



STUDENT NAME:

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(please circle)

MATHEMATICAL METHODS UNIT 3

SAC 1 (Tech-Active)

WILLY WONKERMUND AND THE CHOCOLATE FACTORY

PART 1

Reading Time: 15 minutes

Writing time: 120 minutes

QUESTION AND ANSWER BOOK

<i>Number of Questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
5	5	59
		Total 59

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one school supplied laptop with Mathematica and any number of Mathematica files on a USB.
- Students are NOT permitted to bring into the examination: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 19 pages.

Instructions

- Write your name in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

This page contains only white chocolate.



Charlethan Bucket is a poor, but hopeful, brave, wise and honest boy who lives with his mother and four grandparents. They don't have much money but support each other as best they can. On his way home from school one day, his newspaper route takes him past the gate of the town's mysterious, but highly regarded candy-maker, Willy Wonkermond's Chocolate Factory. As Charlethan peers through the gates, a tinker appears behind him and tells him that nobody ever goes into Wonkermond's Factory, and nobody ever comes out of it either.

Later that night, Charlethan's Grandpa Joe Hill tells him that the tinker was right. Many years ago, other candy makers, especially Phil Schmidtworth, were sending in spies dressed as workers to steal Wonkermond's secret recipes. Finally, Wonkermond laid off all of his workers, closed the factory and vanished. Three years later, the factory started up again, and more candies and chocolates were coming out, but the gates stayed locked, so that no one would steal their recipes. The identities of Wonkermond's new workers remains a mystery.

The next day at school, word, news and announcements spread quickly that a contest has been launched, hidden among five out of billions of Wonkermond's chocolate bars are five golden tickets. The big prize is that those who find these tickets, will receive a lifetime supply of chocolate and get to visit Mr. Wonkermond's long-isolated factory for a free tour. The world then goes crazy, with Wonkermond chocolates being sold out at almost every store.

The first ticket is found by a greedy boy, named Dougustus Gloopericic. The second is found by a spoiled little girl named Verainy Salt, whose father utilized his peanut factory workers to open Wonkermond bars from dawn until dusk. The third is found by a girl named Kristina Beauregarde, who is an avid gum chewer. The fourth is found by a boy named McLean TeeVee, who is obsessed with guns and television, particularly westerns.

While these children claim the tickets in turn, and while various adults go to absurd lengths to find the tickets on their own, Charlethan spends his birthday money from Grandpa Joe Hill to buy two Wonkermond bars, one at a time. Neither bar contains a ticket. But the bond between Charlethan and his Grandfather is all the stronger for trying.

Finally, a Paraguayan television network announces that a millionaire, living in the Andes, has claimed the fifth ticket. With (no more tickets to hunt for, the world gets back to normal. Charlethan is heartbroken at the chance he missed, but is back at school and then work the next day, doing his best to put it all behind him...



Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

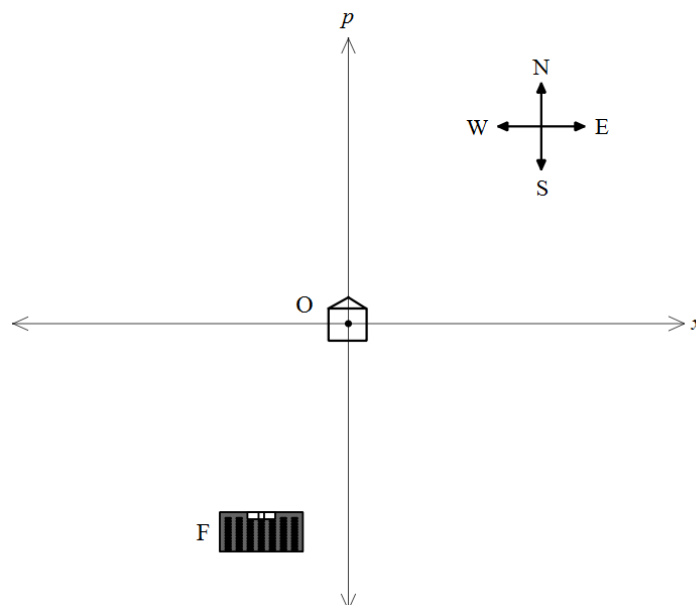
In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (24 marks)

Charlethan is completing his daily rounds, delivering newspapers to nearby homes. As the work is rather dull and Charlethan is a very smart boy, he uses the time to mathematically model the path of his paper route.

