$. Let $f(x) = x^2 + 5x + 4$

= y-intercept : $x = 0$,
$y = (0)^2 + 5(0) + 4$
$y = 4$.
$(0, 4)$,

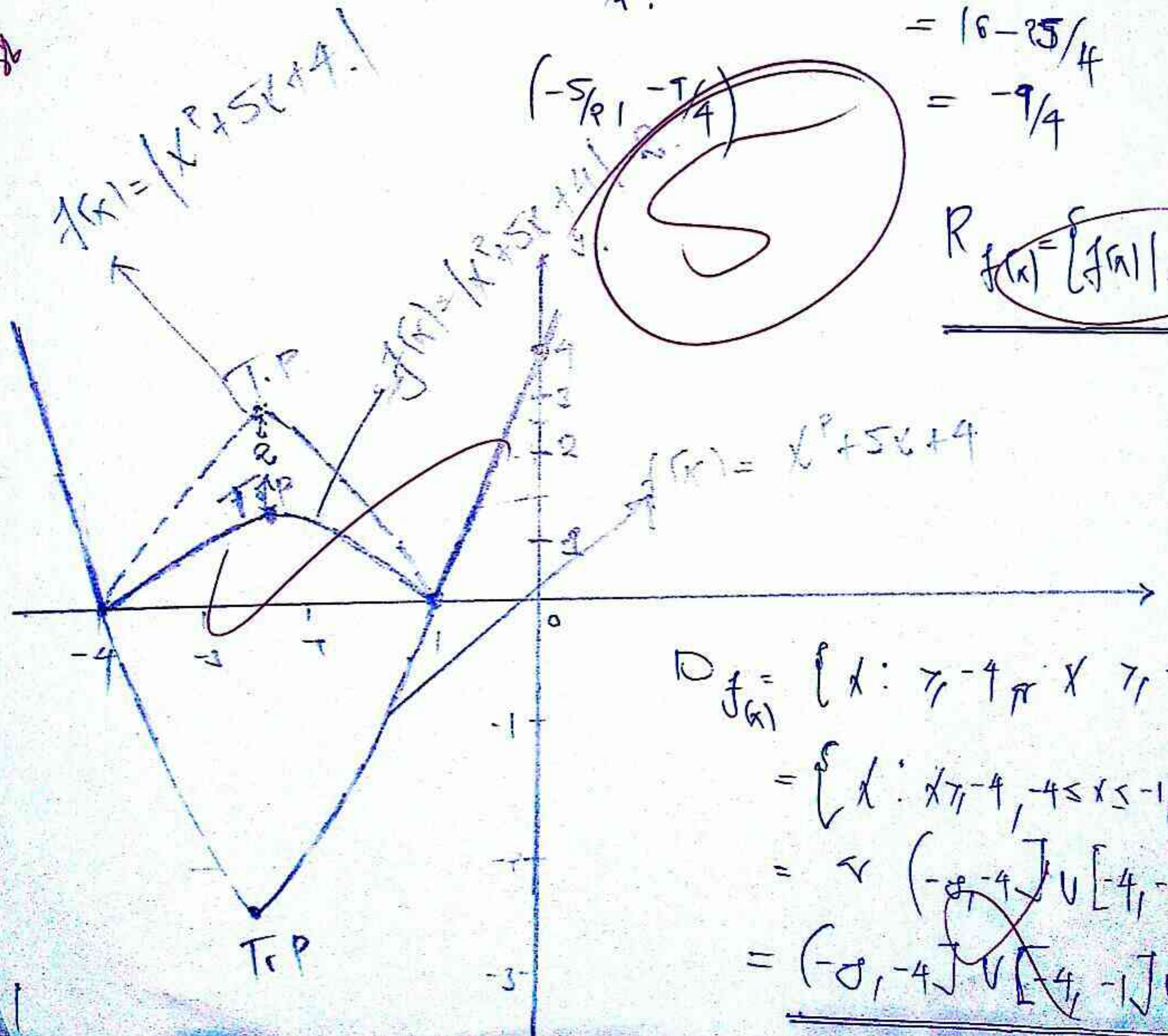
x-intercept, $y = 0$
$x^2 + 5x + 4 = 0$
$x^2 + x + 4x + 4 = 0$
$x(x+1) + 4(x+1) = 0$
$(x+1)(x+4) = 0$
$x + 1 = 0$ v $x + 4 = 0$
$x = -1$, $x = -4$.

$a > 0$, U

Turning point. (T.P) : $x = -b/2a$
$= -5/2(1)$
$= -5/2$.

$y = 4ac - b^2/4a$
$= 4(1)(4) - (5)^2/4(1)$
$= 16 - 25/4$
$= -9/4$

$(-5/2, -9/4)$



S

$R_{f(x)} = \{f(x) \mid f(x) \geq 0$

$D_{f(x)} = \{x : \geq -4$ or $x \geq -1\}$
$= \{x : x \geq -4, -4 \leq x \leq -1, x \geq$
$= $ v $(-\infty, -4] \cup [-4, -1] \cup [1$
$= (-\infty, -4] \cup [-4, -1] \cup [-1,$