Charlethan's home is located at the origin, $O(0,0)$. He walks from his home to the factory and then models his path with the function $p: [-3, \infty) \rightarrow \mathbb{R}$, $p(x) = x^3 + 2x^2 + 1$, where $p(x)$ is the distance north of Charlethan's house and x is the distance east. All distances are measured in km.



The factory where Charlethan collects his papers is located 3 km west of his house at point F .

- a.** State the coordinates of F . 1 mark

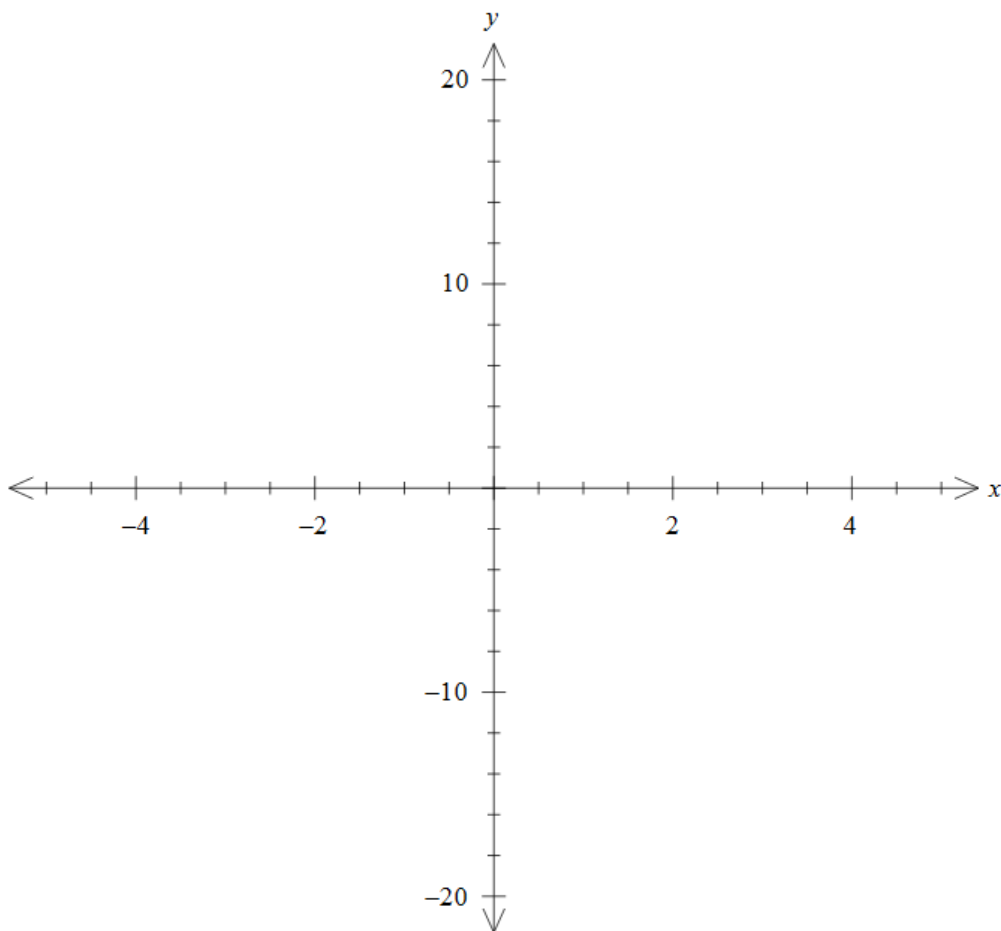
- b.** Find the straight-line distance, in km, between Charlethan's house and the factory. 1 mark

- c. Charlethan knows that there are two points on his route where he is travelling directly east. Find the coordinates of these two points.

2 marks

- d. Sketch the graph of $y = p(x)$ on the set of axes below. Label any axes intercepts, endpoints, and turning points with their coordinates, correct to two decimal places for all non-integer coordinates.

3 marks



Charlethan knows there are two straight roads that intersect the path of his paper route.

- e.** One road, modelled by the equation $q(x)$, is normal to $p(x)$ at $x = -1$.

Show that the rule for this road is given by $q(x) = x + 3$.

2 marks

Charlethan can't quite recall the equation that models the other road, $r(x)$, but he knows that it is tangent to $p(x)$ at $x = a$.

- f.** Show that the rule for $r(x)$, in terms of a , is $r(x) = (3a^2 + 4a)x - 2a^3 - 2a^2 + 1$.

3 marks

- g.** Find the value of a , correct to three decimal places, such that the two roads, $q(x)$ and $r(x)$, intersect as close to Charlethan's house as possible.

3 marks

Having remembered everything about his paper route, Charlethan decides to consider a range of other possibilities.

He considers a generalised function, $s(x) = x^3 + (k+1)x^2 - (k^2-1)x + 1$, where $k \in \mathbb{R}$.

h. Let $k = \frac{1}{2}$.

i. Show that $s(x)$ has exactly one stationary point.

2 marks

ii. Hence find the rule for the inverse function, $s^{-1}(x)$, of $s(x)$.

2 marks

iii. Hence express $s(x)$ in the form $(x-m)^3 + n$, where $m, n \in \mathbb{Q}$.

2 marks

- i. Find the value(s) of k such that $\text{ran}(s^{-1}) = R$.

3 marks

On his way home from his paper rounds, Charlethan discovers a one pound coin in a gutter and buys one more Wonkermond bar, happy that some delicious chocolate will soothe his disappointment of not being able to visit Wonkermond's factory. After leaving the candy store, he overhears two strangers gossiping about the millionaire from Paraguay forging the final ticket. His heart swelling with hope, Charlethan peels back the wrapper and sees gold peeking out at him.



After the resulting fanfare and celebration, a sinister looking man corners Charlethan and introduces himself as Mr. Schmidtworth. "Congratulations on your find boy, you look like you could use some good luck." Schmidtworth begins. "If you want to look after your family properly, perhaps you would be willing to quietly acquire the code to Wonkermond's new product, the everlasting g o b stopper. You'll be set for life if you do, no more paper rounds!" he continued. Charlethan isn't sure what to say so he runs off home.

As he races home to tell his family, he thinks on the offer. Should he take it and lift his family from poverty? He is unsure but puts it out of his mind for now and tells his family the good news. They all break into a surprisingly well choreographed musical number to celebrate. Charlethan shows the ticket to his elderly Grandpa Joe Hill who reads the conditions on the ticket. The tour is the very next day! Charlethan decides to take his grandpa who jumps out of bed for the first time in twenty years and they quickly prepare for the adventure of a lifetime.

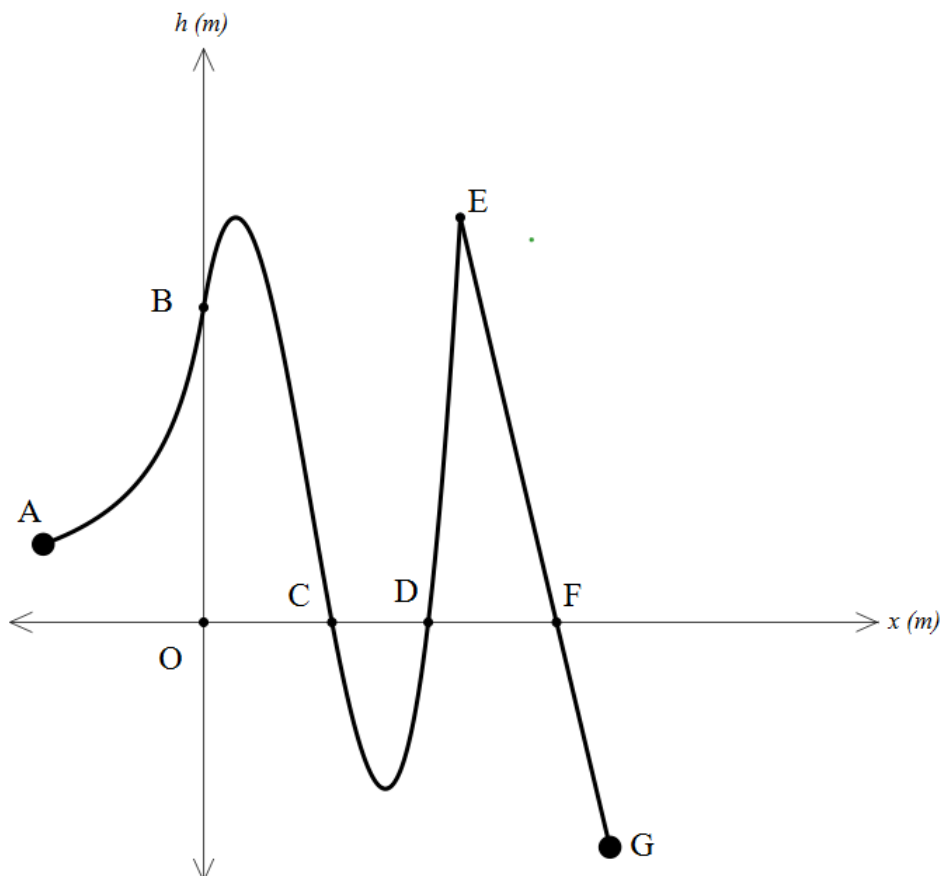
The next day they arrive at the factory and for the first time in years, Wonkermond shows his face and invites them inside. They are asked to sign a contract saying they will follow all instructions and will not take any unauthorised samples.

The very first room they enter is filled with wondrous mathematical creations as well as a river of chocolate winding through the area. Wonkermond tells them they can eat anything in the room, but they mustn't touch the chocolate in the river as they may contaminate it. The children and their chaperones rush into the room to explore...

Question 2 (10 marks)

The chocolate river is fed and maintained by a series of pipes. These pipes carry new chocolate into the river from the factory outside and run parallel to the riverbank.

Charlethan is standing on the riverbank looking at the pipes. From his position, the side-on view of the pipes is shown in the diagram below.



Let h be the vertical position of the pipes above or below the surface of the chocolate river and x be the horizontal distance from Charlethan's current position. All distances are measured in metres.

The pipes are modelled by the function:

$$h(x) = \begin{cases} -3x + c & , \quad 8 < x \leq \frac{38}{3} \\ \frac{1}{4}x^3 - \frac{5}{2}x^2 + \frac{17}{4}x + 7, & 0 \leq x \leq 8 \\ \frac{a}{b-17x} & , \quad -5 \leq x < 0 \end{cases} \text{ , where } a, b, c \in R.$$

- a.** The function h is differentiable at point B where $x = 0$. Show that the values of a and b can be found by solving the simultaneous equations $\frac{a}{b} = 7$ and $\frac{a}{b^2} = \frac{1}{4}$. 2 marks

- b.** State the values of a and b . 1 mark

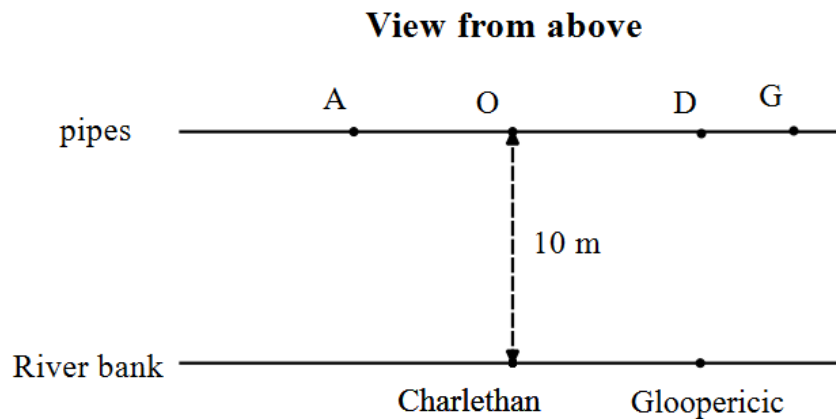
- c.** The function h is continuous at point E, where $x = 8$. Show that $c = 33$. 2 marks

- d.** Find the derivative of $h(x)$. 2 marks

Dougustus Gloopericic ignores Wonkermond's warnings about contaminating the river. He rushes over to the river, opposite to the point D, and bends down to drink from the river. Gloopericic falls in and is pulled towards the end of the pipe at point G.

The pipes run vertically up and down as before but do so from left to right along the horizontal line AG as shown below.

The riverbank that Charlethan and Gloopericic stand on is parallel to the pipes and is 10 metres away, level with $h = 0$.



- e. Find the straight-line distance between the point where Gloopericic falls in and the end of the pipe at point G.

2 marks



The others watch on in horror as Gloopericic is sucked into the pipes. He is briefly stuck in a bend before the pressure builds enough to shoot him through and out of sight. Wonkermond assures them that he will be “fine” and calls for his strange looking assistants to take Mrs Gloopericic to the chocolate vats to collect her son. Soon three short, orange skinned, and green haired creatures rush into the room and begin to sing a song warning the remaining visitors of being too greedy:

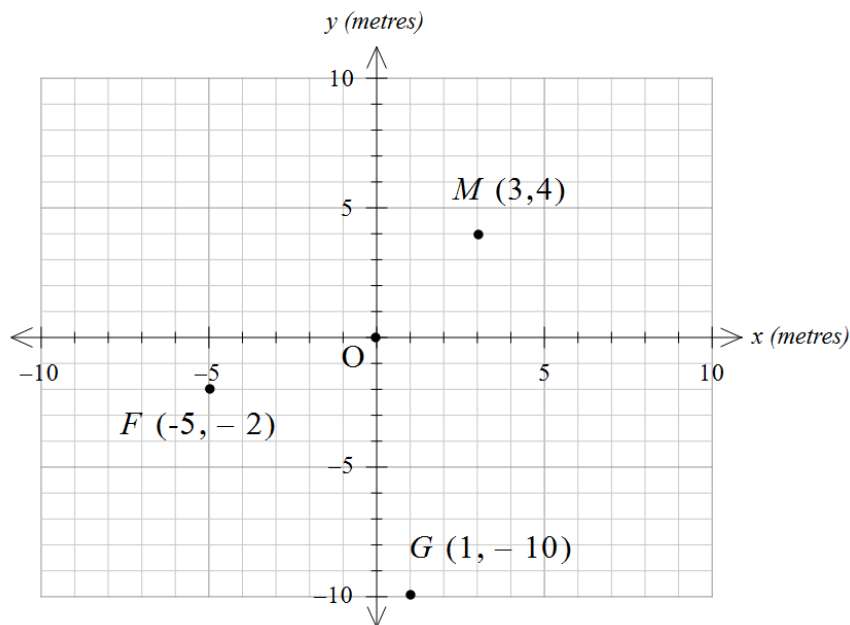
*Oompa Loompa doom-pa-dee-do
I have a perfect puzzle for you
Oompa Loompa doom-pa-dee-dee
If you are wise, you'll listen to me*

*What do you get when you guzzle down sweets?
Ignoring warnings is no good feat.
What do you want with the forbidden treats?
What fate do you think you will meet?*

*Oompa Loompa doom-pa-dee-da
If you're not greedy, you will go far
You will live in happiness too
Like the Oompa Loompa doom-pa-dee-do*

Question 3 (13 marks)

There are three Oompa Loompas. Maydiggie Mugglewump, Gavrilesu Grumbumble, and Finchy Flobblesnitch. They are initially located at points $M(3,4)$, $G(1,-10)$, and $F(-5,-2)$ respectively, surrounding Mrs Glooperic who is located at $O(0,0)$. The diagram below shows the initial positions of the three Oompa Loompas and Mrs Glooperic. The x axis and y axis represent the east-west and north-south directions respectively. All units are measured in metres.



- a. Find the distances, in metres, between the pair of points M and F and the pair of points G and F .

2 marks

- b. Show that the line segments \overline{FM} and \overline{FG} lie on the lines $y = \frac{3}{4}x + \frac{7}{4}$ and $y = -\frac{4}{3}x - \frac{26}{3}$ respectively.

2 marks

- c. Hence state the angle between the line segment \overline{FM} and the line segment \overline{FG} . 1 mark

- d. State the angles $\angle FMG$ and $\angle FGM$. 2 marks

- e. Gavrilescu Grumbumble moves to G_1 so that $\triangle MFG_1$ becomes an equilateral triangle. Find the possible coordinates of G_1 . 3 marks

- f. As they finish their song, Maydiggle dances over to a new point M' . The position of this point can be found by applying, in order, the following transformations to point M :

1. Reflection in the vertical axis.
2. Translation by 2 units in the negative direction of the horizontal axis.
3. Dilation by a factor of $\frac{1}{3}$ from the horizontal axis.

Find the coordinates of M' . 3 marks



The three Oompa Loompas finish their performance and take Mrs Glooperic off to collect her son and leave the factory. The remaining visitors are loaded onto a boat and taken on a somewhat unsettling ride, complete with an ominous serenade by Wonkermond.

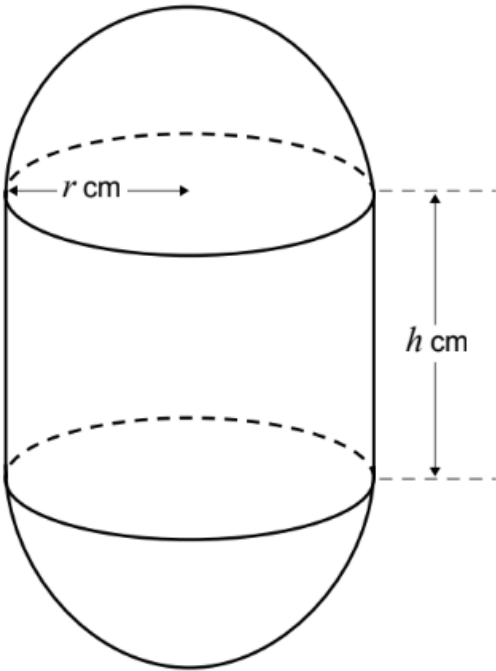
They breathe a sigh of relief as they exit the boat and are taken to see an unfinished invention. Experimental gum that has the taste of a three-course dinner, complete with blueberry pie at the end. Wonkermond describes his current thoughts on the shape of the gum but warns them not to eat any...

The following formulae may be useful for question 4

Volume of a sphere	$V_{\text{sphere}} = \frac{4}{3} \pi r^3$	Volume of a cylinder	$V_{\text{cylinder}} = \pi r^2 h$
Surface area of a sphere	$A_{\text{sphere}} = 4 \pi r^2$		

Question 4 (9 marks)

The prototype chewing gum is to be created in the shape of a right circular cylinder with hemispherical caps as shown below. Let h cm be the height of the cylindrical part of the gum, r cm be the radius of the cylinder and caps, and V cm³ be the volume of the gum.



- a. Show that $V = \frac{\pi r^2}{3}(4r + 3h)$.1 mark

The relationship between h and r is given by $h = \frac{5}{2\pi r^2} - \frac{4r}{3}$.

- b.** Find the volume of the chewing gum. Express your answer in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$. 1 mark

- c.** If the maximum height of the gum is 12cm, find the possible values of r , correct to two decimal places. 2 marks

- d.** The cost of producing one piece of chewing gum depends on the surface area of the gum. If the cylindrical part costs 10 cents per cm^2 and the hemispherical caps cost 15 cents per cm^2 .

Show that the total cost, c dollars, is given by $c = \frac{1}{2r} + \frac{\pi r^2}{3}$. 2 marks

- e.** Hence find the minimum cost of producing a piece of gum and the dimensions of the gum for this minimum cost. Give your answers correct to two decimal places. 3 marks

Kristina Beauregarde can't resist and snatches a piece of the gum. "I have the world record for chewing a piece of gum for three months straight, I can handle this!" she exclaims. "Ohh, I taste tomato soup, oh now its roast chicken and potatoes, yum! And here is the blueberry pie..."

As she finishes the last sentence, she begins to turn blue.

"I warned you, it's not ready yet." remarks Wonkermond, "If you are quick maybe you can stop the transformation."

In order to avoid being transformed into a blueberry, Kristina Beauregarde must transform the bottom cap of the gum into the top cap, halting the effects of the gum. Kristina tries to simplify the transformation by modelling the two-dimensional cross section of the hemispherical caps.

Question 5 (4 marks)

Kristina attempts to transform the function $f(x) = -5 - \sqrt{4x - x^2}$ into the function $g(x) = \sqrt{1 - x^2}$. State a sequence of transformations that would map the $f(x)$ into $g(x)$.



Unfortunately, she is just a little too slow. She swells up into a sphere and turns completely blue. "Not to worry" says Wonkermond, "take her to the juicing room and we should be able to put her right!" 'Should?' thinks Charlethan, but there is no time to think on it further as the Oompa Loompas roll Kristina out of the room, singing as they go.

The others continue on their way towards part 2.

END OF QUESTION AND ANSWER BOOK